Random Matrices
and the Enumeration of Maps

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Random Matrices and the enumeration of maps

Three parts

- Combinatorics – We shall describe what are maps and the problem of enumerating them.

- Random Matrices – For more than thirty years, random matrices have been used to model diverse physical systems (String theory, Quantum field theory, Statistical models on random graphs etc). We shall relate (rigorously) the enumeration of maps and random matrices.

- Probability – Enumerating maps thus becomes a question about estimating matrix integrals. We shall describe the few models which were solved (in particular by large deviation techniques).

A map is a connected graph which is embedded into a surface in such a way that edges do not cross and faces (obtained by cutting the surface along the edges) are homeomorphic to a disk.

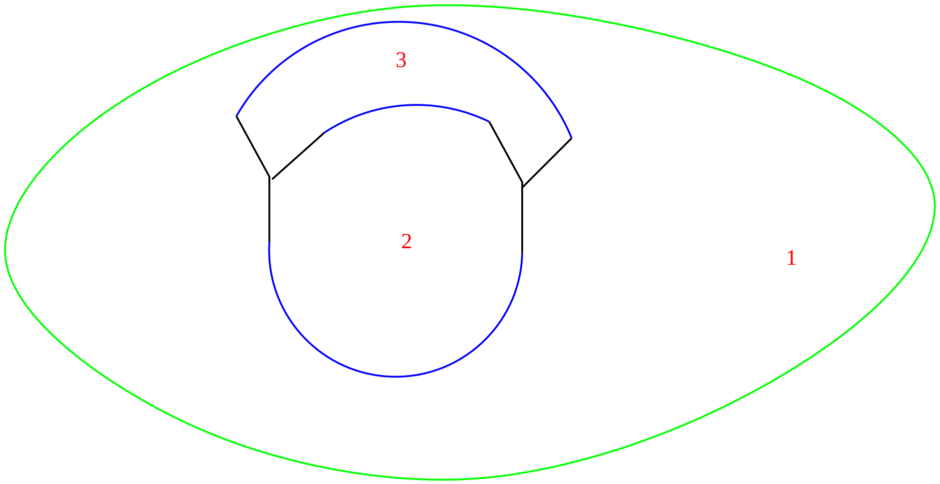
The genus of the map is the genus of the surface in which it is embedded.

$2 - 2g = \sharp$ vertices

$+ \sharp$ faces

- $\sharp$ edges.

2-2g =2+3 -3=2



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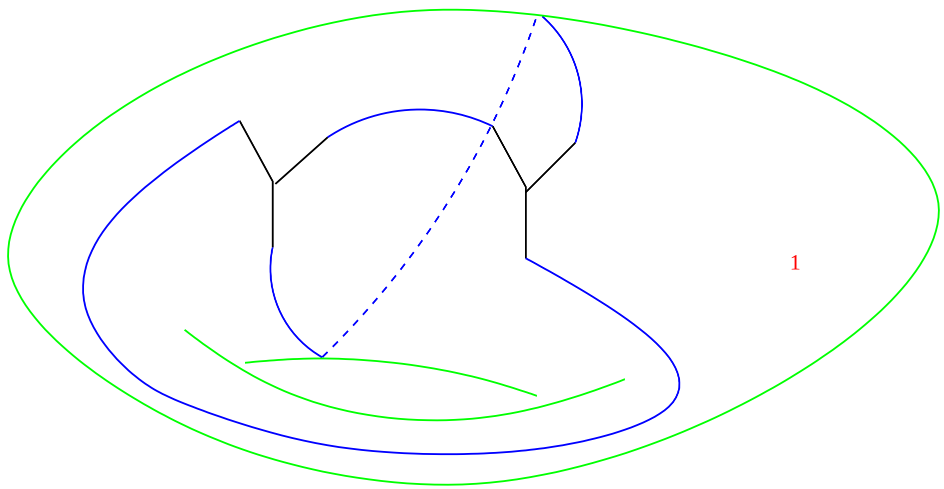

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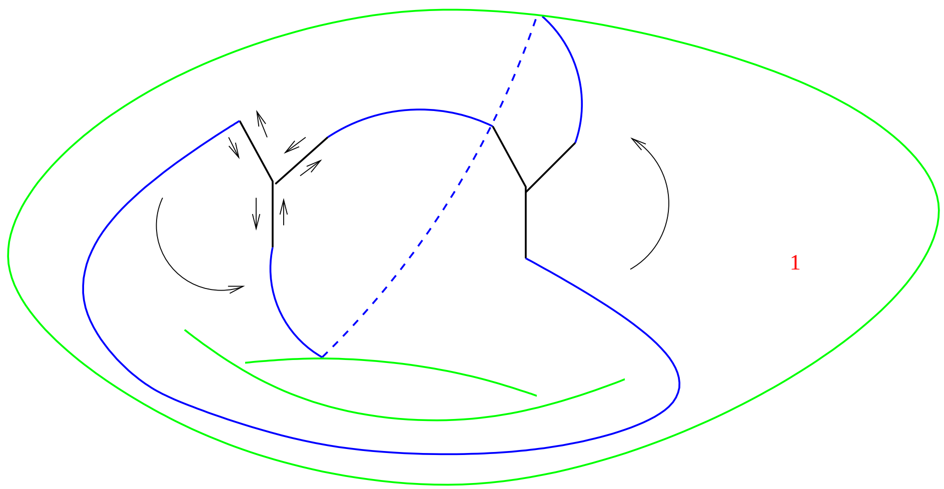


Proposition [Zvonkin,Edmonds,Heffter,Hamilton] Any given cyclic order at the ends of edges of a graph around each vertex uniquely determines the embedding of the graph into a surface, i.e a map.

Problem :

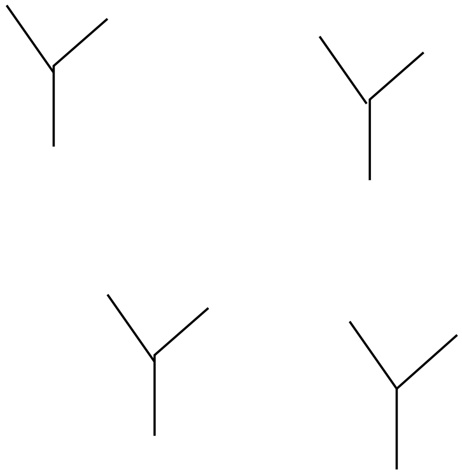
Count the number $M\left((n,d);g\right)$ of maps with genus g and n vertices of valence d.

Example:

$g = 0$,

$n = 4$,

$d = 3$.



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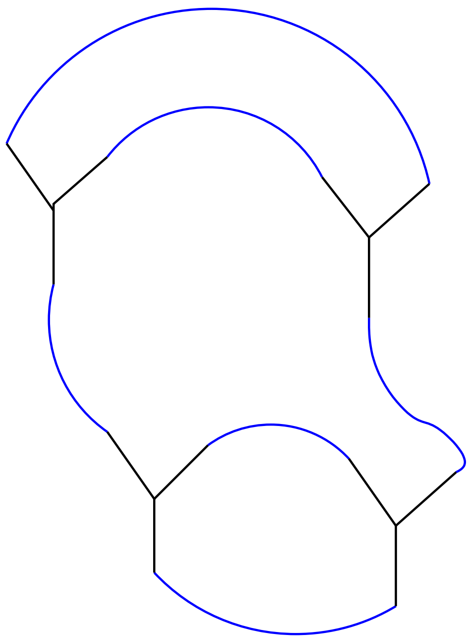
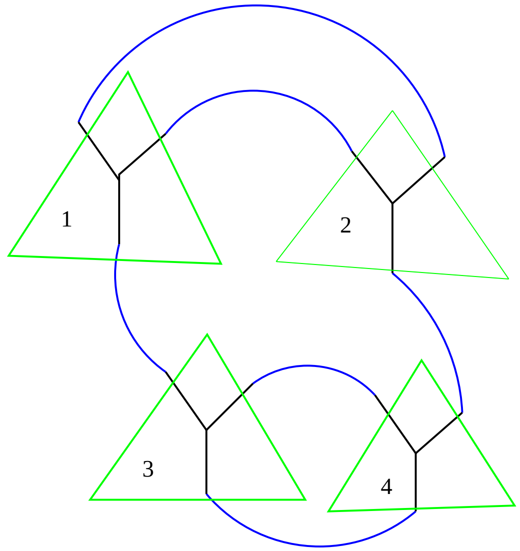


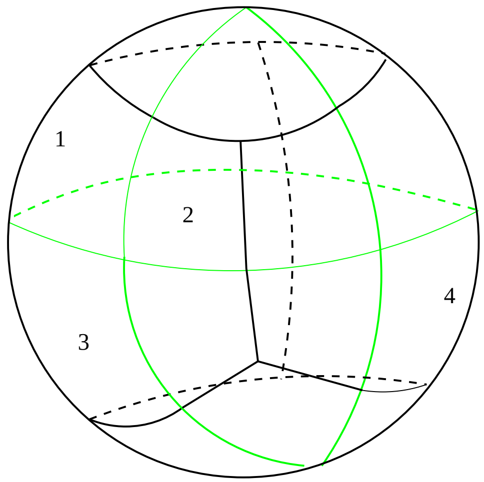
The counting is done up to homeomorphisms.

Dual Problem :

Count the number $M\left((n,d);g\right)$ of ways to cover a surface with genus g with n polygons of degree d.



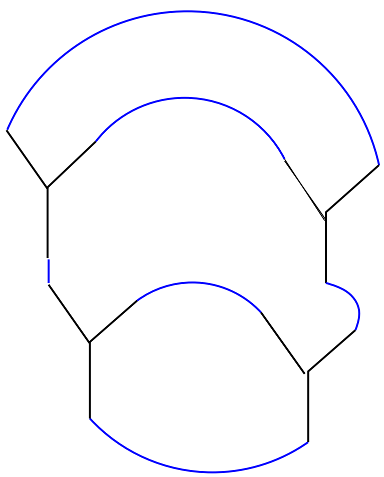
$\Longleftrightarrow$



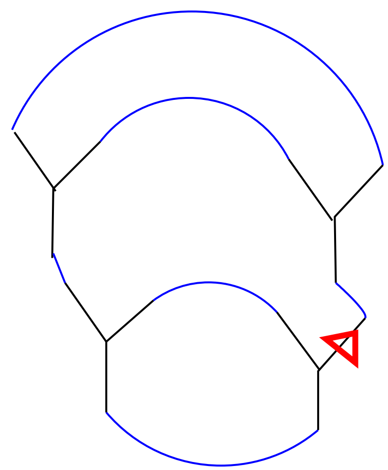
Tutte (60's): count rooted planar maps.

A root= A distinguished oriented edge.

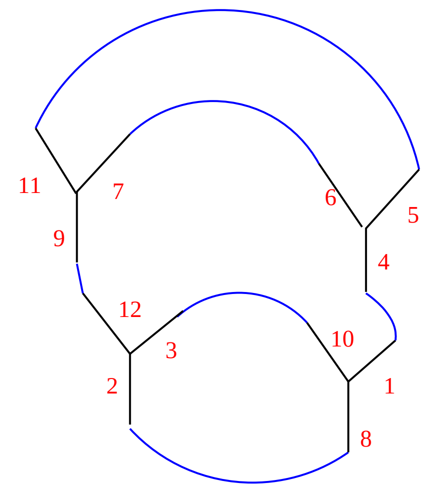
Prescribing a root reduces the number of symmetries;



Unrooted map



Rooted map



Labeled map

A rooted map with n edges has $(2n-1)!$ possible labellings of its half-edges. A map M with n edges has $2n/\sharp\mathrm{Automorphism}(M)$ possible roots.

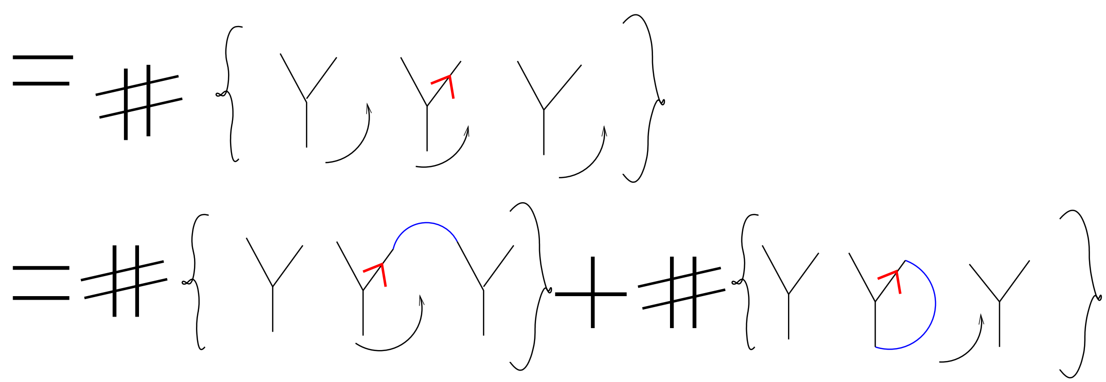
Theorem (Tutte)

$$
\begin{aligned}
M_{\mathrm{root}}((n,3);0) &= \sharp\{\text{rooted triangulations of the sphere with } n \text{ triangles}\} \\
&= 2^{n+1}\frac{(3n)!}{n!(2n+2)!}
\end{aligned}
$$

Idea of the proof:

Surgery on maps =Induction relations on number of maps

$M_{\mathrm{root}}((n,3);0)$



$$= M_{\mathrm{root}}((n-2,3),(1,4);0)+2M_{\mathrm{root}}((n-1,3),(1,1);0)$$

More complicated problems

More general maps – Find the number of rooted maps with genus g and n_1 vertices of degree d_1, ..., n_p vertices of degree d_p.

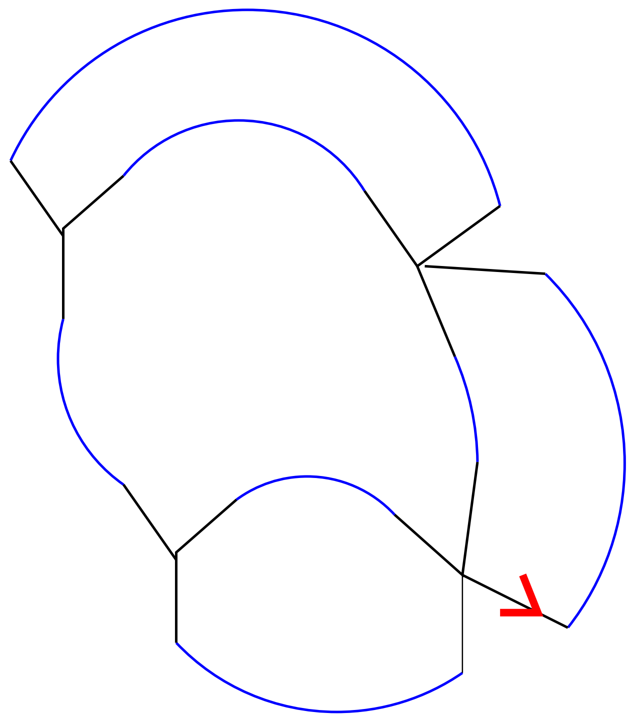
Example

$g = 0,$

$p = 2,$

$n_1 = 2, d_1 = 3,$

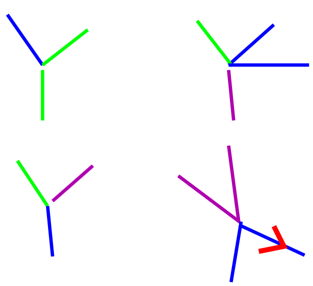
$n_2 = 2, d_2 = 4$

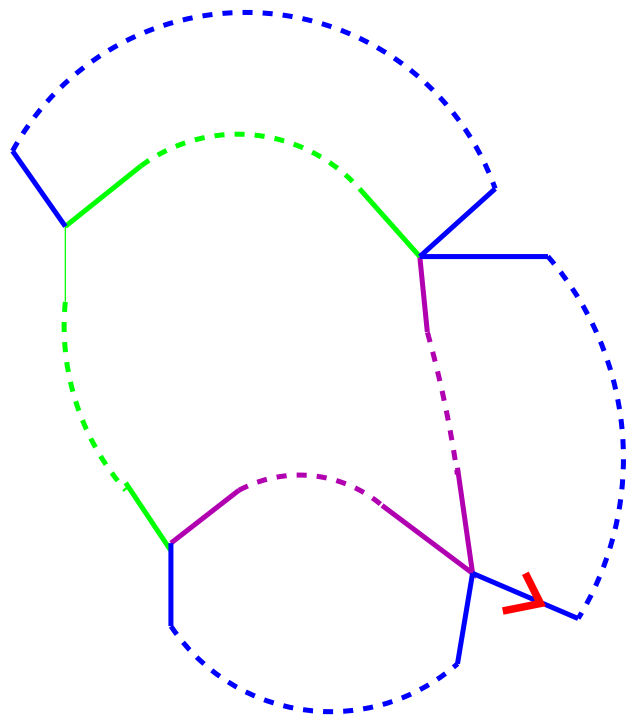


More complicated problems

Colored maps – What if the half-edges are colored ? We replace a vertex with valence d by a vertex with colored half-edges and require that gluing only holds between half-edges of the same color.

Planar map drawn with





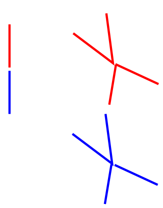
Colored maps: Example of the Ising model on random graphs

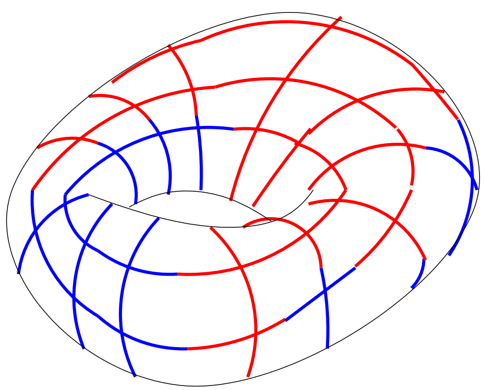
Count the number of maps with genus g, n vertices of valence 4 either blue or red such that the total number of red-blue gluings is equal to m.

Ising model in $\mathbb{Z}^2$: count the number of configurations of spins $(\sigma_i)_{1\leq i\leq n} \in \{-1,+1\}^n$ in a box of size $\sqrt{n} \times \sqrt{n}$ in $\mathbb{Z}^2$ with m nearest neighbours of different signs.

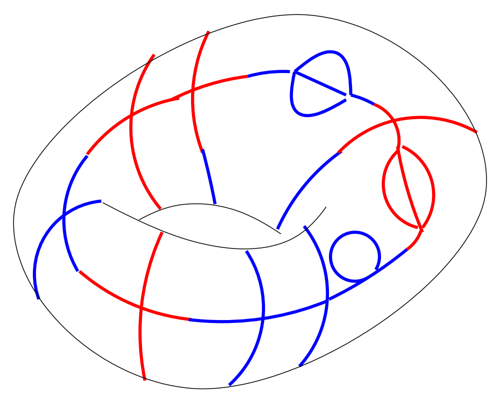
Difference: sum over configurations and underlying graphs.

Map drawn with $g = 1$





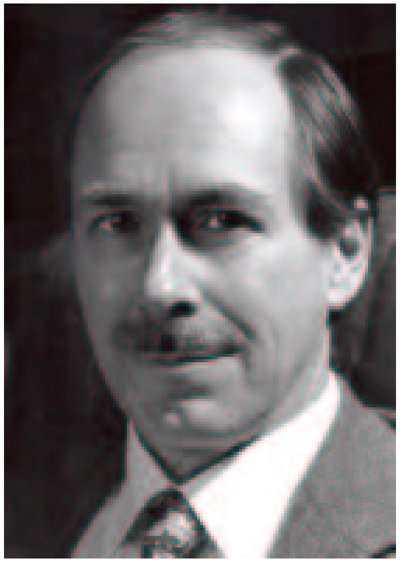
Ising model on the lattice



Ising model on random graphs

Recently, combinatorial methods have been developped to tackle some of these challenges (cf Bousquet-M élou, Schaeffer etc). However, these problems have been studied in physics for more than thirty years by using matrix integrals/matrix models.

’t Hooft noticed in 1974 that matrix integrals are generating functions for the enumeration of maps.



We restrict ourselves here to Gaussian matrices from the GUE.

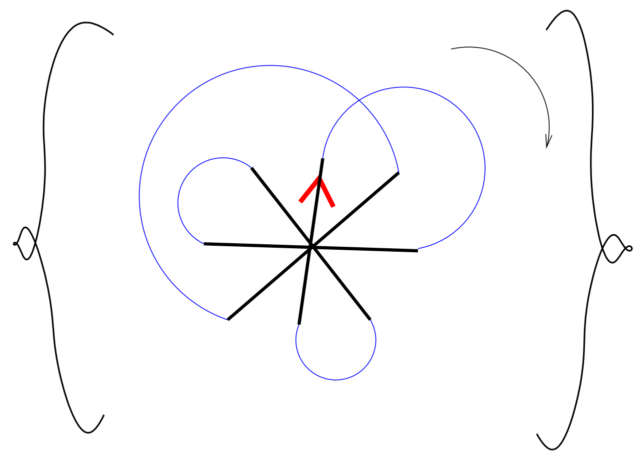
Combinatorial interpretations of Gaussian moments : Feynmann diagrams

Wick formula: If $(G_1, \cdots, G_{2n})$ is a centered Gaussian vector,

$$\mathbb{E}[G_1 G_2 \cdots G_{2n}] = \sum_{\substack{1 \le s_1 < s_2 .. < s_n \le 2n \\ r_i > s_i}} \prod_{j=1}^{n} \mathbb{E}[G_{s_j} G_{r_j}].$$

Example: If for all i, $G_i = G$ follow the standard Gaussian distribution,

$E[G^{2n}] = \sharp$ { number of rooted maps with one vertex with valence $2n$ } $= \sharp$



The Gaussian Unitary Ensemble (GUE)

Let $\mathcal{H}_N = \{A \in \mathcal{M}_{N\times N}(\mathbb{C}); A = A^*\}$. The law μ_N of the GUE is the probability measure on $\mathcal{H}_N$

$$d\mu_N(A) = \frac{1}{Z_N} e^{-\frac{N}{2}\mathrm{tr}(A^2)} dA\,.$$

In other words, $A_{lk} = \bar{A}_{kl}$ for $1 \leq k < l \leq N$ and

$$A_{kl} = (2N)^{-\frac{1}{2}}(g_{kl} + i\tilde{g}_{kl}) \text{ for } \quad k < l, \quad A_{kk} = N^{-\frac{1}{2}} g_{kk}$$

where the $(g_{kl}, \tilde{g}_{kl}, k \leq l)$ are i.i.d standard Gaussian variables;

$$P(d\tilde{g}_{kl}, dg_{kl}, k \leq l) = \frac{1}{(2\pi)^{N^2}} \prod_{1\leq k\leq l\leq N} e^{-\frac{1}{2}(g_{kl})^2} dg_{kl} \prod_{1\leq k<l\leq N} e^{-\frac{1}{2}(\tilde{g}_{kl})^2} d\tilde{g}_{kl}.$$

Combinatorial interpretations of Gaussian Matrix moments ; one matrix

Fact: For all $p \in \mathbb{N}^*$, all $n \in \mathbb{N}$,

$$\int (N\mathrm{tr}(A^p))^n \, d\mu_N(A) = \sum_{F \geq 0} \frac{1}{N^{\frac{pn}{2}-n-F}} G(p,n,F)$$

$$G(p,n,F) = \sharp\{\text{Union of labeled maps with } F \text{ faces and } n \text{ vertices of degree } p\}.$$

Recall that a connected graph can be embedded into a surface with Euler characteristic

$$\chi = 2 - 2g = \sharp\text{vertices} + \sharp\text{faces} - \sharp\text{edges} = n + F - \frac{pn}{2}.$$

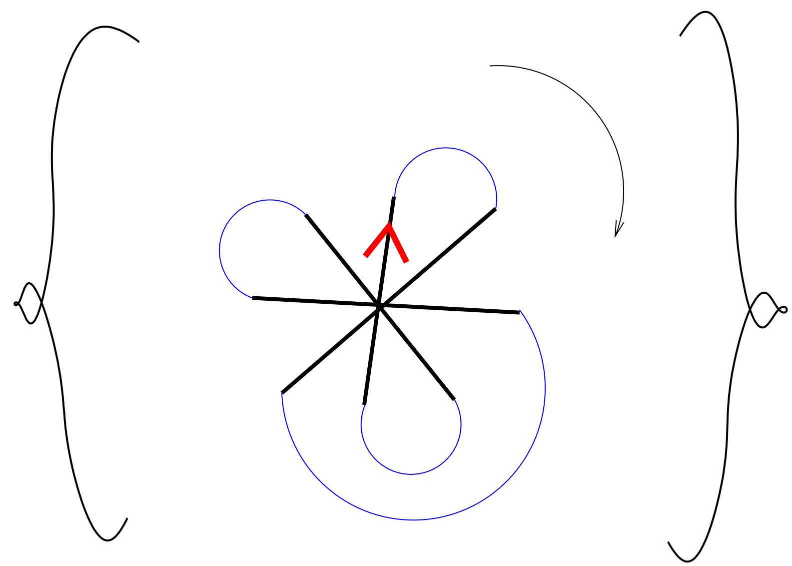
Application of $\int (N\mathrm{tr}(A^p))^n \, d\mu_N(A) = \sum_{F\geq 0} \frac{1}{N^{\frac{np}{2}-n-F}} G(p,n,F)$.

Wigner (1958) already noticed that,

$$\lim_{N\to\infty} \int \frac{1}{N}\mathrm{tr}(A^p) d\mu_N(A) = G(p,1,\frac{p}{2}+1) = \begin{cases} 0 & \text{if p is odd,} \\ C_{\frac{p}{2}} & \text{otherwise,} \end{cases}$$

where $C_{\frac{p}{2}}$ is the Catalan number, i.e. the number of rooted planar ($g = 0$) maps with one vertex of valence p.
(Here, $p = 8$, $F = 5$)

$= \sharp$



$G(p,1,\frac{p}{2}+1) = \int x^p d\sigma(x)$, where σ is the semi-circular law.

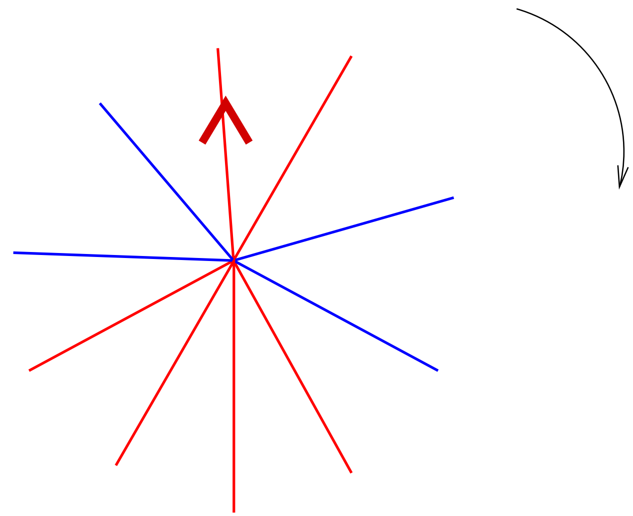
Combinatorial interpretations of Gaussian Matrix moments ; several matrices

Let $m \in \mathbb{N}$. To any monomial $q(X_1, \cdots, X_m) = X_{i_1} \cdots X_{i_p}$, we associate (bijectively) a star of type $q =$

oriented vertex with half-edges of color i_1, i_2, $...i_p$, ordered clockwise, the first half-edge being marked.

Here

$q(X) = {X_1}^2 {X_2}^2 {X_1}^4 {X_2}^2$.



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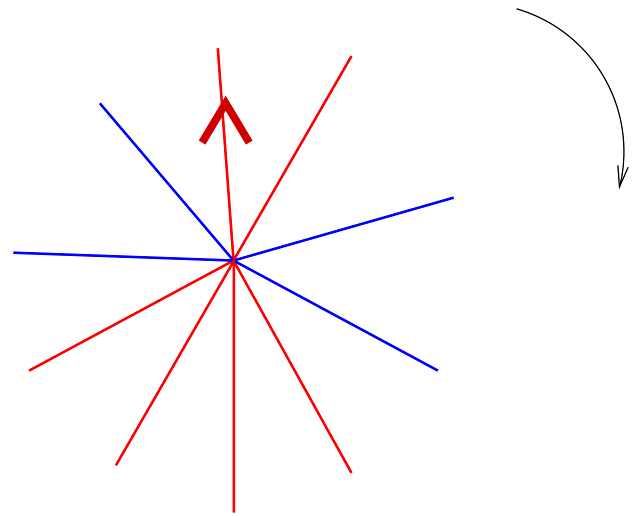


Fact: For any monomial q, all $n \in \mathbb{N}$,

$$\int (N\mathrm{tr}(q(A_1, \cdots, A_m)))^n \, d\mu_N(A_1) \cdots d\mu_N(A_m) = \sum_{F \geq 0} \frac{1}{N^{\frac{pn}{2} - n - F}} G_c(q, n, F),$$

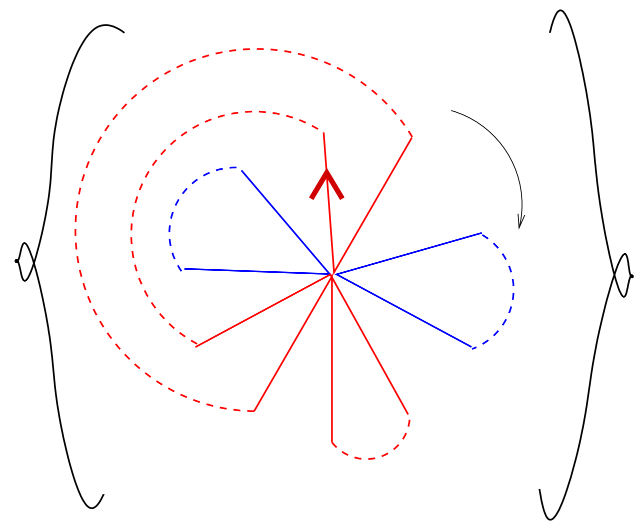
$G_c(q, n, F) = \sharp\{$Union of labeled maps with F faces and n stars of type $q\}$.

Application of $\int (N\mathrm{tr}(q(\mathbf{A})))^n \, d\mu_N^{\otimes m}(\mathbf{A}) = \sum_{F\geq 0} \frac{1}{N^{\frac{np}{2}-n-F}} G_c(q,n,F)$.

Voiculescu (1984) [see also Speicher(1997)]. Let $m \in \mathbb{N}$ and $q(X_1,\cdots,X_m) = X_{i_1}\cdots X_{i_p}$ for $i_1,\cdots,i_p \in \{1,\cdots,m\}$

$$\lim_{N\to\infty} \int \frac{1}{N} \mathrm{tr}(q(A_1,\cdots,A_m)) \, d\mu_N(A_1)\cdots d\mu_N(A_m) := \sigma_m(q)$$

where $\sigma_m(q)$ is the number of planar maps drawn with a star of type q by gluing half-edges of the same color (Here, $q(\mathbf{X}) = X_1{}^2 X_2{}^2 X_1{}^4 X_2{}^2$)

$= \sharp$ 

$\sigma_m =$ law of m free semi-circular variables.

Combinatorial interpretation of Matrix Models

't Hooft (1974) considered generating functions of matrix moments; the matrix models.

Let $m \in \mathbb{N}$ and $(q_1, \cdots, q_n)$ be fixed monomial functions of m non-commutative variables. Let $\mathbf{t} = (t_i)_{1 \leq i \leq n} \in \mathbb{C}^n$ and set $V_{\mathbf{t}}(X_1, \cdots, X_m) = \sum_{i=1}^n t_i q_i(X_1, \cdots, X_m)$.

$$F_N(\mathbf{t}) := \frac{1}{N^2} \log \int e^{-N \operatorname{tr}(V_{\mathbf{t}}(A_1, \cdots, A_m))} d\mu_N(A_1) \cdots d\mu_N(A_m)$$

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$$F_N(\mathbf{t}) := \frac{1}{N^2} \log \int e^{-N \mathrm{tr}(V_{\mathbf{t}}(A_1, \cdots, A_m))} d\mu_N(A_1) \cdots d\mu_N(A_m)$$

$$= \sum_{k_1, .., k_n \in \mathbb{N}} \sum_{g \ge 0} \frac{1}{N^{2g}} \prod_{j=1}^n \frac{(-t_j)^{k_j}}{k_j!} M\left((k_1, q_1), \cdots, (k_n, q_n); g\right)$$

with

$M\left((k_1, q_1), \cdots, (k_n, q_n); g\right) = \sharp\{$Labeled maps of genus g with k_i stars of type $q_i\}$

(Proof: expand exponential + log=connected graph)

Problems

• Problem 1: Can we compute, for reasonnable $V_{\mathbf{t}} = \sum t_i q_i$, $\mathbf{t} = (t_1, \cdots, t_n)$,

$$F(\mathbf{t}) = \lim_{N \to \infty} F_N(\mathbf{t}) \quad ?$$

with

$$F_N(\mathbf{t}) = \frac{1}{N^2} \log \int e^{-N\mathrm{tr}(V_{\mathbf{t}}(A_1,\cdots,A_m))} d\mu_N(A_1)...d\mu_N(A_m)$$

Can we estimate the large N's corrections ?

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$$F_N(\mathbf{t}) = \frac{1}{N^2} \log \int e^{-N\mathrm{tr}(V_{\mathbf{t}}(A_1,\cdots,A_m))} d\mu_N(A_1)...d\mu_N(A_m)$$

Can we estimate the large N's corrections ?

• Problem 2: Are the numbers of maps we want to compute

$$M\left((k_1, q_1), \cdots, (k_n, q_n); 0\right) = \lim_{N\to\infty} (-1)^{k_1+..+k_n} \partial_{t_1}^{k_1} \cdots \partial_{t_n}^{k_n} F_N(\mathbf{t})|_{\mathbf{t}=0}$$

indeed equal to

$$(-1)^{k_1+..+k_n} \partial_{t_1}^{k_1} \cdots \partial_{t_n}^{k_n} [F(\mathbf{t})]|_{\mathbf{t}=0} = (-1)^{k_1+..+k_n} \partial_{t_1}^{k_1} \cdots \partial_{t_n}^{k_n} [\lim_{N\to\infty} F_N(\mathbf{t})]|_{\mathbf{t}=0} ?$$

Same question for the corrections.

Problem 2: From formal to small t_i's expansion

$$F_N(\mathbf{t}) = \frac{1}{N^2}\log\int e^{-N\mathrm{tr}(V(A_1,\cdots,A_m))}d\mu_N(A_1)..d\mu_N(A_m),\ V=\sum_{i=1}^n t_i q_i.$$

Theorem Hypothesis: $\phi_V : (A_1(ij),\cdots,A_m(ij))_{i\le j}\to\mathrm{tr}(V(\mathbf{A}))$ is real-valued. ϕ_V is convex (or we add a cutoff).

For all $\ell\geq 0$, $\exists\,\varepsilon_\ell>0$ so that if $|\mathbf{t}|=\sum_{i=1}^n|t_i|\leq\varepsilon_\ell$,

$$F_N(\mathbf{t})=\sum_{g=0}^{\ell}\frac{1}{N^{2g}}\sum_{k_1,\cdots,k_n}\prod\frac{(-t_i)^{k_i}}{k_i!}M\left((q_1,k_1),\cdots,(q_n,k_n);g\right)+o\left(\frac{1}{N^{2\ell}}\right)$$

$M\left((q_1,k_1),\cdots,(q_n,k_n);g\right)=\sharp\{$maps with genus g with k_i stars of type $q_i\}$

-$m=1$: Ambjórn et al. (95), Albeverio-Pastur-Scherbina (01), Ercolani-McLaughlin (03)

-$m\geq 2$: G.-Maurel-Segala ($\ell\leq 1$ (05)(06)),Maurel-Segala (for all ℓ (06))

Idea of the proof $[g=0]$: non-commutative differential calculus

Take $V_{\mathbf{t}} = \sum_{i=1}^n t_i q_i$. Let $\hat{\mu}_{\mathbf{A}}$ be the empirical distribution

$$\hat{\mu}_A(P) := \int \frac{1}{N}\mathrm{tr}(P(A_1,\cdots,A_m))\,\frac{e^{-N\mathrm{tr}(V_{\mathbf{t}}(A_1,\cdots,A_m))}d\mu_N(A_1)..d\mu_N(A_m).}{e^{N^2 F_N(\mathbf{t})}}$$

for all $P \in \mathbb{C}\langle X_1,\cdots,X_m\rangle$ $[\hat{\mu}_A \in \mathcal{P}(\mathbb{R})$ when $m=1]$.

Fact 1: Let $\partial_i P = \sum_{P=P_1X_iP_2} P_1 \otimes P_2$, $D_i P = \sum_{P=P_1X_iP_2} P_2P_1$. The limit points of $\hat{\mu}_{\mathbf{A}}$ are solution of Schwinger-Dyson's equation

$$\tau_{\mathbf{t}}(X_iQ) = \tau_{\mathbf{t}}\otimes\tau_{\mathbf{t}}(\partial_i Q) - \tau_{\mathbf{t}}(D_iVQ) \quad \forall Q \in \mathbb{C}\langle X_1,\cdots,X_m\rangle \quad \forall i \in \{1,\cdots,m\}.$$

Fact 2: By convexity (or cutoff), $\exists R<\infty$, so that $|\tau(q)| \le R^{\deg q}$.

If $|\mathbf{t}| \le \varepsilon_0$, there is a unique solution

$$\tau_{\mathbf{t}}(q) = \sum_{k_1,\cdots,k_n\in\mathbb{N}} \prod_{i=1}^n \frac{(-t_i)^{k_i}}{k_i!} M((q,1),(q_1,k_1),\cdots,(q_n,k_n);0)$$

[non-commutative derivatives=Tutte's surgery]

Estimating Matrix integrals [Problem 1, $g = 0$]

Compute

$$F(\mathbf{t}) = \lim_{N\to\infty} \frac{1}{N^2} \log \int e^{-N\mathrm{tr}(V(A_1,\cdots,A_m))} d\mu_N(A_1)\cdots d\mu_N(A_m).$$

• When $m = 1$, one can compute the limit of the free energy. Explicit formulae for this limit can be found for triangulations and quadrangulations, the analysis is complicated in general.

• When $m \geq 2$, only few models could be studied. We shall focus on the Ising model on random graphs. However, the result we are going to present extend readily to other models (q-Potts [Zinn Justin (00),G.(04)], chain models [Mehta (87), G.(04)], dually weighted graph models [Kazakov, Staudacher,Wynter (96), G.-Maida (05)] etc)

Computation of the free energy: $m = 1$

Take $V_{\mathbf{t}} = \sum_{i=1}^{D} t_i x^i$. Recall that [up to add a cutoff], for $\mathbf{t}$ small

$$
\begin{aligned}
F(\mathbf{t}) &:= \lim_{N\to\infty} \frac{1}{N^2} \log \int e^{-N\mathrm{tr}(V_{\mathbf{t}}(A))} d\mu_N(A) \\
&= \sum_{k_1,\cdots,k_D \in \mathbb{N}} \prod \frac{(-t_i)^{k_i}}{k_i!} M\left((k_1, x), \cdots, (k_D, x^D); 0\right)
\end{aligned}
$$

On the other hand, diagonalizing A gives (see e.g. Ben Arous-G (97))

$$
\begin{aligned}
F(\mathbf{t}) &= \lim_{N\to\infty} \frac{1}{N^2} \log Z_N^{-1} \int e^{-N\sum_{i=1}^{N} V_{\mathbf{t}}(\lambda_i)} \prod_{i\neq j} |\lambda_i - \lambda_j| e^{-\frac{N}{2}\sum \lambda_i^2} \prod d\lambda_i \\
&= \sup_{\mu\in\mathcal{P}(\mathbb{R})} \{ \int \log|x-y| d\mu(x) d\mu(y) - \int [V_{\mathbf{t}}(x) + \frac{1}{2}x^2] d\mu(x) \} + \mathrm{const.}
\end{aligned}
$$

Explicit solution for $V_{\mathbf{t}}(x) = tx^3$ or tx^4 (Bessis, Itzykson, Zuber (80)).
Complicated in general (see Deift, Kriecherbauer and McLaughlin (98))

Computation of the free energy: $m = 2$. The Ising model on random graphs

We have seen that

$$F(c,h) := \lim_{N\to\infty} \frac{1}{N^2} \log \int e^{-Nc\,\mathrm{tr}(AB) - Nh\,\mathrm{tr}(A^4) - Nh\,\mathrm{tr}(B^4)} d\mu_N(A) d\mu_N(B)$$

$$= \sum_{m,n} \frac{(-c)^m}{m!} \frac{(-h)^n}{n!} \sharp\{\text{labeled planar maps with } n \text{ blue or red vertices}$$

$$\text{of valence 4 and } m \text{ bicolored edges}\}.$$

Mehta (84) [see also Boulatov-Kazakov (87)] used orthogonal polynomial methods to give an explicit formula for $F(c,h)$.

Combinatorial proof of the same formula by Bousquet-Mélou and Schaeffer (02).

The Ising model on random graphs and HCIZ integral

The interacting term in the Ising model is given in terms of the Harish-Chandra-Itzykson-Zuber integral

$$I_N(\lambda,\eta) = \int e^{N\mathrm{tr}(U\mathrm{diag}(\lambda)U^*\mathrm{diag}(\eta))}dU = \frac{\det\left((e^{N\lambda_j\eta_i})_{1\le i,j\le N}\right)}{\prod_{i<j}(\lambda_i-\lambda_j)\prod_{i<j}(\eta_i-\eta_j)}$$

Theorem G-Zeitouni (02), G. (04)[large deviations +stochastic calculus]

Assume $\lim_{N\to\infty}\frac{1}{N}\sum_{i=1}^N \delta_{\lambda_i} = \mu, \lim_{N\to\infty}\frac{1}{N}\sum_{i=1}^N \delta_{\eta_i} = \nu$. Then

$$I(\mu,\nu) = \lim_{N\to\infty}\frac{1}{N^2}\log I_N(\lambda,\eta)$$

exists and an 'explicit' formula is given.

Allows the analysis of the Ising model for general vertices and other models, and also provides asymptotics of Schur functions.

Conclusion

- The relation between matrix integrals and the enumeration of maps can be made rigorous; it provides original formulae for the generating functions of the numbers of interest.
- The analysis of these formulae are a problem on their own (see e.g. Deift, Kriechbauer and McLaughlin, WIP with S. Belinschi).
- Schwinger-Dyson's equations contain all the information.
- There are many other enumerating issues where random matrices are used (see e.g. Diaconis and Gamburd for the enumeration of magic squares)
- There is a huge literature around these subjects in physics which should attract more mathematicians.