


# Spectral analysis of nonstationary signals

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# Outline

1 Introduction: nonstationarity

2 Locally deformed signals: an analysis-based approach

3 Reduction of boundary effects

4 Conclusion

# Stationarity

## Definition (Stationarity)

A random process $X$ is said to be second-order stationary if:

- $\mathbb{E}\{X(t)\} = m_X, \ \forall t \ ,$
- $\mathbb{E}\{X(t)X(\tau)\} = k_X(t-\tau), \ \forall (t,\tau) \ .$

**Spectrum:**

- Gives the distribution over frequencies of the power of $X$.
- Many methods to estimate the spectrum from a single realization of the stationary process $X$ (e.g. Welch method).

# Nonstationarity

The stationarity assumption is often irrelevant to study real-life signals, such as audio signals, or physiological signals.

$\Rightarrow$ **Questions:**

1. Which classes of nonstationarity should we consider?
2. How should we analyze nonstationarity? In particular, how to extend spectral estimation to nonstationary signals?

# Broken stationarity: a class of nonstationarity

Two key ingredients:

1. A zero-mean stationary random process $X$.
2. A deformation operator that breaks stationarity $\mathcal{T}$.

We observe the "deformed" process $Y$ given by:

$$Y = \mathcal{T}X \ .$$

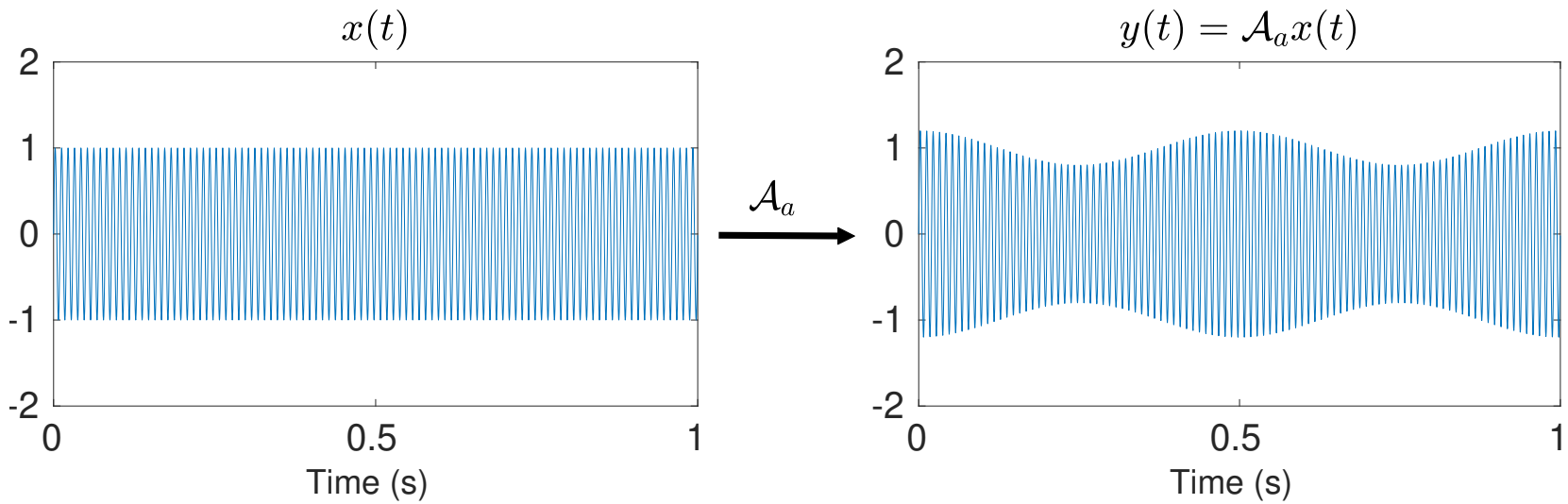
$\Rightarrow$ We limit ourselves to some physically relevant forms of operators.

# Deformation operators

- Amplitude modulation

$$\mathcal{A}_\alpha : \qquad \mathcal{A}_\alpha x(t) = \alpha(t)x(t) \ ,$$

with $\alpha \in C^1$ such that $\forall t,\ 0 < \alpha(t) < \infty$.




- Frequency modulation

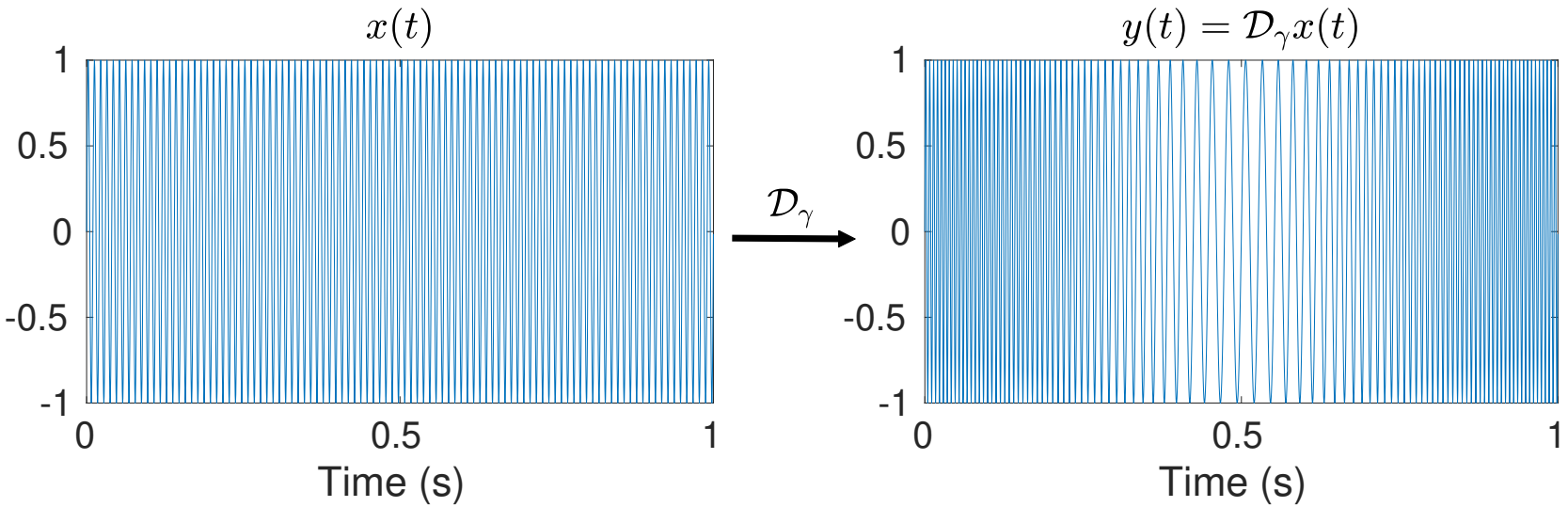
# Deformation operators

- Time warping

$$\mathcal{D}_\gamma : \qquad \mathcal{D}_\gamma x(t) = \sqrt{\gamma'(t)}\, x(\gamma(t)) \ ,$$

where $\gamma \in C^2$ is a strictly increasing function such that

$$0 < \gamma'(t) < \infty, \forall t \ .$$




- Any combination of the above deformations

# Goal: spectral estimation of nonstationary signals

From a single realization of the nonstationary random process $Y$, we aim at estimating simultaneously:

- The spectrum $\mathscr{S}_X$ of the underlying stationary random process $X$,
- The deformation operator $\mathcal{T}$.

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- Wavelet transform and approximation
- Estimation algorithm: JEFAS
- Applications to audio signals
- Multivariate locally deformed signals

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# Model and goal

- **Model**:

$$Y = \mathscr{A}_\alpha \mathscr{D}_\gamma X \ .$$

  where $X$ is a stationary process.

  - Relevant to model physical phenomena, such as Doppler effect, speed variations, or animal vocalizations.

- **Goal**: From a single realization of the process $Y$, estimate simultaneously:
  - the spectrum $\mathscr{S}_X$ of the underlying stationary process $X$,
  - the deformation functions $\alpha$ and $\gamma$.

# Wavelet transform

## Definition (Wavelet transform)

$$\mathcal{W}_x(s,\tau) = \langle x, \psi_{s\tau} \rangle \quad \text{avec} \quad \psi_{s\tau}(t) = 2^{-s/2}\psi\left(2^{-s}(t-\tau)\right) \ ,$$

where $\psi$ is the analysis wavelet.

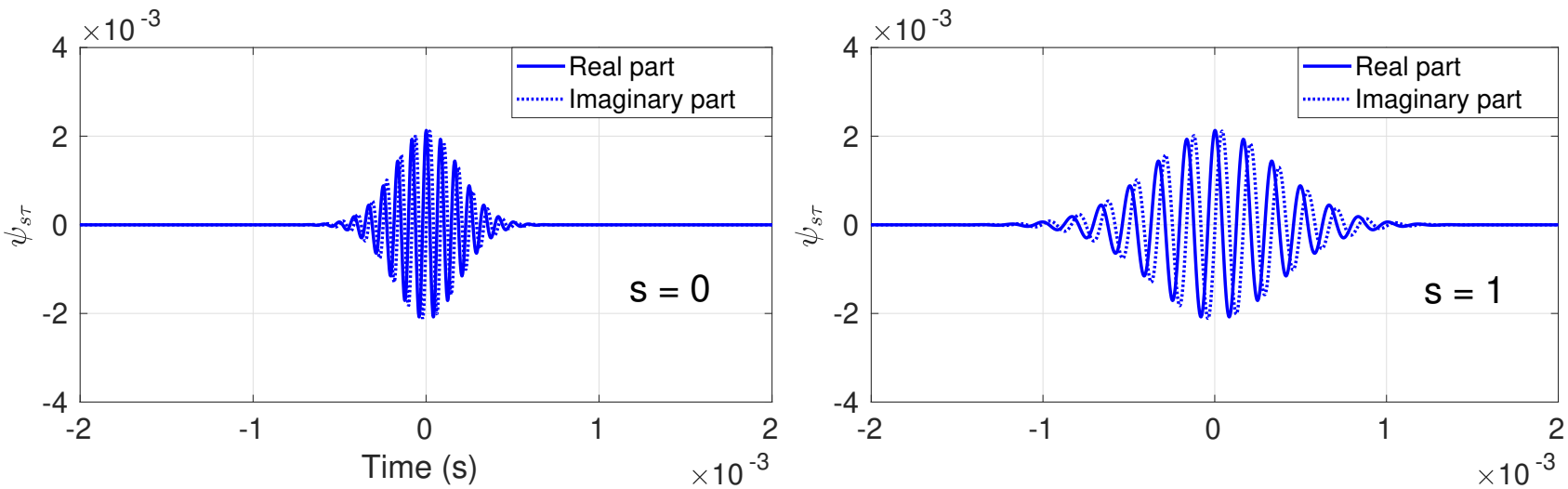
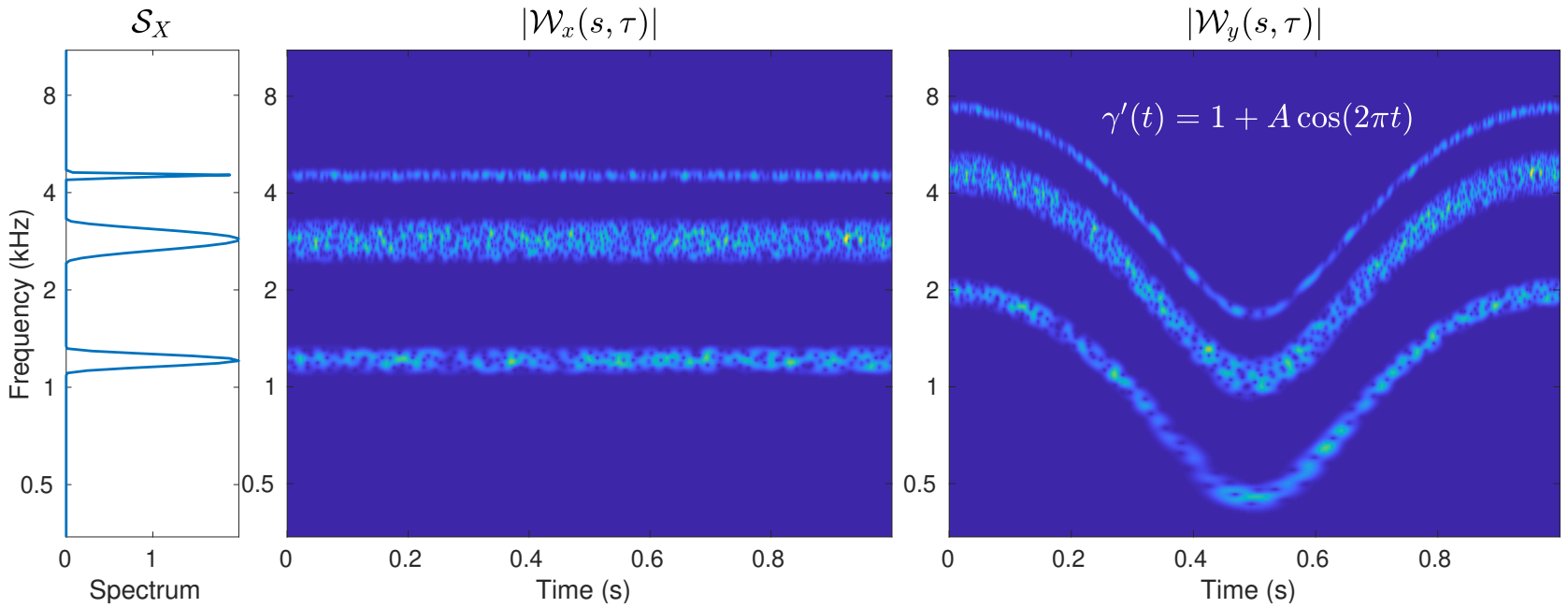



Figure: "Sharp wavelet" for two different values of $s$.

# Approximated behavior




## Approximation theorem

The wavelet transforms of $X$ and $Y$ are approximately related by:

$$\mathcal{W}_Y(s,\tau) \approx \widetilde{\mathcal{W}}_Y(s,\tau) = \alpha(\tau)\mathcal{W}_X\left(s + \log_2(\gamma'(\tau)), \gamma(\tau)\right) \ .$$

The error term $\epsilon = \mathcal{W}_Y - \widetilde{\mathcal{W}}_Y$ is a zero-mean random process, whose variance $\mathbb{E}\left\{|\epsilon(s,\tau)|^2\right\}$ depends on the regularity of $\alpha$ and $\gamma'$, and the speed of decay of $\psi$.

# Estimation algorithm: JEFAS

The JEFAS (*Joint Estimation of Frequency, Amplitude and Spectrum*) algorithm consists in an alternating estimation.

**Initializations:**

- Initialize $\tilde{\mathscr{S}}_X^{(0)}$.
- Initialize the amplitude modulation by $\tilde{\theta}_1^{(0)} = 1, \forall \tau$.
- $k \leftarrow 1$

**while** stopping criterion = FALSE **do**

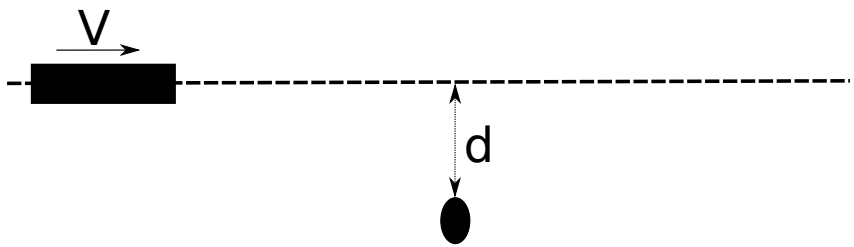
- Time warping: Estimate $\tilde{\theta}_2^{(k)}$ by ML, $\forall \tau$.
- Amplitude modulation: Estimate $\tilde{\theta}_1^{(k)}$ by ML, $\forall \tau$.
- Spectrum: Estimate $\tilde{\mathscr{S}}_X^{(k)}$ from the "rectified" wavelet transform.
- $k \leftarrow k + 1$

**end while**

# Doppler effect

**Assumptions:**

- A source emits a stationary sound.
- The source follows a uniform linear motion, at speed $V$.
- From a fixed station, we record the sound emitted by the source.

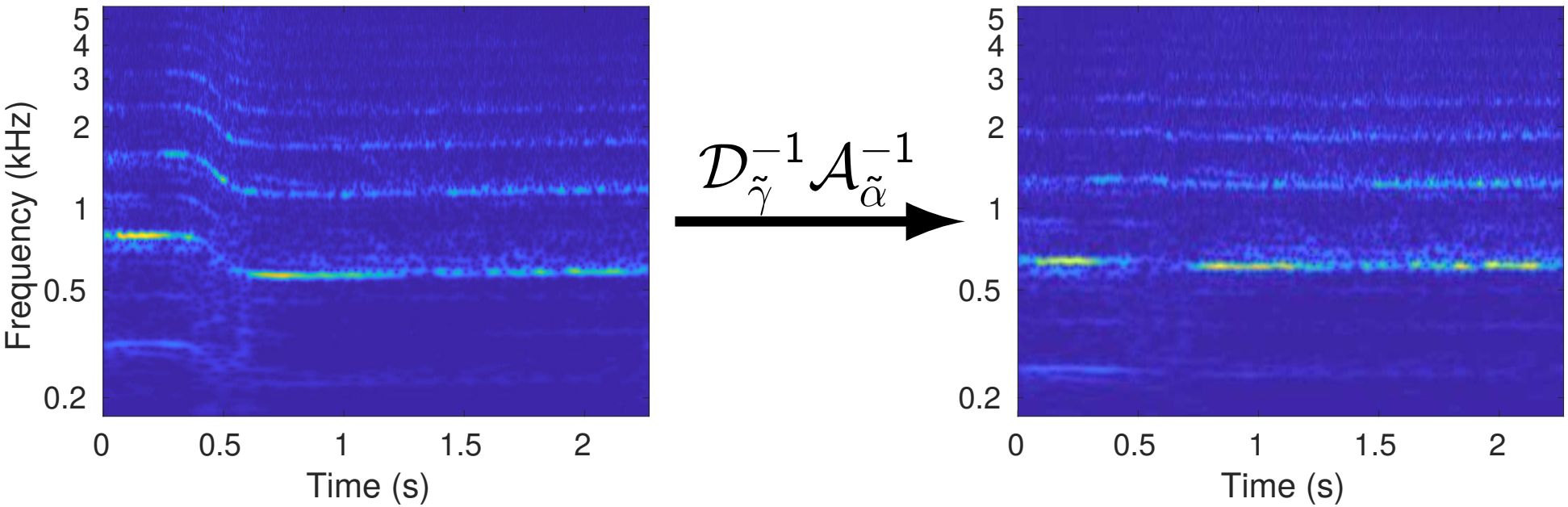


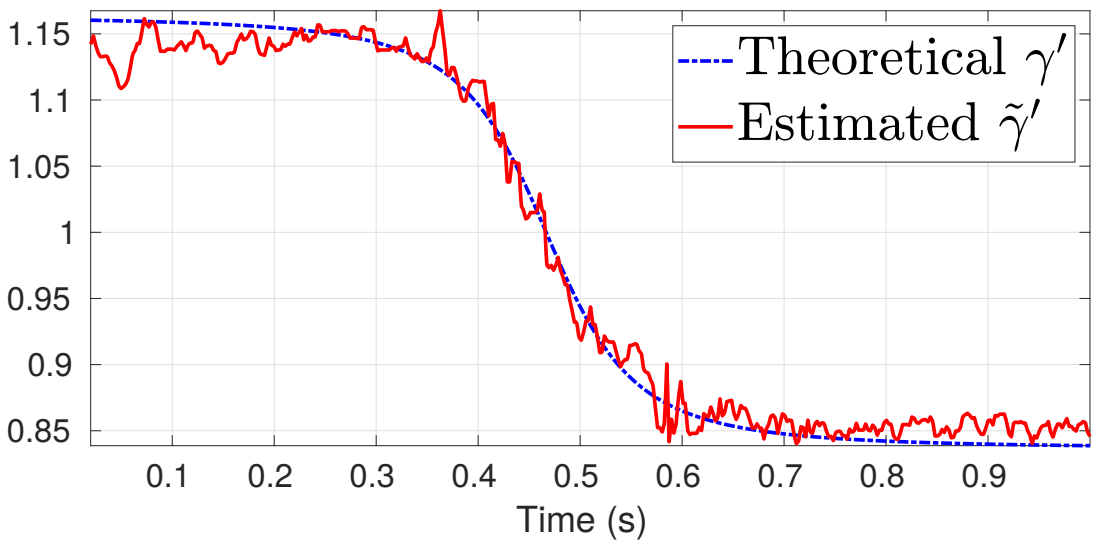

$\Rightarrow$ Due to the Doppler effect, the sound we receive is time-warped, with:

$$\gamma'(t) = \frac{c^2}{c^2 - V^2}\left(1 - \frac{V^2 t}{\sqrt{d^2(c^2 - V^2) + (cVt)^2}}\right).$$

An example:

# Doppler effect




Comparison to the theoretical function with: $d = 5$ m and $V = 54$ m/s.

# Spectral analysis of a broadband wind sound

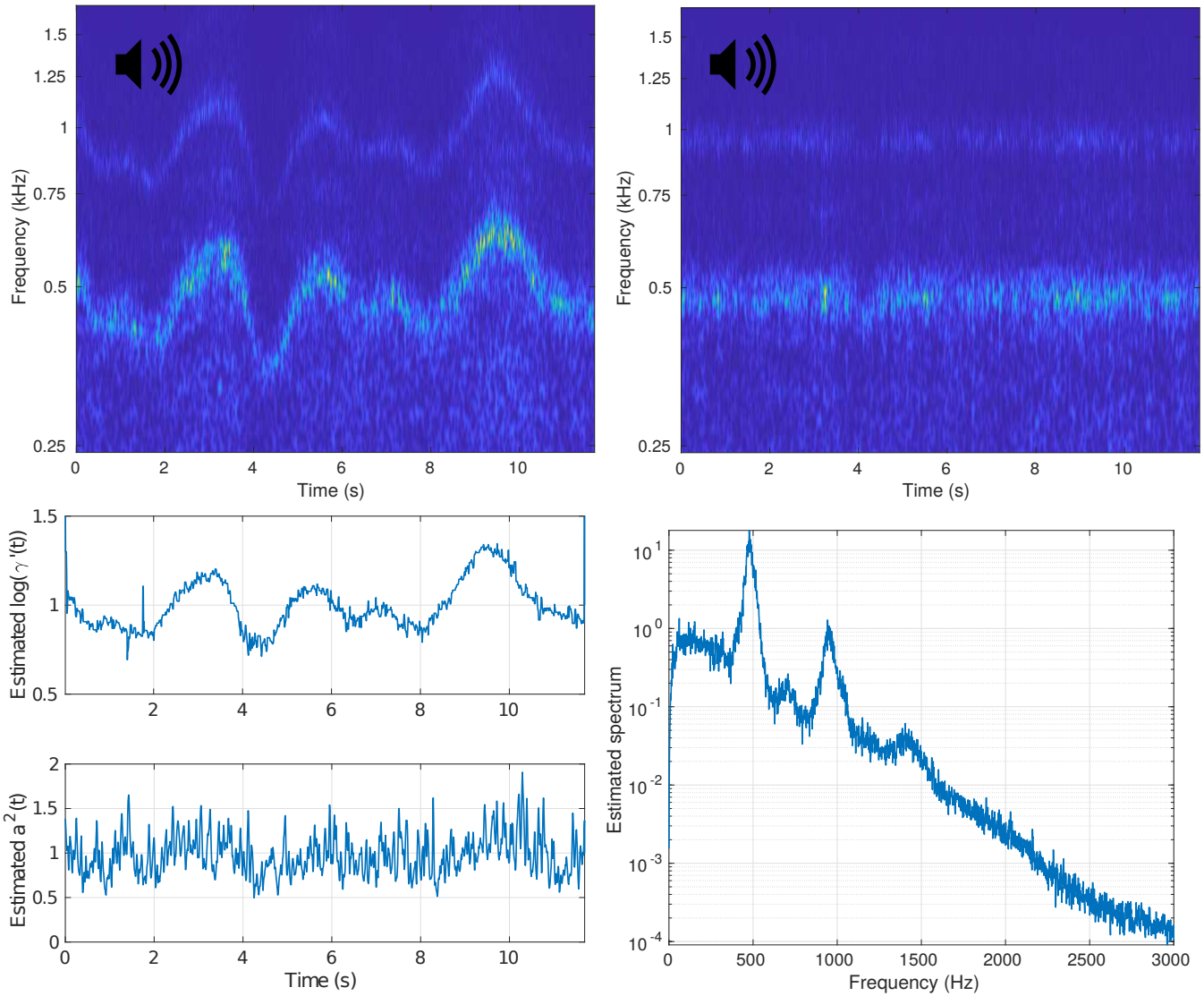



Figure: Top: Scalograms of the original signal (left) and the estimated stationary signal (right). Bottom left: estimated time warping and amplitude modulation. Bottom right: estimated spectrum.

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  - Model
  - Estimation algorithm: JEFAS-BSS
  - Illustrations on synthetic mixtures

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# Model and goal (determined case)

- **Model**: Multivariate extension of the previous model, i.e.:

$$\begin{pmatrix} Z_1 \\ \vdots \\ Z_N \end{pmatrix} = \begin{pmatrix} \mathcal{A}_{\alpha_{11}} & \cdots & \mathcal{A}_{\alpha_{1N}} \\ \vdots & \ddots & \vdots \\ \mathcal{A}_{\alpha_{N1}} & \cdots & \mathcal{A}_{\alpha_{NN}} \end{pmatrix} \begin{pmatrix} \mathcal{D}_{\gamma_1} & & \\ & \ddots & \\ & & \mathcal{D}_{\gamma_N} \end{pmatrix} \begin{pmatrix} X_1 \\ \vdots \\ X_N \end{pmatrix}$$

  where $X_j$ are uncorrelated stationary processes.

  - Relevant to model linear mixtures of different nonstationary sources.

- **Goal**: From a single realization of the multivariate process $(Z_1, \ldots, Z_N)$, estimate simultaneously:
  - the spectra $\mathscr{S}_{X_j}$ of the underlying stationary processes $X_j$,
  - the deformation functions $\alpha_{ij}$ and $\gamma_j$.

# Reformulation of the problem

**Spectral estimation problem**

$\Leftrightarrow$

**Doubly nonstationary Blind Source Separation (BSS)**

## Mixture model

At time $t$, we observe the multivariate signal $\mathbf{z}(t) \in \mathbb{R}^N$

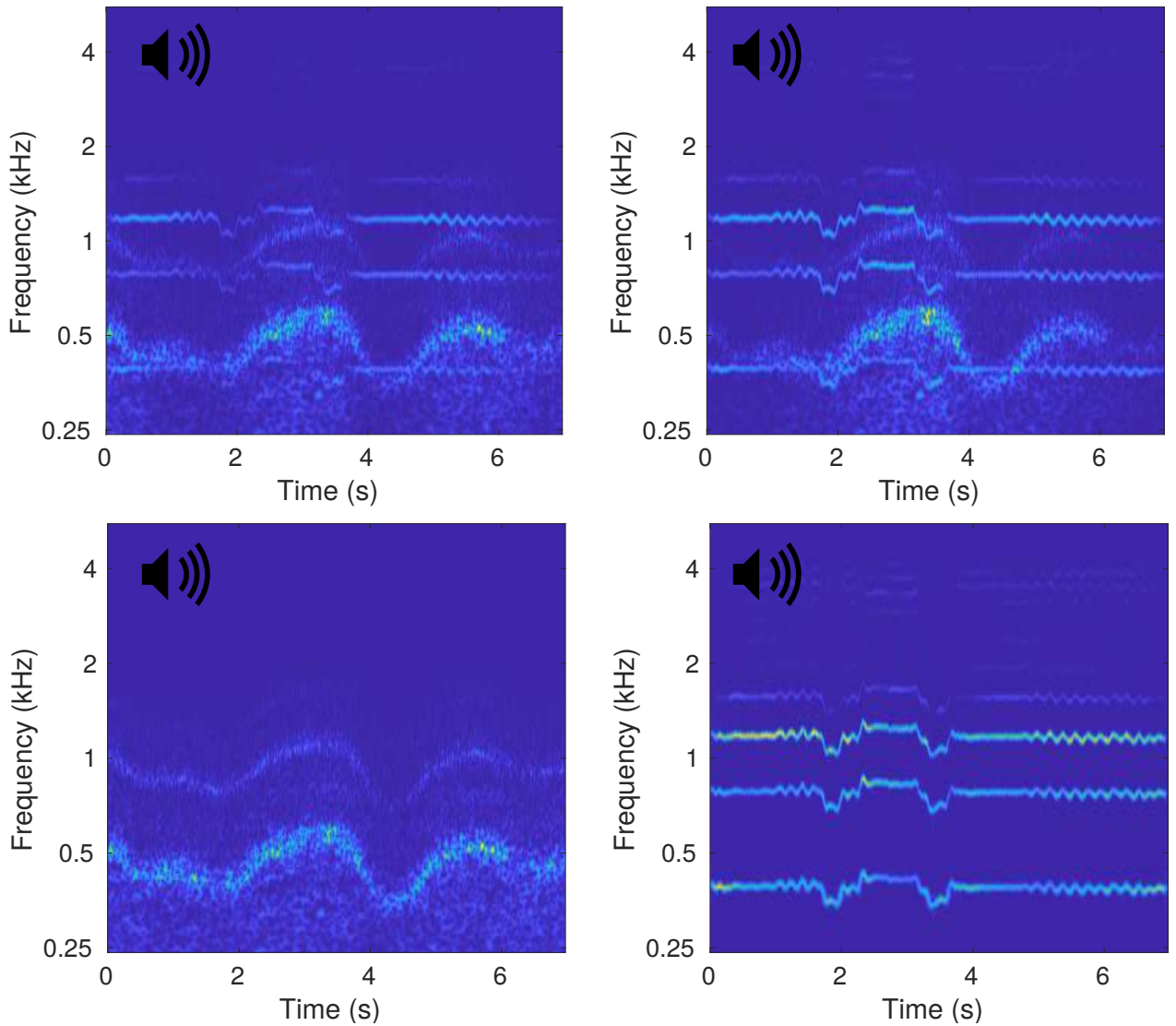
$$\mathbf{z}(t) = \mathbf{A}(t)\mathbf{y}(t) \ ,$$

where:

- $\mathbf{A}(t) \in \mathbb{R}^{N \times N}$ is the mixing matrix,
- $\mathbf{y}(t) \in \mathbb{R}^N$ is such that

$$y_i(t) = \mathcal{D}_{\gamma_i} x_i(t) \ .$$

# Illustration on a synthetic mixture of audio signals



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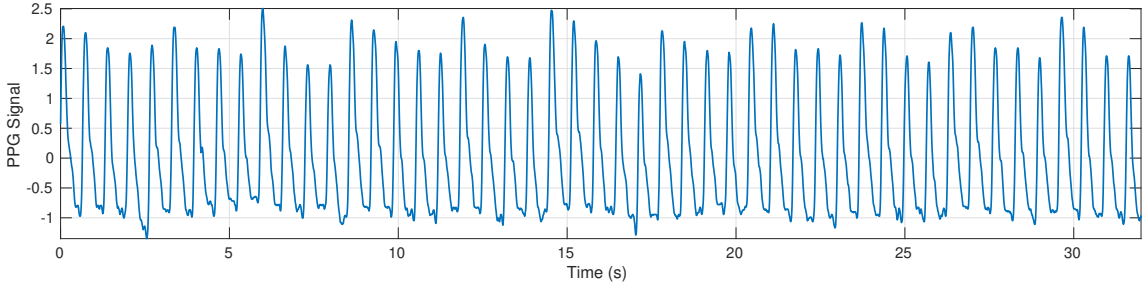
3 Reduction of boundary effects
- Motivations
- Boundary effects
- A boundary-effect reduction algorithm
- Numerical results

4 Conclusion

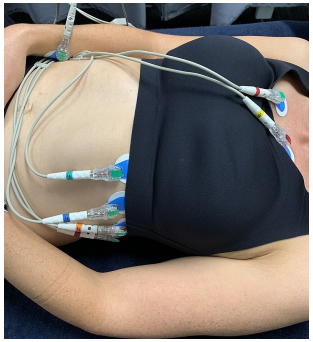
# Biomedical signals: Cardiac signals

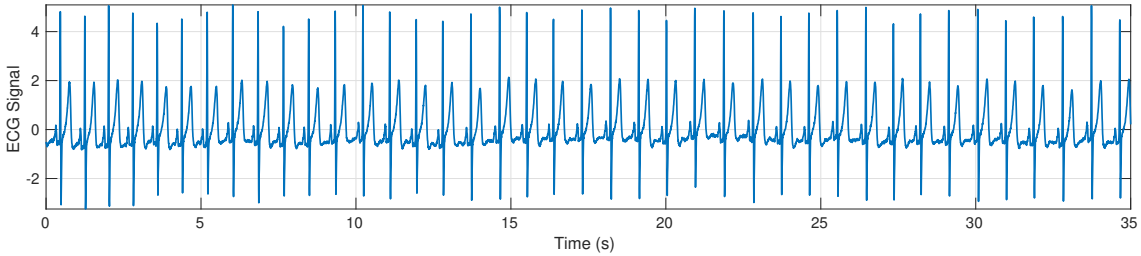
- Blood volume changes $\leftrightarrow$ Photoplethysmogram (PPG)



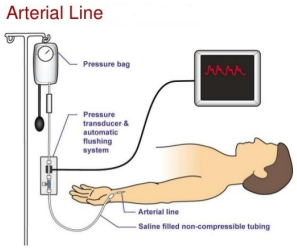


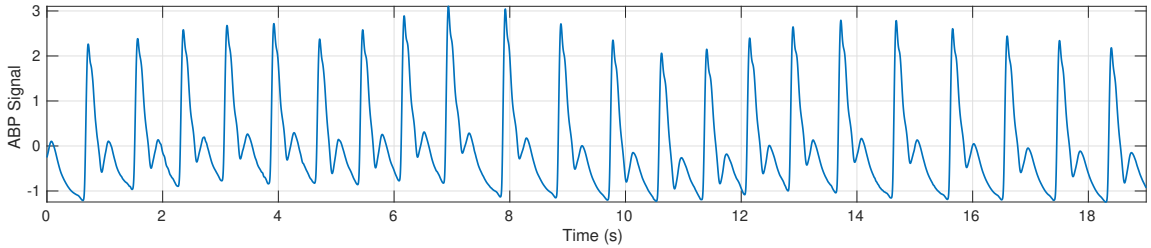

- Electrical activity of the heart $\leftrightarrow$ Electrocardiogram (ECG)





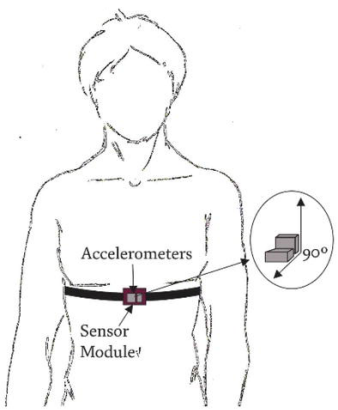

- Arterial blood pressure (ABP)

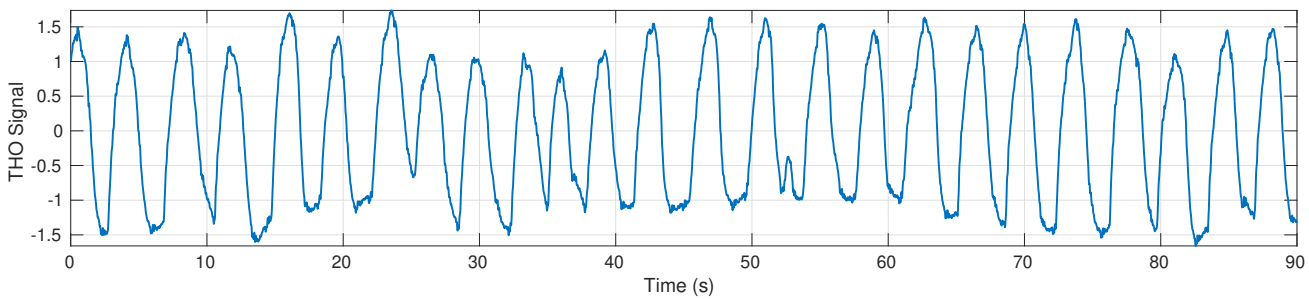


# Biomedical signals: Respiratory signals

- Thoracic movement recorded by a piezoelectric sensor (THO)



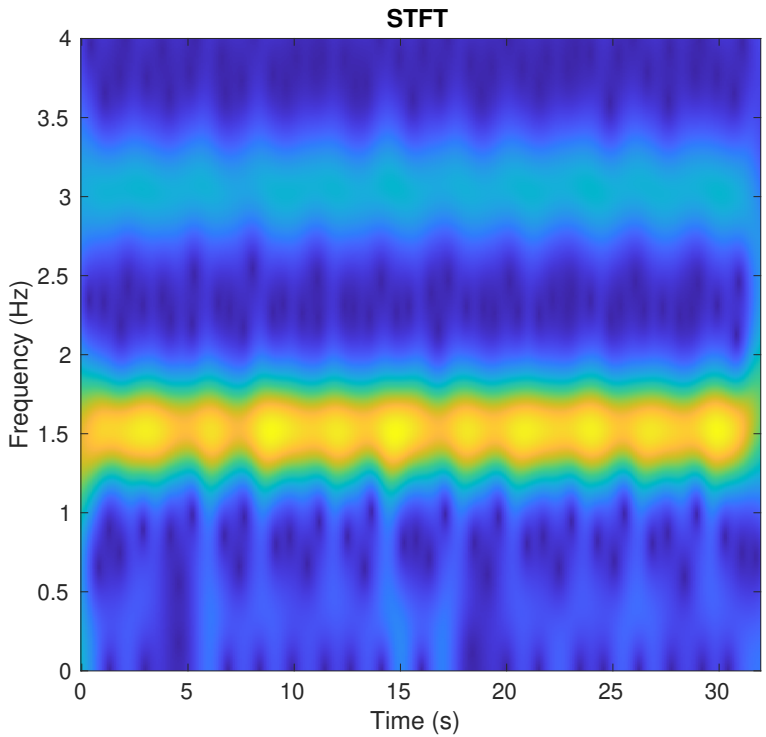



# Time-frequency analysis

**Goal:** Visualize the evolution of the instantaneous frequencies forming the signal.

⇒ Time-frequency representations.

- Short-Time Fourier Transform (STFT);
- Synchrosqueezing Transform (SST);
- Reassignment (RS);
- ...

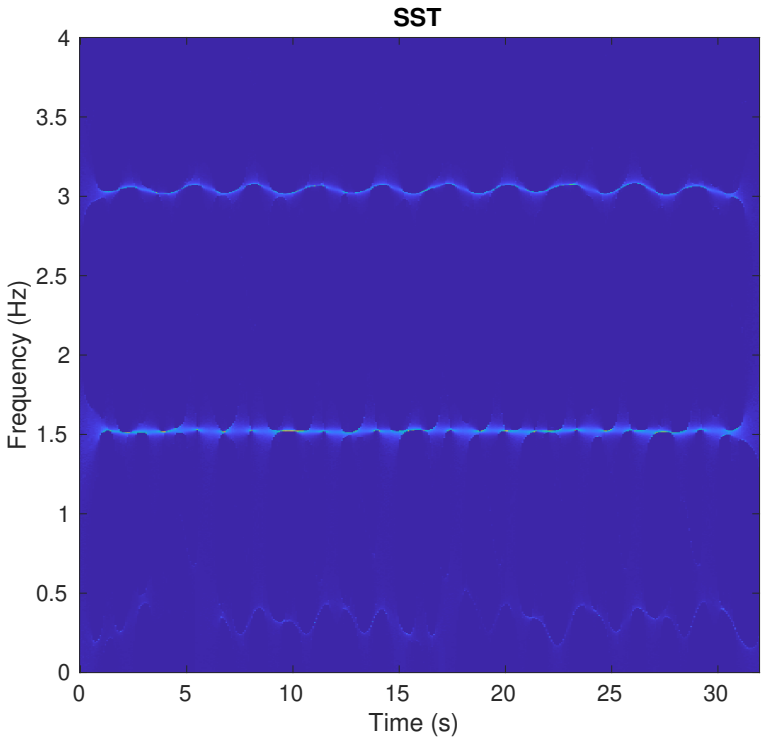


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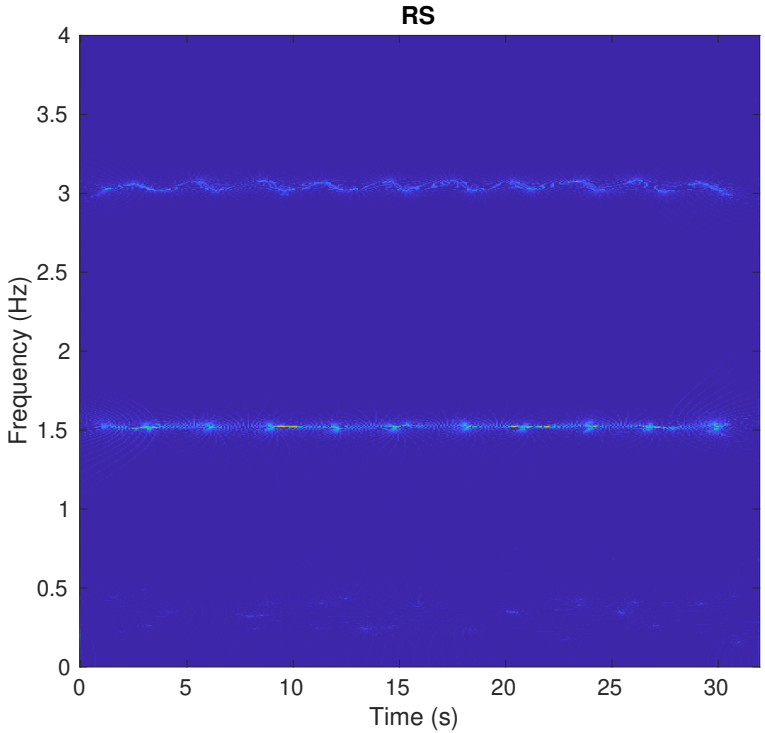


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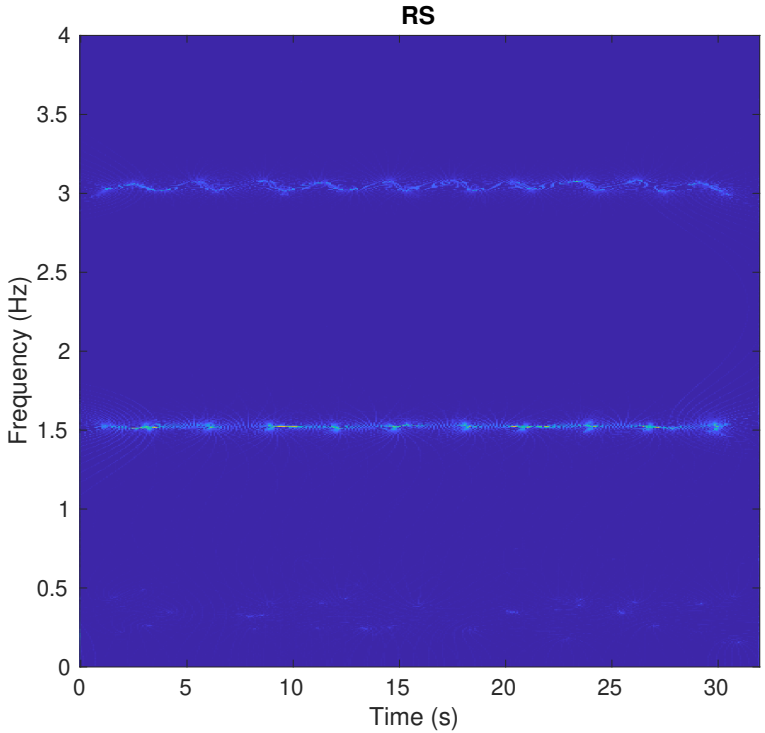


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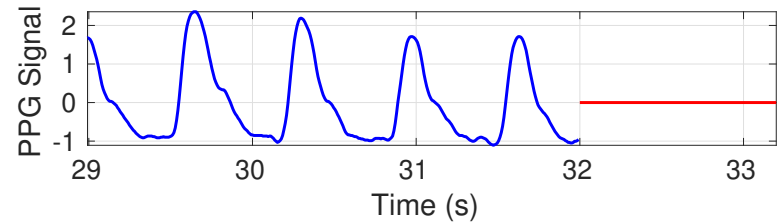
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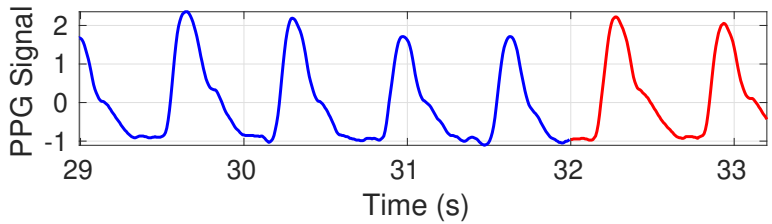


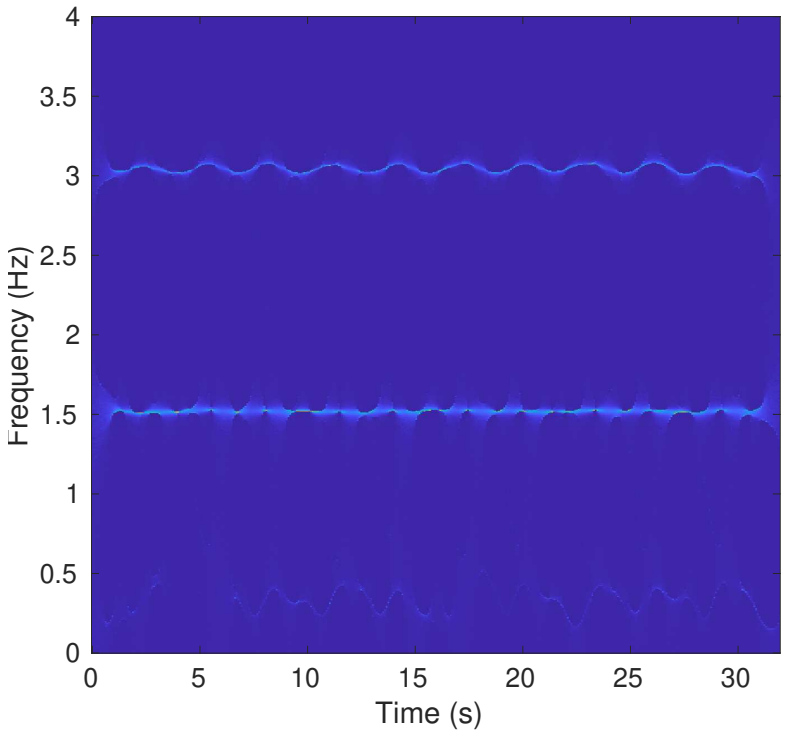
# Boundary effects

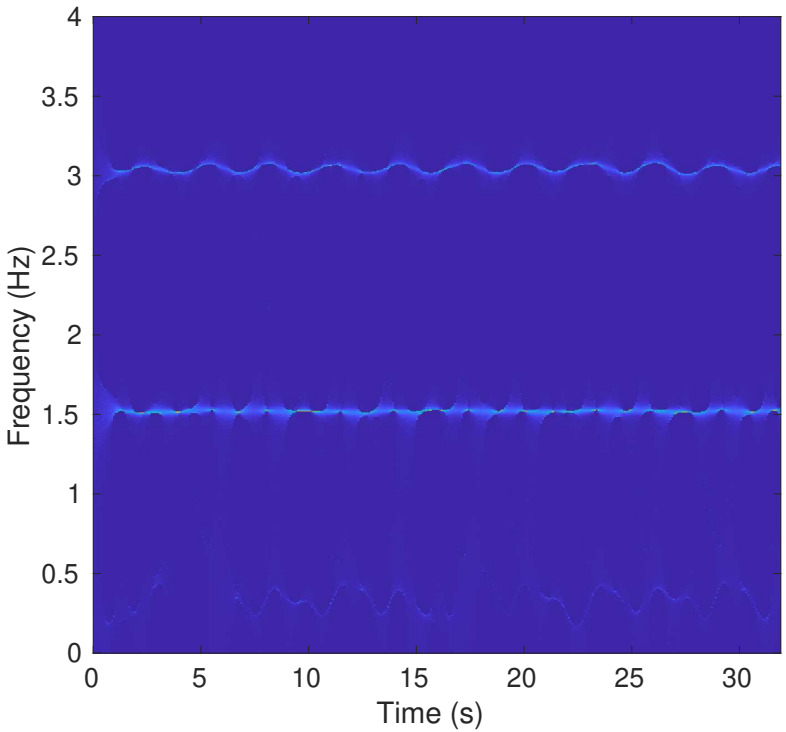
- Time-frequency analysis is based on windowing techniques.
- Time-frequency representations are thus sensitive to boundaries.
- The accuracy of these representation is deteriorated near boundaries.

$\Longrightarrow$ **Boundary effects**

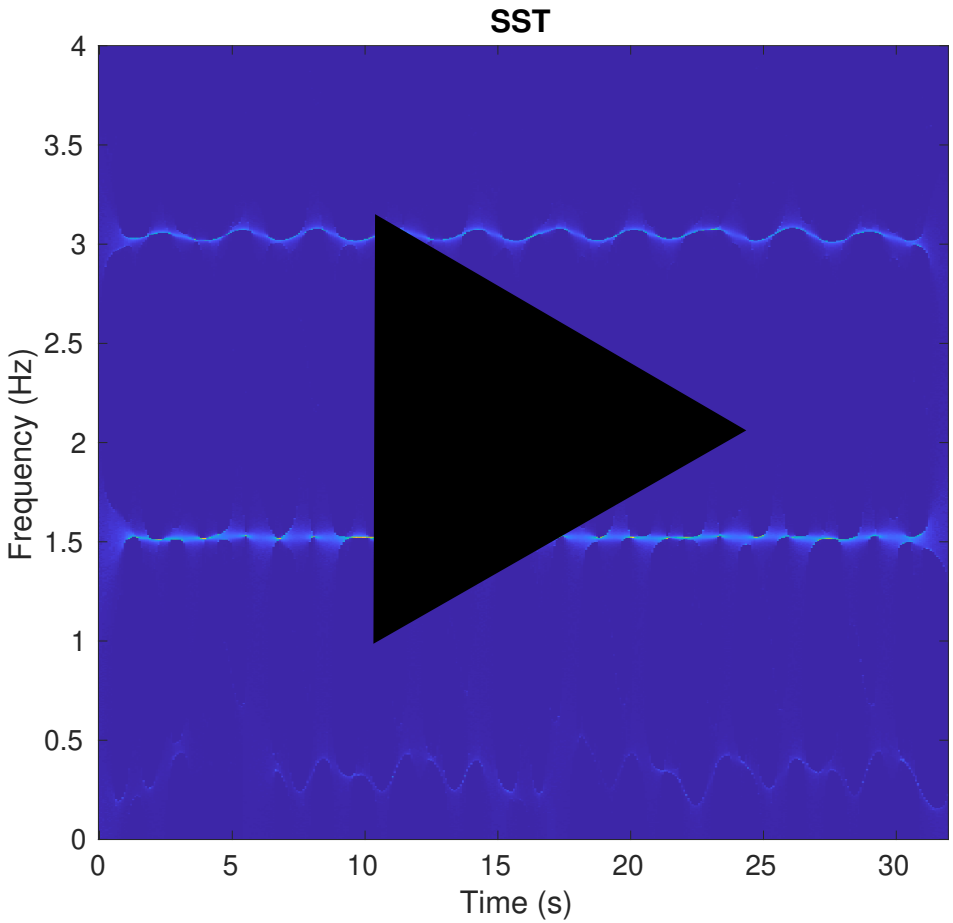


# Real-time time-frequency analysis




**Goal:** Being able to display real-time boundary-free time-frequency representations.

# Strategy

**Two-step strategy:**

1. Forecast a sufficiently long extension of the signal to push signal's boundaries;
2. Perform a time-frequency analysis on the extended signal, until the current time.

$\Rightarrow$ How can we forecast the biomedical signals that interest us?

# Extension step

**Idea:** Take advantage of the shape of the signals we consider to establish a **fast** and **accurate** forecasting method.

## Adaptive Harmonic Model

- $f_s$ is the sampling frequency;
- $\mathbf{x} \in \mathbb{R}^N$ is the observed signal;
- $\mathbf{w} \in \mathbb{R}^N$ is a Gaussian white noise;
- $\mathbf{z} \in \mathbb{R}^N$ is an oscillatory deterministic signal such that

$$\mathbf{z}[n] = \sum_{j=1}^{J} a_j\left(\frac{n}{f_s}\right) \cos\left(2\pi\phi_j\left(\frac{n}{f_s}\right)\right),$$

where $a_j\left(\frac{n}{f_s}\right)$ and $\phi_j'\left(\frac{n}{f_s}\right)$ describe how large and fast the signal oscillates at time $\frac{n}{f_s}$.

Then, we assume that

$$\mathbf{x} = \mathbf{z} + \sigma\mathbf{w}\,.$$

# Extension step

To determine a prediction of the signal of length $L$, we estimate the dynamical behavior of the oscillating signal from its previous values.

1. We extract a set $\{\mathbf{x}_1, \ldots, \mathbf{x}_K\}$ of $K$ subsignals of length $M > L$ from $\mathbf{x}$.
2. Establishing a dynamical model consists in determining the relationship that binds $\mathbf{x}_{k+1}$ to $\mathbf{x}_k$, that is finding a function $f$ so that

$$\mathbf{x}_{k+1} = f(\mathbf{x}_k), \quad \forall k \in \{1, \ldots, K-1\} .$$

# Extension step

Here, we consider here a naive dynamical model, assuming that we have

$$\mathbf{x}_{k+1} = \mathbf{A}\mathbf{x}_k \,, \quad \forall k \in \{1, \ldots, K-1\} \,,$$

where $\mathbf{A} \in \mathbb{R}^{M \times M}$.

- Classical strategy in the study of dynamical systems: *linearization of a nonlinear system*.
- Fast and simple estimation of the parameters characterizing the dynamical model. Indeed:

$$\tilde{\mathbf{A}} = \mathbf{Y}\mathbf{X}^T(\mathbf{X}\mathbf{X}^T)^{-1} \,.$$
Least square estimator

  where $\mathbf{X} = \begin{pmatrix} \mathbf{x}_1 & \cdots & \mathbf{x}_{K-1} \end{pmatrix}$ and $\mathbf{Y} = \begin{pmatrix} \mathbf{x}_2 & \cdots & \mathbf{x}_K \end{pmatrix}$.
- The forecasting of $\mathbf{x}_{k+\ell}$ is obtained by recursively applying the linear relation:

$$\tilde{\mathbf{x}}_{K+\ell} = \underbrace{\tilde{\mathbf{A}}\tilde{\mathbf{A}} \cdots \tilde{\mathbf{A}}}_{\ell \text{ times}} \mathbf{x}_K = \tilde{\mathbf{A}}^\ell \mathbf{x}_K \,.$$

$\Longrightarrow$ Fast forecasting strategy. Is it accurate?

# Theoretical forecasting error

In the case where $\mathbf{x}$ follows the AHM model, we asymptotically bound the forecasting error $\epsilon$ defined by

$$\epsilon[n] = \tilde{\mathbf{x}}[n] - \mathbf{z}[n] \ .$$

In particular, we focus on

1. The bias, such that

$$\mu[n] = \mathbb{E}\{\epsilon[n]\} \ ;$$

2. The variance, given by

$$\gamma[n] = \mathbb{E}\{(\epsilon[n] - \mu[n])^2\} \ .$$

# Theoretical forecasting error

## Theorem (Asymptotic behavior the the forecasting error)

- *The first-order moment of the forecasting error at time $n \geq N$ satisfies*

$$|\mu[n]| \leq a_0^{(n)}\sigma^2 + \frac{1}{K}\left(\frac{a_1^{(n)}}{\sigma^2} + a_2^{(n)}\right) + o\left(\frac{1}{K}\right)$$

*as $K \to \infty$, where $a_0^{(n)}$, $a_1^{(n)}$, and $a_2^{(n)}$ are positive quantities, independent of $K$ and $\sigma$.*

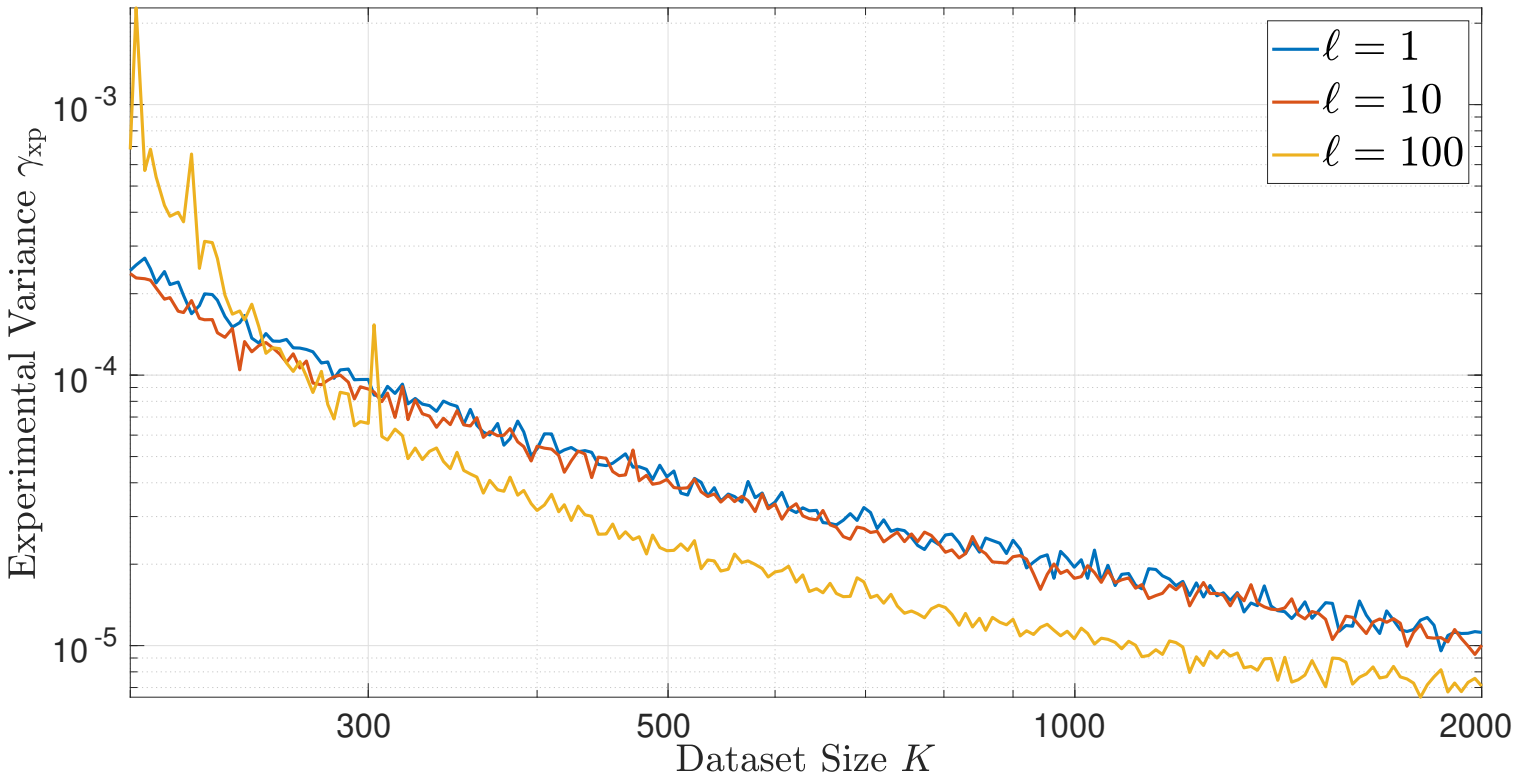
- *Its second-order moment satisfies:*

$$|\gamma[n]| \leq c_0^{(n)}\sigma^2 + \frac{1}{K}\left(\frac{c_1^{(n)}}{\sigma^2} + c_2^{(n)} + c_3^{(n)}\sigma^2\right) + o\left(\frac{1}{K}\right)$$

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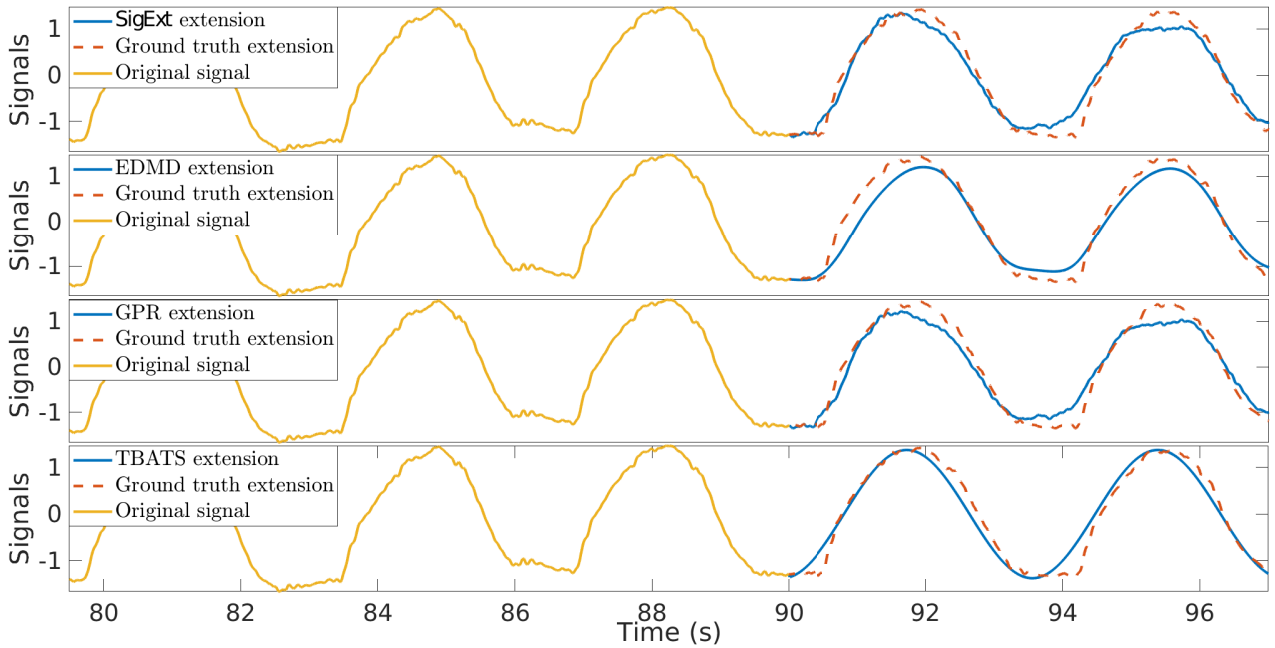
# Illustration of the theoretical results on a simulated signal

- Evolution of the experimental forecasting variance in function of the dataset size for three different values the forecasting sample index $\ell$.



# Applications to Real Physiological Signals

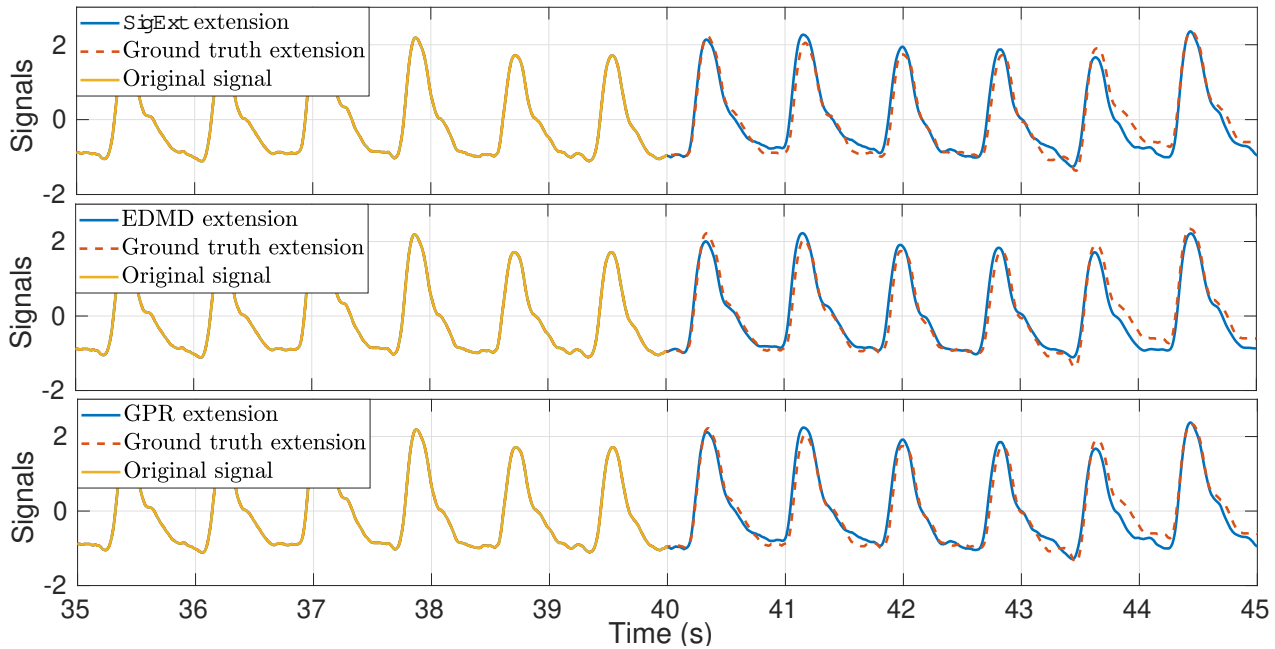
## Respiratory signal




| Extension method | MSE | Performance index $D$ (mean $\pm$ SD) | | | |
|---|---|---|---|---|---|
| | | STFT | SST | RS | ConceFT |
| **SigExt** | 0.292 $\pm$4.438 | 0.370 $\pm$0.623 | 0.408 $\pm$0.436 | 0.866 $\pm$0.879 | 0.423 $\pm$0.344 |
| Symmetric | 0.044 $\pm$0.111 | 1.162 $\pm$0.893 | 1.173 $\pm$0.886 | 1.022 $\pm$0.281 | 1.144 $\pm$0.579 |
| EDMD | 0.026 $\pm$0.112 | 0.359 $\pm$0.266 | 0.422 $\pm$0.282 | 0.828 $\pm$0.248 | 0.449 $\pm$0.296 |
| GPR | 0.331 $\pm$4.858 | 0.391 $\pm$0.853 | 0.411 $\pm$0.406 | 0.897 $\pm$1.140 | 0.430 $\pm$0.364 |

# Applications to Real Physiological Signals

## PPG signal




| Extension method | MSE | Performance index $D$ (mean $\pm$ SD) | | | |
|---|---|---|---|---|---|
| | | STFT | SST | ConceFT | RS |
| **SigExt** | 0.018 $\pm$0.014 | 0.280 $\pm$0.107 | 0.309 $\pm$0.112 | 0.367 $\pm$0.183 | 0.534 $\pm$0.160 |
| Symmetric | 0.037 $\pm$0.007 | 1.168 $\pm$0.390 | 1.209 $\pm$0.340 | 1.310 $\pm$0.140 | 0.983 $\pm$0.304 |
| EDMD | 0.012 $\pm$0.005 | 0.289 $\pm$0.126 | 0.319 $\pm$0.134 | 0.375 $\pm$0.163 | 0.503 $\pm$0.163 |
| GPR | 0.018 $\pm$0.013 | 0.276 $\pm$0.106 | 0.303 $\pm$0.110 | 0.361 $\pm$0.165 | 0.544 $\pm$0.157 |

# Real-Time Implementation

- $f_s$: sampling frequency
- $t_{\text{forecast}}$: time to compute the forecasting of $L$ oncoming samples
- $t_{\text{SST}}$: time to compute one column of the synchrosqueezing transform
- $H$: hop size (number of samples between each successive column of the synchrosqueezing transform)

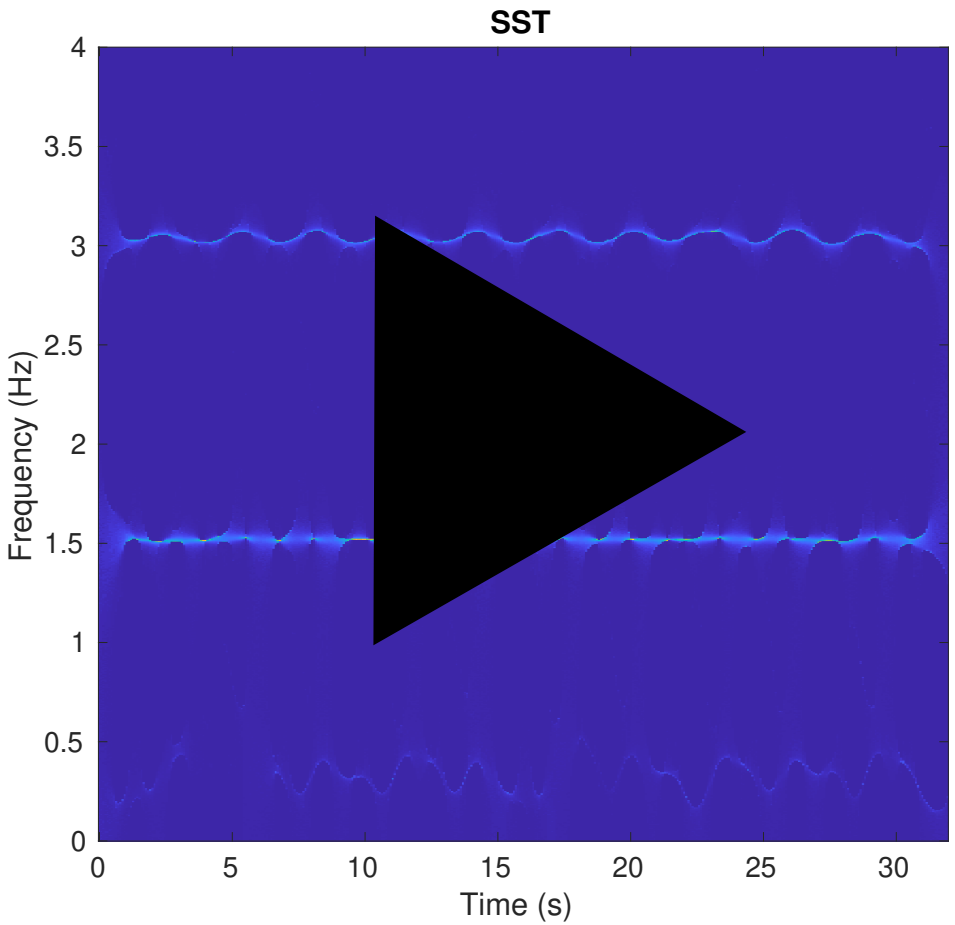
A general rule to determine the acceptable values of $H$ for real-time implementation of update the boundary-free TF representation is

$$t_{\text{forecast}} + \left\lceil \frac{L}{H} \right\rceil t_{\text{SST}} < \frac{H}{f_s} \, .$$

# Applications to Real Physiological Signals

## PPG signal : real-time implementation

In this example, taking $H \geq 8$ samples is sufficient to ensure the feasibility of real-time implementation. It thus allows a maximum overlap of 98.4% of the window length.



1 Introduction: nonstationarity

2 Locally deformed signals: an analysis-based approach

3 Reduction of boundary effects

4 Conclusion

# Conclusion

**Summary:**

- Broken stationarities:

$$Y = \mathcal{T} X$$

  - Locally deformed signals
  - Multivariate locally deformed signals
  - Locally harmonic signals
- Spectral estimation: Simultaneous estimation of the spectrum $\mathscr{S}_X$ and the deformation operator $\mathcal{T}$.
- Reduction of boundary effects: Forecasting of the variation of the deformation operator $\mathcal{T}$.

# Questions?

# Cross synthesis

$y_1(t) = \alpha_1(t)x_1(\gamma_1(t)) \Longrightarrow$ Spectrum estimation of $\mathscr{S}_{x_1}$

Cross synthesis:
$y_{1\to 2}(t) = \tilde{\alpha}_2(t)\tilde{x}_1(\tilde{\gamma}_2(t))$

$y_2(t) = \alpha_2(t)x_2(\gamma_2(t)) \Longrightarrow$ Deformations estimations of $\alpha_2$ and $\gamma_2$

| | | **Deformation** | | |
|---|---|---|---|---|
| | | Song | Wind | Formula 1 |
| **Spectrum** | Song | | | |
| | Wind | | | |
| | Formula 1 | | | |