



Synthèse temps-échelle de signaux non stationnaires

Adrien Meynard, Bruno Torrèsani

GRETSI, Nancy
7 septembre 2022

Outline

- 1 Introduction : nonstationarity
- 2 Locally deformed signals : an analysis-based approach
- 3 Locally deformed signals : a synthesis-based approach
- 4 Conclusion and perspectives

Stationarity

Definition (Stationarity)

A random process X is said to be second-order stationary if :

- $\mathbb{E}\{X(t)\} = m_X, \forall t$,
- $\mathbb{E}\{X(t)X(\tau)\} = k_X(t - \tau), \forall(t, \tau)$.

Spectrum :

- Gives the distribution over frequencies of the power of X .
- Many methods to estimate the spectrum from a single realization of the stationary process X (e.g. Welch method).

Nonstationarity

The stationarity assumption is often irrelevant to study real-life signals, such as audio signals, or physiological signals.

⇒ **Questions :**

- 1 Which classes of nonstationarity should we consider ?
- 2 How should we analyze nonstationarity ? In particular, how to extend spectral estimation to nonstationary signals ?

Broken stationarity : a class of nonstationarity¹

Two key ingredients :

- 1 A zero-mean stationary random process X .
- 2 A deformation operator that breaks stationarity $\mathcal{T}$.

We observe the « deformed » process Y given by :

$$Y = \mathcal{T}X .$$

⇒ We limit ourselves to some physically relevant forms of operators.

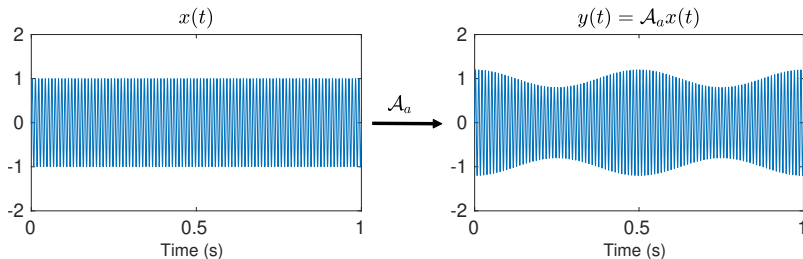
1. H. Omer. *Modèles de déformation de processus stochastiques généralisés. Application à l'estimation des non stationnarités dans les signaux audio.*
PhD thesis, Aix-Marseille Université, 2015

Deformation operators

■ Amplitude modulation

$$\mathcal{A}_\alpha : \quad \mathcal{A}_\alpha x(t) = \alpha(t)x(t) ,$$

with $\alpha \in C^1$ such that $\forall t, 0 < c_\alpha \leq \alpha(t) \leq C_\alpha < \infty$.



■ Frequency modulation

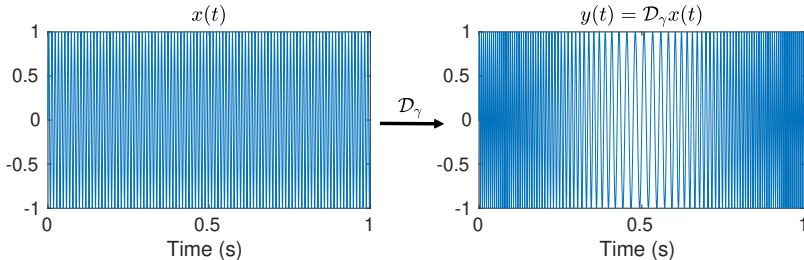
Deformation operators

■ Time warping

$$\mathcal{D}_\gamma : \quad \mathcal{D}_\gamma x(t) = \sqrt{\gamma'(t)} x(\gamma(t)) ,$$

where $\gamma \in C^2$ is a strictly increasing function such that

$$0 < c_\gamma \leq \gamma'(t) \leq C_\gamma < \infty, \forall t .$$



■ Any combination of the above deformations

Goal : spectral estimation of nonstationary signals

From a single realization of the nonstationary random process Y , we aim at estimating simultaneously :

- The spectrum $\mathcal{S}_X$ of the underlying stationary random process X ,
- The deformation operator $\mathcal{T}$.

- 1 Introduction : nonstationarity
- 2 Locally deformed signals : an analysis-based approach
 - Model
 - Wavelet transform and approximation
 - Estimation algorithm : JEFAS
- 3 Locally deformed signals : a synthesis-based approach
- 4 Conclusion and perspectives

Model and goal

■ Model :

$$Y = \mathcal{A}_\alpha \mathcal{D}_\gamma X .$$

where X is a stationary process.

- Relevant to model physical phenomena, such as Doppler effect, speed variations, or animal vocalizations.

■ Goal : From a single realization of the process Y , estimate simultaneously :

- the spectrum $\mathcal{S}_X$ of the underlying stationary process X ,
- the deformation functions α and γ .

Wavelet transform

Definition (Wavelet transform)

$$\mathcal{W}_x(s, \tau) = \langle x, \psi_{s\tau} \rangle \quad \text{avec} \quad \psi_{s\tau}(t) = 2^{-s/2} \psi(2^{-s}(t - \tau)) ,$$

where ψ is the analysis wavelet.

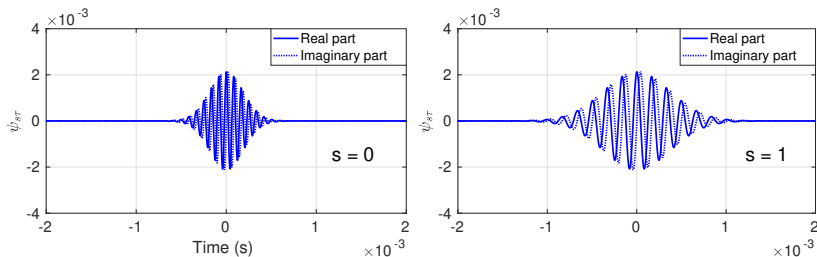
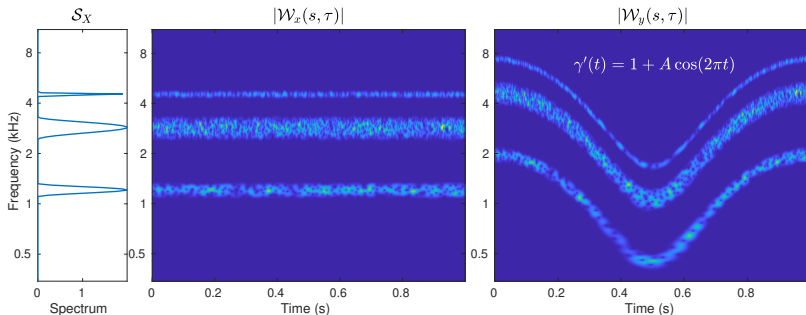


FIGURE – « Sharp wavelet » for two different values of s .

Approximated behavior



Approximation theorem

The wavelet transforms of X and Y are approximately related by :

$$\mathcal{W}_Y(s, \tau) \approx \widetilde{\mathcal{W}}_Y(s, \tau) = \alpha(\tau) \mathcal{W}_X(s + \log_2(\gamma'(\tau)), \gamma(\tau)) .$$

The error term $\epsilon = \mathcal{W}_Y - \widetilde{\mathcal{W}}_Y$ is a zero-mean random process, whose variance $\mathbb{E} \{ |\epsilon(s, \tau)|^2 \}$ depends on the regularity of α and γ' , and the speed of decay of ψ .

Estimation algorithm : JEFAS

The **JEFAS** (*Joint Estimation of Frequency, Amplitude and Spectrum*) algorithm consists in an alternating estimation.

Initializations :

- Initialize the power spectrum estimate.
- Initialize the amplitude modulation by a constant.
- $k \leftarrow 1$

while stopping criterion = FALSE do

- **Time warping** : Estimate $\tilde{\alpha}^{(k)}$ by ML, $\forall \tau$.
- **Amplitude modulation** : Estimate $\tilde{\gamma}^{(k)}$ by ML, $\forall \tau$.
- **Spectrum** : Estimate $\tilde{\mathcal{J}}_X^{(k)}$ from the “rectified” wavelet transform.
- $k \leftarrow k + 1$

end while

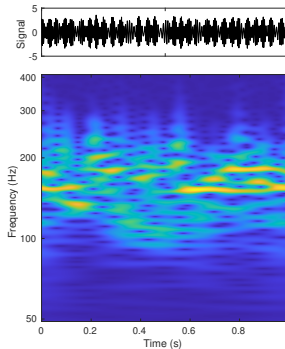
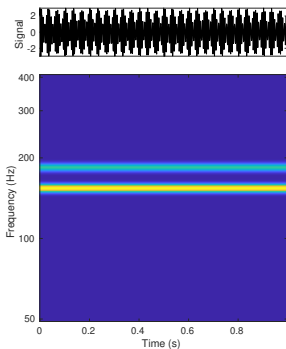
- 1 Introduction : nonstationarity
- 2 Locally deformed signals : an analysis-based approach
- 3 Locally deformed signals : a synthesis-based approach
 - Motivations and model
 - Estimation algorithm : JEFAS-S
 - Illustrations
- 4 Conclusion and perspectives

Locally harmonic signal

Signal of the form :

$$= A_1 \cos(2\pi\xi_1\gamma(t)) + A_2 \cos(2\pi\xi_2\gamma(t)) ,$$

where γ' is the **fast varying instantaneous frequency**.



- **JEFAS** : Interference patterns on the wavelet transform.
 - ⇒ Approximated behavior does not hold.
 - ⇒ JEFAS diverges.

Spectral estimation : Analysis vs. Synthesis

1 Analysis-based approach $\Rightarrow$ JEFAS and JEFAS-BSS

Model of nonstationarity :

locally time-warped signals, of the form :

$$Y = \mathcal{D}_{\gamma} X ,$$

where X is an arbitrary stationary process.

Spectral estimation : Analysis vs. Synthesis

1 Analysis-based approach $\Rightarrow$ JEFAS and JEFAS-BSS

Model of nonstationarity :

locally time-warped signals, of the form :

$$= \mathcal{D}_\gamma X ,$$

where X is an arbitrary stationary process.

Spectral estimation : Analysis vs. Synthesis

2 Synthesis-based approach $\Rightarrow$ JEFAS-S

Synthesis model $\Leftrightarrow$ Reconstruction formula :

$$y(t) = \text{Re} \left(\sum_s (\psi_s * \mathbf{W}_s)(t) \right) + \epsilon(t) ,$$

where $\mathbf{W}_s(t)$ are random time-scale coefficients, and $\epsilon(t)$ is a noise.

► Discretization of the problem :

$$\mathbf{y} = \text{Re} \left(\sum_{n=1}^N \Psi_n \mathbf{w}_n \right) + \epsilon ,$$

where $\mathbf{w}_n$ is the n -th column of the time-scale representation.

Bayesian inference

- Likelihood : $\epsilon \sim \mathcal{N}(0, \sigma^2 \mathbf{I})$

$$p(\mathbf{y}|\mathbf{w}_n) = \mathcal{N} \left(\text{Re} \left(\sum_{n=1}^N \Psi_n \mathbf{w}_n \right), \sigma^2 \mathbf{I} \right)$$

- Prior : on the synthesis coefficients $\mathbf{w}_n$

- ▶ Uncorrelated vectors such that :

$$\mathbf{w}_n \sim \mathcal{CN}_c(0, \mathbf{C}_n) .$$

- ▶ Covariance matrices $\mathbf{C}_n$ are translated versions of reference covariance function c :

$$[\mathbf{C}_n]_{mm'} = [\mathbf{C}(\theta_n)]_{mm'} = c(s_m + \theta_n, s_{m'} + \theta_n) ,$$

where $\theta_n \in \mathbb{R}$ is the shift parameter.

Estimation strategy

- ▶ Expectation Maximization (EM) principle where :
 - θ is the parameter,
 - w_n is the latent variable.

EM steps

The update at iteration k relies on the following two steps :

1 Time-scale representation update



Maximum *a posteriori* estimation :

$$\tilde{w}_n^{(k)} = \frac{1}{2} \mathbf{C} \left(\tilde{\theta}_n^{(k-1)} \right) \Psi_n^H \mathbf{C}_y \left(\tilde{\theta}^{(k-1)} \right)^{-1} \mathbf{y} .$$

2 Nonstationarity parameter update :

$$\tilde{\theta}_n^{(k)} = \arg \max_{\theta_n} \mathcal{L}(\theta_n) - \frac{1}{2} \text{Tr} \left(\mathbf{C}(\theta_n)^{-1} \Gamma_n \left(\tilde{\theta}^{(k-1)} \right) \right) ,$$

Algorithm : JEFAS-S

Initialization : estimate $\tilde{\theta}^{(0)}$ and $\tilde{\mathcal{J}}^{(0)}$ using JEFAS.

• $k \leftarrow 1$

while stopping criterion = FALSE **do**

• Time-scale representation estimation : $\tilde{\mathbf{w}}_n^{(k)}$.

• Time-warping parameter estimation : $\tilde{\theta}^{(k)}$.

• Spectrum estimation : $\tilde{\mathcal{J}}_X^{(k)}$.

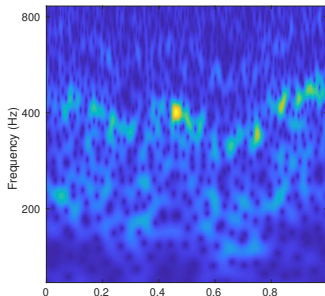
• $k \leftarrow k + 1$.

end while

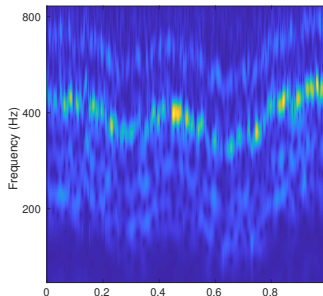
- Alternating estimation $\Rightarrow$ Similar to JEFAS.
- Convergence ensured by the EM principle.
- Additional estimation of the time-scale representation $\Rightarrow$ JEFAS-S is slower to converge than JEFAS.

Broadband synthetic signal

Wavelet transform



Estimated adapted time-scale representation



- **Time-warping parameter estimation** : not improved with respect to JEFAS.
- **Allows denoising** : improvement of **7.06 dB** of the Signal to Noise Ratio between the measurements $\mathbf{y}$ and the reconstructed signal $\tilde{\mathbf{y}}_0$.

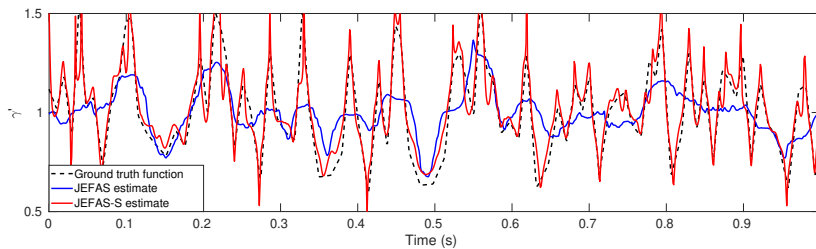
Locally harmonic signal

Signal of the form :

$$y(t) = A_1 \cos(2\pi\xi_1\gamma(t)) + A_2 \cos(2\pi\xi_2\gamma(t)) ,$$

where γ' is the (normalized) **fast varying instantaneous frequency**.

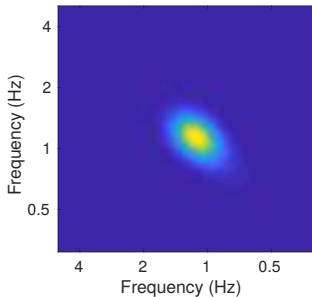
JEFAS-S : Prior of uncorrelation between $\mathbf{w}_n \Rightarrow$ No interference $\Rightarrow$ JEFAS-S converges.



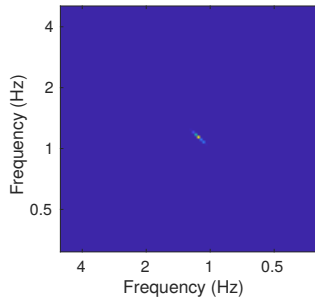
Locally harmonic signal

**Covariance
matrices**

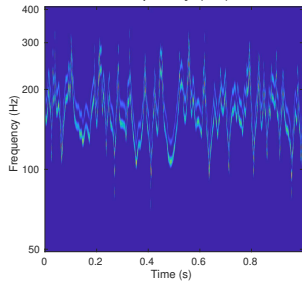
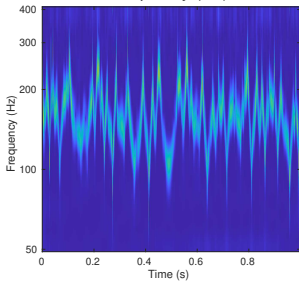
Wavelet-like prior



Sharp prior



**Time-scale
representations**



- 1 Introduction : nonstationarity
- 2 Locally deformed signals : an analysis-based approach
- 3 Locally deformed signals : a synthesis-based approach
- 4 Conclusion and perspectives

Conclusion

Summary :

■ Broken stationarities :

$$Y = \mathcal{T}X$$

- ▶ Locally deformed signals
- ▶ Multivariate locally deformed signals
- ▶ Locally harmonic signals

■ Spectral estimation : Simultaneous estimation of the spectrum $\mathcal{S}_X$ and the deformation operator $\mathcal{T}$.

Two estimation strategies :

- Analysis-based approaches
- Synthesis-based approaches