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Large deviations for the extreme eigenvalues of a deformed GOE: a functional equation approach

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Abstract. We establish a large deviation principle for the smallest eigenvalue of a random matrix model composed of the sum of a GOE matrix and a diagonal matrix with an outlier. Our result generalizes and unifies previously studied cases.

Résumé. Nous établissons un principe de grandes déviations pour la plus petite valeur propre d'un modèle de matrice aléatoire composé de la somme d'une matrice GOE et d'une matrice diagonale avec un outlier. Notre résultat généralise et unifie quelques cas étudiés précédemment.

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Contents

1. Introduction

1.1. Setting of the problem

In this paper, we study the asymptotic behavior of the smallest eigenvalue of a sequence of random matrices $\{X_N\}_{N\in\mathbb{N}}$ where X_N is given by the sum of two $N\times N$ matrices: a matrix G_N taken from the Gaussian Orthogonal Ensemble (GOE) of variance t, and a deterministic matrix B_N whose empirical eigenvalue distribution converges weakly towards a probability measure ν. We also allow the deterministic matrix to have an outlier Λ below the support of ν.

In this setting, the typical behavior of the spectrum of $X_N = G_N + B_N$ is well known. The empirical measure of the eigenvalues of X_N converges towards the free convolution $\sigma_t \boxplus \nu$ where σ_t is the semicircular distribution with variance t [6, 24]. This probability measure was described in detail in [7]. Moreover, it was proved in [10] that the smallest eigenvalue $\lambda_1(X_N)$ of X_N converges towards a limit $\ell^{\Lambda}_{\nu,t}$ which is equal to the left boundary $\ell_{\nu,t}$ of the support of $\sigma_t \boxplus \nu$ if and only if Λ is above some threshold, a phenomenon called the Ben-Arous-Baik-Péché (BBP) transition (see Theorem 1.1).

Our aim is to study the large deviations for the law of $\lambda_1(X_N)$, the smallest eigenvalue of X_N. Our study is motivated by a problem arising in the theory of spin glasses. It is well known in the physics literature that the equilibrium properties of the p-spin glass model are related to the topology of the energy landscape of a random function called the Thouless-Anderson-Palmer (TAP) free energy [23]. In the case of the pure Ising p-spin glass, the Hessian of the TAP free energy contains a projector term, which may produce a negative eigenvalue at low enough temperature [9, 21, 22]. By the Kac-Rice formula, computing the (annealed) number of local minima therefore requires studying the large deviations of the smallest eigenvalue of a random matrix of the form considered in this paper.

Several other works have studied large deviation principles (LDPs) for the extreme eigenvalues of GOE matrices. The article [19], which examined a rank-one deformation of the GOE, was extended in [5] to finite rank deformations, and in [8], which proved an LDP for the joint distribution of extreme eigenvalues and their associated eigenvectors. The article [13] investigated the extreme eigenvalues of Wigner matrices, namely self-adjoint random matrices with entries independent above the diagonal. They showed that if the entries are sharp sub-Gaussian in the sense that their Laplace transform is bounded above by the Laplace transform of the Gaussian variable with the same variance, the large deviations are the same as for Gaussian entries. This came as a surprise, as large deviations for the smallest eigenvalue of Wigner matrices are generally very sensitive to the distribution of the entries, so that even their speed depends on their tail. For distributions with tails heavier than Gaussian, an LDP was established in [2], showing the speed and the rate function depends on the tail decay rate. In contrast, the large deviations when the entries follow general sub-Gaussian distributions have the same speed as for Gaussian variables, but their rate function is non-universal and in general is different from the Gaussian one below a certain threshold, a transition indicating eigenvector localization phenomena [3, 11]. Large deviations were also investigated for more structured matrices. They were first studied for the sum of two unitarily invariant models [16] and more recently for matrices with a variance profile, highlighting the impact of the variance profile on the rate function [12, 18]. In [20], an LDP was obtained for $G_N + B_N$ where B_N is a diagonal matrix without outliers, namely when $\Lambda = \ell_\nu$.

The strategy used in [20] is based on a tilt by spherical integrals introduced in [13] and subsequently developed in diverse contexts in [3, 11, 12, 14, 16]. However, this strategy can only be used to study large deviations below the limiting outlier $\ell^{\Lambda}_{\nu,t}$. On the other hand, [19] studied the case where $\nu = \delta_0$, namely when B_N has only one non trivial eigenvalue Λ. Large deviations were then obtained thanks to an exact estimation of the density due to the asymptotics of spherical integrals derived in [15]. In this paper, we prove a large deviation principle for $\lambda_1(X_N)$, unifying the results of [19] (where $\nu = \delta_0$) and [20] (where no outlier is permitted, namely $\Lambda = \ell_\nu$). Our approach is new and consists of estimating the density of the law of the smallest eigenvalue by exhibiting a fixed point functional equation for the large deviation rate function.

1.2. Main result

The GOE of size $N\times N$ with variance parameter $t>0$ is the random symmetric matrix $G_N = (G_{ij})_{1\le i,j\le N}$ defined as follows: $G_{ji} = G_{ij}$ for all i,j, and the family $(G_{ij})_{1\le i\le j\le N}$ consists of independent centered Gaussian random variables such that for every $i \le j$,

var:GOEt

$$\mathrm{Var}(G_{ij}) = \frac{t}{N}\big(1 + 1_{i=j}\big). \tag{1.1}$$

Equivalently, G_N has density proportional to $\exp\big(-\frac{N}{4t}\mathrm{Tr}(G^2)\big)$ with respect to Lebesgue measure on the space $\mathcal{S}_N(\mathbb{R})$ of $N\times N$ symmetric matrices with real entries. It is well known, see e.g [25], that the empirical measure of the eigenvalues of G_N converges almost surely towards the semi-circle law σ_t given by

$$\sigma_t(\mathrm{d}x) = \frac{1_{[-2\sqrt{t},2\sqrt{t}]}}{2\pi\sqrt{t}}\sqrt{4t-x^2}\mathrm{d}x\,.$$

We make the following assumptions on the sequence of matrices $(B_N)_{N\in\mathbb{N}}$:

mption:B *Assumptions* 1.1. Let $B_N\in\mathcal{S}_N(\mathbb{R})$ and write $b_1^N\le\cdots\le b_N^N$ for its eigenvalues. Assume:

1. There exists $\nu\in\mathcal{P}(\mathbb{R})$ with $\inf\mathrm{supp}(\nu)>-\infty$ such that, letting $\mu_{B_N} := \frac{1}{N}\sum_{i=1}^N \delta_{b_i^N}$,

eq:a_imu
$$\lim_{N\to\infty} d(\mu_{B_N},\nu) = 0. \tag{1.2}$$

2. Let $\ell_\nu := \inf\mathrm{supp}(\nu)$. There exists $\Lambda\in\mathbb{R}$ such that

eq:Lambdalim
$$b_1^N \underset{N\to\infty}{\longrightarrow} \Lambda \le \ell_\nu. \tag{1.3}$$

3. The second smallest eigenvalue satisfies

conv2
$$b_2^N \underset{N\to\infty}{\longrightarrow} \ell_\nu. \tag{1.4}$$

4. $\sup_N |b_N^N| < \infty$.

We denote by $\eta_\nu:[0,1]\to\mathbb{R}$ the (right-continuous) quantile function of ν, i.e. the generalized inverse of $F_\nu(x) = \nu((-\infty,x])$, so that $\nu = (\eta_\nu)_\#\mathrm{Unif}[0,1]$.

Under this assumption, the empirical measure and the smallest eigenvalue of $X_N = B_N + G_N$ converge almost surely. We now describe their limits more precisely. First, it was shown by Voiculescu [6, 24], see also [1, Corollary 5.4.11], that the empirical measure of the eigenvalues of $X_N + G_N$ converges almost surely towards the free convolution $\nu\boxplus\sigma_t$. Following [7] we can define uniquely this probability measure by giving a formula for its Stieltjes transform. If we denote by G_μ the Stieltjes transform of a probability measure μ on $\mathbb{R}$ given, for z outside the support of μ, by

$$G_\mu(z) = \int\frac{\mathrm{d}\mu(x)}{z-x}\,,$$

then, the Stieltjes transform $G_{\nu\boxplus\sigma_t}$ of $\sigma_t\boxplus\nu$ is defined for every $z\in\mathbb{C}^+$ by

freeco
$$G_{\nu\boxplus\sigma_t}(z) = G_\nu(\omega_{\nu,t}(z))\,. \tag{1.5}$$

Here, $\omega_{\nu,t}$ is the subordination function defined as the functional inverse of

$$H_{\nu,t} : \left|\begin{array}{rcl}\overline{\Omega_{\nu,t}} & \longrightarrow & \mathbb{C}^+\cup\mathbb{R}\\ z & \longmapsto & z+tG_\nu(z)\end{array}\right.,$$

with $\Omega_{\nu,t} := \{u+iv\in\mathbb{C}^+ : v > v_t(u)\}$ where for every $u\in\mathbb{R}$, we set

eq:defvu
$$v_t(u) := \inf\Big\{v\in\mathbb{R}^+ : \int\frac{\mathrm{d}\nu(\lambda)}{|\lambda-u|^2+v^2}\le\frac{1}{t}\Big\}. \tag{1.6}$$

We discuss free convolution and some of its properties in Appendix A.

We denote by $\ell_{\nu,t}$ (resp. ℓ_ν) the left boundary point of the support of $\nu\boxplus\sigma_t$ (resp. of ν). Note that $H_{\nu,t}$ is increasing on $(-\infty,\omega_{\nu,t}(\ell_{\nu,t}))$ and decreasing on $(\omega_{\nu,t}(\ell_{\nu,t}),\ell_\nu)$, so that the equation $H_{\nu,t}(x)=\lambda$ has at most two solutions smaller than ℓ_ν, with the smallest being $\omega_{\nu,t}(\lambda)$.

When $-\Lambda$ is large enough, it is well-known that the smallest eigenvalue $\lambda_1(X_N)$ of $X_N = G_N + B_N$ escapes the bulk, becoming smaller than $\ell_{\nu,t}$. Throughout the paper, we denote

$$\ell_{\nu,t}^\Lambda := \begin{cases} H_{\nu,t}(\Lambda) & \text{if } \Lambda\le\omega_{\nu,t}(\ell_{\nu,t})\\ \ell_{\nu,t} & \text{if } \Lambda\ge\omega_{\nu,t}(\ell_{\nu,t}).\end{cases}$$

We can state the following Theorem:

convth **Theorem 1.1.** *The smallest eigenvalue of $X_N = G_N + B_N$ converges almost surely towards $\ell^\Lambda_{\nu,t}$ when N goes to infinity. Moreover the map $\Lambda \in (-\infty, \ell_\nu) \mapsto \ell^\Lambda_{\nu,t}$ and $\nu \mapsto \ell^\Lambda_{\nu,t}$ are continuous (if $\mathcal{P}(\mathbb{R})$ is endowed with the weak topology).*

The convergence of the smallest eigenvalue was studied in [10, Theorem 5.1] in the case where G_N is a GUE matrix rather than a GOE matrix. They show that the spectrum of $G_N + B_N$ concentrates in a neighborhood of the support of $\sigma_t \boxplus \mu_{B_N}$, from which outliers can be characterized as the image by $H_{\nu,t}$ of the outliers of B_N by (1.5). This extends to the case where G_N follows the GOE, see e.g [4, Theorem 2.5]. The continuity statements are proven in Lemma A.2.

Let $\Lambda \le \ell_\nu$ and $\lambda \le \ell_{\nu,t}$. Throughout the paper, when $\lambda \in (H_{\nu,t}(\ell_\nu), \ell_{\nu,t}]$, we denote $\omega^*_{\nu,t}(\lambda)$ the unique solution in $[\omega_{\nu,t}(\ell_{\nu,t}), \ell_\nu)$ of the equation $H_{\nu,t}(\gamma) = \lambda$. Note that when $\lambda \ge H_{\nu,t}(\Lambda)$ and $\Lambda \in [\omega_{\nu,t}(\ell_{\nu,t}), \ell_\nu)$, $\lambda \in (H_{\nu,t}(\ell_\nu), \ell_{\nu,t}]$ since $H_{\nu,t}$ is decreasing on $[\omega_{\nu,t}(\ell_{\nu,t}), \ell_\nu)$ (see e.g Lemma A.2). Define

eq:defg
$$\gamma_\Lambda(\lambda) := \begin{cases} \Lambda & \text{if } \Lambda \le \omega_{\nu,t}(\ell_{\nu,t}), \text{ or } \Lambda \ge \omega_{\nu,t}(\ell_{\nu,t}) \text{ and } \lambda \le H_{\nu,t}(\Lambda), \\ \omega^*_{\nu,t}(\lambda) & \text{if } \lambda \ge H_{\nu,t}(\Lambda) \text{ and } \Lambda \in [\omega_{\nu,t}(\ell_{\nu,t}), \ell_\nu). \end{cases} \tag{1.7}$$

Notice that $\gamma_\Lambda(\lambda) = \min(\Lambda, \omega^*_{\mu,t}(\lambda))$ (see Figure 1).

For every probability measure ν on $\mathbb{R}$, we let S_ν be the logarithmic potential defined by

def:Snu1
$$S_\nu : x \in \mathbb{R} \setminus \mathrm{supp}(\nu) \mapsto -\int \log|x-\lambda| \mathrm{d}\nu(\lambda). \tag{1.8}$$

We set $I^\Lambda_{\nu,t}$ to be infinite on $(\ell_{\nu,t}, +\infty)$ and otherwise be given by

formula
$$I^\Lambda_{\nu,t}(\lambda) = \frac{1}{2}(S_\nu(\omega_{\nu,t}(\lambda)) - S_\nu(\gamma_\Lambda(\lambda))) + \frac{1}{4t}((\lambda - \gamma_\Lambda(\lambda))^2 - (\lambda - \omega_{\nu,t}(\lambda))^2). \tag{1.9}$$

Then, the main result of our paper is the following large deviation principle.

orem:LDP **Theorem 1.2** (LDP for the smallest eigenvalue, GOE case.)**.** *Let $t > 0$. Let $(B_N)_{N \in \mathbb{N}}$ be a sequence of matrices of size $N \times N$ satisfying Assumptions 1.1. Let ν and Λ be as in (1.2) and (1.3). Let G_N be a sequence of GOE matrices of size $N \times N$ and variance t.*

Then, the smallest eigenvalue of $G_N + B_N$ satisfies a large deviation principle at speed N with good rate function $I^\Lambda_{\nu,t}$. Moreover $I^\Lambda_{\nu,t}$ is strictly convex on $(-\infty, \ell_{\nu,t}]$ and attains its minimum value at $\ell^\Lambda_{\nu,t}$.

Further properties of the rate function are discussed in Lemma 3.1.

Remark 1.2. Assumption 1.1 (4) should not be necessary.

Remark 1.3. Our result can be applied when B_N has no outlier by taking $\Lambda = \ell_\nu$, thus recovering the result of [20], and in the case where $\nu = \delta_0$, thus recovering the result of [19]. This is detailed in Section 4.

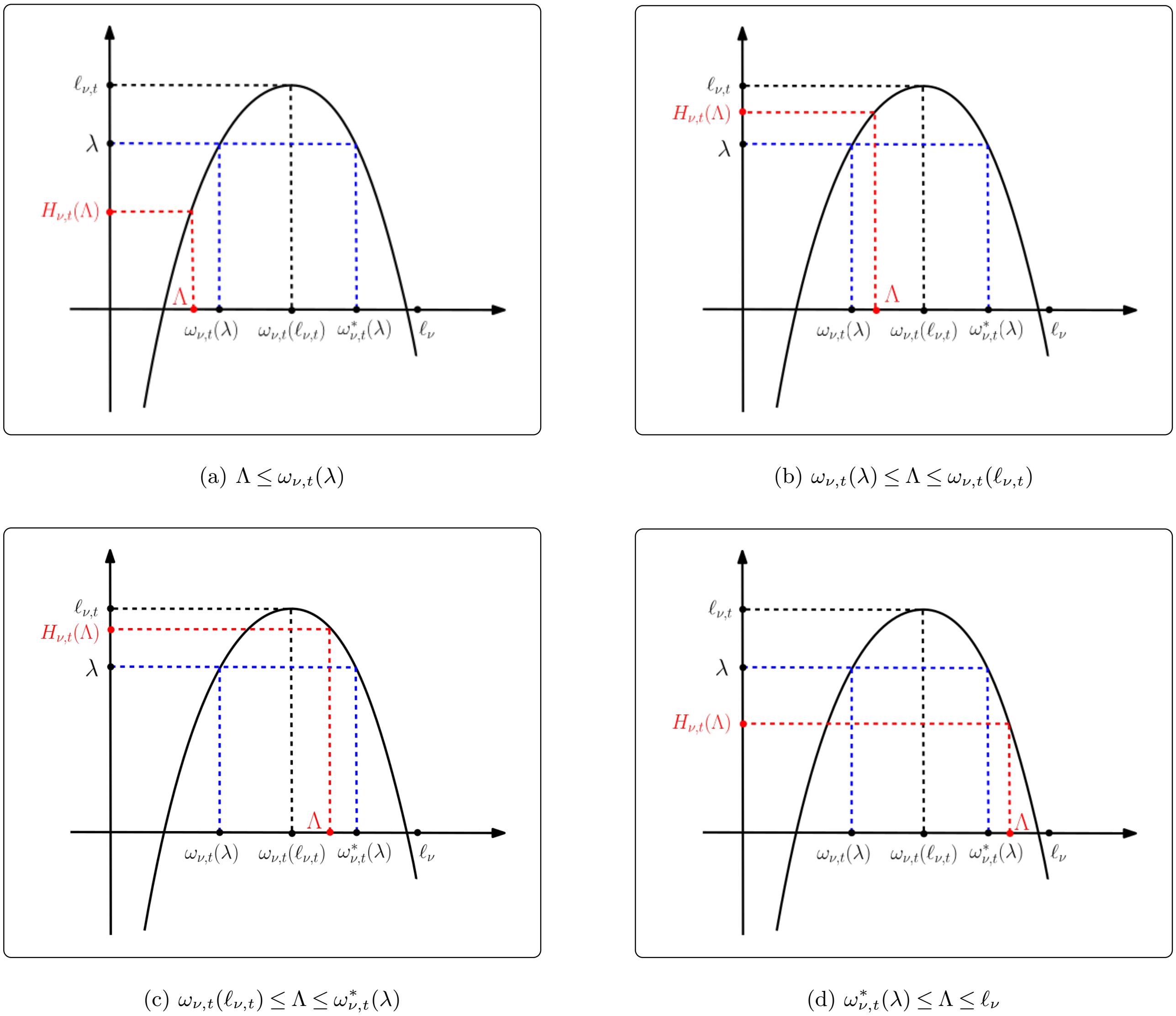
Our result extends readily to the case where G_N is a GUE matrix, namely when G_N is Hermitian, with independent entries above the diagonal which are complex centered Gaussian variables with independent real and imaginary part with variance $t/2N$, and real on the diagonal with variance t/N.

m:LDPGUE **Theorem 1.3** (LDP for the smallest eigenvalue, GUE case)**.** *Let $t > 0$. Let $(B_N)_{N \in \mathbb{N}}$ be a sequence of matrices of size $N \times N$ satisfying Assumptions 1.1. Let ν and Λ be as in (1.2) and (1.3). Let G_N be a sequence of GUE matrices of size $N \times N$ and variance t. Then, the smallest eigenvalue of $G_N + B_N$ satisfies a large deviation principle at speed $2N$ with good rate function $I^\Lambda_{\nu,t}$.*

We will discuss in Section 5 this generalization. Another natural generalization concerns the case where B_N as several outliers, we then make the following conjecture

Conjecture 1.4. Assume that B_N satisfies (1.2) and (1.3), but instead of (1.4), the kth smallest eigenvalues converge towards $\Lambda_1 \le \Lambda_2 \le \cdots \le \Lambda_k \le \ell_\nu$ whereas the $(k+1)$th converges towards ℓ_ν. Let G_N be a GOE matrix. We then conjecture that the kth smallest eigenvalues $\lambda_1 \le \lambda_2 \le \cdots \le \lambda_k$ of $G_N + B_N$ satisfy a large deviation principle with speed N and good rate function $\sum_{i=1}^k I^{\Lambda_i}_{\nu,t}(\lambda_i)$.

In the case where the other eigenvalues of B_N vanish, this conjecture agrees with [14, Proposition 2.7]. Moreover, it agrees with the convergence of the outliers of $B_N + G_N$ towards $(\ell^{\Lambda_i}_{\nu,t})_{1 \le i \le k}$. Note that by the



(a) $\Lambda \leq \omega_{\nu,t}(\lambda)$

(b) $\omega_{\nu,t}(\lambda) \leq \Lambda \leq \omega_{\nu,t}(\ell_{\nu,t})$

(c) $\omega_{\nu,t}(\ell_{\nu,t}) \leq \Lambda \leq \omega^*_{\nu,t}(\lambda)$

(d) $\omega^*_{\nu,t}(\lambda) \leq \Lambda \leq \ell_\nu$

e:3cases

Fig 1: The graph of the function $H_{\nu,t}$

contraction principle, the resulting rate function for λ_1 would then be given by

$$I_{\nu,t}^{\Lambda_1,\dots,\Lambda_k}(x) = \inf\left\{\sum_{i=1}^{k} I_{\nu,t}^{\Lambda_i}(\lambda_i), \lambda_1 = x \leq \lambda_2 \leq \cdots \leq \lambda_k\right\}$$

which is equal to $I_{\nu,t}^{\Lambda_1}(x)$ if $x \leq \ell_{\nu,t}^{\Lambda_2}$ but to $\sum_{j\leq i} I_{\nu,t}^{\Lambda_j}(x)$ if $x \in [\ell_{\nu,t}^{\Lambda_i}, \ell_{\nu,t}^{\Lambda_{i+1}}]$ by monotonicity of the rate function on this segment. We believe than the same approach than the one developed in this article should apply, but it is not straightforward.

1.3. Notations

For the reader's convenience, we collect in this section all notations used in this paper, some of which have been defined already.

- Let $S^{N-1} \subset \mathbb{R}^N$ denote the unit sphere, and let $\mathcal{S}_N(\mathbb{R})$ denote the space of real symmetric $N \times N$ matrices.
- We denote by $\mathcal{P}(\mathbb{R})$ the set of probability measures on $\mathbb{R}$. We denote by $\eta_\nu : [0,1] \to \mathbb{R}$ the (right-continuous) quantile function of ν, i.e. the generalized inverse of $F_\nu(x) = \nu((-\infty, x])$.

- For any $\nu \in \mathcal{P}(\mathbb{R})$, we denote by $\mathrm{supp}(\nu)$ the support of ν and ℓ_ν the infimum of $\mathrm{supp}(\nu)$. For $t > 0$ we denote $\ell_{\nu,t} := \ell_{\nu \boxplus \sigma_t}$.
- For any $\nu \in \mathcal{P}(\mathbb{R})$, we let S_ν be the logarithmic potential defined by

$$S_\nu : x \in \mathbb{R} \setminus \mathrm{supp}(\nu) \mapsto -\int \log|x-\lambda| \mathrm{d}\nu(\lambda).$$

We let $\mathbb{C}^+ = \{z = x+iy : y > 0\}$. Define for $\nu \in \mathcal{P}(\mathbb{R})$ with support with endpoints a, b,

def:Gnu

$$G_\nu : \mathbb{C}^+ \cup \mathbb{R} \setminus [a,b] : z \mapsto \int \frac{\mathrm{d}\nu(x)}{z-x}. \tag{1.10}$$

We then let $G_\nu(a) = \lim_{x \uparrow a} G_\nu(x)$ and $G_\nu(b) = \lim_{x \downarrow b} G_\nu(x)$. One can show that $G_\nu|_{\mathbb{R}}$ is a bijection from $\mathbb{R} \setminus [a,b]$ to $(G_\nu(a), G_\nu(b)) \setminus \{0\}$. We will write

def:Knu

$$K_\nu : (G_\nu(a), G_\nu(b)) \setminus \{0\} \to \mathbb{R} \setminus [a,b] \tag{1.11}$$

its inverse.
- For every $\nu \in \mathcal{P}(\mathbb{R})$, we let

def:rhonut

$$H_{\nu,t}(\Lambda) = \Lambda + tG_\nu(\Lambda) \tag{1.12}$$

and

def:ellnut

$$\ell^\Lambda_{\nu,t} = \begin{cases} H_{\nu,t}(\Lambda) & \text{if } \Lambda \leq \omega_{\nu,t}(\ell_{\nu,t}) \\ \ell_{\nu,t} & \text{if } \Lambda \geq \omega_{\nu,t}(\ell_{\nu,t}). \end{cases} \tag{1.13}$$

- For every $\nu \in \mathcal{P}(\mathbb{R})$ and $\lambda \in [H_{\nu,t}(\ell_\nu), \ell_{\nu,t}]$, we denote $\omega^*_{\nu,t}(\lambda)$ the unique solution γ of

def:omega*

$$H_{\nu,t}(\gamma) = \lambda, \quad \gamma \in [\omega_{\nu,t}(\ell_{\nu,t}), \ell_\nu]. \tag{1.14}$$

- We denote $(e_1, \ldots, e_N)$ the canonical basis of $\mathbb{R}^N$.
- We let d be the bounded Lipschitz distance on probability measure on $\mathbb{R}$ defined for all $\mu, \nu \in \mathcal{P}(\mathbb{R})$ by

def:distance

$$d(\mu,\nu) = \sup \left| \int f \mathrm{d}(\mu - \nu) \right|, \tag{1.15}$$

where the supremum is taken over bounded 1-Lipschitz functions $f : \mathbb{R} \to \mathbb{R}$ such that $\|f\|_\infty + \|f\|_L \leq 1$ where

$$\|f\|_L := \sup_{x \neq y} \left| \frac{f(x) - f(y)}{x - y} \right|, \qquad \|f\|_\infty = \sup_{x \in \mathbb{R}} |f(x)|.$$

- For every $p \geq 1$, we denote Δ^p the p-simplex

def:Deltap

$$\Delta^p := \{(y_1, \ldots, y_p) \in [0,1]^p : y_1 + \cdots + y_p = 1\}. \tag{1.16}$$

- $\lambda^N = (\lambda_1, \ldots, \lambda_N)$ denotes a N dimensional vector in $\mathbb{R}^N$ and $\mathrm{d}\lambda^N = \mathrm{d}\lambda_1 \cdots \mathrm{d}\lambda_N$.
- $O(A)$ (resp. $o(A)$) designates a function so that there exists a universal constant C such that $|O(A)| \leq C|A|$ (resp. $o(A)/|A|$ goes to zero as A goes to zero or infinity (according to the context)). $o_A(1)$ is a quantity going to zero when A goes to infinity or zero, depending on the context.
- If u is a vector in $\mathbb{R}^k$ or $\mathbb{C}^k$, $\|u\| = \sqrt{\sum |u_i|^2}$, and $\|u\|_\infty = \max |u_i|$. If E is a vector subspace of $\mathbb{R}^k$ or $\mathbb{C}^k$,we denote by $\|v\|_E$ the ℓ_2 norm of the orthogonal projection of v on E.

1.4. Sketch of the proof

of ideas

We assume in these heuristics that the matrix B_N has eigenvalues $\Lambda := \eta_0 < \eta_1 < \cdots < \eta_p$ with multiplicities 1, $N_1, \ldots, N_p$, such that for every $i = 1, \ldots, p$, N_i/N converges to $\alpha_i > 0$. We outline the arguments to derive the weak large deviation principle, namely to estimate the probability that the smallest eigenvalue $\lambda_1(X_N)$ of

$X_N = G_N + B_N$ is close to a given $x \le \ell_{\nu,t}$. Let $\varepsilon > 0$. The probability that $\lambda_1(X_N)$ lies in $(x-\varepsilon, x+\varepsilon)$ is given by

$$\mathbb{P}(\lambda_1(X_N) \in (x-\varepsilon, x+\varepsilon)) = \frac{1}{Z_N^t} \int_{\lambda_1(X_N)\in(x-\varepsilon,x+\varepsilon)} e^{-\frac{N}{4t}\operatorname{Tr}((X_N - B_N)^2)}\, \mathrm{d}X_N,$$

where $\mathrm{d}X_N$ is the Lebesgue measure on symmetric matrices of size $N \times N$ and Z_N^t is the normalizing constant. Let us denote by $\lambda_1 \le \cdots \le \lambda_N$ the eigenvalues of X_N and $(v_1, \ldots, v_N)$ the associated orthonormal eigenvectors. We show in Lemma 2.1 that one can split the term in the density into

$$\operatorname{Tr}((X_N - B_N)^2) = \lambda_1^2 - 2\lambda_1\langle v_1, B_N v_1\rangle + 2\langle v_1, B_N^2 v_1\rangle - \langle v_1, B_N v_1\rangle^2 + \operatorname{Tr}((p_{v_1}^{\perp}(X_N - B_N)p_{v_1}^{\perp})^2),$$

where $p_{v_1}^{\perp}$ stands for the projection onto the ortho-complement of v_1. For every $\lambda \in \{\Lambda = \eta_0, \eta_1 \ldots, \eta_p\}$, let us denote E_λ the associated eigenspace of B_N and, for a vector $v \in \mathbb{R}^N$, we write $\|v\|_{E_\lambda}$ the Euclidean norm of the projection of v on E_λ. We then observe that $\langle v_1, B_N v_1\rangle$ and $\langle v_1, B_N^2 v_1\rangle$ only depend on $Y_N(v_1) := (\|v_1\|^2_{E_{\eta_0}}, \ldots, \|v_1\|^2_{E_{\eta_p}})$ and λ_1. In fact

$$\frac{1}{4t}(-2\lambda_1\langle v_1, B_N v_1\rangle + 2\langle v_1, B_N^2 v_1\rangle - \langle v_1, B_N v_1\rangle^2) =: L^{\Lambda}_{\nu,t}(\lambda_1, Y_N(v_1))$$

is a polynomial $L^{\Lambda}_{\nu,t}$ in λ_1 and $Y_N(v_1)$, see Lemma 2.1. Performing the change of variables $X_N \mapsto (\lambda_1, \ldots, \lambda_N, v_1, O)$, we find a constant C_N^t such that

heu1 (1.17)

$$P_N(x,\varepsilon) := \int_{\lambda_1\in(x-\varepsilon,x+\varepsilon)} e^{-\frac{N}{4t}\operatorname{Tr}((X_N - B_N)^2)}\, \mathrm{d}X_N = C_N^t \int_{\lambda_1\in(x-\varepsilon,x+\varepsilon)} \mathrm{d}\lambda_1\, \mathrm{d}v_1\, e^{-NL_{\nu,t}(\lambda_1, Y_N(v_1)) - N\frac{\lambda_1^2}{4t}}$$
$$\times \int \prod_{j>1}|\lambda_1 - \lambda_j| e^{-\frac{N}{4t}\operatorname{Tr}((OD(\lambda)O^T - p_{v_1}^{\perp}B_N p_{v_1}^{\perp})^2)} \prod_{2\le i<j\le N}|\lambda_i - \lambda_j| 1_{\lambda_1\le\ldots\le\lambda_N}\, \mathrm{d}\lambda_2\cdots\mathrm{d}\lambda_N\, \mathrm{d}O,$$

where $D(\lambda) = \operatorname{diag}(\lambda_2, \ldots, \lambda_N)$, O follows the Haar measure on $v_1^{\perp}$. Notice that $OD(\lambda)O^T$ follows the law of the $(N-1)\times(N-1)$ GOE matrix $\sqrt{(N-1)/N}G_{N-1}$ (living in $v_1^{\perp}$) under the distribution

def:prob

$$\mathbb{P}_{N-1}(\mathrm{d}\lambda, \mathrm{d}O) := \frac{1}{Z_{N-1}^{t(N-1)/N}} e^{-\frac{N}{4t}\operatorname{Tr}((OD(\lambda)O^T)^2)} \prod_{2\le i<j\le N}|\lambda_i - \lambda_j| 1_{\lambda_2\le\cdots\le\lambda_N}\, \mathrm{d}\lambda_2\cdots\mathrm{d}\lambda_N\, \mathrm{d}O. \tag{1.18}$$

Putting everything together, if $\mathbb{P}_{p_{v_1}^{\perp}B_N p_{v_1}^{\perp}}$ designates the law of $X_{N-1} := p_{v_1}^{\perp}B_N p_{v_1}^{\perp} + \sqrt{(N-1)/N}G_{N-1}$, we find that $P_N(x,\varepsilon)$ equals

$$C_N^t Z_{N-1}^{t(N-1)/N} \int_{\lambda_1\in(x-\varepsilon,x+\varepsilon)} \mathrm{d}\lambda_1\, \mathrm{d}v_1\, e^{-NL^{\Lambda}_{\nu,t}(\lambda_1, Y_N(v_1)) - N\frac{\lambda_1^2}{4t}} \int_{\lambda_2\ge\lambda_1} \prod_{j>1}|\lambda_1 - \lambda_j| \mathrm{d}\mathbb{P}_{p_{v_1}^{\perp}B_N p_{v_1}^{\perp}}(\lambda_2, \ldots, \lambda_N). \tag{1.19}$$

We observe that X_{N-1} is a one-dimensional perturbation of the matrix $X_N = G_N + B_N$ and therefore that the empirical measure of its eigenvalues converges towards $\nu \boxplus \sigma_t$ almost surely. Restricting ourselves to this set, we can approximate $\prod_{j>1}|\lambda_1 - \lambda_j|$ by $\exp(N\int \log|x-y|\mathrm{d}\nu\boxplus\sigma_t)$ in the above integral. Fix $Y = (y_0, \ldots, y_p) \in \Delta^{p+1}$, where Δ^{p+1} is the $(p+1)$-simplex (1.16). Restricting the domain of integration to vectors $v_1 \in \mathbb{R}^N$ such that $Y_N(v_1)$ is close to Y and using the large deviation principle for $Y_N(v_1)$ when v_1 follows the uniform law on the sphere stated in Lemma 2.2, we roughly obtain the lower bound

lk

$$\liminf_{N\to\infty} \frac{1}{N} \log \frac{1}{Z_N^t} \int_{\lambda_1(X_N)\in(x-\varepsilon,x+\varepsilon)} e^{-\frac{N}{4t}\operatorname{Tr}((X_N - B_N)^2)}\, \mathrm{d}X_N \gtrsim -J^{\Lambda}_{\nu,t}(x,Y) + \int \log|\lambda - x|\, \mathrm{d}(\nu\boxplus\sigma_t)(\lambda)$$
$$- \frac{x^2}{4t} - C_t + \liminf_{N\to\infty}\frac{1}{N}\log \mathbb{P}_{B'_{N-1}}(\lambda_2 \ge x), \tag{1.20}$$

where B'_{N-1} is the $(N-1)\times(N-1)$ matrix living in $v_1^{\perp}$ where it equals $p_{v_1}^{\perp}B_N p_{v_1}^{\perp}$, C_t is a finite constant and

$$J^{\Lambda}_{\nu,t} : (\lambda, Y) \in (-\infty, \ell_{\nu,t}] \times \Delta^{p+1} \mapsto L^{\Lambda}_{\nu,t}(\lambda, Y) - \frac{1}{2}\sum_{k=1}^{p} \alpha_k \log(y_k/\alpha_k).$$

By Courant-Fischer's theorem (see Lemma 2.3), when $Y_N(v_1) \simeq Y$, the matrix B'_{N-1} has a unique outlier approximately equal to $\Lambda_1 := \Phi(\Lambda, Y)$, which is defined as the smallest solution in $[\Lambda, \eta_1]$ of the equation

$$\frac{y_0}{\Lambda - \Lambda_1} + \sum_{k=1}^{p} \frac{y_k}{\eta_k - \Lambda_1} = 0.$$

Note that it always exists when y_0 and y_1 are positive since the function of Λ_1 on the left hand side then goes from $-\infty$ to $+\infty$. We will see it is then unique. Moreover, the spectrum of B'_{N-1} converges to $\nu = \sum_{k=1}^{p} \alpha_k \delta_{\eta_k}$. As a consequence, we see that the last term in (1.20) is also the probability of deviation of the smallest eigenvalue λ_2 above x of a $(N-1) \times (N-1)$ matrix given by the sum of a GOE matrix and a matrix B'_{N-1} with empirical measure of the eigenvalues converging to ν and with outlier $\Phi(\Lambda, Y)$. We set

heu2

$$F^{-}_{\nu,t}(\Lambda, x) := -\lim_{\varepsilon \to 0} \liminf_{N \to \infty} \frac{1}{N} \log \mathbb{P}(\lambda_1(X_N) \in (x - \varepsilon, x + \varepsilon)). \tag{1.21}$$

As a consequence of the previous considerations, we obtain the functional inequality

$$-F^{-}_{\nu,t}(\Lambda, x) \gtrsim \sup_{Y \in \Delta^{p+1}} \left\{ -J^{\Lambda}_{\nu,t}(x, Y) - C_t + \int \log|\lambda - x| \, \mathrm{d}(\nu \boxplus \sigma_t)(\lambda) - \frac{x^2}{4t} - \inf_{y \geq x} F^{-}_{\nu,t}(\Phi(\Lambda, Y), y) \right\},$$

for some constant C_t depending only on t. In other words, we find

heu3

$$F^{-}_{\nu,t}(\Lambda, x) \lesssim \inf_{Y \in \Delta^{p+1}} \left\{ J^{\Lambda}_{\nu,t}(x, Y) + C_t - \int \log|\lambda - x| \, \mathrm{d}(\nu \boxplus \sigma_t)(\lambda) + \frac{x^2}{4t} + \inf_{y \geq x} F^{-}_{\nu,t}(\Phi(\Lambda, Y), y) \right\}. \tag{1.22}$$

One can define $F^{+}_{\nu,t}$ similarly but with $\limsup$ instead of $\liminf$ and show that

$$F^{+}_{\nu,t}(\Lambda, x) \gtrsim \inf_{Y \in \Delta^{p+1}} \left\{ J^{\Lambda}_{\nu,t}(x, Y) + C_t - \int \log|\lambda - x| \, \mathrm{d}(\nu \boxplus \sigma_t)(\lambda) + \frac{x^2}{4t} + \inf_{y \geq x} F^{+}_{\nu,t}(\Phi(\Lambda, Y), y) \right\}.$$

We then solve the above functional inequalities. We first show that $F^{+}_{\nu,t}$ and $F^{-}_{\nu,t}$ vanish at $\ell^{\Lambda}_{\nu,t}$, are decreasing on $(-\infty, \ell^{\Lambda}_{\nu,t}]$ and increasing on $(\ell^{\Lambda}_{\nu,t}, \ell_{\nu,t}]$. Therefore $\inf_{y \geq x} F^{\pm}_{\nu,t}(\Phi(\Lambda, Y), y)$ vanish except when $x \geq \ell^{\Phi(\Lambda,Y)}_{\nu,t}$, namely when $\omega_{\nu,t}(x) \geq \Phi(\Lambda, Y)$. Then $\inf_{y \geq x} F^{\pm}_{\nu,t}(\Phi(\Lambda, Y), y) = F^{\pm}_{\nu,t}(\Phi(\Lambda, Y), x)$. We then prove that $F^{+}_{\nu,t}, F^{-}_{\nu,t}$ are in fact equal to $I^{\Lambda}_{\nu,t}$ which satisfies the functional equation

$$I^{\Lambda}_{\nu,t}(x) = \inf_{Y \in \Delta^{p+1}} \left\{ J^{\Lambda}_{\nu,t}(x, Y) + C_t - \int \log|y - x| \mathrm{d}(\nu \boxplus \sigma_t)(y) + \frac{x^2}{4t} + I^{\Phi(\Lambda,Y)}_{\nu,t}(x) 1_{\omega_{\nu,t}(x) \geq \Phi(\Lambda,Y)} \right\}.$$

This establishes the weak LDP which can easily be upgraded to a full LDP. We finally extend the result to general matrices that satisfy Assumptions 1.1. This is achieved by bounding B_N from below and above by two matrices that have a finite number of eigenvalues, and by using monotonicity arguments.

2. Large deviations in the pure point case

Let B_N be a $N \times N$ diagonal matrix with eigenvalues Λ with multiplicity 1 and $\eta_1, \ldots, \eta_p$ with multiplicity $N_1, \cdots, N_p$ such that for every $i \in \{1, \ldots, p\}$, N_i/N converges towards $\alpha_i > 0$. Let G_N be a GOE matrix of size $N \times N$ and variance $t > 0$. We assume that $\Lambda = \eta_0 \leq \eta_1 < \eta_2 < \cdots < \eta_p$ and denote

def:nu

$$\nu := \sum_{i=1}^{p} \alpha_i \delta_{\eta_i}. \tag{2.1}$$

Throughout this section we denote by $\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_N$ the eigenvalues of $X_N = B_N + G_N$. We denote in short $\lambda^N = (\lambda_1, \ldots, \lambda_N)$.

Let Δ^{p+1} be the $(p+1)$-simplex as defined in (1.16). We recall that $\nu \boxplus \sigma_t$ is the limiting distribution of the spectrum of $G_N + B_N$, namely the almost sure limit of $\frac{1}{N} \sum_{i=1}^{N} \delta_{\lambda_i}$, and $\ell_{\nu,t}$ denotes the leftmost point in the support of $\nu \boxplus \sigma_t$. The goal of this section is to prove the following large deviation principle, which specializes Theorem 1.2 to the case where B_N has a fixed number of distinct eigenvalues. Let $I^{\Lambda}_{\nu,t}$ be as defined in (1.9).

LDP:dis *Proposition* 2.1 (LDP in the pure point case). Let ν be as in (2.1) and $\Lambda \le \ell_\nu = \inf_{1\le k\le p} \eta_k$.

1. The map $I^\Lambda_{\nu,t}$ is a good rate function which is strictly convex and vanishes at $\ell^\Lambda_{\nu,t}$.
2. Let λ_1 be the smallest eigenvalue of $X_N = G_N + B_N$. Then the law of λ_1 satisfies a large deviation principle with speed N and rate function $I^\Lambda_{\nu,t}$.

It is easy to see that $I^\Lambda_{\nu,t}$ is a good rate function. Indeed, it is continuous on $(-\infty, \ell_{\nu,t})$ since S_ν is continuous on $(-\infty, \ell_\nu)$, $\omega_{\nu,t}$ is continuous and increasing on $(-\infty, \ell_{\nu,t})$, with values in $(-\infty, \omega_{\nu,t}(\ell_{\nu,t})]$, and $\omega^*_{\nu,t}$ is continuous with values in $[\omega_{\nu,t}(\ell_{\nu,t}), \ell_\nu]$. This shows that $I^\Lambda_{\nu,t}$ is continuous except possibly at $\ell_{\nu,t}$ where it is lower semi-continuous. Moreover, $I^\Lambda_{\nu,t}(\ell^\Lambda_{\nu,t}) = 0$.Indeed, if $\Lambda \le \omega_{\nu,t}(\ell_{\nu,t})$, then $\ell^\Lambda_{\nu,t} = H_{\nu,t}(\Lambda)$ and $\omega_{\nu,t}(\ell^\Lambda_{\nu,t}) = \Lambda = \gamma_\Lambda(\ell^\Lambda_{\nu,t})$. Besides, if $\Lambda \ge \omega_{\nu,t}(\ell_{\nu,t})$, $\ell^\Lambda_{\nu,t} = \ell_{\nu,t}$ and $\omega_{\nu,t}(\ell_{\nu,t}) = \omega^*_{\nu,t}(\ell_{\nu,t}) = \gamma_\Lambda(\ell^\Lambda_{\nu,t})$. Let us show that its level sets are compact. As $x \to -\infty$, we notice that $\omega_{\nu,t}(x) \sim x$ whereas $\omega^*_{\nu,t}(x)$ and $\gamma_\Lambda(x)$ stay bounded. It follows that

infinity
$$I^\Lambda_{\nu,t}(x) = \frac{1}{2} S_\nu(\omega_{\nu,t}(x)) + \frac{x^2}{4t} + o(x^2) = \frac{x^2}{4t} + o(x^2) \underset{x\to-\infty}{\longrightarrow} +\infty, \tag{2.2}$$

where $o(x^2)/x^2$ goes to zero as x goes to infinity. Hence, the level sets of $I^\Lambda_{\nu,t}$ are compact. The convexity of $I^\Lambda_{\nu,t}$ is established in Lemma 3.1, along with the fact that the minimum is achieved at $\ell^\Lambda_{\nu,t}$. It follows in particular that $I^\Lambda_{\nu,t}$ is non-negative. Hence, $I^\Lambda_{\nu,t}$ is a convex good rate function.

The fact that λ_1 is exponentially tight is clear since B_N is uniformly bounded and the smallest eigenvalue of the GOE is well known to be exponentially tight [1, Section 2.6.2]. We more precisely find that (see e.g the proof of [13, Lemma 1.8]), since the entries are Gaussian, the operator norm of X_N, namely $\|X_N\|_{\mathrm{op}} = \sup_{\|v\|=1} |\langle u, X_N u\rangle|$, is such that there exists a finite constant C such that for N large enough,

expt
$$\mathbb{P}(\|X_N\|_{\mathrm{op}} \ge K) \le e^{-N(K-C)}. \tag{2.3}$$

The main point of the proof of Proposition 2.1, on which we will focus for the remainder of this section, is therefore to prove a weak large deviation principle, namely that for every $x < \ell_{\nu,t}$,

wldp
$$-I^\Lambda_{\nu,t}(x) \le \lim_{\delta\to 0} \liminf_{N\to\infty} \frac{1}{N} \log \mathbb{P}(|\lambda_1 - x| \le \delta) \le \lim_{\delta\to 0} \limsup_{N\to\infty} \frac{1}{N} \log \mathbb{P}(|\lambda_1 - x| \le \delta) \le -I^\Lambda_{\nu,t}(x). \tag{2.4}$$

In fact, the large deviation principle (2.4) is easy to prove for $x > \ell_{\nu,t}$.Indeed, by concentration of measure, we know that since Gaussian variables satisfy log-Sobolev inequality with a constant independent from their mean, that there exists positive universal constant C_1, C_2 so that for every $\zeta > 0$, see e.g [17, Corollary 1.4(b)],

con0
$$\mathbb{P}\Big(d\Big(\frac{1}{N}\sum_{i=1}^N \delta_{\lambda_i}, \mathbb{E}\Big[\frac{1}{N}\sum_{i=1}^N \delta_{\lambda_i}\Big]\Big) \ge \zeta\Big) \le \frac{C_1}{\zeta^{3/2}} e^{-C_2 N^2 \zeta^2}, \tag{2.5}$$

where d is the distance defined in (1.15). Furthermore, since $\mathbb{E}[\frac{1}{N}\sum \delta_{\lambda_i}]$ converges weakly towards $\nu_\alpha \boxplus \sigma_t$, $d(\mathbb{E}[\frac{1}{N}\sum \delta_{\lambda_i}], \nu_\alpha \boxplus \sigma_t)$ goes to zero as N goes to infinity. Therefore, for every $\zeta > 0$, there exists $C_\zeta > 0$ such that for N large enough,

con1
$$\mathbb{P}\Big(d\Big(\frac{1}{N}\sum_{i=1}^N \delta_{\lambda_i}, \nu_\alpha \boxplus \sigma_t\Big) \ge \zeta\Big) \le e^{-C_{\zeta/2} N^2}, \tag{2.6}$$

with $\nu_\alpha = \sum \alpha_i \delta_{\eta_i}$. However, if for some $\varepsilon > 0$, $x > \ell_{\nu,t} + 2\varepsilon$, there exists $\zeta > 0$ small enough so that $\{|\lambda_1 - x| \le \varepsilon\} \subset \{d\Big(\frac{1}{N}\sum \delta_{\lambda_i}, \nu_\alpha \boxplus \sigma_t\Big) \ge \zeta\}$ from which we conclude from (2.6) that

$$\lim_{\delta\to 0} \limsup_{N\to\infty} \frac{1}{N} \log \mathbb{P}(|\lambda_1 - x| \le \delta) \le -\infty.$$

We do not need to prove (2.4) at $x = \ell_{\nu,t}$ either. Indeed, if $\ell_{\nu,t} = \ell^\Lambda_{\nu,t}$, then Theorem 1.1 shows that for every $\delta > 0$,

$$\limsup_{N\to\infty} \frac{1}{N} \log \mathbb{P}(|\lambda_1 - \ell^\Lambda_{\nu,t}| \le \delta) = \liminf_{N\to\infty} \frac{1}{N} \log \mathbb{P}(|\lambda_1 - \ell^\Lambda_{\nu,t}| \le \delta) = 0,$$

which corresponds to the announced statement, since by Proposition 2.1, $I^{\Lambda}_{\nu,t}(\ell^{\Lambda}_{\nu,t})=0$. If $\ell_{\nu,t}>\ell^{\Lambda}_{\nu,t}$, the monotonicity of the large deviation rate function proven in Lemma B.1 implies that for every $\delta>0$ small enough, every $y<\ell_{\nu,t}$ so that $|y-\ell_{\nu,t}|<|y-\ell^{\Lambda}_{\nu,t}|/2$,

$$\limsup_{N\to\infty}\frac{1}{N}\log\mathbb{P}(|\lambda_1-\ell_{\nu,t}|\le\delta)\le\limsup_{N\to\infty}\frac{1}{N}\log\mathbb{P}(|\lambda_1-y|\le\delta)$$

so that equation (2.4) at $y<\ell_{\nu,t}$ gives, after taking the limit $\delta\to0$,

$$\limsup_{\delta\to0}\limsup_{N\to\infty}\frac{1}{N}\log\mathbb{P}(|\lambda_1-\ell_{\nu,t}|\le\delta)\le -I^{\Lambda}_{\nu,t}(y)$$

and the left-continuity of $I^{\Lambda}_{\nu,t}$ at $\ell_{\nu,t}$ implies by letting y going to $\ell_{\nu,t}$ that

bo

$$\limsup_{N\to\infty}\frac{1}{N}\log\mathbb{P}(|\lambda_1-\ell_{\nu,t}|\le\delta)\le -I^{\Lambda}_{\nu,t}(\ell_{\nu,t})\,. \tag{2.7}$$

Similarly for the lower bound, for every $\delta>0$ and $\gamma\in(0,\delta/4)$, we have

$$\liminf_{N\to\infty}\frac{1}{N}\log\mathbb{P}(|\lambda_1-\ell_{\nu,t}|\le\delta)\ge\liminf_{N\to\infty}\frac{1}{N}\log\mathbb{P}(|\lambda_1-\ell_{\nu,t}-\delta/2|\le\gamma),$$

which implies by equation (2.4), after letting N going to infinity and γ to zero, that

$$\liminf_{N\to\infty}\frac{1}{N}\log\mathbb{P}(|\lambda_1-\ell_{\nu,t}|\le\delta)\ge -I^{\Lambda}_{\nu,t}(\ell_{\nu,t}-\delta/2).$$

Equation (2.4) at $x=\ell_{\nu,t}>\ell^{\Lambda}_{\nu,t}$, then follows from the left-continuity of $I^{\Lambda}_{\nu,t}$ at $\ell_{\nu,t}$ (and (2.7)).

Hence, the rest of this section is dedicated to the proof of (2.4). As announced in the heuristics, the proof of (2.4) starts by factorizing the terms depending on λ_1.

2.1. *Projection onto the first eigenvector*

Let v_1 be an eigenvector of X_N of norm 1 associated to λ_1. One can first project onto $v_1^{\perp}$ using the following lemma:

ojection

Lemma 2.1. *Let $X,B\in\mathcal{S}_N(\mathbb{R})$. Let λ_1 be an eigenvalue of X and $v_1\in S^{N-1}$ be an associated eigenvector. One may write*

plitting

$$\mathrm{Tr}((X-B)^2)=\lambda_1^2-2\lambda_1\langle v_1,Bv_1\rangle+2\langle v_1,B^2v_1\rangle-\langle v_1,Bv_1\rangle^2+\mathrm{Tr}((p^{\perp}_{v_1}(X-B)p^{\perp}_{v_1})^2), \tag{2.8}$$

where $p^{\perp}_{v_1}$ is the projection onto the orthocomplement of v_1. Moreover, let $(\Lambda,\eta_i,1\le i\le p)$ be the eigenvalues of B. Denote by w_0 a unit eigenvector of B associated to the eigenvalue Λ and $P_{E_{\eta_k}}$ the orthogonal projection onto the eigenspace E_{η_k} corresponding to the eigenvalue η_k for $k\in\{1,\dots,p\}$. Denote by $Y^N=(Y^N_0,\dots,Y^N_p)$ the vector in Δ^{p+1} given by

eq:defY

$$Y^N_0=\langle v_1,w_0\rangle^2,\quad Y^N_k=\|P_{E_{\eta_k}}v_1\|^2,\quad 1\le k\le p. \tag{2.9}$$

Then,

equi

$$-2\lambda_1\langle v_1,Bv_1\rangle+2\langle v_1,B^2v_1\rangle-\langle v_1,Bv_1\rangle^2=4tL^{\Lambda}_{\nu,t}(\lambda_1,Y^N) \tag{2.10}$$

with $L^{\Lambda}_{\nu,t}:(-\infty,\ell_{\nu,t}]\times\Delta^{p+1}\to\mathbb{R}$, given for all $x\in(-\infty,\ell_{\nu,t}]$ and $Y\in\Delta^{p+1}$ by

defL

$$L^{\Lambda}_{\nu,t}(x,Y):=-\frac{x}{2t}\Big(\Lambda y_0+\sum_{k=1}^{p}\eta_k y_k\Big)+\frac{1}{2t}\Big(\sum_{k=1}^{p}\eta_k^2y_k+\Lambda^2y_0\Big)-\frac{1}{4t}\Big(\Lambda y_0+\sum_{k=1}^{p}\eta_k y_k\Big)^2. \tag{2.11}$$

Proof. Let $v_1, \dots, v_N$ be a family of eigenvectors of X. By writing the matrix $X - B$ in the basis $v_1, \dots, v_N$, one can check that

$$\mathrm{Tr}((X-B)^2) = \langle v_1, (X-B)v_1\rangle^2 + 2\sum_{j>1} \langle v_1, (X-B)v_j\rangle^2 + \mathrm{Tr}\Big((p_{v_1}^{\perp}(X-B)p_{v_1}^{\perp})^2\Big). \tag{2.12}$$

One may then write

$$(X-B)v_1 = \sum_{j=1}^{N} \langle v_j, (X-B)v_1\rangle v_j.$$

Therefore

$$\langle v_1, (X-B)^2 v_1\rangle = \sum_{j=1}^{N} \langle v_j, (X-B)v_1\rangle^2$$

which allows one to rewrite (2.12) as

$$\mathrm{Tr}((X-B)^2) = 2\langle v_1, (X-B)^2 v_1\rangle - \langle v_1, (X-B)v_1\rangle^2 + \mathrm{Tr}\Big((p_{v_1}^{\perp}(X-B)p_{v_1}^{\perp})^2\Big).$$

One can finally check using $Xv_1 = \lambda_1 v_1$ that

$$2\langle v_1, (X-B)^2 v_1\rangle - \langle v_1, (X-B)v_1\rangle^2 = \lambda_1^2 - 2\lambda_1 \langle v_1, Bv_1\rangle + 2\langle B^2 v_1, v_1\rangle - \langle v_1, Bv_1\rangle^2. \tag{2.13}$$

Equality (2.8) follows. To prove (2.10), observe that

$$\langle v_1, B_N v_1\rangle = \Lambda \langle v_1, w_0\rangle^2 + \sum_{k=1}^{p} \eta_k \|P_{E_{\eta_k}} v_1\|^2 = \Lambda Y_0^N + \sum_{k=1}^{p} \eta_k Y_k^N, \tag{2.14}$$

$$\langle v_1, B_N^2 v_1\rangle = \Lambda^2 \langle v_1, w_0\rangle^2 + \sum_{k=1}^{p} \eta_k^2 \|P_{E_{\eta_k}} v_1\|^2 = \Lambda^2 Y_0^N + \sum_{k=1}^{p} \eta_k^2 Y_k^N. \tag{2.15}$$

Equality (2.10) follows by plugging (2.14) and (2.15) in (2.13).

□

2.2. LDP for Dirichlet variables

One can observe that, for v_1 uniformly distributed on S^{N-1}, the random variable Y^N defined in (2.9) follows a Dirichlet law with parameter $(\frac{1}{2}, \frac{N_1}{2}, \dots, \frac{N_p}{2})$ which are such that N_i/N goes to $\alpha_i > 0$ for every $i = 1, \dots, p$. Using the explicit density of the Dirichlet distribution, it is easy to prove by Laplace's method [14, Theorem 3.3] that Y^N satisfies the following LDP:

Lemma 2.2. *Let v_1 be uniformly distributed on S^{N-1}. Define*

$$Y_0^N = \langle v_1, w_0\rangle^2, \quad Y_k^N = \|P_{E_{\eta_k}} v_1\|^2, \quad 1 \le k \le p. \tag{2.16}$$

Then, the sequence of random variables (Y^N) satisfies a large deviation principle at speed N and with good rate function given by

$$I_\nu : Y = (y_0, \dots, y_p) \in \Delta^{p+1} \mapsto -\frac{1}{2}\sum_{k=1}^{p} \alpha_k \log(y_k/\alpha_k). \tag{2.17}$$

2.3. Outliers of $p_{v_1}^{\perp} B_N p_{v_1}^{\perp}$

As outlined in Subsection 1.4, the proof involves computing the probability that the smallest eigenvalue of $G_{N-1} + B'_{N-1}$ is larger than x, where G_{N-1} is a GOE matrix of size $(N-1)\times(N-1)$ on the orthocomplement of v_1 and B'_{N-1} is the $(N-1)\times(N-1)$ matrix given by $p_{v_1}^{\perp} B_N p_{v_1}^{\perp}$ in $v_1^{\perp}$. Thus, we first study the outliers of the matrix B'_{N-1} when v_1 is such that Y_N is equal to some Y, and then invoke known results on the BBP transition. Here, $x \le \ell_{\nu,t}$ and $Y \in \Delta^{p+1}$ are given. We define $\Phi(\Lambda,\cdot):\Delta^{p+1}\to\mathbb{R}$ by setting for all $Y\in\Delta^{p+1}$,

def:PhiL

$$\Phi(\Lambda,Y)=\begin{cases}\Lambda & \text{if } y_0=0,\\ \Lambda_1 & \text{if } y_0\in(0,1], y_1\in(0,1],\\ \eta_1 & \text{if } y_1=0 \text{ and } y_0\neq 0,\end{cases} \tag{2.18}$$

where $\Lambda_1\in[\Lambda,\eta_1]$ is the unique solution in $[\Lambda,\eta_1]$ of the equation

eq:sm

$$\frac{y_0}{\Lambda-\Lambda_1}+\sum_{k=1}^{p}\frac{y_k}{\eta_k-\Lambda_1}=0. \tag{2.19}$$

Observe that Λ_1 is always uniquely defined when y_0 and y_1 are positive. Indeed, because $y_0=1-\sum_{k=1}^p y_k$, (2.19) is equivalent to

eq:sm2

$$f(\Lambda_1)=1 \qquad\text{with}\qquad f(x)=\sum_{k=1}^{p} y_k\frac{\eta_k-\Lambda}{\eta_k-x}\,. \tag{2.20}$$

f is continuous on $[\Lambda,\eta_1]$ with values $f([\Lambda,\eta_1])=[\sum_{k=1}^p y_k,(+\infty)1_{y_1>0}+(\sum_{k=2}^p y_k\frac{\eta_k-\Lambda}{\eta_k-\eta_1})1_{y_1=0}]$. Hence, since $\sum_{k=1}^p y_k=1-y_0<1$ if $y_0>0$, we see that (2.20) has at least a solution when y_0 and y_1 are positive by the intermediate value theorem. Because f is strictly increasing on $[\Lambda,\eta_1]$, this solution is unique. Note also that $y_0=1$ implies $y_1=0$ and therefore $\Phi(\Lambda,Y)=\eta_1$. Moreover, when $y_1=0$, there might be no solution if $\sum_{k=2}^p y_k\frac{\eta_k-\Lambda}{\eta_k-\eta_1}<1$, for instance if y_0 is big enough.

spectrum **Lemma 2.3.** *Let B_N be a $N\times N$ diagonal matrix with eigenvalues $\eta_0:=\Lambda<\eta_1<\cdots<\eta_p$ with multiplicity $N_0=1,N_1,\cdots,N_p$ such that for every $i\in\{1,\dots,p\}$, $N_i\ge 2$. Let $v_1\in\mathbb{R}^N$ be a unit vector and B'_{N-1} be the $(N-1)\times(N-1)$ matrix given by the restriction of B_N to the orthocomplement of v_1. Moreover:*

1. *Let $\kappa>0$. Let $\Phi(\Lambda,\cdot)$ be given by (2.18). Then $\Phi(\Lambda,\cdot)$ is continuous on $[0,1]\times[\kappa,1]\times[0,1]^{p-1}\cap\Delta^{p+1}$.*
2. *Assume $v_1\notin\cup_{k=0}^p E_{\eta_k}$. For each $k=0,\dots,p$, set*

eq:v proj

$$y_k(v_1):=\|P_{E_{\eta_k}}v_1\|^2,\qquad Y(v_1):=(y_0(v_1),\dots,y_p(v_1)). \tag{2.21}$$

Assume $y_0(v_1)>0$. Then the matrix B'_{N-1} has eigenvalues $\Lambda_1<\cdots<\Lambda_p$ of multiplicity 1, and $\eta_1,\dots,\eta_p$ of multiplicities $N_1-1,\dots,N_p-1$ (where we adopt the convention that multiplicities are added when eigenvalues coincide). Moreover, for every $k=1,\dots,p$, $\Lambda_k\in(\eta_{k-1},\eta_k]$, and $\Lambda_1=\Phi(\Lambda,Y(v_1))$.

3. *If $v_1\in E_{\eta_k}$ for some $k=0,\dots,p$, then the matrix B'_{N-1} has eigenvalues η_i with multiplicity N_i for every $i\in\{0,\dots,p\}\setminus\{k\}$, and η_k with multiplicity N_k-1. Moreover, the smallest eigenvalue Λ_1 of B'_{N-1} equals $\Phi(\Lambda,Y(v_1))$.*

Proof. Let us prove (1). Let us first show the continuity of $\Phi(\Lambda,Y)$ on $Y\subset[\varepsilon,1]\times[\kappa,1]\times[0,1]^{p-1}\cap\Delta^{p+1}$ for some $\varepsilon>0$. We see from (2.19) that Λ_1 stays away from Λ and η_1. In fact, because $y_0\ge\varepsilon$ and $\sum y_i\le 1$

$$\frac{\varepsilon}{\Lambda_1-\Lambda}\le\frac{y_0}{\Lambda_1-\Lambda}=\sum_{k=1}^{p}\frac{y_k}{\eta_k-\Lambda_1}\le\frac{1}{\eta_1-\Lambda_1}$$

implies that $\Lambda_1-\Lambda\ge\varepsilon(\eta_1-\Lambda_1)$ and therefore that $\Lambda_1-\Lambda\ge\varepsilon(1+\varepsilon)^{-1}(\eta_1-\Lambda)$. The same argument shows that since $y_1\ge\kappa$ and $y_0\le 1$,

$$\frac{1}{\Lambda_1-\Lambda}\ge\frac{y_0}{\Lambda_1-\Lambda}=\sum_{k=1}^{p}\frac{y_k}{\eta_k-\Lambda_1}\ge\frac{y_1}{\eta_1-\Lambda_1}\ge\frac{\kappa}{\eta_1-\Lambda_1}$$

shows that $\eta_1-\Lambda_1\ge\kappa(1+\kappa)^{-1}(\eta_1-\Lambda)$. Hence, Λ_1 is at distance of order ε of Λ and κ of η_1. Thus, if $Y,Y'\in[\varepsilon,1]\times[\kappa,1]\times[0,1]^{p-1}\cap\Delta^{p+1}$, we easily deduce from (2.19) that $\Phi(\Lambda,Y)$ is the zero of a smooth function and hence that there exists a finite constant $C(\varepsilon,\kappa)$ so that

$$|\Phi(\Lambda,Y)-\Phi(\Lambda,Y')|\le C(\varepsilon,\kappa)\sum_{i=1}^{p}|y_i-y'_i|\,,$$

which implies the continuity of $\Phi(\Lambda, .)$ on $[\varepsilon, 1] \times [\kappa, 1] \times [0,1]^{p-1} \cap \Delta^{p+1}$. When y_0 goes to zero, we have since the η_i's are bounded above by η_p and $\sum y_k = 1 - y_0$,

$$\frac{y_0}{\Lambda_1 - \Lambda} = \sum_{k=1}^{p} \frac{y_k}{\eta_k - \Lambda_1} \geq \frac{1 - y_0}{\eta_p - \Lambda_1}$$

which insures that $\Lambda_1 - \Lambda \leq y_0(1-y_0)^{-1}(\eta_p - \Lambda)$ goes to zero as y_0 goes to zero. It is clear that $\Phi(\Lambda, Y)$ was extended by continuity to Λ when y_0 goes to zero, which in turn completes the proof of the continuity of $\Phi(\Lambda, .)$ on $[0,1] \times [\kappa, 1] \times [0,1]^{p-1} \cap \Delta^{p+1}$.

We next turn to the proof of (2). For each k, the eigenspace of B'_{N-1} associated with η_k is

$$E'_{\eta_k} = E_{\eta_k} \cap v_1^{\perp}.$$

Hence $\dim(E'_{\eta_k})$ is either N_k (if $P_{E_{\eta_k}}(v_1) = 0$) or $N_k - 1$ (if $P_{E_{\eta_k}}(v_1) \neq 0$). In particular, since $P_{E_{\eta_0}}(v_1) \neq 0$, we have $E'_{\eta_0} = \{0\}$ and $\Lambda (= \eta_0)$ is not an eigenvalue of B'_{N-1}.

Let $\lambda'_1 \leq \cdots \leq \lambda'_{N-1}$ be the eigenvalues of B'_{N-1}, and $\lambda_1 \leq \cdots \leq \lambda_N$ those of B_N. By the Courant–Fischer theorem,

$$\lambda_1 \leq \lambda'_1 \leq \cdots \leq \lambda'_{N-1} \leq \lambda_N.$$

Thus, for each $k = 1, \ldots, p$, there is at most one eigenvalue of B'_{N-1} in (η_{k-1}, η_k); when it exists we denote it by Λ_k. If η_k has multiplicity $N_k - 1$ in B'_{N-1}, then such a Λ_k lies in (η_{k-1}, η_k); otherwise, with a slight abuse of notation, we set $\Lambda_k := \eta_k$.

By the Courant-Fischer theorem, Λ_1 can be expressed as

$$\Lambda_1 = \inf_{v \in \mathbb{R}^N : \|v\|_2 = 1, \langle v, v_1 \rangle = 0} \langle v, B_N v \rangle.$$

Let $u \in \mathbb{R}^N$ be a minimizer in the above problem. By variational calculus, there exist $C_1, C \in \mathbb{R}$ such that

$$B_N u = C u + C_1 v_1.$$

Since $\langle u, v_1 \rangle = 0$, we have

$$\langle u, B_N u \rangle = C = \Lambda_1.$$

Since $\Lambda_1 \notin \{\lambda_1, \ldots, \lambda_N\}$, we deduce that for every $i = 1, \ldots, N$, if we denote by w_i the eigenvector of B_N with eigenvalue λ_i and for a vector $r \in S^{N-1}$ $r_i = \langle r, w_i \rangle$,

$$u_i = C_1 \frac{(v_1)_i}{\lambda_i - \Lambda_1}.$$

Thus, Λ_1 satisfies

$$\langle u, v_1 \rangle = C_1 \sum_{i=1}^{N} \frac{(v_1)_i^2}{\lambda_i - \Lambda_1} = 0,$$

which can be written, since $C_1 \neq 0$ as u has norm one,

$$\frac{y_0(v_1)}{\Lambda_1 - \Lambda} + \sum_{k=1}^{p} \frac{y_k(v_1)}{\Lambda_1 - \eta_k} = 0.$$

Λ_1 is thus the unique solution of (2.19) as claimed. Item (3) is clear. □

We next study the typical behavior of the smallest eigenvalue of $p_{v_1}^{\perp}(G_N + B_N)p_{v_1}^{\perp}$ in $v_1^{\perp}$, when v_1 is as in Lemma 2.3. It is a straightforward application of Lemma 2.3 and Theorem 1.1 as this matrix is the sum of a GOE matrix and the matrix B'_{N-1} with spectrum asymptotically distributed according to ν and outlier $\Phi(\Lambda, Y(v_1))$ as in Lemma 2.3.

ue proj2 **Lemma 2.4** (Smallest eigenvalue of the remainder)**.** *Let $\Lambda = \eta_0 \leq \eta_1 < \eta_2 \cdots < \eta_p$. Let B_N be a $N \times N$ diagonal matrix with eigenvalue Λ with multiplicity 1 and $\eta_1, \ldots, \eta_p$ of multiplicities N_i such that N_i/N goes to $\alpha_i > 0$. Let $\nu := \sum_{k=1}^{p} \alpha_k \delta_{\eta_k}$. Let v_1 and $Y(v_1)$ be as in Lemma 2.3. Then the smallest eigenvalue of $p_{v_1}^{\perp}(G_N + B_N)p_{v_1}^{\perp}$ in $v_1^{\perp}$ converges towards $H_{\nu,t}(\Phi(\Lambda, Y(v_1)))$ if $\Phi(\Lambda, Y(v_1)) \leq \omega_{\nu,t}(\ell_{\nu,t})$ and to $\ell_{\nu,t}$ otherwise.*

2.4. A functional inequality for the large deviation upper bound

We denote for $\alpha \in \Delta^p$ and $\delta > 0$, $\mathcal{B}_\delta^N(\alpha)$ the ℓ_∞-ball

$$\mathcal{B}_\delta^N(\alpha) = \Big\{ \mathsf{M} = (M_1, \dots, M_p) \in (\mathbb{N})^p : \sup_{1 \le i \le p} \Big| \frac{M_i}{N} - \alpha_i \Big| \le \delta, \quad \sum_{i=1}^p M_i = N - 1 \Big\}.$$

We shall assume without loss of generality that δ is strictly smaller than $\min \alpha_i$ so that the integers M_i are strictly positive. Let B_N be a diagonal matrix with eigenvalues Λ with multiplicity 1, and $\eta_1, \dots, \eta_p$ of respective multiplicities $\mathsf{M} = (M_1, \dots, M_p)$. Let $\mathsf{M} \in \mathcal{B}_\delta^N(\alpha)$. For every $\Lambda \le \ell_\nu$, let us denote $\mathbb{P}_{N,\Lambda,\mathsf{M}}$ the law of $X_N = G_N + B_N$ and

$$P_{N,\Lambda,\mathsf{M}}(x, \varepsilon) := \mathbb{P}_{N,\Lambda,\mathsf{M}}(|\lambda_1(X_N) - x| \le \varepsilon),$$

Recall that for $\alpha \in \Delta^p$, we denote $\nu = \sum \alpha_i \delta_{\eta_i}$. For all $\Lambda \le \ell_\nu$ and $x \le \ell_{\nu,t}$, let us define

eq:defF+

$$F_{\nu,t}^+(\Lambda, x) := - \limsup_{\Lambda' \to \Lambda} \lim_{\varepsilon \downarrow 0} \lim_{\delta \downarrow 0} \limsup_{N \to \infty} \sup_{\mathsf{M} \in \mathcal{B}_\delta^N(\alpha)} \frac{1}{N} \log P_{N,\Lambda',\mathsf{M}}(x, \varepsilon). \tag{2.22}$$

mark:lsc

Remark 2.1. Let us remark that, by construction, $\Lambda \mapsto F_{\nu,t}^+(\Lambda, x)$ is lower semicontinuous. Indeed, if $f : \mathbb{R} \to \mathbb{R}$ is a measurable function, the function

$$g : x \in \mathbb{R} \mapsto \limsup_{y \to x} f(y) = \lim_{\varepsilon \to 0} \sup_{B(x,\varepsilon)} f$$

is upper semicontinuous.

Recall the map Φ defined in (2.18). Let us define

def:C_t

$$C_t := \frac{1}{2} - \frac{1}{2} \ln t. \tag{2.23}$$

Recall the map $L_{\nu,t}^\Lambda$ defined in (2.11) and the map $I_\nu : Y \in \Delta^{p+1} \mapsto -\frac{1}{2} \sum_{k=1}^p \alpha_k \log(y_k/\alpha_k)$ from Lemma 2.2. We set

ef:JnutL

$$J_{\nu,t}^\Lambda : (x, Y) \in (-\infty, \ell_{\nu,t}] \times \Delta^{p+1} \mapsto L_{\nu,t}^\Lambda(x, Y) + I_\nu(Y), \tag{2.24}$$

and, for $(\Lambda, x, Y) \in (-\infty, \ell_\nu] \times (-\infty, \ell_{\nu,t}] \times \Delta^{p+1}$, we define

defK

$$K_{\nu,t}(\Lambda, x, Y) = J_{\nu,t}^\Lambda(x, Y) - C_t - \int \log|t - x| \mathrm{d}(\nu \boxplus \sigma_t)(t) + \frac{x^2}{4t}. \tag{2.25}$$

We denote $f(x^-) = \lim_{\varepsilon \downarrow 0} f(x - \varepsilon)$. The goal of this section is to prove the following:

er bound

Lemma 2.5 (Functional upper bound)**.** *Let $\Lambda \le \ell_\nu$. Let $\Phi(\Lambda, \cdot)$ be as in (2.18) and $\ell_{\nu,t}^\Lambda$ be as in (1.13). For every $x < \ell_{\nu,t}$, we have*

eq:ub

$$F_{\nu,t}^+(\Lambda, x) \ge \inf_{Y \in \Delta^{p+1}} \Big\{ K_{\nu,t}(\Lambda, x, Y) + F_{\nu,t}^+(\Phi(\Lambda, Y), x^-) 1_{\omega_{\nu,t}(x) > \Phi(\Lambda, Y)} \Big\} \tag{2.26}$$

Moreover, $x \mapsto F_{\nu,t}^+(\Lambda, x)$ decreases on $(-\infty, \ell_{\nu,t}^\Lambda)$ and increases on $(\ell_{\nu,t}^\Lambda, \ell_{\nu,t})$, while it vanishes at $\ell_{\nu,t}^\Lambda$. It is bounded from below by $c(x - \ell_{\nu,t}^\Lambda)^2$ for some $c > 0$.

Proof. The properties of $F_{\nu,t}^+$ listed at the end of the lemma are proven in Lemma B.2 and Lemma B.1, and we focus below on the proof of the functional inequality (2.26). We fix $\alpha \in \Delta^p \cap (0,1]^p$ and denote by $\nu_\alpha = \sum \alpha_i \delta_{\eta_i}$. For $\delta > 0$ and $\mathsf{M} \in \mathcal{B}_\delta^N(\alpha)$ we denote by $\nu_{\frac{\mathsf{M}}{N}} = \frac{1}{N} \delta_\Lambda + \sum \frac{M_i}{N} \delta_{\eta_i}$ the empirical measure of the eigenvalues of a diagonal matrix $B_N(\mathsf{M})$ with eigenvalues η_i with multiplicity M_i, $\mathsf{M}=(M_1,\dots,M_p)$, Λ with multiplicity 1. Let us first prove that for every $\delta>0$, there exists $C_\delta>0$ such that, for N large enough, for every $\mathsf{M}\in\mathcal{B}_\delta^N(\alpha)$,

con12
$$\mathbb{P}_{N,\Lambda,\mathsf{M}}\Big(d\Big(\frac{1}{N}\sum_{i=1}^N \delta_{x_i},\nu_\alpha\boxplus\sigma_t\Big)\geq 8\|\eta\|_\infty\sqrt{\delta p}\Big)\leq e^{-\frac{1}{2}C_\delta N^2}\,. \tag{2.27}$$

We observe that (2.27) is uniform in $\mathsf{M}\in\mathcal{B}_\delta^N(\alpha)$. In fact, recall from (2.5) that there are two universal positive constants C_1,C_2 so that for every $\zeta>0$, for every $\mathsf{M}\in\mathbb{N}^p$ so that $\sum M_i=N-1$,

con02
$$\mathbb{P}_{N,\Lambda,\mathsf{M}}\Big(d\Big(\frac{1}{N}\sum_{i=1}^N\delta_{\lambda_i},\mathbb{E}_{\mathbb{P}_{N,\Lambda,\mathsf{M}}}\Big[\frac{1}{N}\sum_{i=1}^N\delta_{\lambda_i}\Big]\Big)\geq\zeta\Big)\leq\frac{C_1}{\zeta^{3/2}}e^{-C_2N^2\zeta^2}\,, \tag{2.28}$$

On the other hand, for every $\mathsf{M}\in\mathcal{B}_\delta^N(\alpha)$, for N large enough, $d(\mathbb{E}_{\mathbb{P}_{N,\Lambda,\mathsf{M}}}[\frac{1}{N}\sum\delta_{\lambda_i}],\nu_{\frac{\mathsf{M}}{N}}\boxplus\sigma_t)$ goes to zero. We claim that this convergence is uniform. Indeed, by the Cauchy Schwarz and Hoffman-Wielandt inequalities [1, Lemma 2.1.19], for every Lipschitz function f, if X_N,Y_N are two self-adjoint matrices,

$$\Big|\frac{1}{N}\mathrm{Tr}(f(X_N+G_N))-\frac{1}{N}\mathrm{Tr}(f(Y_N+G_N))\Big|\leq\|f\|_L\sqrt{\frac{1}{N}\mathrm{Tr}(X_N-Y_N)^2},$$

where we recall that $\|f\|_L$ stands for the Lipschitz seminorm of f. As a consequence,

con200
$$d\left(\frac{1}{N}\sum_{i=1}^N\delta_{\lambda_i(G_N+B_N(\mathsf{M}))},\frac{1}{N}\sum_{i=1}^N\delta_{\lambda_i(G_N+B_N(\mathsf{M}'))}\right)\leq\sqrt{\frac{1}{N}\mathrm{Tr}((B_N(\mathsf{M})-B_N(\mathsf{M}'))^2)}. \tag{2.29}$$

It follows that

con20
$$d\left(\frac{1}{N}\sum_{i=1}^N\delta_{\lambda_i(G_N+B_N(\mathsf{M}))},\frac{1}{N}\sum_{i=1}^N\delta_{\lambda_i(G_N+B_N(\mathsf{M}'))}\right)\leq\|\eta\|_\infty\sqrt{\frac{2}{N}\sum_{i=1}^p|M_i-M_i'|}\leq 2\|\eta\|_\infty\sqrt{\delta p}. \tag{2.30}$$

This control also holds in the large N limit, yielding $d(\nu_{\frac{\mathsf{M}}{N-1}}\boxplus\sigma_t,\nu_{\frac{\mathsf{M}'}{N-1}}\boxplus\sigma_t)\leq 2\|\eta\|_\infty\sqrt{\delta p}$. Moreover, (2.30) also holds after taking the expectation. As a consequence, we deduce that uniformly on $\mathsf{M}\in\mathcal{B}_\delta^N(\alpha)$, for N large enough,

bobo
$$d\Big(\mathbb{E}_{\mathbb{P}_{N,\Lambda,\mathsf{M}}}\Big[\frac{1}{N}\sum_{i=1}^N\delta_{\lambda_i}\Big],\nu_\alpha\boxplus\sigma_t\Big)\leq 4\|\eta\|_\infty\sqrt{\delta p}, \tag{2.31}$$

from which we get (2.27) uniformly from (2.28) by taking $\zeta=4\|\eta\|_\infty\sqrt{\delta p}$.

Hereafter α is fixed and we denote in short ν for ν_α. Moreover, since the sequence $(B_N)_{N\in\mathbb{N}}$ is bounded uniformly for the operator norm, there exists L_0 finite ($L_0=(2+\|B_N\|_{\mathrm{op}})^2$) such that for every $L>L_0$, by [1, (2.6.21)]

con2
$$\sup_{\mathsf{M}\in\mathcal{B}_\delta^N(\alpha)}\mathbb{P}_{N,\Lambda,\mathsf{M}}\Big(\frac{1}{N}\sum\lambda_i^2\geq L\Big)\leq e^{-(L-L_0)N^2}\,. \tag{2.32}$$

Therefore, if for $L>L_0$ and $\delta>0$ fixed, we let

$$K_{L,\delta}=\Big\{\lambda^N\in\mathbb{R}^N:\frac{1}{N}\sum\lambda_i^2\leq L, d\Big(\frac{1}{N}\sum\delta_{\lambda_i},\nu_\alpha\boxplus\sigma_t\Big)\leq\delta\Big\},$$

then we deduce from (2.32) and (2.27) that there exists some positive real number $c(\delta,L)$ so that for N large enough $\mathbb{P}_{N,\Lambda,\mathsf{M}}(K_{L,\delta}^c)\leq e^{-c(L,\delta)N^2}$ and therefore

eq:loc
$$P_{N,\Lambda,\mathsf{M}}(x,\varepsilon)=\mathbb{P}_{N,\Lambda,\mathsf{M}}\Big(|\lambda_1(X_N)-x|\leq\varepsilon,\lambda^N(X_N)\in K_{L,\delta}\Big)+O(e^{-c(L,\delta)N^2}) \tag{2.33}$$

where $O(e^{-c(L,\delta)N^2})$ is bounded above by $e^{-c(L,\delta)N^2}$. We denote in the following

$$\mathcal{C}_{L,\delta}(x,\varepsilon) := \{\lambda^N \in K_{L,\delta} : \lambda_1 < \lambda_2 < \cdots < \lambda_N, |\lambda_1 - x| \le \varepsilon\}.$$

By (2.33), one needs to upper bound

$$P^{L,\delta}_{N,\Lambda,\mathsf{M}}(x,\varepsilon) := \mathbb{P}_{N,\Lambda,\mathsf{M}}\Big(\lambda^N(X_N) \in \mathcal{C}_{L,\delta}(x,\varepsilon)\Big).$$

Let us perform the change of variables $X_N \mapsto (\lambda_1,\dots,\lambda_N,O_N)$ where $\lambda_1 < \ldots < \lambda_N$ are the eigenvalues of X_N and O_N the corresponding orthogonal matrix so that $X_N = O_N D_N(\lambda) O_N^T$ with $D_N(\lambda)$ the diagonal matrix with entries $(\lambda_i)_{1\le i\le N}$. We find

$$P^{L,\delta}_{N,\Lambda,\mathsf{M}}(x,\varepsilon) = \frac{1}{Z_N^t}\int_{\lambda^N\in\mathcal{C}_{L,\delta}(x,\varepsilon)} \prod_{i<j}|\lambda_i-\lambda_j| 1_{\lambda_1<\ldots<\lambda_N} e^{-\frac{N}{4t}\mathrm{Tr}((O_N D_N(\lambda)O_N^T - B_N)^2)} \mathrm{d}O_N \mathrm{d}\lambda^N,$$

where $\mathrm{d}O_N$ is the Haar measure on $O_N(\mathbb{R})$ and Z_N^t is the normalizing constant of the deformed GOE measure, i.e. the value of the same integral when the indicator of $\{\lambda^N \in C_{L,\delta}(x,\varepsilon)\}$ is replaced by 1, $\mathrm{d}\lambda^N$ denotes $\mathrm{d}\lambda_1\ldots\mathrm{d}\lambda_N$ in short. Applying Lemma 2.1 (see (2.10) and (2.11)) gives, by the definition (2.11) of $L^{\Lambda}_{\nu,t}$,

$$P^{L,\delta}_{N,\Lambda,\mathsf{M}}(x,\varepsilon) = \int_{S^{N-1}} \mathrm{d}v_1 \int \mathrm{d}\mathbb{P}_{v_1}(O_{N-1}) \mathbb{M}(\mathcal{C}_{L,\delta}(x,\varepsilon), v_1, O_{N-1})$$

$$\mathbb{M}(A,v,O) = \frac{1}{Z_N^t}\int_{\lambda^N\in A}\prod_{i<j}|\lambda_i-\lambda_j| e^{-NL^{\Lambda}_{\nu,t}(\lambda_1,Y(v)) - \frac{N}{4t}\lambda_1^2 - \frac{N}{4t}\mathrm{Tr}(OD_{N-1}(\lambda)O^T - B'_{N-1})^2} \mathrm{d}\lambda^N$$

where $\mathbb{P}_{v_1}$ is the Haar measure on the orthocomplement of v_1, B'_{N-1} the projection of B_N onto the orthocomplement of v_1 and $D_{N-1}(\lambda)$ is the diagonal matrix with entries $(\lambda_2,\dots,\lambda_N)$.

Let $\chi,\kappa,\zeta > 0$. We denote $V_0 = \{v \in S^{N-1} : y_1(v) = \|P_{E_{\eta_1}} v\|^2 < 2\kappa\}$ where we recall that $P_{E_{\eta_k}} v$ is the orthogonal projection of the vector v onto the eigenspace E_{η_k} of B_N for the eigenvalue η_k and $\|v\|_{E_{\eta_k}} = \|P_{E_{\eta_k}} v\|$ its Euclidean norm. Since I_ν is lower semicontinuous, one can cover V_0^c by a finite union of measurable sets $\cup_{1\le k\le M} V_{Y_k,\zeta_k,\chi}$ where for $Y = (y_0,\dots,y_p) \in \Delta^{p+1}$, and $\zeta,\chi > 0$,

eq:defEY

$$V_{Y,\zeta,\chi} := \Big\{ v \in S^{N-1} : \max_{0\le k\le p} \big| \|P_{E_{\eta_k}} v\|^2 - y_k \big| \le \zeta, I_\nu((\|P_{E_{\eta_k}} v\|^2)_{0\le k\le p}) \ge I_\nu((y_k)_{0\le k\le p}) - \chi \Big\} \tag{2.34}$$

and $(Y_k)_{1\le k\le M}$ is a family of points of Δ^{p+1}. We can assume from the start that $\zeta_j \le \zeta \le \kappa$ for every j for some $\zeta > 0$ to be chosen later and so that $Y_j(1) \ge \kappa > 0$. We denote in short V_k for $V_{Y_k,\zeta_k,\chi}$. We therefore find

exp

$$P^{L,\delta}_{N,\Lambda,\mathsf{M}}(x,\varepsilon) \le \sum_{k=0}^{M} \int_{v_1\in V_k} \mathrm{d}v_1 \int \mathrm{d}\mathbb{P}_{v_1}(O_{N-1}) \mathbb{M}(\mathcal{C}_{L,\delta}(x,\varepsilon), v_1, O_{N-1}) \tag{2.35}$$

We next estimate each term in the above right hand side. For every $L \in (2,\infty)$, let h_L be a continuously differentiable function with values in $[0,1]$, derivative uniformly bounded by one, equal to one for $|x| \le L/2$ and to zero for $|x| \ge L$. We set

$$f_L(x) := h_L(x)\log(\max(|x|, L^{-1})) + 4\frac{L^{-2}}{\log L}(L_0 + 1).$$

Observe that $\log|x| \le h_L(x)\log(\max(|x|, L^{-1}))$ if $|x| \le L/2$. Therefore, by Chebyshev's inequality,

$$\begin{aligned}\sum_{j>1}\log|\lambda_1-\lambda_j| &\le \sum_{j>1} h_L(\lambda_1-\lambda_j)\log(\max(|\lambda_1-\lambda_j|, L^{-1}) + \sum_{j:|\lambda_i-\lambda_1|>L/2}\log(|\lambda_1-\lambda_j|) \\ &\le \sum_{j>1} h_L(\lambda_1-\lambda_j)\log(\max(|\lambda_1-\lambda_j|, L^{-1}) + \frac{L^{-2}}{\log L}\sum_{j:|\lambda_i-\lambda_1|>L/2}|\lambda_1-\lambda_j|^2\end{aligned}$$

Therefore, on $\{d(\frac{1}{N}\sum\delta_{\lambda_i},\nu\boxplus\sigma_t)\le\delta\}\cap\{\frac{1}{N}\sum\lambda_i^2\le L_0+1\}\cap\{|\lambda_1-x|\le\varepsilon\}$ with $|x|+\varepsilon\le\sqrt{L_0}$,

$$\prod_{j>1}|\lambda_1-\lambda_j|\le\exp\Big\{\sum_{i=1}^N f_L(\lambda_1-\lambda_i)\Big\}\le\exp\Big\{N\int f_L(\lambda_1-y)\mathrm{d}(\sigma_t\boxplus\nu)(y)+NC_L\delta\Big\},$$

where we finally used that f_L is Lipschitz to find a finite constant C_L such that the last inequality holds. Finally, if $x+\varepsilon<\ell_{\nu,t}$, and therefore is at positive distance from the support of $\sigma_t\boxplus\nu$, because $\sigma_t\boxplus\nu$ is compactly supported,

approx

$$\int f_L(\lambda_1-y)\mathrm{d}(\sigma_t\boxplus\nu)(y)=\int\log|x-y|\mathrm{d}(\sigma_t\boxplus\nu)(y)+O(\varepsilon)+O(L^{-1})\,. \tag{2.36}$$

Similarly $\lambda_1^2/4t$ is very close to $x^2/4t$. As a consequence, we find that

eq2

$$\mathbb{M}(\mathcal{C}_{L,\delta}(x,\varepsilon),v_1,O_{N-1})\le e^{N(\int\log|x-y|\mathrm{d}(\sigma_t\boxplus\nu)(y)-\frac{x^2}{4t}+o(1))}\mathbb{L}(\mathcal{C}_{L,\delta}(x,\varepsilon),v_1,O_{N-1}) \tag{2.37}$$

with $o(1)=C_L\delta+O(\varepsilon)+O(L^{-1})$ and

$$\mathbb{L}(A,v_1,O_{N-1}):=\frac{1}{Z_N^t}\int_{\lambda^N\in A}\prod_{2\le i<j}|\lambda_i-\lambda_j|e^{-NL^{\Lambda}_{\nu,t}(\lambda_1,Y(v_1))-\frac{N}{4t}\mathrm{Tr}(O_{N-1}D_{N-1}(\lambda)O^T_{N-1}-B'_{N-1})^2}\mathrm{d}\lambda^N\,.$$

Let us finally remark that $O_{N-1}D_{N-1}(\lambda)O^T_{N-1}$ follows the law of a GOE of dimension $N-1$ and variance $t(N-1)/N$ under $\mathbb{P}_{N-1}(\mathrm{d}\lambda,\mathrm{d}O_{N-1})$ defined in (1.18) so that

eq3

$$\int\mathrm{d}\mathbb{P}_{v_1}(O_{N-1})\int_{\lambda_2\le\cdots\le\lambda_N}\prod_{2\le i<j}|\lambda_i-\lambda_j|e^{-\frac{N}{4t}\mathrm{Tr}((O_{N-1}D_{N-1}(\lambda)O^T_{N-1}-B'_{N-1})^2)}\mathrm{d}\lambda_2\ldots\mathrm{d}\lambda_N=Z_{N-1}^{t\frac{N-1}{N}}\,. \tag{2.38}$$

Moreover, Z_N^t can be exactly computed by Selberg integral, see e.g [1, (2.5.11)], resulting, for every $t>0$ and $N\in\mathbb{N}$, in

$$Z_N^t=N!\Big(\frac{2t}{N}\Big)^{\frac{N(N+1)}{4}}(2\pi)^N\prod_{j=0}^{N-1}\frac{\Gamma((j+1)/2)}{\Gamma(1/2)},$$

from which the limit of $\frac{1}{N}\log\frac{Z_{N-1}^{t\frac{N-1}{N}}}{Z_N^t}$ is easily computed to be equal to C_t as in (2.23).

We next estimate the first term in the right hand side of (2.35). Putting together (2.37) and (2.38) we see that there is finite constants C,C' so that

$$\int_{v_1\in V_0}\mathrm{d}v_1\int\mathrm{d}\mathbb{P}_{v_1}(O_{N-1})\mathbb{M}(\mathcal{C}_{L,\delta}(x,\varepsilon),v_1,O_{N-1})\le e^{CN}\mathbb{P}(y_1(v_1)\le2\kappa)\le e^{-C'N}\Big(\frac{2\kappa}{\alpha_1}\Big)^{\frac12\alpha_1N}$$

where we finally used Lemma 2.2 and assumed N large enough. The constant C comes from the supremum of the mass of $\mathbb{M}(\mathcal{C}_{L,\delta}(x,\varepsilon),v_1,O_{N-1})$ over v_1 which is easily seen to be at most of order CN by the previous discussion, whereas the constant C' integrates C and the term in the rate function for the large deviation principle for $Y(v_1)$ given in Lemma 2.2 which does not depend on κ.

We now estimate the next terms in the right hand side of (2.35) and fix $k\in\{1,\ldots,M\}$. When $v_1\in V_k=V_{Y_k,\zeta_k,\chi}$ and $|\lambda_1-x|\le\varepsilon$, $L^{\Lambda}_{\nu,t}(\lambda_1,Y(v_1))$ is at a distance from $L^{\Lambda}_{\nu,t}(x,Y_k)$ of order $\zeta_k+\varepsilon$. Moreover, $y_1(v_1)\ge\kappa$ if $\zeta_k\le\zeta\le\kappa$. By Lemma 2.3, the matrix B'_{N-1} has eigenvalues $\eta_1,\ldots,\eta_p$ of respective multiplicities between M_k-1 and M_k with p possible outliers $\Lambda_1\le\cdots\le\Lambda_p$ with for every $k=2,\ldots,p$, $\Lambda_k\in[\eta_{k-1},\eta_k]$. Moreover, by the continuity of Λ_1 proven in Lemma 2.3 (1) and (2), Λ_1 is at distance at most of order $o_\zeta(1)$ going to zero with ζ from $\Phi(\Lambda,Y_k)$. Note that $o_\zeta(1)$ can be chosen uniformly as Y belongs to a compact set and hence Φ is uniformly continuous. We assume it is smaller than $\varepsilon/2$. We can bound B'_{N-1} from above by a matrix $B''_{N-1}+\varepsilon/2I$, that is $B''_{N-1}+\varepsilon/2I-B'_{N-1}$ is positive semidefinite, in $v_1^\perp$ with the same eigenspaces, the outlier $\Phi(\Lambda,Y_k)$ and eigenvalues η_i with multiplicity $M'_i=M_i$ except for η_1 which has multiplicity $M'_1=M_1-1$. We therefore get

$$\int_{v_1\in V_k}\mathrm{d}v_1\int\mathrm{d}\mathbb{P}_{v_1}(O_{N-1})\mathbb{M}(\mathcal{C}_{L,\delta}(x,\varepsilon),v_1,O_{N-1})$$

jk
$$\le e^{N\left(\int \log|x-y|\mathrm{d}(\sigma_t\boxplus\nu)(y)-\frac{x^2}{4t}-L^{\Lambda}_{\nu,t}(x,Y_k)+o(1)\right)}\frac{Z^{t\frac{N-1}{N}}_{N-1}}{Z^t_N}\int_{v_1\in V_k}\mathrm{d}v_1\mathbb{K}(x,\varepsilon,v_1) \tag{2.39}$$

where $o(1)=C_L\delta+O(\varepsilon)+O(L^{-1})$ and $\mathbb{K}(x,\varepsilon,v_1)$ is equal to

$$
\begin{aligned}
&= \left(Z^{t\frac{N-1}{N}}_{N-1}\right)^{-1}\int \mathrm{d}\mathbb{P}_{v_1}(O_{N-1})\int_{x-2\varepsilon\le\lambda_2\le\cdots\le\lambda_N}\prod_{2\le i<j}|\lambda_i-\lambda_j|e^{-\frac{N}{4t}\mathrm{Tr}((O_{N-1}D_{N-1}(\lambda)O^T_{N-1}-B'_{N-1})^2)}\mathrm{d}\lambda^{N-1}\\
&= \mathbb{P}_{N-1,\Lambda_1,\mathsf{M}'}(\lambda_1(\sqrt{N(N-1)^{-1}}G_{N-1}+B'_{N-1})\ge x-2\varepsilon)
\end{aligned}
$$

d-1
$$\le \mathbb{P}_{N-1,\Lambda_1,\mathsf{M}'}(\lambda_1(G_{N-1}+B''_{N-1})\ge x-3\varepsilon)=:\mathbb{K}(x,\varepsilon,Y_k) \tag{2.40}$$

where we denoted $\mathrm{d}\lambda^{N-1}=\mathrm{d}\lambda_2\ldots\mathrm{d}\lambda_N$, and in the last line we incorporated the error due to the small change in the variance (which can be removed after rescaling and boundedness of B'_N) in an additional error of order ε for N large enough. Note here that $\mathbb{K}(x,\varepsilon,Y_k)$ only depends on Y_k and x. Observe that Theorem 1.1 and Lemma 2.3 imply that, if the empirical measure of the λ_i is at distance smaller than δ from $\nu\boxplus\sigma_t$, the smallest eigenvalue of $G_{N-1}+B''_{N-1}$ is close to $H_{\nu,t}(\Phi(\Lambda,Y_k))$ if $\Phi(\Lambda,Y_k)<\omega_{\nu,t}(\ell_{\nu,t})$ and close to $\ell_{\nu,t}$ otherwise, with some error going to zero when N goes to infinity and then δ to zero. We therefore have three cases:

- If $\Phi(\Lambda,Y_k)\ge\omega_{\nu,t}(\ell_{\nu,t})$, then the smallest eigenvalue of $G_{N-1}+B''_{N-1}$ converges towards $\ell_{\nu,t}$ and $\mathbb{P}_{N-1,\Lambda_1,\mathsf{M}'}(\lambda_1(G_{N-1}+B''_{N-1})\ge x-3\varepsilon)$ goes to one as soon as $x\le\ell_{\nu,t}$ and ε is small enough.

- If $x\le H_{\nu,t}(\Phi(\Lambda,Y_k))$ and $\Phi(\Lambda,Y_k)\le\omega_{\nu,t}(\ell_{\nu,t})$, then the smallest eigenvalue of $G_{N-1}+B''_{N-1}$ goes to $H_{\nu,t}(\Phi(\Lambda,Y_k))$ when N goes to infinity and is therefore greater than $x-3\varepsilon$. Hence, we see that $\mathbb{P}_{N-1,\Lambda_1,\mathsf{M}'}(\lambda_1(G_{N-1}+B''_{N-1})\ge x-2\varepsilon)$ goes to one as N goes to infinity.

- If $x>H_{\nu,t}(\Phi(\Lambda,Y_k))$ and $\Phi(\Lambda,Y_k)\le\omega_{\nu,t}(\ell_{\nu,t})$, then $\mathbb{P}_{N-1,\Lambda_1,\mathsf{M}'}(\lambda_1(G_{N-1}+B''_{N-1})\ge x-3\varepsilon)$ goes to zero exponentially fast, hence this term contributes to the probability of large deviations and needs to be evaluated. Observe that $\lambda_1(G_{N-1}+B''_N)\le\ell_{\nu,t}+\varepsilon$ with overwhelming probability by (2.27). We can cut the compact set $[x-4\varepsilon,\ell_{\nu,t}]$ into a finite number K of intervals of size γ to deduce that

$$\mathbb{P}_{N-1,\Lambda_1,\mathsf{M}'}(\lambda_1(G_{N-1}+B''_N)\ge x-3\varepsilon)\le\sum_{p=1}^{K}P_{N-1,\Phi(\Lambda,Y_k),\mathsf{M}'}(x-4\varepsilon+p\gamma,\gamma)\,.$$

According to the proof of Lemma B.1, there exists $c>0$ such that

$$P_{N-1,\Phi(\Lambda,Y_k),\mathsf{M}'}(x-4\varepsilon+p\gamma,\gamma)\le 2N^2P_{N-1,\Phi(\Lambda,Y_k),\mathsf{M}'}(x-4\varepsilon,\gamma)+2N^2e^{-cN^2}$$

so that we obtain

$$\mathbb{P}_{N-1,\Lambda_1,\mathsf{M}'}(\lambda_1(G_{N-1}+B''_N)\ge x-3\varepsilon)\le 2KN^2P_{N-1,\Phi(\Lambda,Y_k),\mathsf{M}'}(x-4\varepsilon,\gamma)+2N^2Ke^{-cN^2}\,.$$

Hence, for $x>H_{\nu,t}(\Phi(\Lambda,Y_k))$ and $\Phi(\Lambda,Y_k)\le\omega_{\nu,t}(\ell_{\nu,t})$, after taking the large N limit and γ to zero, we find

$$\limsup_{N\to\infty}\frac{1}{N}\log\mathbb{P}_{N-1,\Lambda_1,\mathsf{M}'}(\lambda_1(G_{N-1}+B''_N)\ge x-3\varepsilon)\le -F^+_{\nu,t}(\Phi(\Lambda,Y_k),x-4\varepsilon)\,.$$

To conclude we need to first take the supremum over $\mathsf{M}\in\mathcal{B}^N_\delta(\alpha)$, so $\mathsf{M}'\in\mathcal{B}^{N-1}_\delta(\alpha)$ before the large N limit. We have to be careful because ν depends implicitly on M/N and Y_N is not equal to Y but is close to it. But the condition $\{\Phi(\Lambda,Y)\le\omega_{\nu,t}(\ell_{\nu,t})\}\cap\{x\ge H_{\nu,t}(\Phi(\Lambda,Y))\}=\{\omega_{\nu,t}(x)\ge\Phi(\Lambda,Y)\}$ is continuous in ν (as $\omega_{\nu,t}(x)$ is) and Y. Moreover, $F^+_{\nu,t}(\Phi(\Lambda,Y),x-4\varepsilon)$ is lower semicontinuous in Y. The argument above is therefore valid uniformly on $\mathsf{M}\in\mathcal{B}^N_\delta(\alpha)$ as long as we are in the interior of the set $\{\omega_{\nu,t}(x)\ge\Phi(\Lambda,Y)\}$ for δ small enough. We therefore conclude that, with $o_\zeta(1)$ going to zero with ζ,

Bk
$$\lim_{\delta\downarrow 0}\limsup_{N\to\infty}\sup_{\substack{v_1\in V_k\\ M\in\mathcal{B}^N_\delta(\alpha)}}\frac{1}{N}\log\mathbb{K}(x,\varepsilon,v_1)\le -F^+_{\nu,t}(\Phi(\Lambda,Y_k),x-4\varepsilon)1_{\omega_{\nu,t}(x)>\Phi(\Lambda,Y_k)}+o_\zeta(1). \tag{2.41}$$

By Lemma 2.2, for every k,

$$\limsup_{N\to\infty}\frac{1}{N}\log\mathbb{P}(V_k)\le -I_\nu(Y_k)+\chi.$$

We therefore deduce from (2.35), (2.39), (2.41) that, with $o_{\chi,\kappa,\varepsilon}(1)$ going to zero when χ,κ,ε go to zero,

$$\begin{aligned}\lim_{\delta\downarrow 0}\limsup_{N\to\infty}&\sup_{\mathsf{M}\in\mathcal{B}_\delta^N(\alpha)}\frac{1}{N}\log P_{N,\Lambda,\mathsf{M}}(x,\varepsilon)\\ &\le\max\Big\{\sup_k\Big\{-L^\Lambda_{\nu,t}(x,Y_k)-I_\nu(Y_k)-F^+_{\nu,t}(\Phi(\Lambda,Y_k),x-4\varepsilon)1_{\omega_{\nu,t}(x)>\Phi(\Lambda,Y_k)}\Big\}\\ &\qquad+\int\log|\lambda-x|\mathrm{d}(\nu\boxplus\sigma_t)(\lambda)-\frac{x^2}{4t}-C_t+o_{\chi,\kappa,\varepsilon}(1),\frac{1}{2}\alpha_1\log\frac{\kappa}{\alpha_1}+C'\Big\}.\end{aligned}$$

Letting ε go to zero, we obtain

$$\begin{aligned}\lim_{\varepsilon\downarrow 0}\lim_{\delta\downarrow 0}\limsup_{N\to\infty}&\sup_{\mathsf{M}\in\mathcal{B}_\delta^N(\alpha)}\frac{1}{N}\log P_{N,\Lambda,\mathsf{M}}(x,\varepsilon)\\ &\le\max\Big\{\sup_k\Big\{-K_{\nu,t}(\Lambda,x,Y_k)-F^+_{\nu,t}(\Phi(\Lambda,Y_k),x^-)1_{\omega_{\nu,t}(x)\ge\Phi(\Lambda,Y_k)}\Big\}+o_{\chi,\kappa}(1),\frac{1}{2}\alpha_1\log\frac{\kappa}{\alpha_1}+C'\Big\}\\ &\le\max\Big\{\sup_{Y\in\Delta^{p+1}}\Big\{-K_{\nu,t}(\Lambda,x,Y)-F^+_{\nu,t}(\Phi(\Lambda,Y),x^-)1_{\omega_{\nu,t}(x)>\Phi(\Lambda,Y)}\Big\}+o_{\chi,\kappa}(1),\frac{1}{2}\alpha_1\log\frac{\kappa}{\alpha_1}+C'\Big\}.\end{aligned}$$

We can finally let χ, κ go to zero . Now, we take the supremum over $\Lambda\in[\Lambda'-\varepsilon,\Lambda'+\varepsilon]$ on both sides. In the right-hand side it can be interchanged with the supremum over Y. The map $J^\Lambda_{\nu,t}(\lambda,Y)=L^\Lambda_{\nu,t}(\lambda,Y)+I_\nu(Y)$ is uniformly continuous in Λ. Moreover

$$\lim_{\varepsilon\downarrow 0}\inf_{\Lambda\in[\Lambda'-\varepsilon,\Lambda'+\varepsilon]}F^+_{\nu,t}(\Phi(\Lambda,Y),x^-)1_{\omega_{\nu,t}(x)>\Phi(\Lambda,Y)}\ge F^+_{\nu,t}(\Phi(\Lambda',Y),x^-)1_{\omega_{\nu,t}(x)>\Phi(\Lambda',Y)},$$

by lower semi-continuity of $F^+_{\nu,t}(.,x^-)$, and the continuity of $\Lambda\mapsto\Phi(\Lambda,Y)$. Note that the boundary point when $\omega_{\nu,t}(x)=\Phi(\Lambda,Y)$ is also such that $\omega_{\nu,t}(x^-)=\Phi(\Lambda,Y)$ by continuity of $\omega_{\nu,t}$ and then since $F^+_{\nu,t}(\omega_{\nu,t}(x),x)$ vanishes at every x, so that the above quantity vanishes at this point. We therefore conclude that (2.26) holds. □

2.5. *A functional inequality for the large deviation lower bound*

For every $\Lambda\le\ell_\nu$ and $x\le\ell_{\nu,t}$, define

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$$F^-_{\nu,t}(\Lambda,x):=-\liminf_{\Lambda'\to\Lambda}\lim_{\varepsilon\downarrow 0}\lim_{\delta\downarrow 0}\liminf_{N\to\infty}\inf_{\mathsf{M}\in\mathcal{B}_\delta^N(\alpha)}\frac{1}{N}\log P_{N,\Lambda',\mathsf{M}}(x,\varepsilon).\tag{2.42}$$

Denoting $f(x^+)=\lim_{\varepsilon\downarrow 0}f(x+\varepsilon)$, our goal is to prove the following:

er bound

Lemma 2.6 (Functional lower bound)**.** *Let $\Lambda\le\ell_\nu$. Let $\Phi(\Lambda,\cdot)$ be as in (2.18) and $\ell^\Lambda_{\nu,t}$ be as in (1.13). For every $x<\ell_{\nu,t}$, we have*

eq:lb

$$F^-_{\nu,t}(\Lambda,x)\le\inf_{Y\in\Delta^{p+1}}\Big\{K_{\nu,t}(\Lambda,x,Y)+F^-_{\nu,t}(\Phi(\Lambda,Y),x^+)1_{\omega_{\nu,t}(x)\ge\Phi(\Lambda,Y)}\Big\}.\tag{2.43}$$

Moreover, $0\le F^+_{\nu,t}\le F^-_{\nu,t}$. Furthermore, $x\mapsto F^-_{\nu,t}(\Lambda,x)$ decreases on $(-\infty,\ell^\Lambda_{\nu,t})$ and increases on $(\ell^\Lambda_{\nu,t},\ell_{\nu,t})$, while it vanishes at $\ell^\Lambda_{\nu,t}$. It is bounded from below by $c(x-\ell^\Lambda_{\nu,t})^2$ for some $c>0$.

Proof. The only remaining task is to prove (2.43), as the rest of the lemma is established in Lemma B.2. Let $x\in(-\infty,\ell_{\nu,t})$ and $\varepsilon>0$.

Again B_N has eigenvalues η_i with multiplicity M_i and Λ' with multiplicity 1. Let $Y \in \Delta^{p+1}$ and $\kappa, \zeta, \chi > 0$. We may and shall assume that $y_1 > 2\kappa$ since we saw in the upper bound that small y_1 have exponentially small contributions. We also can take ζ small enough so that y_1 is bounded below by κ uniformly on $V_{Y,\zeta,\chi}$ defined in (2.34). One can bound $P_{N,\Lambda',\mathsf{M}}(x,\varepsilon)$ from below by

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$$(2.44)\quad P_{N,\Lambda',\mathsf{M}}(x,\varepsilon) \geq \frac{1}{Z_N^t} \int_{V_{Y,\zeta,\chi}} \mathrm{d}v_1 \int \mathrm{d}\mathbb{P}_{v_1}(v_2,\dots,v_N)$$
$$\int_{B(\delta,\varepsilon,x)} \exp\Big\{ -NL_{\nu,t}^{\Lambda'}(\lambda_1, Y(v_1)) - \frac{N}{4t}\lambda_1^2 \Big\} \prod_{1\leq i<j} |\lambda_i - \lambda_j| \exp\Big\{ -\frac{N}{4t} \mathrm{Tr}\Big(\big(p_{v_1}^{\perp}(OD(\lambda)O^T - B)\,p_{v_1}^{\perp}\big)^2 \Big) \Big\} \mathrm{d}\lambda^N.$$

where $B(\delta,\varepsilon,x) := \{|\lambda_1 - x| \leq \varepsilon\} \cap C(\delta,\varepsilon,x)$ with, if $\nu = \sum \alpha_i \delta_{\eta_i}$,

$$C(\delta,\varepsilon,x) := \{x + 2\varepsilon < \lambda_2 < \cdots < \lambda_N\} \cap \Big\{ d\Big(\frac{1}{N-1}\sum_{i=2}^N \delta_{\lambda_i}, \nu \boxplus \sigma_t\Big) \leq 2\delta \Big\} \cap \Big\{ \frac{1}{N-1}\sum_{i=2}^N \lambda_i^2 \leq L_0 + 1 \Big\}.$$

Using the continuity of $L_{\nu,t}^{\Lambda'}$ and the fact that λ_1 is at distance at least ε from the rest of the spectrum (if $x + 2\varepsilon < \ell_{\nu,t}$) so that, by arguments similar to those used in (2.36), with $o_\delta(1)$ going to zero as δ goes to zero (while $\delta \ll \varepsilon$ and L_0 is fixed)

$$\frac{1}{N}\sum_{i=2}^N \log|\lambda_1 - \lambda_i| = \int \log|\lambda_1 - x| \mathrm{d}(\nu \boxplus \sigma_t)(x) + o_\delta(1) \quad \text{on } B(\delta,\varepsilon,x),$$

we find some $o_{\varepsilon,\delta}(1)$ going to zero when $\delta \ll \varepsilon$ go to zero, x

$$P_{N,\Lambda',\mathsf{M}}(x,\varepsilon) \geq \frac{2\varepsilon Z_{N-1}^{t(1-N^{-1})}}{Z_N^t} e^{N(\int \log|x-\lambda| \mathrm{d}(\sigma_t \boxplus \nu)(\lambda) - L_{\nu,t}^{\Lambda'}(x,Y) - \frac{1}{4t}x^2 + o_{\varepsilon,\delta}(1))}$$
$$\times \int_{V_{Y,\zeta,\varepsilon}} \mathrm{d}v_1 \mathbb{P}(\lambda(G_{N-1} + B'_{N-1}) \in C(\delta,\varepsilon,x))$$

where finally we compared the remaining probability to that of a perturbed GOE matrix in $v_1^{\perp}$ as in (2.40). We recall that B'_{N-1}, as a $(N-1)\times(N-1)$ matrix in $v_1^{\perp}$, has a unique outlier in $[\Lambda, \ell_\nu]$, which is close to $\Phi(\Lambda', Y)$, and eigenvalues η_i with multiplicities in $\{M_i, M_i - 1\}$. Let B''_{N-1} be the matrix with outlier $\Phi(\Lambda', Y)$ and eigenvalues η_i with multiplicity $M'_i = M_i$ except for η_p with multiplicity $M'_p = M_p - 1$ (with the same eigenspaces as B'_{N-1}). Then $B'_{N-1} \geq B''_{N-1} - \varepsilon$ for N large enough so that

$$\begin{aligned} \mathbb{P}(\lambda(G_{N-1} + B'_{N-1}) \in C(\delta,\varepsilon,x)) &\geq \mathbb{P}(\lambda_1(G_{N-1} + B'_{N-1}) \geq x + 2\varepsilon) \\ &\quad - \mathbb{P}\Big(d\Big(\frac{1}{N-1}\sum_{i=2}^N \delta_{\lambda_i}, \nu \boxplus \sigma_t\Big) \geq \delta \Big) - \mathbb{P}\Big(\frac{1}{N-1}\sum_{i=2}^N \lambda_i^2 \geq L_0 + 1 \Big) \\ &\geq \mathbb{P}(\lambda_1(G_{N-1} + B''_{N-1}) \geq x + 3\varepsilon) - e^{-cN^2}, \end{aligned}$$

where we finally used (2.27) and (2.32). c depends on δ but we can choose δ going to zero with N so that $cN^2 \gg N$. As for the proof of the upper bound, the right-hand side goes to 1 for ε small enough except when $\omega_{\nu,t}(x) > \Phi(\Lambda, Y)$. Notice that B'_{N-1} satisfies the same assumptions as B_N except that its outlier is now going to $\Phi(\Lambda', Y)$ (when κ goes to zero) so that, for every $\Lambda < \ell_\nu$, for every $Y \in \Delta^{p+1}$,

$$\lim_{\delta\to 0} \liminf_{N\to\infty} \inf_{\mathsf{M} \in B_\delta^N(\alpha)} \frac{1}{N} \log P_{N,\Lambda',\mathsf{M}}(x,\varepsilon) \geq -K_{\nu,t}(\Lambda', x, Y) - F_{\nu,t}^{-}(\Phi(\Lambda', Y), x + 3\varepsilon) + o(\varepsilon, \kappa, \varepsilon'),$$

where we took δ to zero. We can then let ε going to zero and take the supremum over all possible Y. We can finally take the infimum over $\Lambda' \in [\Lambda - \varepsilon, \Lambda + \varepsilon]$ and use the upper semi-continuity of $F_{\nu,t}^{-}$. This completes the proof of (2.43) as soon as $x + 3\varepsilon < \ell_{\nu,t}$, which is true for ε small enough as long as $x < \ell_{\nu,t}$. $\square$

2.6. Solving the functional equation for $\Lambda \geq \omega_{\nu,t}(\lambda)$

Recall that for $\Lambda \leq \ell_\nu$ and $\lambda \leq \ell_{\nu,t}$,

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$$\gamma_\Lambda(\lambda) := \begin{cases} \Lambda & \text{if } \Lambda \leq \omega_{\nu,t}(\ell_{\nu,t}), \text{ or } \Lambda \geq \omega_{\nu,t}(\ell_{\nu,t}) \text{ and } \lambda \leq H_{\nu,t}(\Lambda) \\ \omega^*_{\nu,t}(\lambda) & \text{if } \Lambda \geq \omega_{\nu,t}(\ell_{\nu,t}) \text{ and } \lambda \geq H_{\nu,t}(\Lambda), \end{cases} \tag{2.45}$$

where $\omega^*_{\nu,t}(\lambda)$ is as in (1.14) and $H_{\nu,t}(\Lambda)$ as in (1.12). Also define

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$$I^\Lambda_{\nu,t}(\lambda,\gamma) := \frac{1}{2}(S_\nu(\omega_{\nu,t}(\lambda)) - S_\nu(\gamma)) + \frac{1}{4t}((\lambda-\gamma)^2 - (\lambda - \omega_{\nu,t}(\lambda))^2). \tag{2.46}$$

and recall $I^\Lambda_{\nu,t}(\lambda) := I^\Lambda_{\nu,t}(\lambda, \gamma_\Lambda(\lambda))$. We first prove that $F^+_{\nu,t}$ and $F^-_{\nu,t}$ equal $I^\Lambda_{\nu,t}$ in the simplest case where the functional equations (2.26) and (2.43) are such that the infimum is taken at Y so that the term depending on $F^+_{\nu,t}$ or $F^-_{\nu,t}$ vanishes.

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Lemma 2.7. *Let $H_{\nu,t}(\Lambda)$ be as in (1.12). We have that for every $\lambda < \ell_{\nu,t}$ and $\Lambda \leq \ell_\nu$,*

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$$F^+_{\nu,t}(\Lambda,\lambda) \leq F^-_{\nu,t}(\Lambda,\lambda) \leq I^\Lambda_{\nu,t}(\lambda). \tag{2.47}$$

Moreover, if $\Lambda \geq \omega_{\nu,t}(\lambda)$, then

equality

$$F^-_{\nu,t}(\Lambda,\lambda) = F^+_{\nu,t}(\Lambda,\lambda) = I^\Lambda_{\nu,t}(\lambda) = \inf K_{\nu,t}(\Lambda,\lambda,\cdot). \tag{2.48}$$

The condition $\Lambda \geq \omega_{\nu,t}(\lambda)$ corresponds to cases (a) and (b) in Figure 2. A central step in the proof of Lemma 2.7 is to characterize the minimizers of $K_{\nu,t}(\Lambda,\lambda,\cdot)$, equivalently those of the function $J^\Lambda_{\nu,t}(\lambda,\cdot)$ defined in (2.24). In cases (a) and (b), the minimizer Y of $J^\Lambda_{\nu,t}(\lambda,\cdot)$ can be identified explicitly; this relies crucially on the fact that $\Phi(\Lambda, Y) \geq \omega_{\nu,t}(\ell_{\nu,t})$, where $\Phi(\Lambda, Y)$ is given by (2.18). Once this minimizer is determined, we complete the proof of Lemma 2.7.

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Lemma 2.8. *Fix $\lambda \in (-\infty, \ell_{\nu,t}]$. Let $H_{\nu,t}(\Lambda)$ be as in (1.12) and $\omega^*_{\nu,t}$ be as in (1.14).*

1. *For every $\Lambda \leq \ell_\nu$ and with either $\gamma = \omega^*_{\nu,t}(\lambda)$ or $\gamma = \Lambda$, let $Y(\gamma) := (y_0(\gamma), y_1(\gamma), \ldots, y_p(\gamma))$ be given by*

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$$y_i(\gamma) := \frac{t\alpha_i}{(\eta_i - \omega_{\nu,t}(\lambda))(\eta_i - \gamma)}, \quad \textit{for every } 1 \leq i \leq p, \tag{2.49}$$

and $y_0(\gamma) = 1 - \sum_{i=1}^p y_i(\gamma)$. The local minimizers of $J^\Lambda_{\nu,t}(\lambda,.)$ are included in $\{Y(\omega^*_{\nu,t}(\lambda)), Y(\Lambda)\}$ whenever $y_0(\Lambda) \geq 0$ or equal to $Y(\omega^*_{\nu,t}(\lambda)) = (0, (y_i(\omega^*_{\nu,t}(\lambda)))_{1\leq i\leq p})$ whenever $y_0(\Lambda) < 0$. Moreover, for $\gamma = \omega^*_{\nu,t}(\lambda)$ and $\gamma = \Lambda$, we have

eq:valueYK

$$K_{\nu,t}(\Lambda,\lambda,Y(\gamma)) = I^\Lambda_{\nu,t}(\lambda,\gamma). \tag{2.50}$$

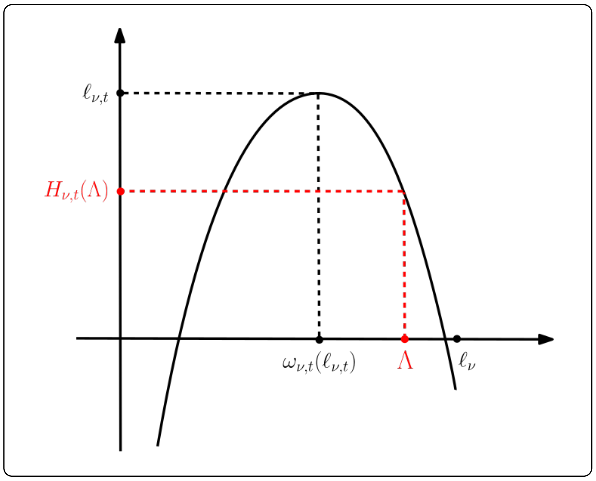
2. *The function $J^\Lambda_{\nu,t}(\lambda,\cdot)$ admits a unique minimizer $Y \in \Delta^{p+1}$ in the following cases:*
   - *If $\Lambda \leq \omega_{\nu,t}(\ell_{\nu,t})$ and $\lambda \leq H_{\nu,t}(\Lambda)$, or $\Lambda \geq \omega_{\nu,t}(\ell_{\nu,t})$ and $\lambda \leq H_{\nu,t}(\Lambda)$, or $\Lambda < \omega_{\nu,t}(\ell_{\nu,t})$ and $\lambda = \ell_{\nu,t}$, then the minimizer is given by $Y(\Lambda)$ and*

$$y_0(\Lambda) = \begin{cases} 1 + tG'_\nu(\Lambda) & \textit{if } \omega_{\nu,t}(\lambda) = \Lambda \\ \frac{\lambda - H_{\nu,t}(\Lambda)}{\omega_{\nu,t}(\lambda) - \Lambda} & \textit{if } \omega_{\nu,t}(\lambda) \neq \Lambda. \end{cases}$$

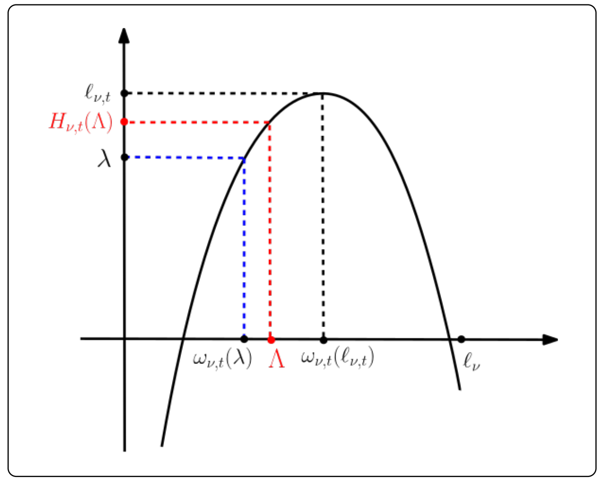
   - *If $\Lambda \geq \omega_{\nu,t}(\ell_{\nu,t})$ and $\lambda \in (H_{\nu,t}(\Lambda), \ell_{\nu,t}]$, then the unique minimizer is $Y(\omega^*_{\nu,t}(\lambda))$ with $y_0(\omega^*_{\nu,t}(\lambda))$ equal to zero.*

Proof. Step 1: existence of a minimizer. By Cauchy-Schwarz inequality, since $t > 0$ and $Y \in \Delta^{p+1}$,

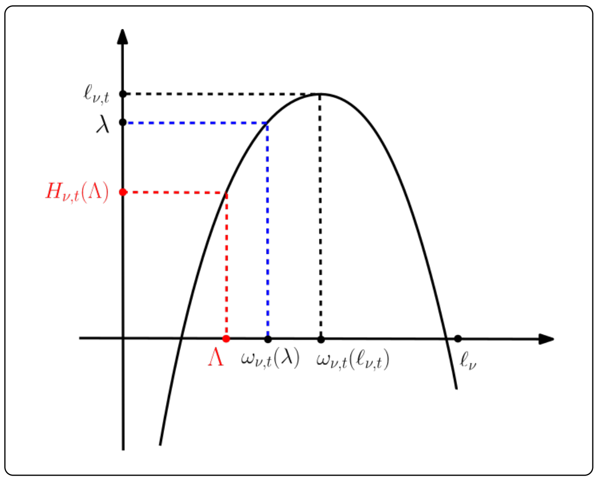
$$J^\Lambda_{\nu,t}(\lambda, Y) \geq -\frac{\lambda}{2t}\Big(\Lambda y_0 + \sum_{k=1}^p \eta_k y_k\Big) + \frac{1}{4t}\Big(\sum_{k=1}^p \eta_k^2 y_k + \Lambda^2 y_0\Big) - \frac{1}{2}\sum_{k=1}^p \alpha_k \log(y_k/\alpha_k)\,.$$



(a) $\Lambda \geq \omega_{\nu,t}(\ell_{\nu,t})$



(b) $\Lambda \leq \omega_{\nu,t}(\ell_{\nu,t})$ and $\lambda \leq H_{\nu,t}(\Lambda)$



(c) $\Lambda < \omega_{\nu,t}(\lambda)$

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Fig 2: The graph of the function $H_{\nu,t}$

Thus, since the last term is non-negative by Jensen's inequality, one can see by completing the square that

$$J^{\Lambda}_{\nu,t}(\lambda, Y) \geq \frac{1}{4t}\Big(\sum_{k=1}^{p}(\eta_k - \lambda)^2 y_k + (\Lambda-\lambda)^2 y_0\Big) - \frac{1}{4t}\lambda^2 \geq -\frac{1}{4t}\lambda^2.$$

Therefore $J^{\Lambda}_{\nu,t}(x,\cdot)$ is uniformly bounded from below on Δ^{p+1}. By Lemma 2.2, $Y \in (1-y_0)\Delta^p \mapsto I_\nu(Y)$ is lower semicontinuous. Moreover,

$$Y \in (1-y_0)\Delta^p \mapsto \sum_{k=1}^{p} \eta_k y_k \quad \text{ and } \quad Y \in (1-y_0)\Delta^p \mapsto \sum_{k=1}^{p} \eta_k^2 y_k$$

are continuous. Thus, $J^{\Lambda}_{\nu,t}(x,\cdot)$ is lower semicontinuous and since it is bounded from below, it achieves its minimal value. The function $J^{\Lambda}_{\nu,t}(x,\cdot)$ can be differentiated and its critical points analyzed. We describe them in the following.

Step 2: interior critical points. Let $Y \in \Delta^{p+1}$ be a critical point of $J^{\Lambda}_{\nu,t}(x,\cdot)$ such that $y_0 > 0$. By variational calculus, there exist constants R_1, R_2 such that for every $k \in \{1,\dots,p\}$

$$-x\eta_k + \eta_k^2 - R_1\eta_k - \frac{t\alpha_k}{y_k} = R_2$$

and

$$-x\Lambda + \Lambda^2 - R_1\Lambda = R_2.$$

Therefore, for every $k = 1,\dots,p$,

$$y_k = \frac{t\alpha_k}{\eta_k^2 - x\eta_k - R_1\eta_k - R_2},$$

where R_1 and R_2 are such that

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$$\begin{cases} y_0 + \sum_{i=1}^p y_i = 1 \\ \Lambda y_0 + \sum_{i=1}^p \eta_i y_i = R_1 \\ -x\Lambda + \Lambda^2 - R_1\Lambda = R_2. \end{cases} \tag{2.51}$$

One may write Y in the form

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$$y_k = \frac{t\alpha_k}{(\eta_k - \gamma_1)(\eta_k - \gamma_2)} \quad \text{for every } k = 1,\dots,p \tag{2.52}$$

with constants $\gamma_1, \gamma_2 \in \mathbb{C}$ satisfying $\gamma_1 + \gamma_2 = x + R_1$ and $\gamma_1\gamma_2 = -R_2$. The last equation in (2.51) therefore gives

$$(\Lambda - \gamma_1)(\Lambda - \gamma_2) = 0.$$

It follows that $\gamma_1 = \Lambda$ or $\gamma_2 = \Lambda$ and $\gamma_1, \gamma_2 \in \mathbb{R}$. We assume without loss of generality that $\gamma_1 \le \gamma_2$. For $\gamma \in \{\gamma_1, \gamma_2\}$ we set for every $k = 1,\dots,p$,

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$$y_k(\gamma) = \frac{t\alpha_k}{(\eta_k - \Lambda)(\eta_k - \gamma)}, \qquad y_0(\gamma) = 1 - \sum_{k=1}^p y_k(\gamma)\,. \tag{2.53}$$

Then, the previous argument shows that y is either equal to $y(\gamma_1)$ or $y(\gamma_2)$. Note that $y_k(\gamma)$ is nonnegative for every k such that γ belongs to $(-\infty, \ell_\nu)$ since $\Lambda \le \ell_\nu \le \eta_k$. If $\gamma \neq \Lambda$, the first equation of (2.51) gives

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$$\sum_{k=1}^p y_k(\gamma) = \frac{t}{\Lambda - \gamma}(G_\nu(\gamma) - G_\nu(\Lambda)) = 1 - y_0(\gamma), \tag{2.54}$$

otherwise if $\gamma = \Lambda$, we have

$$1 + tG_\nu'(\Lambda) = y_0(\Lambda).$$

As a consequence,

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$$y_0(\gamma) := \begin{cases} 1 + tG_\nu'(\Lambda) & \text{if } \gamma = \Lambda \\ \frac{H_{\nu,t}(\gamma) - H_{\nu,t}(\Lambda)}{\gamma - \Lambda} & \text{if } \gamma \neq \Lambda. \end{cases} \tag{2.55}$$

Furthermore the second equation of (2.51) gives

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$$\begin{aligned} \gamma + \Lambda - x = R_1 &= \Lambda y_0(\gamma) + \frac{t}{\gamma - \Lambda}\sum_{k=1}^p \eta_k\alpha_k\Big(\frac{1}{\eta_k - \gamma} - \frac{1}{\eta_k - \Lambda}\Big) \\ &= \Lambda y_0(\gamma) - \frac{t}{\gamma - \Lambda}\Big(\gamma G_\nu(\gamma) - \Lambda G_\nu(\Lambda)\Big) \\ &= \Lambda y_0(\gamma) - tG_\nu(\gamma) - \frac{t\Lambda}{\gamma - \Lambda}(G_\nu(\gamma) - G_\nu(\Lambda)) \\ &= \Lambda - tG_\nu(\gamma) \end{aligned} \tag{2.56}$$

where we finally used (2.54). Thus we find

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$$H_{\nu,t}(\gamma) = x, \quad \gamma < \ell_\nu. \tag{2.57}$$

By Remark A.1, whenever $x < \ell_{\nu,t}$, the above equation has two distinct solutions $\gamma_1 < \gamma_2$ in $(-\infty, \ell_\nu)$ and moreover $\gamma_1 = \omega_{\nu,t}(x)$ and $\gamma_2 = \omega^*_{\nu,t}(x)$. If $x = \ell_{\nu,t}$, then Equation (2.57) has a unique solution $\gamma = \omega_{\nu,t}(\ell_{\nu,t})$. Observe that if $\Lambda > \omega_{\nu,t}(\ell_{\nu,t})$ and $x > H_{\nu,t}(\Lambda)$, then for $\gamma \in \{\gamma_1, \gamma_2\}$, $y_0(\gamma) < 0$ by (2.55), showing that $Y(\gamma)$ is not a critical point in Δ^{p+1}. Let us now assume that $\Lambda \leq \omega_{\nu,t}(\ell_{\nu,t})$ or $\Lambda \geq \omega_{\nu,t}(\ell_{\nu,t})$ and $x \leq H_{\nu,t}(\Lambda)$. We check whether $y_0(\gamma)$ belongs to $[0,1]$. Note that we always have $y_0(\gamma) < 1$. Moreover, in the case $\gamma = \Lambda$,

$$y_0 = 1 + tG'_\nu(\Lambda),$$

which is strictly positive if and only if $\Lambda < \omega_{\nu,t}(\ell_{\nu,t})$. Therefore if $x = H_{\nu,t}(\Lambda)$ (giving $\gamma_1 = \gamma_2 = \Lambda$) and $\Lambda < \omega_{\nu,t}(\ell_{\nu,t})$, then $y_0 \in (0,1)$. If $\Lambda \notin \{\gamma_1, \gamma_2\}$, then Equation (2.55) shows that

$$y_0(\gamma) = \frac{x - H_{\nu,t}(\Lambda)}{\gamma - \Lambda}$$

and $y_0(\gamma)$ is positive only if

- $\Lambda < \omega_{\nu,t}(\ell_{\nu,t})$ (since $H_{\nu,t}$ is increasing on $(-\infty, \omega_{\nu,t}(\ell_{\nu,t}))$),
- or $\Lambda > \omega_{\nu,t}(\ell_{\nu,t})$ and $x < H_{\nu,t}(\Lambda)$.

Note that $y_0(\gamma) \leq 1$ by Definition (2.54) since α and Λ are smaller or equal to ℓ_ν. Hence, in this range of parameters $Y(\gamma_i)_{i=1,2}$ are critical points in Δ^{p+1}, but none of them belong to Δ^{p+1} otherwise. In the following we compute the value of $J^\Lambda_{\nu,t}(x,\cdot)$ for every of these two solutions in order to select the minimizer. Let $\gamma \in \{\gamma_1, \gamma_2\}$. One computes as in (2.56)

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$$M(\gamma) := \Lambda y_0(\gamma) + \sum_{i=1}^p \eta_i y_i(\gamma) = \Lambda - tG_\nu(\gamma) = \Lambda + \gamma - x, \tag{2.58}$$

Moreover,

$$\begin{aligned}
\sigma(\gamma) &:= \Lambda^2 y_0(\gamma) + \sum_{k=1}^p \eta_k^2 y_k(\gamma) \\
&= \Lambda^2 y_0(\gamma) + t\sum_{k=1}^p \frac{\alpha_k}{(\eta_k - \gamma)(\eta_k - \Lambda)}\left((\eta_k - \gamma)(\eta_k - \Lambda) + (\gamma + \Lambda)\eta_k - \gamma\Lambda\right) \\
&= \Lambda^2 y_0(\gamma) + t + (\gamma + \Lambda)\sum \eta_k y_k(\gamma) - \gamma\Lambda(1 - y_0(\gamma)) \\
&= t + (\gamma + \Lambda)M(\gamma) - \gamma\Lambda.
\end{aligned}$$

Moreover, since $\sum \alpha_k = 1$,

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$$I_\nu(Y(\gamma)) = -\frac{1}{2}\sum_{k=1}^p \alpha_k \ln\left(\frac{t}{(\eta_k - \gamma)(\eta_k - \Lambda)}\right) = -\frac{1}{2}\left(S_\nu(\gamma) + S_\nu(\Lambda)\right) - \frac{1}{2}\ln t\,. \tag{2.59}$$

Thus, we find

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$$\begin{aligned}
J^\Lambda_{\nu,t}(x, Y(\gamma)) &= L^\Lambda_{\nu,t}(x, Y(\gamma)) + I_\nu(Y(\gamma)) = -\frac{x}{2t}M(\gamma) + \frac{1}{2t}\sigma(\gamma) - \frac{1}{4t}M(\gamma)^2 + I_\nu(Y(\gamma)) \\
&= \frac{1}{4t}(-2x - M(\gamma) + 2(\gamma + \Lambda))M(\gamma) + \frac{1}{2} - \frac{\gamma\Lambda}{2t} - \frac{1}{2}(S_\nu(\Lambda) + S_\nu(\gamma)) - \frac{1}{2}\ln t \\
&= \frac{1}{4t}((\gamma + \Lambda - x)^2 - 2\gamma\Lambda) + \frac{1}{2} - \frac{1}{2}(S_\nu(\Lambda) + S_\nu(\gamma)) - \frac{1}{2}\ln t \\
&= \frac{1}{4t}(\gamma^2 + \Lambda^2 + x^2 - 2\gamma x - 2\Lambda x) + \frac{1}{2} - \frac{1}{2}(S_\nu(\Lambda) + S_\nu(\gamma)) - \frac{1}{2}\ln t.
\end{aligned} \tag{2.60}$$

Define the map

$$\phi:\gamma\in(-\infty,\ell_\nu)\mapsto J(x,Y(\gamma))=\frac{1}{4t}(\gamma^2+\Lambda^2+x^2-2\gamma x-2\Lambda x)+\frac{1}{2}-\frac{1}{2}(S_\nu(\Lambda)+S_\nu(\gamma))-\frac{1}{2}\ln t.$$

One can compute that for all $\gamma\in(-\infty,\ell_\nu)$,

$$\phi'(\gamma)=\frac{1}{2t}(H_{\nu,t}(\gamma)-x).$$

Therefore ϕ is strictly increasing on $[\gamma_1,\gamma_2]$ and therefore since we assumed $x\neq\ell_{\nu,t}$,

$$J^\Lambda_{\nu,t}(x,Y(\gamma_1))=\phi(\gamma_1)<\phi(\gamma_2)=J^\Lambda_{\nu,t}(x,Y(\gamma_2)).$$

We deduce that when $x\neq\ell_{\nu,t}$, $Y(\gamma_2)$ is not a minimizer of $J^\Lambda_{\nu,t}(x,\cdot)$.

Step 3: study of the critical points on the boundary. We next consider the critical points so that y_0 equal to zero. Let $Y\in\Delta^p$ be a critical point of $J^\Lambda_{\nu,t}(x,0,\cdot)$. By variational calculus, there exist R_1,R_2 such that

$$-x\eta_k+\eta_k^2-R_1\eta_k-\frac{t\alpha_k}{y_k}=R_2,\quad\text{for every } 1\le k\le p,$$

where $R_1=\sum\eta_k y_k$. Therefore

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$$y_k=\frac{t\alpha_k}{(\eta_k-\gamma_1)(\eta_k-\gamma_2)},\quad\text{for every } 1\le k\le p \tag{2.61}$$

where $\gamma_1,\gamma_2\in\mathbb{C}$ satisfy $\gamma_1+\gamma_2=x+R_1$ and $\gamma_1\gamma_2=-R_2$. The equation $\sum y_k=1$ gives

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$$H_{\nu,t}(\gamma_1)=H_{\nu,t}(\gamma_2). \tag{2.62}$$

Moreover

$$\sum_{k=1}^p\eta_k y_k=-tG_\nu(\gamma_2)+\gamma_1=R_1=\gamma_1+\gamma_2-x.$$

Inserting (2.62), this shows that

$$H_{\nu,t}(\gamma_1)=H_{\nu,t}(\gamma_2)=x.$$

Therefore $\gamma_1=\omega_{\nu,t}(x)$ and $\gamma_2=\omega^*_{\nu,t}(x)$ where $\omega^*_{\nu,t}(x)$ is as in (1.14). Let us now compute $J^\Lambda_{\nu,t}(x,0,Y)$. Let $M(\gamma):=\sum\eta_k y_k(\gamma)=\gamma_1+\gamma_2-x$. We have

$$\sum_{k=1}^p\eta_k^2 y_k=t+(\gamma_1+\gamma_2)M(\gamma)-\gamma_1\gamma_2.$$

Therefore we get

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$$\begin{aligned}J^\Lambda_{\nu,t}(x,0,Y)&=\frac{1}{4t}(-2x-M(\gamma)+2(\gamma_1+\gamma_2))M(\gamma)+\frac{1}{2}-\frac{\gamma_1\gamma_2}{2t}-\frac{1}{2}(S_\nu(\gamma_1)+S_\nu(\gamma_2))-\frac{1}{2}\ln t\\&=\frac{1}{4t}(\gamma_1^2+\gamma_2^2+x^2-2\gamma_1x-2\gamma_2x)+\frac{1}{2}-\frac{1}{2}(S_\nu(\gamma_1)+S_\nu(\gamma_2))-\frac{1}{2}\ln t.\end{aligned} \tag{2.63}$$

Step 4: Identification of the minimizer.

- If $\Lambda>\omega_{\nu,t}(\ell_{\nu,t})$ and $x>H_{\nu,t}(\Lambda)$, then by Step 2, there is no critical point with $y_0>0$ and the only minimizer is therefore given by $Y(\omega^*_{\nu,t}(x))$ where $Y(\gamma)$ is as defined in (2.49).
- If $\Lambda\le\omega_{\nu,t}(\ell_{\nu,t})$ and $x\le H_{\nu,t}(\Lambda)$, or $\Lambda\ge\omega_{\nu,t}(\ell_{\nu,t})$ and $x\le H_{\nu,t}(\Lambda)$, then the minimizer is either $Y(\gamma_1)$ defined in (2.53) or $(0,Y)$ defined in (2.61). We claim it is $Y(\gamma_1)$. In fact, recall

$$J^\Lambda_{\nu,t}(x,0,Y)=\frac{1}{4t}(\gamma_1^2+\gamma_2^2+x^2-2\gamma_1x-2\gamma_2x)+\frac{1}{2}-\frac{1}{2}(S_\nu(\gamma_1)+S_\nu(\gamma_2))-\frac{1}{2}\ln t,$$

while

$$J^{\Lambda}_{\nu,t}(x, Y(\gamma_1)) = \frac{1}{4t}(\gamma_1^2 + \Lambda^2 + x^2 - 2\gamma_1 x - 2\Lambda x) + \frac{1}{2} - \frac{1}{2}(S_\nu(\gamma_1) + S_\nu(\Lambda)) - \frac{1}{2}\ln t.$$

Define

$$\psi : \gamma \in (-\infty, \ell_\nu] \mapsto \frac{1}{4t}(\gamma_1^2 + \gamma^2 + x^2 - 2\gamma_1 x - 2\gamma x) + \frac{1}{2} - \frac{1}{2}(S_\nu(\gamma_1) + S_\nu(\gamma)) - \frac{1}{2}\ln t$$

Note that for all $\gamma \in (-\infty, \ell_\nu)$,

$$\psi'(\gamma) = \frac{1}{2t}(H_{\nu,t}(\gamma) - x).$$

Hence ψ is strictly increasing on $[\gamma_1, \gamma_2]$. Recall that $\Lambda \in [\gamma_1, \gamma_2]$. Therefore if $\Lambda < \gamma_2$, then

$$J^{\Lambda}_{\nu,t}(x, 0, Y) = \psi(\gamma_2) > \psi(\Lambda) = J^{\Lambda}_{\nu,t}(x, Y(\gamma_1))$$

which implies that $Y(\gamma_1)$ is the unique minimizer of $J^{\Lambda}_{\nu,t}(x, \cdot)$. If $\Lambda = \gamma_2$, then $Y(\gamma_1)$ is again the unique minimizer of $J^{\Lambda}_{\nu,t}(x, \cdot)$. We can also conclude in the case where $\Lambda < \omega_{\nu,t}(\ell_{\nu,t})$ and $x = \ell_{\nu,t}$. In this case, one can notice that $\gamma_2 = \gamma_1 = \omega_{\nu,t}(\ell_{\nu,t})$. Moreover ψ is strictly increasing on $(-\infty, \gamma_1)$. Therefore

$$J^{\Lambda}_{\nu,t}(x, 0, Y) = \psi(\gamma_1) > J^{\Lambda}_{\nu,t}(x, Y(\Lambda)) = \psi(\Lambda).$$

Therefore $Y(\Lambda)$ is the unique minimizer of $J^{\Lambda}_{\nu,t}(x, \cdot)$.

Step 5: Proof of (2.50). Let $Y(\gamma)$ be as in (2.49). Recall that by (2.60) and (2.63), for $\gamma \in \{\omega^*_{\nu,t}(x), \Lambda\}$,

$$J^{\Lambda}_{\nu,t}(x, Y(\gamma)) = \frac{1}{4t}(\gamma_1^2 + \gamma^2 + x^2 - 2\gamma_1 x - 2\gamma x) + \frac{1}{2} - \frac{1}{2}(S_\nu(\gamma_1) + S_\nu(\gamma)) - \frac{1}{2}\ln t.$$

By Lemma A.3,

$$-\int \log|y - x| \mathrm{d}(\nu \boxplus \sigma_t)(y) = S_{\nu\boxplus\sigma_t}(t) = S_\nu(\gamma_1) - \frac{1}{2t}(\gamma_1^2 + x^2 - 2x\gamma_1).$$

Therefore, combining this with (2.60) and recalling that $C_t = \frac{1}{2} - \frac{1}{2}\ln t$, we obtain

$$\begin{aligned} K_{\nu,t}(\Lambda, x, Y(\gamma)) &= \frac{1}{4t}(\gamma_1^2 + \gamma^2 + x^2 - 2\gamma_1 x - 2\gamma x) - \frac{1}{2}(S_\nu(\gamma_1) + S_\nu(\gamma)) + S_\nu(\gamma_1) \\ &\quad - \frac{1}{4t}(2\gamma_1^2 + 2x^2 - 4x\gamma_1 - x^2) \\ &= \frac{1}{2}(S_\nu(\gamma_1) - S_\nu(\gamma)) + \frac{1}{4t}(\gamma^2 - \gamma_1^2 - 2\gamma x + 2\gamma_1 x) \\ &= \frac{1}{2}(S_\nu(\gamma_1) - S_\nu(\gamma)) + \frac{1}{4t}((\gamma - x)^2 - (\gamma_1 - x)^2) = I^{\Lambda}_{\nu,t}(x, \gamma), \end{aligned}$$

which proves (2.50) by recalling (2.46). □

Before giving the proof of Lemma 2.7 we establish the following:

lue proj **Lemma 2.9.** *Let $H_{\nu,t}(\Lambda)$ be as in (1.12) and $\omega^*_{\nu,t}$ be as in (1.14).*

1. *Let $x \in (-\infty, \ell_{\nu,t}]$. If $\Lambda < \omega_{\nu,t}(\ell_{\nu,t})$, or $\Lambda > \omega_{\nu,t}(\ell_{\nu,t})$ and $x < H_{\nu,t}(\Lambda)$, let $Y(\Lambda) = (y_0(\Lambda), \ldots, y_p(\Lambda)) \in \Delta^{p+1}$ be given by*

$$y_k(\Lambda) = \frac{t\alpha_k}{(\eta_k - \omega_{\nu,t}(x))(\eta_k - \Lambda)}, \quad \text{for every } k = 1, \ldots, p, \quad y_0(\Lambda) = 1 - \sum_{k=1}^{p} y_k(\Lambda)$$

   Then $y_0(\Lambda)$ is strictly positive. Moreover, $\Phi(\Lambda, Y(\Lambda)) = \omega^*_{\nu,t}(x)$.
2. *Let $Y = (0, y_1, \ldots, y_p) \in \Delta^{p+1}$. Then, $\Phi(\Lambda, Y) = \Lambda$.*

Proof. Let us prove (1). By Lemma 2.3 and the proof of the previous lemma, it is enough to prove that $\Phi(\Lambda, Y(\Lambda)) = \omega^*_{\nu,t}(x)$ where we recall that $\Phi(\Lambda, Y)$ is defined as the smallest solution Λ_1 in (2.19). We denote in short $\alpha = \omega_{\nu,t}(x)$. Since $y_0(\Lambda) \neq 0$, $\Lambda_1 \neq \Lambda$. We claim that also $\Lambda_1 \neq \alpha$ because

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$$\sum_{k=0}^{p} \frac{y_k}{\eta_k - \alpha} \neq 0, \tag{2.64}$$

where we denote $\eta_0 := \Lambda$. Indeed, using that for every real numbers $a \neq b$ and $x \notin \{a, b\}$,

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$$\frac{1}{(x-a)^2(x-b)} = -\frac{1}{(a-b)^2}\left(\frac{1}{x-a} + \frac{b-a}{(x-a)^2} - \frac{1}{x-b}\right), \tag{2.65}$$

and setting $a = \alpha$ and $b = \Lambda$, one gets

$$\begin{aligned}
\sum_{k=0}^{p} \frac{y_k}{\eta_k - \alpha} &= \frac{y_0}{\Lambda - \alpha} - \frac{t}{(\Lambda-\alpha)^2}(-G_\nu(\alpha) + G_\nu(\Lambda) - (\Lambda - \alpha)G'_\nu(\alpha)) \\
&= -\frac{1}{(\Lambda-\alpha)^2}(H_{\nu,t}(\alpha) - H_{\nu,t}(\Lambda) - tG_\nu(\alpha) + tG_\nu(\Lambda) - t(\Lambda-\alpha)G'_\nu(\alpha)) \\
&= \frac{1 + tG'_\nu(\alpha)}{\Lambda - \alpha}.
\end{aligned}$$

Since $x < \ell_{\nu,t}$, we have $\alpha = \omega_{\nu,t}(x) < \omega_{\nu,t}(\ell_{\nu,t})$ and therefore $1 + tG'_\nu(\alpha) > 0$, which yields

$$\sum_{k=0}^{p} \frac{y_k}{\eta_k - \alpha} \neq 0,$$

which proves (2.64). We can thus take $\Lambda_1 \notin \{\Lambda, \alpha\}$. Then, by partial fraction decomposition,

$$\begin{aligned}
&\sum_{k=1}^{p} \frac{\alpha_k}{(\eta_k - \Lambda_1)(\eta_k - \Lambda)(\eta_k - \alpha)} \\
&\qquad = -\frac{1}{(\Lambda_1-\Lambda)(\Lambda_1-\alpha)}G_\nu(\Lambda_1) - \frac{1}{(\Lambda-\alpha)(\Lambda-\Lambda_1)}G_\nu(\Lambda) - \frac{1}{(\Lambda_1-\alpha)(\Lambda-\alpha)}G_\nu(\alpha),
\end{aligned}$$

which implies

$$\frac{y_0}{\Lambda - \Lambda_1} - t\frac{1}{(\Lambda-\Lambda_1)(\alpha-\Lambda_1)}G_\nu(\Lambda_1) + t\frac{1}{(\alpha-\Lambda)(\Lambda-\Lambda_1)}G_\nu(\Lambda) - t\frac{1}{(\alpha-\Lambda_1)(\alpha-\Lambda)}G_\nu(\alpha) = 0.$$

Multiplying by $(\Lambda - \Lambda_1)(\alpha - \Lambda_1)(\alpha - \Lambda)$ gives

$$y_0(\alpha - \Lambda_1)(\alpha - \Lambda) - t(\alpha - \Lambda)G_\nu(\Lambda_1) + t(\alpha - \Lambda_1)G_\nu(\Lambda) - t(\Lambda - \Lambda_1)G_\nu(\alpha) = 0.$$

Recalling that $y_0(\alpha - \Lambda) = x - H_{\nu,t}(\Lambda)$, this can be simplified into

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$$\begin{aligned}
&(x - H_{\nu,t}(\Lambda))(\alpha - \Lambda_1) - t(\alpha - \Lambda)G_\nu(\Lambda_1) + t(\alpha - \Lambda_1)G_\nu(\Lambda) - t(\Lambda - \Lambda_1)G_\nu(\alpha) = 0 \\
\iff\ &(\alpha - \Lambda)(\alpha - \Lambda_1) + tG_\nu(\alpha)(\alpha - \Lambda) - tG_\nu(\Lambda_1)(\alpha - \Lambda) = 0 \\
\iff\ &(\alpha - \Lambda_1) + tG_\nu(\alpha) - tG_\nu(\Lambda_1) = 0 \\
\iff\ &H_{\nu,t}(\Lambda_1) = \lambda. \qquad (2.66)
\end{aligned}$$

Since $\Lambda_1 \neq \alpha$, it follows that Λ_1 equals $\omega^*_{\nu,t}(x)$. The proof of (2) is immediate. $\square$

We can now give the proof of Lemma 2.7.

Proof of Lemma 2.7. Let us prove the upper bound (2.47). Let $Y(\gamma)\in\Delta^{p+1}$ be given by (2.49) with γ as in (2.45). By (2.50), we know that $K_{\nu,t}(\Lambda,x,Y(\gamma))=I^{\Lambda}_{\nu,t}(x)$. Using (2.43), and taking $Y=Y(\gamma)$ in the infimum yields

$$F^{-}_{\nu,t}(\Lambda,x)\le K_{\nu,t}(\Lambda,x,Y(\gamma))+F^{-}_{\nu,t}(\Phi(\Lambda,Y),x^{+})1_{\omega_{\nu,t}(x)\ge\Phi(\Lambda,Y(\gamma))}.$$

As shown in Lemma 2.9, $\Phi(\Lambda,Y(\gamma))\ge\omega_{\nu,t}(\ell_{\nu,t})\ge\omega_{\nu,t}(x)$ so that the last term in the right-hand side vanishes. It follows that

$$F^{-}_{\nu,t}(\Lambda,x)\le K_{\nu,t}(\Lambda,x,Y(\gamma))=I^{\Lambda}_{\nu,t}(x)$$

where we finally used (2.50). Using $F^{+}_{\nu,t}\le F^{-}_{\nu,t}$ we get the desired upper bounds. Let us now assume that $\Lambda\ge\omega_{\nu,t}(x)$ or $\Lambda\le\omega_{\nu,t}(\ell_{\nu,t})$ and $x\le H_{\nu,t}(\Lambda)$. Since $F^{+}_{\nu,t}\ge0$, one has

$$F^{+}_{\nu,t}(\Lambda,x)\ge\inf_{Y}K_{\nu,t}(\Lambda,x,Y).$$

By Lemma 2.8, in the range of parameters that we consider, $K_{\nu,t}(\Lambda,x,.)$ and $J^{\Lambda}_{\nu,t}$ have a unique minimizer $Y(\gamma)$ with $\gamma=\gamma_{\Lambda}(x)$. Hence

$$F^{+}_{\nu,t}(\Lambda,x)\ge K_{\nu,t}(\Lambda,x,Y(\gamma))=I^{\Lambda}_{\nu,t}(x),$$

which gives $F^{+}_{\nu,t}(\Lambda,x)=I^{\Lambda}_{\nu,t}(x)$ using (2.47). Since $F^{-}_{\nu,t}(\Lambda,x)\ge F^{+}_{\nu,t}(\Lambda,x)$ we get that $F^{-}_{\nu,t}(\Lambda,x)=I^{\Lambda}_{\nu,t}(x)$. ☐

2.7. Solving the functional inequalities

In the next lemma, we prove that the equality (2.48) in fact holds for any $\Lambda\le\ell_{\nu}$ and $x\le\ell_{\nu,t}$. In view of Lemma 2.7, it only remains to prove it for $\Lambda<\omega_{\nu,t}(x)$, i.e in case (c) of Figure 1. This will complete the proof of the weak large deviation principle (2.4).

lemma:F+ **Lemma 2.10.** *Let $\Lambda\le\ell_{\nu}$ and $x<\ell_{\nu,t}$. Let $I^{\Lambda}_{\nu,t}(x)$ be as in (2.46). We have*

$$F^{-}_{\nu,t}(\Lambda,x)=F^{+}_{\nu,t}(\Lambda,x)=I^{\Lambda}_{\nu,t}(x).$$

To prove Lemma 2.10, we first demonstrate that $I^{\Lambda}_{\nu,t}$ satisfies the functional equation necessary for it to be equal to $F^{\pm}_{\nu,t}$, as suggested by (2.26) and (2.43).

a:solves **Lemma 2.11.** *Let $\Lambda\le\ell_{\nu}$. Let $\Phi(\Lambda,Y)$ be as in (2.19), $H_{\nu,t}(\Lambda)$ be as in (1.12) and $\omega^{*}_{\nu,t}$ be as (1.14). Let $I^{\Lambda}_{\nu,t}$ be as in (2.46) and K as in (2.25). Let $x<\ell_{\nu,t}$. Let $\omega^{*}_{\nu,t}(x)$ be the unique solution γ in $[\omega_{\nu,t}(\ell_{\nu,t}),\ell_{\nu})$ of the equation $H_{\nu,t}(\gamma)=x$. Then, the function $G(Y):=K_{\nu,t}(\Lambda,x,Y)+I^{\Phi(\Lambda,Y)}_{\nu,t}(x)1_{\omega_{\nu,t}(x)\ge\Phi(\Lambda,Y)}$ admits a unique minimizer $Y^{*}\in\Delta^{p+1}$.*

1. *Moreover,*

$$\Phi(\Lambda,Y^{*})\ge\omega_{\nu,t}(\ell_{\nu,t})$$

and

$$I^{\Lambda}_{\nu,t}(x)=\inf_{Y\in\Delta^{p+1}}G(Y)=K_{\nu,t}(\Lambda,x,Y^{*}).$$

2. *Furthermore,*
   - *(a) If $\Lambda\le\omega_{\nu,t}(\ell_{\nu,t})$, or $\Lambda\ge\omega_{\nu,t}(\ell_{\nu,t})$ and $x\le H_{\nu,t}(\Lambda)$, then $Y^{*}=Y(\Lambda)$ with $Y(\Lambda)$ as defined in (2.49),*
   - *(b) If $\Lambda\ge\omega_{\nu,t}(\ell_{\nu,t})$ and $x\in(H_{\nu,t}(\Lambda),\ell_{\nu,t}]$, then y^{*}_{0} vanishes. Moreover, $Y^{*}=Y(\omega^{*}_{\nu,t}(x))$ with $Y(\omega^{*}_{\nu,t}(x))$ as defined in (2.49).*

Remark 2.2.

- In view of our proof, conditionally to λ_1 being in a δ-neighborhood of x, $Y(v_1)$ concentrates in a ε_{δ}- neighborhood of the minimizers of $K_{\nu,t}(\Lambda,x,\cdot)+I^{\Phi(\Lambda,\cdot)}_{\nu,t}(x)1_{\omega_{\nu,t}(x)\ge\Phi(\Lambda,\cdot)}$, namely Y^{*}, for some ε_{δ} going to zero with δ. Indeed, the contribution of the complement of this set is exponentially small. Obtaining large deviations for the joint distribution of $(\lambda_1,Y(v_1))$ may be possible but would require additional work.

- When $\omega_{\nu,t}(x) = \Phi(\Lambda, Y)$, $I_{\nu,t}^{\Phi(\Lambda,Y)}(x) = 0$ so that in the definition of G we can equivalently take the indicator function $\omega_{\nu,t}(x) \geq \Phi(\Lambda, Y)$ or $\omega_{\nu,t}(x) > \Phi(\Lambda, Y)$.

Proof. Since $I_{\nu,t}^{\Phi(\Lambda,Y)} \geq 0$, we deduce from the first step of the proof of Lemma 2.8 that G is bounded from below (by $K_{\nu,t}$). We claim that it is lower semicontinuous in $Y \in \Delta^{p+1}$. We already checked in the previous proof that $J_{\nu,t}^{\Lambda}(x, .)$ is lower semicontinuous, and so is $K_{\nu,t}(\Lambda, x, .)$ by Definition (2.25). Moreover $Y \mapsto \Phi(\Lambda, Y)$ is continuous on $y_1 \geq \kappa$ for every $\kappa > 0$ by Lemma 2.3. Recall that when y_1 goes to zero, $\Phi(\Lambda, Y)$ goes to η_1 which is larger than $\omega_{\nu,t}(x)$ unless $\ell_{\nu,t}^{\Lambda} = \ell_{\nu,t} = \eta_1$ and x goes to $\ell_{\nu,t}$. Since we assume $x < \ell_{\nu,t}$, we may and shall assume $y_1 > \kappa_x$ for some $\kappa_x > 0$ and hence $\Phi(x, .)$ continuous. Furthermore, when $\Phi(x, Y) \leq \omega_{\nu,t}(x)$, we have by Lemma 2.7

defI
$$I_{\nu,t}^{\Phi(\Lambda,Y)}(x) = \frac{1}{2}(S_\nu(\omega_{\nu,t}(x)) - S_\nu(\Phi(\Lambda, Y))) + \frac{1}{4t}((x - \Phi(\Lambda, Y))^2 - (x - \omega_{\nu,t}(x))^2). \tag{2.67}$$

Since S_ν is continuous on $(-\infty, \ell_\nu)$ and $\Phi(\Lambda, Y)(x) \leq \ell_{\nu,t} \leq \ell_\nu$, the lower semi-continuity of

$$Y \mapsto I_{\nu,t}^{\Phi(\Lambda,Y)}(x)$$

follows. The lower semi-continuity of $Y \mapsto I_{\nu,t}^{\Phi(\Lambda,Y)}(x) 1_{\Phi(\Lambda,Y) < \omega_{\nu,t}(x)}$ then follows from the fact that $I_{\nu,t}^{\Phi(\Lambda,Y)}(x)$ vanishes when $\Phi(\Lambda, Y) = \omega_{\nu,t}(x)$. It follows that G is bounded from below and lower semicontinuous , and hence achieves its minimal value. Next, we characterize the minimizers of G.

Step 1: study of the local minima such that $\Phi(\Lambda, \cdot) \geq \omega_{\nu,t}(x)$. On the set $\{Y \in \Delta^{p+1} : \Phi(\Lambda, Y) \geq \omega_{\nu,t}(x)\}$, the last term in G vanishes and therefore the minimizers of G on this set are the minimizers Y of $J_{\nu,t}^{\Lambda}(x, \cdot)$ that satisfy $\Phi(\Lambda, Y) \geq \omega_{\nu,t}(x)$.

The minimizers of $J_{\nu,t}^{\Lambda}(x, \cdot)$ are in $\{Y(\Lambda), Y(\omega_{\nu,t}^*(x))\}$ by Lemma 2.8. In view of Lemma 2.9, $\Phi(\Lambda, Y(\Lambda)) = \omega_{\nu,t}^*(x) \geq \omega_{\nu,t}(x)$. Besides $\Phi(\Lambda, Y(\omega_{\nu,t}^*(x))) = \Lambda$. Therefore the possible minimizers of G on the set $\{Y \in \Delta^{p+1} : \Phi(\Lambda, Y) \geq \omega_{\nu,t}(x)\}$ are $Y(\Lambda)$, and $Y(\omega_{\nu,t}^*(x))$ whenever $\Lambda \geq \omega_{\nu,t}(x)$. Additionally, (2.50) shows that the value of $G(Y(\gamma))$ for $\gamma \in \{\omega_{\nu,t}^*(x), \Lambda\}$ is $I_{\nu,t}^{\Lambda}(x, \gamma)$.

Step 2: study of the critical points such that $\Phi(\Lambda, \cdot) < \omega_{\nu,t}(x)$ and $y_0 > 0$ Let $Y \in \Delta^{p+1}$ be a critical point of G such that $y_0 > 0$ and

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$$\Phi(\Lambda, Y) < \omega_{\nu,t}(x). \tag{2.68}$$

By variational calculus similar to the proof of Lemma 2.8, but with the additional term $I_{\nu,t}^{\Phi(\Lambda,Y)}(x)$ given by (2.67), there exist real numbers R_1, R_2 and C_0 such that for $k \in \{1, \ldots, p\}$,

$$-x\eta_k + \eta_k^2 - R_1\eta_k - \frac{t\alpha_k}{y_k} - C_0\frac{1}{\eta_k - \Lambda_1} = R_2$$

and

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$$-x\Lambda + \Lambda^2 - R_1\Lambda - \frac{C_0}{\Lambda - \Lambda_1} = R_2, \tag{2.69}$$

with $R_1 = \Lambda y_0 + \sum \eta_k y_k$, and $\Lambda_1 = \Phi(\Lambda, Y)$. The new term in C_0 comes from the derivative of $I_{\nu,t}^{\Phi(\Lambda,Y)}$ and the remark that

$$\partial_{y_k}(I_{\nu,t}^{\Phi(\Lambda,Y)}(x)) = \partial_{\Lambda'} I_{\nu,t}^{\Lambda'}(x)|_{\Lambda' = \Phi(\Lambda,Y)} \partial_{y_k}\Phi(\Lambda, Y)$$

whereas by Equation (2.19),

$$\partial_{y_k}\Phi(\Lambda, Y) = \left(\frac{y_0}{(\Lambda - \Phi(\Lambda, Y))^2} - \sum_{k=1}^{p} \frac{y_k}{(\eta_k - \Phi(\Lambda, Y))^2}\right)^{-1} \frac{1}{\Phi(\Lambda, Y) - \eta_k} =: \frac{C_0}{\partial_{\Lambda'} I_{\nu,t}^{\Lambda'}(x)|_{\Lambda' = \Phi(\Lambda,Y)}} \frac{1}{\Lambda_1 - \eta_k}.$$

It follows that for every $k = 1, \ldots, p$,

defy

$$y_k = \frac{t\alpha_k}{\eta_k^2 - (x + R_1)\eta_k - R_2 - \frac{C_0}{\eta_k - \Lambda_1}} = \frac{t\alpha_k(\eta_k - \Lambda_1)}{(\eta_k^2 - (x + R_1)\eta_k - R_2)(\eta_k - \Lambda_1) - C_0}. \tag{2.70}$$

According to (2.69), Λ is one of the roots of the denominator in the last display. Let us write this denominator as

defiP

$$P(\eta_k) := (\eta_k^2 - (x + R_1)\eta_k - R_2)(\eta_k - \Lambda_1) - C_0 = (\eta_k - \gamma_1)(\eta_k - \gamma_2)(\eta_k - \Lambda), \tag{2.71}$$

with $\gamma_2, \gamma_1 \in \mathbb{C}$. Identifying the coefficient in front of η_k^2, we observe that

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$$\Lambda + \gamma_1 + \gamma_2 = x + R_1 + \Lambda_1. \tag{2.72}$$

- Assume that $\Lambda \neq \gamma_1 \neq \gamma_2$. By definition of $\Phi(\Lambda, Y) = \Lambda_1$, see (2.19), we have

eq:eqy0

$$\frac{y_0}{\Lambda - \Lambda_1} + \sum_{k=1}^{p} \frac{t\alpha_k}{(\eta_k - \Lambda)(\eta_k - \gamma_1)(\eta_k - \gamma_2)} = 0. \tag{2.73}$$

Using the partial fraction decomposition

$$\frac{1}{(\eta_k - \Lambda)(\eta_k - \gamma_1)(\eta_k - \gamma_2)} = \frac{1}{(\Lambda - \gamma_1)(\Lambda - \gamma_2)} \frac{1}{\eta_k - \Lambda} + \frac{1}{(\gamma_1 - \Lambda)(\gamma_1 - \gamma_2)} \frac{1}{\eta_k - \gamma_1}$$
$$+ \frac{1}{(\gamma_2 - \Lambda)(\gamma_2 - \gamma_1)} \frac{1}{\eta_k - \gamma_2},$$

one may rewrite (2.73) as

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$$\frac{y_0}{\Lambda - \Lambda_1} - \frac{1}{(\Lambda - \gamma_1)(\Lambda - \gamma_2)} G_\nu(\Lambda) + \frac{1}{(\Lambda - \gamma_1)(\gamma_1 - \gamma_2)} G_\nu(\gamma_1) - \frac{1}{(\Lambda - \gamma_2)(\gamma_1 - \gamma_2)} G_\nu(\gamma_2) = 0. \tag{2.74}$$

The condition $\sum_{k=0}^{p} y_k = 1$ reads

$$y_0 + t \sum_{k=1}^{p} \alpha_k \frac{\eta_k - \Lambda_1}{(\eta_k - \Lambda)(\eta_k - \gamma_1)(\eta_k - \gamma_2)} = 1,$$

which gives

$$t \sum_{k=1}^{p} \frac{\alpha_k}{(\eta_k - \gamma_1)(\eta_k - \gamma_2)} + t(\Lambda - \Lambda_1) \sum_{k=1}^{p} \frac{\alpha_k}{(\eta_k - \Lambda)(\eta_k - \gamma_1)(\eta_k - \gamma_2)} = 1 - y_0.$$

Hence, using (2.73), one gets

eq:=1

$$\sum_{k=1}^{p} \frac{t\alpha_k}{(\eta_k - \gamma_1)(\eta_k - \gamma_2)} = 1, \tag{2.75}$$

which gives

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$$\frac{1}{\gamma_1 - \gamma_2}(-tG_\nu(\gamma_1) + tG_\nu(\gamma_2)) = 1 \Leftrightarrow H_{\nu,t}(\gamma_1) = H_{\nu,t}(\gamma_2). \tag{2.76}$$

In view of (2.72), we have

$$\Lambda y_0 + t \sum_{k=1}^{p} \frac{\alpha_k(\eta_k - \Lambda_1)\eta_k}{(\eta_k - \Lambda)(\eta_k - \gamma_1)(\eta_k - \gamma_2)} = R_1 = \Lambda + \gamma_1 + \gamma_2 - x - \Lambda_1,$$

which implies

$$\Lambda y_0 + t \sum_{k=1}^{p} \frac{\alpha_k(\eta_k - \Lambda_1)}{(\eta_k - \gamma_1)(\eta_k - \gamma_2)} + \Lambda(1 - y_0) = \Lambda + \gamma_1 + \gamma_2 - x - \Lambda_1.$$

Simplifying further we obtain

$$-tG_\nu(\gamma_2)+(\gamma_1-\Lambda_1)=\gamma_1+\gamma_2-x-\Lambda_1.$$

Therefore, inserting (2.76), we conclude that

$$H_{\nu,t}(\gamma_2)=H_{\nu,t}(\gamma_1)=x.$$

Therefore $\gamma_1=\omega_{\nu,t}(x)$ and $\gamma_2=\omega^*_{\nu,t}(x)$. Moreover we can compute using (2.74) that

$$\frac{y_0}{\Lambda-\Lambda_1}(\Lambda-\gamma_1)(\Lambda-\gamma_2)=\Lambda+tG_\nu(\Lambda)-x$$

which implies

q:y0here

$$\Lambda_1=\Lambda-y_0\frac{(\Lambda-\gamma_1)(\Lambda-\gamma_2)}{H_{\nu,t}(\Lambda)-x}\,. \tag{2.77}$$

- Assume that $\Lambda=\gamma_1=\gamma_2$. Then proceeding as in (2.75), we get

$$\sum_{k=1}^{p}\frac{t\alpha_k}{(\eta_k-\Lambda)^2}=1,$$

implying that $\Lambda=\omega_{\nu,t}(\ell_{\nu,t})$. In particular, $\Phi(\Lambda,Y)\geq\omega_{\nu,t}(x)$, which contradicts (2.68).

- Assume that $\Lambda=\gamma_1$ and $\gamma_1\neq\gamma_2$. Proceeding as above we easily get $H_{\nu,t}(\Lambda)=H_{\nu,t}(\gamma_2)=x$, from which it is inferred that $\Lambda=\omega_{\nu,t}(x)$, implying that $\Phi(\Lambda,Y)\geq\omega_{\nu,t}(x)$, thus contradicting (2.68).
- Assume that $\Lambda\neq\gamma_1$ and $\gamma_1=\gamma_2=\gamma$. Then we get

$$\sum_{k=1}^{p}\frac{t\alpha_k}{(\eta_k-\gamma)^2}=1$$

and $H_{\nu,t}(\gamma)=x$, which implies that $x=\ell_{\nu,t}$ and $\gamma=\omega_{\nu,t}(\ell_{\nu,t})$, contradicting the assumption that $x<\ell_{\nu,t}$.

We conclude from (2.70) and (2.71) and the above discussion that if Y is a critical point of G satisfying (2.68), then it is of the form

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$$y_k=\frac{t\alpha_k(\eta_k-\Lambda_1)}{(\eta_k-\Lambda)(\eta_k-\omega_{\nu,t}(x))(\eta_k-\omega^*_{\nu,t}(x))},\quad\text{for every } 1\leq k\leq p, \tag{2.78}$$

for some $\Lambda_1\in(\Lambda,\ell_\nu)$ such that letting $y_0:=1-\sum_{k=1}^p y_k$, the relations (2.77), (2.73) hold. In the sequel we denote $\gamma_1:=\omega_{\nu,t}(x)$ and $\gamma_2:=\omega^*_{\nu,t}(x)$. We next show that this critical point will never be a minimizer of G.

Step 3: study of the minimizers. A minimizer Y of G is either such that $\Phi(\Lambda,Y)<\omega_{\nu,t}(x)$ or a critical point of $J^\Lambda_{\nu,t}(x,\cdot)$ such that $\Phi(\Lambda,Y)\geq\omega_{\nu,t}(x)$. Therefore it is either of the form (2.78) or of the form $Y(\gamma)$, for $\gamma\in\{\Lambda,\omega^*_{\nu,t}(x)\}$ by Lemma 2.8. We will study these minimizers in every different regimes and show that G in fact has a unique minimizer and it is the same as that of $J^\Lambda_{\nu,t}$, except in the previously undecided case $\Lambda<\omega_{\nu,t}(x)$, where it is now given by $Y(\Lambda)$. This will complete the proof of the Lemma.

- Assume that $\Lambda\geq\omega_{\nu,t}(x)$. Then, since $\Phi(\Lambda,Y)\geq\Lambda\geq\omega_{\nu,t}(x)$,

$$G(Y)=K_{\nu,t}(\Lambda,x,Y)=J^\Lambda_{\nu,t}(x,Y)-\int\log|x-y|\mathrm{d}(\nu\boxplus\sigma_t)(y)+\frac{x^2}{4t}-C_t.$$

The function $G(Y)$ is therefore uniquely minimized by the unique minimizer of $J^\Lambda_{\nu,t}(x,\cdot)$ described in Lemma 2.8, which is equal to $Y(\Lambda)$ if $x\leq\ell^\Lambda_{\nu,t}$ and $Y(\omega^*_{\nu,t}(x))$ otherwise. This proves the second part of Lemma 2.11(2)(a) and Lemma 2.11(2)(b).

- Assume that $\Lambda<\omega_{\nu,t}(x)$ and $x=\ell_{\nu,t}$. By Lemma 2.8, the minimum of $J^\Lambda_{\nu,t}(\ell_{\nu,t},\cdot)$ is uniquely attained at $Y=Y(\Lambda)$. Moreover $\Phi(\Lambda,Y(\Lambda))=\omega^*_{\nu,t}(\ell_{\nu,t})=\omega_{\nu,t}(\ell_{\nu,t})$. Therefore $G(Y(\Lambda))=I^\Lambda_{\nu,t}(\ell_{\nu,t})$. Hence, using $I^\Lambda_{\nu,t}\geq 0$ and $I^{\Phi(\Lambda,Y(\Lambda))}_{\nu,t}(\ell_{\nu,t})=0$, we see that if Y is a minimizer of G, then $J^\Lambda_{\nu,t}(\ell_{\nu,t},Y)=J^\Lambda_{\nu,t}(\ell_{\nu,t},Y(\Lambda))$. Therefore, $Y=Y(\Lambda)$.

- Assume that $\Lambda < \omega_{\nu,t}(x)$ and $x < \ell_{\nu,t}$. Then, for every $Y \in \Delta^p$, $\Phi(\Lambda,(0,Y)) = \Lambda$ and

$$G((0,Y)) = J^{\Lambda}_{\nu,t}(x,(0,Y)) - \int \log|y-x|\mathrm{d}(\nu \boxplus \sigma_t)(y) + \frac{x^2}{4t} + I^{\Lambda}_{\nu,t}(x)1_{\Lambda<\omega_{\nu,t}(x)} - C_t.$$

Since the term depending on $I^{\Lambda}_{\nu,t}$ does not depend on Y we conclude that $G(0,\cdot)$ is minimal at the minimum of $J^{\Lambda}_{\nu,t}(x,(0,\cdot))$, namely at $(y_1(\omega^*_{\nu,t}(x)),\ldots,y_p(\omega^*_{\nu,t}(x)))$ and therefore

$$\inf_{Y\in\Delta^p} G((0,Y)) = I^{\omega^*_{\nu,t}(x)}_{\nu,t}(x) + I^{\Lambda}_{\nu,t}(x).$$

One can check that for every $\lambda < \ell_{\nu,t}$,

$$\frac{\partial}{\partial\lambda} I^{\omega^*_{\nu,t}(\lambda)}_{\nu,t}(\lambda) = \frac{1}{2t}(\omega_{\nu,t}(\lambda) - \omega^*_{\nu,t}(\lambda)) < 0.$$

Since $I^{\omega^*_{\nu,t}(\ell_{\nu,t})}_{\nu,t}(\ell_{\nu,t}) = 0$ we deduce that for all $x < \ell_{\nu,t}$,

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$$I^{\omega^*_{\nu,t}(x)}_{\nu,t}(x) > 0 \tag{2.79}$$

and therefore

$$\inf_{Y\in\Delta^p} G((0,Y)) > I^{\Lambda}_{\nu,t}(x) = G(Y(\Lambda))$$

with $Y(\Lambda) \in \Delta^{p+1}$. It follows that minimizers of G are not of the form $(0,Y)$ with $Y \in \Delta^p$.

Let us prove that $Y(\Lambda)$ uniquely minimizes G. Since $\Lambda \le \omega_{\nu,t}(\ell_{\nu,t})$, $Y(\Lambda) \in \Delta^{p+1}$. By (2.50), we have $G(Y(\Lambda)) = I^{\Lambda}_{\nu,t}(x)$. Therefore the minimizer of G is either $Y(\Lambda)$ or (y_0,Y) of the form (2.78).

Let (y_0,Y) be of the form (2.78). Let us compute $G(y_0,Y)$. Define $M(x) := \Lambda y_0 + \sum_{k=1}^p \eta_k y_k$. Recall that $M(x) = \Lambda + \gamma_1 + \gamma_2 - x - \Lambda_1$. We have

$$\begin{aligned}\sigma(x) := \Lambda^2 y_0 + \sum_{k=1}^p \eta_k^2 y_k &= \Lambda^2 y_0 + t - (\Lambda_1 - \Lambda)tG_\nu(\Lambda) + (\gamma_1+\gamma_2)M(x) - \Lambda(\gamma_1+\gamma_2)y_0 - \gamma_1\gamma_2(1-y_0)\\ &= (\Lambda-\gamma_1)(\Lambda-\gamma_2)y_0 - \gamma_1\gamma_2 - (\Lambda_1-\Lambda)tG_\nu(\Lambda) + (\gamma_1+\gamma_2)M(x)\\ &= \gamma_1\gamma_2 + (\gamma_1+\gamma_2)M(x) + (\Lambda-\Lambda_1)(\Lambda-x).\end{aligned}$$

where we inserted (2.77). It follows that

$$\begin{aligned}G(y_0,Y) = \frac{1}{2} - \frac{1}{2}\ln t + \frac{1}{4t}(-2x - M(x) + 2(\gamma_1+\gamma_2))M(x) - \frac{\gamma_1\gamma_2}{2t} + \frac{1}{2t}(\Lambda-\Lambda_1)(\Lambda-x)\\ - \frac{1}{2}(S_\nu(\Lambda) + S_\nu(\gamma_1) + S_\nu(\gamma_2) - S_\nu(\Lambda_1)) + \frac{1}{2}(S_\nu(\gamma_1) - S_\nu(\Lambda_1)) + \frac{1}{4t}(\Lambda_1^2 - 2x\Lambda_1 - \gamma_1^2 + 2x\gamma_1)\\ = -\frac{1}{2}(S_\nu(\gamma_2) + S_\nu(\Lambda)) + \frac{Z}{4t} + \frac{1}{2} - \frac{1}{2}\ln t,\end{aligned}$$

with

$$\begin{aligned}Z &:= (-2x - M(x) + 2(\gamma_1+\gamma_2))M(x) - 2\gamma_1\gamma_2 + 2(\Lambda^2 + \Lambda_1 x - \Lambda x - \Lambda_1 x) + \Lambda_1^2 - 2x\Lambda_1 - \gamma_1^2 + 2x\gamma_1\\ &= \Lambda^2 + x^2 + \gamma_2^2 - 2\gamma_2 x - 2x\Lambda.\end{aligned}$$

We therefore find that

al value

$$G(y_0,Y) = \frac{1}{4t}(\Lambda^2 + x^2 + \gamma_2^2 - 2\gamma_2 x - 2x\Lambda) - \frac{1}{2}(S_\nu(\gamma_2) + S_\nu(\Lambda)) + \frac{1}{2} - \frac{1}{2}\ln t. \tag{2.80}$$

We next show that this is greater than $G(Y(\Lambda))$. Let us define

phigamma

$$\phi : \gamma \in (-\infty,\ell_\nu) \mapsto \frac{1}{4t}(\Lambda^2 + x^2 + \gamma^2 - 2\gamma x - 2x\Lambda) - \frac{1}{2}(S_\nu(\gamma) + S_\nu(\Lambda)) + \frac{1}{2} - \frac{1}{2}\ln t. \tag{2.81}$$

For all $\gamma < \ell_\nu$, one has

$$\phi'(\gamma) = \frac{H_{\nu,t}(\gamma) - x}{2t}.$$

Therefore ϕ is strictly increasing on $[\gamma_1, \gamma_2]$. It follows that if $\gamma_1 < \gamma_2$,

$$\phi(\gamma_2) = G(y_0, Y) > G(Y(\Lambda)) = \phi(\gamma_1)$$

and $Y(\Lambda)$ is the unique minimizer of G. If $\gamma_1 = \gamma_2$, then $Y(\Lambda)$ is still the unique minimizer of G. This proves the first part of Lemma 2.11 (2)(a) and completes the proof of the lemma.

□

We finally prove Lemma 2.10 by iterating the functional inequality (2.26) and showing that after a finite number of steps it does not depend on $F^+_{\nu,t}$, so that $F^+_{\nu,t}$ is bounded from below by $I^\Lambda_{\nu,t}$.

Proof of Lemma 2.10. By Lemma 2.7, we only need to consider the case where $\Lambda < \omega_{\nu,t}(x)$ and $x < \ell_{\nu,t}$ which we assume in the sequel. Recall

$$F^+_{\nu,t}(\Lambda, x) \ge \inf_{Y \in \Delta^{p+1}} \{K_{\nu,t}(\Lambda, x, Y) + F^+_{\nu,t}(\Phi(\Lambda, Y), x^-) 1_{\Phi(\Lambda,Y) < \omega_{\nu,t}(x)}\}.$$

Recall from Lemma B.2 that $x \mapsto F^+_{\nu,t}(\Phi(\Lambda, Y), x^-)$ increases on $\{x < \ell_{\nu,t} : \Phi(\Lambda, Y) < \omega_{\nu,t}(x)\}$. Hence, for $\varepsilon > 0$ small enough,

eq:ine
$$F^+_{\nu,t}(\Lambda, x) \ge \inf_{Y \in \Delta^{p+1}} \{K_{\nu,t}(\Lambda, x, Y) + F^+_{\nu,t}(\Phi(\Lambda, Y), x - \varepsilon) 1_{\Phi(\Lambda,Y) < \omega_{\nu,t}(x)}\}, \tag{2.82}$$

By iteration define $\Phi_0(\Lambda, Y) = \Lambda, \Phi_1(\Lambda, Y) = \Phi(\Lambda, Y)$ and for each $k \ge 1$,

$$\Phi_k(\Lambda, Y_1, \ldots, Y_k) := \Phi(\Phi_{k-1}(\Lambda, Y_1, \ldots, Y_{k-1}), Y_k) \quad \text{for all } Y_1, \ldots, Y_k \in \Delta^{p+1}.$$

Recall that for every Λ, Y, $\Phi(\Lambda, Y) \ge \Lambda$. Consequently for every $k \ge 1$

tf
$$\bigcap_{i=1}^{k} \{Y_1, \ldots, Y_k \in (\Delta^{p+1})^k : \Phi_i(\Lambda, Y_1, \ldots, Y_i) < \omega_{\nu,t}(x)\} = \{Y_1, \ldots, Y_k \in (\Delta^{p+1})^k : \Phi_k(\Lambda, Y_1, \ldots, Y_k) < \omega_{\nu,t}(x)\}. \tag{2.83}$$

Also recall that by Lemma 2.8, $\inf K_{\nu,t}(\Phi_{k-1}(\Lambda, Y_1, \ldots, Y_{k-1}), x, \cdot) = 0$ when $\Phi_{k-1}(\Lambda, Y_1, \ldots, Y_{k-1}) = \omega_{\nu,t}(x)$. Let $n \ge 1$. Applying (2.82) n times for ε small enough, gives

opt
$$\begin{aligned} F^+_{\nu,t}(\Lambda, x) \ge \inf_{Y_1,\ldots,Y_n} \Big(&\sum_{k=1}^{n} K_{\nu,t}(\Phi_{k-1}(\Lambda, Y_1, \ldots, Y_{k-1}), x - (k-1)\varepsilon, Y_k) 1_{\cap_{\ell=1}^{k-1}\{\Phi_\ell(\Lambda,Y_1,\ldots,Y_\ell) < \omega_{\nu,t}(x-(\ell-1)\varepsilon)\}} \\ &+ F^+_{\nu,t}(\Phi_n(\Lambda, Y_1, \ldots, Y_n), x - n\varepsilon) 1_{\cap_{\ell=1}^{n}\{\Phi_\ell(\Lambda,Y_1,\ldots,Y_\ell) < \omega_{\nu,t}(x-(\ell-1)\varepsilon)\}} \Big). \end{aligned} \tag{2.84}$$

Recall that $\Lambda \mapsto F^+_{\nu,t}(\Lambda, x)$ is lower semicontinuous and $K_{\nu,t}(., x, Y)$ is continuous so that the above minimum is achieved on the compact set $(\Delta^{p+1})^n$. Let $(Y_1^\varepsilon, \ldots, Y_n^\varepsilon)$ be a minimizer. By compactness, one can extract a subsequence $(Y_1^{\varepsilon_l}, \ldots, Y_n^{\varepsilon_l})$ converging as ε goes to zero to some $(Y_1, \ldots, Y_n) \in \Delta^{p+1}$.

- Assume that $\Phi_n(\Lambda, Y_1, \ldots, Y_n) > \omega_{\nu,t}(x)$. Then, by continuity of Φ and $\omega_{\nu,t}$, for ε small enough, $\Phi_n(\Lambda, Y_1^\varepsilon, \ldots, Y_n^\varepsilon) > \omega_{\nu,t}(x - (n-1)\varepsilon)$. Hence, in this case, by letting ε go to zero and using (2.83), we find

$$\begin{aligned} F^+_{\nu,t}(\Lambda, x) &\ge K_{\nu,t}(\Lambda, x, Y_1) + \sum_{k=2}^{n} K_{\nu,t}(\Phi_{k-1}(\Lambda, Y_1, \ldots, Y_{k-1}), x, Y_k) 1_{\{\Phi_{k-1}(\Lambda,Y_1,\ldots,Y_{k-1}) < \omega_{\nu,t}(x)\}} \\ &\quad + I^{\Phi_n(\Lambda,Y_1,\ldots,Y_n)}_{\nu,t}(x) 1_{\{\Phi_n(\Lambda,Y_1,\ldots,Y_n) < \omega_{\nu,t}(x)\}} \\ &\ge \inf_{Y_1,\ldots,Y_n} \Big(K_{\nu,t}(\Lambda, x, Y_1) + \sum_{k=2}^{n} K_{\nu,t}(\Phi_{k-1}(\Lambda, Y_1, \ldots, Y_{k-1}), x, Y_k) 1_{\{\Phi_{k-1}(\Lambda,Y_1,\ldots,Y_{k-1}) < \omega_{\nu,t}(x)\}} \\ &\quad + I^{\Phi_n(\Lambda,Y_1,\ldots,Y_n)}_{\nu,t}(x) 1_{\{\Phi_n(\Lambda,Y_1,\ldots,Y_n) < \omega_{\nu,t}(x)\}} \Big), \end{aligned}$$

where in the first line we kept the last term and replaced $F^+_{\nu,t}$ by $I_{\nu,t}$ since it is multiplied by zero. We can then take the infimum over Y_n. By Lemma 2.11,

$$\inf_{Y_n}\{K_{\nu,t}(\Phi_{n-1}(\Lambda,Y_1,\dots,Y_{n-1}),x,Y_n)1_{\{\Phi_{n-1}(\Lambda,Y_1,\dots,Y_{n-1})<\omega_{\nu,t}(x)\}}+I_{\nu,t}^{\Phi_n(\Lambda,Y_1,\dots,Y_n)}(x)1_{\{\Phi_n(\Lambda,Y_1,\dots,Y_n)<\omega_{\nu,t}(x)\}}\}$$

$$=I_{\nu,t}^{\Phi_{n-1}(\Lambda,Y_1,\dots,Y_{n-1})}(x)1_{\{\Phi_{n-1}(\Lambda,Y_1,\dots,Y_{n-1})<\omega_{\nu,t}(x)\}}$$

and proceeding inductively we deduce that

$$F^+_{\nu,t}(\Lambda,x)\ge I^{\Lambda}_{\nu,t}(x).$$

• We show that it is impossible that $\Phi_n(\Lambda,Y_1,\dots,Y_n)\le\omega_{\nu,t}(x)$ if n is large enough by contradiction. We first notice that by definition for every Λ,Y,

bv

$$\frac{y_0}{\Phi(\Lambda,Y)-\Lambda}=\sum_{i=1}^{p}\frac{y_i}{\eta_i-\Phi(\Lambda,Y)}. \tag{2.85}$$

We can assume without loss of generality that Λ is such that $\eta_i-\Phi(\Lambda,Y)>\gamma$ for some $\gamma>0$; otherwise if $\Phi(\Lambda,Y)$ is close to ℓ_ν, then $\Phi(\Lambda,Y)$ would be greater than $\omega_{\nu,t}(x)$, placing us in the previously discussed scenario. Hence, by (2.85),

$$\Phi(\Lambda,Y)\ge\Lambda+\gamma y_0\,.$$

We therefore have two cases, being given some small $\delta>0$,

- Either $y_0\ge\delta/\gamma$, and then

lb2

$$\Phi(\Lambda,Y)\ge\Lambda+\delta\,. \tag{2.86}$$

- Or $y_0\le\delta/\gamma$. Then denote by $\tilde Y$ the element of Δ^{p+1} such that $\tilde y_0=0$ and $\tilde y_i=y_i(1-y_0)^{-1}$ for $i\in\{1,\dots,p\}$. The continuity of $K_{\nu,t}(\Lambda,x,.)$ implies that for δ small enough, uniformly on Y,Λ,x,

$$K_{\nu,t}(\Lambda,x,Y)=K_{\nu,t}(\Lambda,x,\tilde Y)+O(\delta)\ge\min_{Y\in\Delta^p}\{K_{\nu,t}(\Lambda,x,(0,Y))\}+O(\delta).$$

By Lemma 2.8, the minimum of $K_{\nu,t}(\Lambda,x,\cdot)$ over elements of the form $(0,Y)$ is taken at $Y=Y(\omega^*_{\nu,t}(x))$ and therefore

lb1

$$\min_{Y\in\Delta^p}\{K_{\nu,t}(\Lambda,x,(0,Y))\}\ge I_{\nu,t}^{\omega^*_{\nu,t}(x)}(x). \tag{2.87}$$

Notice that the right-hand side does not depend on Λ and moreover by (2.79), $I_{\nu,t}^{\omega^*_{\nu,t}(x)}(x)>0$. We hereafter choose δ small enough such that

$$\kappa(x):=\min_{Y\in\Delta^p}\{K_{\nu,t}(\Lambda,x,(0,Y))\}+O(\delta)$$

is strictly positive and independent of Λ.

Returning to (2.84), and recalling that $(Y_1^\varepsilon,\dots,Y_n^\varepsilon)$ is a minimizer, and that $F^+_{\nu,t}$ is lower semicontinuous in the variable Λ while $K_{\nu,t}$ and $\omega_{\nu,t}$ are continuous, we obtain the following: if again $(Y_1,\cdots,Y_n)$ denotes a limiting point of the optimizing sequence $(Y_1^\varepsilon,\dots,Y_n^\varepsilon)$, then

$$\begin{aligned}F^+_{\nu,t}(\Lambda,x)\ge&\sum_{k=1}^{n}K_{\nu,t}(\Phi_{k-1}(\Lambda,Y_1,\dots,Y_{k-1}),x,Y_k)1_{\{\Phi_{k-1}(\Lambda,Y_1,\dots,Y_{k-1})<\omega_{\nu,t}(x)\}}\\&+F^+_{\nu,t}(\Phi_n(\Lambda,Y_1,\dots,Y_n),x^-)1_{\{\Phi_n(\Lambda,Y_1,\dots,Y_n)<\omega_{\nu,t}(x)\}}.\end{aligned}$$

From the above discussion, we see from (2.86) that if there are ℓ indices m such that $(Y_m)_0\ge\delta/\gamma$ then $\Phi_n(\Lambda,Y_1,\dots,Y_n)\ge\Lambda+\ell\delta$ which implies that $\Phi_n(\Lambda,Y_1,\dots,Y_n)>\omega_{\nu,t}(x)$ if ℓ is greater than some constant, which contradicts our assumption. Therefore, we conclude that there exists ℓ_0 finite such that for $n-\ell_0$ indices i, $(Y_i)_0 \leq \delta/\gamma$ *** I replaced y_0 by δ ***. But then, since $K_{\nu,t}$ and $F^+_{\nu,t}$ are non-negative, (2.87) implies

$$F^+_{\nu,t}(\Lambda,x) \geq (n-\ell_0)\kappa(x),$$

which is impossible for n large enough since $F^+_{\nu,t}(\Lambda,x)$ is finite. Therefore, we deduce that there exists n large enough, such that for every $k \geq n$, $\Phi_k(\Lambda,Y_1,\dots,Y_k) > \omega_{\nu,t}(x)$. We can argue as in the first case to conclude that

$$F^+_{\nu,t}(\Lambda,x) \geq I^\Lambda_{\nu,t}(x).$$

Using the upper bound (2.47), this concludes the proof of Lemma 2.10. □

3. Large deviations in the general case

In this section, we suppose that the sequence of matrices (B_N) satisfies Assumption 1.1. The measure ν is now allowed to have a continuous support. We first remark that we may restrict ourselves to the case where $b_1^N = \Lambda$ and $b_2^N = \ell_{nu}$ up to replace B_N by another matrix B'_N satisfying this hypothesis, so that they are equal except for these two entries. In fact, we then see that $\|B_N - B'_N\|_{op}$ goes to zero with N so that also $\lambda_1(G_N+B_N) - \lambda_1(G_N+B'_N)$ goes to zero with N everywhere. Thus, large deviations for the distribution of $\lambda_1(G_N+B'_N)$ implies the large deviation for the law of $\lambda_1(G_N+B'_N)$ since they are exponentially equivalent [**?**][Theorem 4.2.13].

3.1. Properties of the rate function

propgrf

We first prove the approximation and convexity properties of the rate function $I^\Lambda_{\nu,t}$ defined in (1.9). Recall that η_ν is the transport map such that $\nu = \eta_\nu \# 1_{[0,1]}dx$.

onvexity

Lemma 3.1 (Properties of the rate function)**.** *1. (Continuity) Let ν be a probability measure on $\mathbb{R}$ with bounded support. Let $(\nu_n)_{n\geq 1}$ be a sequence of probability measures on $\mathbb{R}$ such that (η_{ν_n}) converges uniformly towards η_ν. Then for every $x \in (-\infty,\ell_{\nu,t})$,*

$$\lim_{n\to\infty} I^\Lambda_{\nu_n,t}(x) = I^\Lambda_{\nu,t}(x).$$

2. (Convexity) Let ν be a probability measure on $\mathbb{R}$ with support bounded from below. The map $I^\Lambda_{\nu,t}$ is strictly convex on $(-\infty,\ell_{\nu,t})$ and achieves its minimum uniquely, at $H_{\nu,t}(\Lambda) = \Lambda + tG_\nu(\Lambda)$ if $\Lambda \leq \omega_{\nu,t}(\ell_{\nu,t})$ or $\ell_{\nu,t}$ if $\Lambda \geq \omega_{\nu,t}(\ell_{\nu,t})$.

3. (Bounds) Moreover, for every $\varepsilon > 0$, there exists a positive constant C_ε so that for $x \leq \ell_{\nu,t} - \varepsilon$,

lowerq

$$I^\Lambda_{\nu,t}(x) \geq C_\varepsilon (x - \ell^\Lambda_{\nu,t})^2\,. \tag{3.1}$$

At infinity, we have $I^\Lambda_{\nu,t}(x) \simeq \frac{\lambda^2}{4}$.

Proof.

Step 1: proof of continuity It is easy to check that, if $|\eta_\nu| \vee |x| \vee |\Lambda| \leq M$, if $|\eta_{\nu_n} - \eta_\nu| \leq \varepsilon$, and if $x \leq \ell_{\nu,t} - \kappa$, then there exist constants $C = C(M,t) > 0$ and $c = c(M,t,\kappa) > 0$ such that

$$|\ell_{\nu,t} - \ell_{\nu_n,t}| \leq C\varepsilon, \qquad |G_\nu(x) - G_{\nu_n}(x)| \leq C\kappa^{-2}\varepsilon, \qquad |H_{\nu,t}(x) - H_{\nu_n,t}(x)| \leq C\kappa^{-2}\varepsilon,$$

$$H'_{\nu,t}(\omega_{\nu,t}(x)) \geq c,$$

and therefore

$$|\omega_{\nu,t}(x) - \omega_{\nu_n,t}(x)| + |\omega^*_{\nu,t}(x) - \omega^*_{\nu_n,t}(x)| \leq \frac{C}{c}\,\varepsilon.$$

See also Lemma A.2. By continuity of $S_\nu(x)$ for $x < \ell_\nu$ (since ν has compact support), we deduce that $I^\Lambda_{\nu_n,t} \to I^\Lambda_{\nu,t}$ uniformly on compact subsets of $(-\infty,\ell_{\nu,t})$, hence away from $\ell_{\nu,t}$.

Step 2: proof of convexity Let ν be a probability measure on $\mathbb{R}$ with support bounded from below. Assume first that $\Lambda \le \omega_{\nu,t}(\ell_{\nu,t})$ so that $\gamma_\Lambda(x)=\Lambda$. A direct derivation shows that

$$\partial_x I^{\Lambda}_{\nu,t}(x) = \frac{1}{2t}(\omega_{\nu,t}(x)-\Lambda).$$

This implies that for all $x<\ell_{\nu,t}$,

partial2

$$\partial_x^2 I^{\Lambda}_{\nu,t}(x) = \frac{1}{2t}\omega'_{\nu,t}(x) > 0\,. \tag{3.2}$$

Indeed the term in the r.h.s is clearly positive since

$$\omega'_{\nu,t}(x) = \frac{1}{1+tG'_\nu(\omega_{\nu,t}(x))} > 0$$

on $(-\infty,\ell_{\nu,t})$ since $\omega_{\nu,t}(\ell_{\nu,t})$ satisfies

$$\omega_{\nu,t}(\ell_{\nu,t}) = \inf\Big\{x : \int \frac{\mathrm{d}\nu(\lambda)}{(\lambda-x)^2} \ge \frac{1}{t}\Big\}.$$

It follows that $I^{\Lambda}_{\nu,t}$ is strictly convex when $\Lambda \le \omega_{\nu,t}(\ell_{\nu,t})$. Moreover, since $\omega_{\nu,t}(H_{\nu,t}(\Lambda))=\Lambda$, we find $\partial_x I^{\Lambda}_{\nu,t}(H_{\nu,t}(\Lambda))=0$. Thus, $I^{\Lambda}_{\nu,t}$ attains its minimum uniquely at $\ell^{\Lambda}_{\nu,t}=H_{\nu,t}(\Lambda)$.

Now assume that $\Lambda \ge \omega_{\nu,t}(\ell_{\nu,t})$. For all $x<H_{\nu,t}(\Lambda)$, we have still $\gamma_\Lambda(x)=\Lambda$ and so as above,

ty

$$\partial_x I^{\Lambda}_{\nu,t}(x) = \frac{1}{2t}(\omega_{\nu,t}(x)-\Lambda) \le 0. \tag{3.3}$$

Moreover, for all $x\in[H_{\nu,t}(\Lambda),\ell_{\nu,t})$, we have $\gamma_\Lambda(x)=\omega^*_{\nu,t}(x)$ and we find

eq:F2

$$\partial_x I^{\Lambda}_{\nu,t}(x) = \frac{1}{2t}(\omega_{\nu,t}(x)-\omega^*_{\nu,t}(x)) \le 0. \tag{3.4}$$

It follows that for all $x\in(H_{\nu,t}(\Lambda),\ell_{\nu,t})$,

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$$\partial_x^2 I^{\Lambda}_{\nu,t}(x) = \frac{1}{2t}\Big(\frac{1}{1+tG'_\nu(\omega_{\nu,t}(x))} - \frac{1}{1+tG'_\nu(\omega^*_{\nu,t}(x))}\Big) > 0, \tag{3.5}$$

since $\omega_{\nu,t}(x)<\omega_{\nu,t}(\ell_{\nu,t})<\omega^*_{\nu,t}(x)$. Together with (3.2), this proves that $I^{\Lambda}_{\nu,t}$ is strictly convex and decreasing. Thus it reaches its minimum uniquely at $\ell^{\Lambda}_{\nu,t}=\ell_{\nu,t}$. Finally, the bound (3.1) is a direct consequence from uniform positive lower bounds on $\partial_x^2 I^{\Lambda}_{\nu,t}(x)$ that are easily deduced from (3.2) and (3.5) which degenerate only at $x=\ell_{\nu,t}$. The behaviour at infinity was already derived in (2.2). $\square$

3.2. Proof of Theorem 1.2

Proof of Theorem 1.2. Fix $\varepsilon>0$ small enough so that $\varepsilon<\ell_\nu-\Lambda$. Since ν has at most countably many atoms, for each such ε we can choose a shift $s\in(0,\varepsilon)$ such that

$$\nu\big(\{\ell_\nu+s+k\varepsilon\}\big)=0 \qquad \forall k\in\mathbb{Z}.$$

Define the maps

$$D^-_\varepsilon(x) := \ell_\nu+s+\varepsilon\Big\lfloor\frac{x-(\ell_\nu+s)}{\varepsilon}\Big\rfloor, \qquad D^+_\varepsilon(x) := \ell_\nu+s+\varepsilon\Big\lceil\frac{x-(\ell_\nu+s)}{\varepsilon}\Big\rceil,$$

and the discretized matrices

$$B^-_{N,\varepsilon} := \mathrm{diag}\big(\Lambda, D^-_\varepsilon(b^N_2),\dots,D^-_\varepsilon(b^N_N)\big), \qquad B^+_{N,\varepsilon} := \mathrm{diag}\big(\Lambda, D^+_\varepsilon(b^N_2),\dots,D^+_\varepsilon(b^N_N)\big).$$

Then $B^-_{N,\varepsilon}\le B_N\le B^+_{N,\varepsilon}$ and $\|B^\pm_{N,\varepsilon}-B_N\|_{op}\le\varepsilon$. Let $\nu^\pm_\varepsilon := (D^\pm_\varepsilon)_\#\nu$. By the choice of s, the discontinuity points of $D^\pm_\varepsilon$ have 0 measure under ν, hence $(D^\pm_\varepsilon)_\#\mu_{B_N}\Rightarrow\nu^\pm_\varepsilon$ weakly.

Since $B^-_{N,\varepsilon} \le B_N \le B^+_{N,\varepsilon}$, we have almost surely

$$\lambda_1(B^-_{N,\varepsilon}+G_N) \le \lambda_1(B_N+G_N) \le \lambda_1(B^+_{N,\varepsilon}+G_N),$$

and hence for all $x \in \mathbb{R}$,

sandwich

$$\mathbb{P}\big(\lambda_1(B^-_{N,\varepsilon}+G_N)\ge x\big) \le \mathbb{P}\big(\lambda_1(B_N+G_N)\ge x\big) \le \mathbb{P}\big(\lambda_1(B^+_{N,\varepsilon}+G_N)\ge x\big). \tag{3.6}$$

Consequently we can use Proposition 2.1 to deduce from (3.6) that

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$$-\inf_{[x,+\infty)} I^{\Lambda}_{\nu^-_\varepsilon,t} \le \liminf_{N\to\infty}\frac{1}{N}\log\mathbb{P}\left(\lambda_1(B_N+G_N)\ge x\right) \le \limsup_{N\to\infty}\frac{1}{N}\log\mathbb{P}\left(\lambda_1(B_N+G_N)\ge x\right) \le -\inf_{[x,+\infty)} I^{\Lambda}_{\nu^+_\varepsilon,t}\,. \tag{3.7}$$

We now observe that the function $D^{\pm}_{\varepsilon}(\eta_\nu)-\eta_\nu$ converges uniformly towards zero so that $I^{\Lambda}_{\nu^{\pm}_\varepsilon,t}-I^{\Lambda}_{\nu,t}$ goes to zero by the first point of Lemma 3.1. Therefore, since $I^{\Lambda}_{\nu^{\pm}_\varepsilon,t}$ goes to infinity like $x^2/4t$, we also deduce that

$$\lim_{\varepsilon\to 0}\inf_{[x,+\infty)} I^{\Lambda}_{\nu^{\pm}_\varepsilon,t} = \inf_{[x,+\infty)} I^{\Lambda}_{\nu,t}$$

and the same when the infimum is taken on $(x,+\infty)$. Therefore, letting ε going to zero, (3.7) implies that

lim

$$-\inf_{[x,+\infty)} I^{\Lambda}_{\nu,t} \le \liminf_{N\to\infty}\frac{1}{N}\log\mathbb{P}\left(\lambda_1(B_N+G_N)\ge x\right) \le \limsup_{N\to\infty}\frac{1}{N}\log\mathbb{P}\left(\lambda_1(B_N+G_N)\ge x\right) \le -\inf_{[x,+\infty)} I^{\Lambda}_{\nu,t}. \tag{3.8}$$

Let $\delta>0$ be such that $x+\delta\le \ell_\nu$. It follows that since $I^{\Lambda}_{\nu,t}$ is strictly increasing above $\ell^{\Lambda}_{\nu,t}$ by Lemma 3.1, for $\kappa>0$ small and N large enough

$$\begin{aligned}\mathbb{P}\left(\lambda_1(B_N+G_N)\ge x+\delta\right) &\le e^{-N\inf_{(-\infty,x+\delta]}(I^{\Lambda}_{\nu,t}-\kappa)} \le e^{-N\inf_{(-\infty,x]}(I^{\Lambda}_{\nu,t}+\kappa)}\\ &\le e^{-N\kappa/2}\mathbb{P}\left(\lambda_1(B_N+G_N)\ge x\right).\end{aligned}$$

Therefore

$$\mathbb{P}\left(\lambda_1(B_N+G_N)\ge x\right) = \mathbb{P}\left(\lambda_1(B_N+G_N)\in(x,x+\delta)\right)(1+o(1))\,.$$

As a consequence, we deduce from (3.8) that for $x\ge \ell^{\Lambda}_{\nu,t}$,

wer

$$\begin{aligned}-\inf_{(x,x+\delta)} I^{\Lambda}_{\nu,t} &\le \liminf_{N\to\infty}\frac{1}{N}\log\mathbb{P}\left(\lambda_1(B_N+G_N)\in[x,x+\delta)\right)\\ &\le \limsup_{N\to\infty}\frac{1}{N}\log\mathbb{P}\left(\lambda_1(B_N+G_N)\in[x,x+\delta)\right) \le -\inf_{[x,x+\delta)} I^{\Lambda}_{\nu,t}.\end{aligned} \tag{3.9}$$

Since $I^{\Lambda}_{\nu,t}$ is lower semicontinuous, we have

$$\lim_{\delta\to0}\inf_{(x,x+\delta)} I^{\Lambda}_{\nu,t} = \lim_{\delta\to0}\inf_{[x,x+\delta)} I^{\Lambda}_{\nu,t} = I^{\Lambda}_{\nu,t}(x).$$

Hence, letting δ going to zero, (3.9) gives the weak large deviations estimate above $\ell^{\Lambda}_{\nu,t}$. Similarly we find by the above comparison that

$$-\inf_{(-\infty,x]} I_{\nu^+_\epsilon,\Lambda} \le \liminf_{N\to\infty}\frac{1}{N}\log\mathbb{P}\left(\lambda_1(B_N+G_N)\le x\right) \le \limsup_{N\to\infty}\frac{1}{N}\log\mathbb{P}\left(\lambda_1(B_N+G_N)\le x\right) \le -\inf_{(-\infty,x]} I_{\nu^-_\varepsilon,\Lambda}.$$

We can also use the convergence of $I^{\Lambda}_{\nu^{\pm}_\varepsilon,t}$ towards $I^{\Lambda}_{\nu^{\pm},t}$ and the fact that $I^{\Lambda}_{\nu,t}$ is strictly decreasing below $\ell^{\Lambda}_{\nu,t}$ to deduce the weak large deviation principle below $\ell^{\Lambda}_{\nu,t}$. This proves the weak LDP everywhere. This result can be upgraded to a full LDP using exponential tightness as in the proof of Proposition 2.1 and the fact that $I^{\Lambda}_{\nu,t}$ is a good rate function. Lemma 3.1 allows to complete the proof of Theorem 1.2.

□

4. Comparison with prior results

prior

In this section we show that our rate function agrees with [19, 20].

4.1. The case of the GOE with an outlier

We first consider the case where $B_N = \Lambda e_1 e_1^T$ for $\Lambda < 0$ and compare our result with [19], more precisely the formula obtained in second version on the arxiv of this paper. Let G_N be a GOE of variance $t > 0$. We study the smallest eigenvalue of $G_N + \Lambda e_1 e_1^T$. Note that $\sigma_t = \sigma_t \boxplus \delta_0$. Therefore, the characteristic associated to the Burgers equation, see Remark A.1, is

$$H_{0,t}(x) = x + \frac{t}{x}.$$

Let $\omega_{0,t}$ be the inverse of $H_{0,t}$ on $(-\infty, -2\sqrt{t}]$, namely

$$\omega_{0,t} : \lambda \in (-\infty, -2\sqrt{t}] \mapsto \frac{\lambda - \sqrt{\lambda^2 - 4t}}{2}.$$

Notice that the support of σ_t is $[-2\sqrt{t}, 2\sqrt{t}]$. With the normalization of [19], taking $\beta = 1$ gives a LDP for the smallest eigenvalue of $G_N + \Lambda e_1 e_1^T$ where G_N is GOE of variance $t = 1/2$. In the sequel we take $t = 1/2$. Denote $\rho^\Lambda_{0,1/2} := \Lambda + \frac{1}{2\Lambda}$. Let J_Λ be the rate function found in [19], given if $\Lambda \leq -\frac{1}{\sqrt{2}}$ by

$$J_\Lambda(x) := \begin{cases} M_\Lambda(x) & \text{if } x \leq -\sqrt{2} \\ +\infty & \text{if } x > -\sqrt{2}, \end{cases}$$

where

$$M_\Lambda(x) := \frac{1}{2}\int_x^{\rho^\Lambda_{0,1/2}} \sqrt{z^2 - 2}\mathrm{d}z - \Lambda(x - \rho^\Lambda_{0,1/2}) + \frac{1}{4}(x^2 - (\rho^\Lambda_{0,1/2})^2)$$

and if $\Lambda \geq -\frac{1}{\sqrt{2}}$ by

$$J_\Lambda(x) := \begin{cases} N_\Lambda(x) & \text{if } x \leq \rho^\Lambda_{0,1/2} \\ \int_x^{-\sqrt{2}} \sqrt{z^2 - 2}\mathrm{d}z & \text{if } \rho^\Lambda_{0,1/2} \leq x \leq -\sqrt{2} \\ +\infty & \text{if } x > -\sqrt{2}, \end{cases}$$

where

$$N_\Lambda(x) := \frac{1}{2}\int_x^{-\sqrt{2}} \sqrt{z^2 - 2}\mathrm{d}z - \Lambda x + \frac{1}{4}x^2 + \frac{1}{4} + \frac{1}{4}\log 2 + \frac{\Lambda^2}{2} + \frac{1}{2}\ln|\Lambda|.$$

Lemma 4.1. *We have*

$$J_\Lambda = I^\Lambda_{0,1/2}$$

where we denoted in short $I^\Lambda_{0,1/2}$ for $I^\Lambda_{\delta_0,1/2}$.

Proof. For all $x < -\sqrt{2}$, let $\omega_{0,1/2}(x) \leq \omega^*_{0,1/2}(x)$ be the two solutions in $\mathbb{R}^-$ of the equation $\gamma + \frac{1}{2\gamma} = x$, namely

$$\omega_{0,1/2}(x) = \frac{x - \sqrt{x^2 - 2}}{2} \quad \text{and} \quad \omega^*_{0,1/2}(x) = \frac{x + \sqrt{x^2 - 2}}{2}.$$

Let $x < -\sqrt{2}$. Assume that $\Lambda \leq -\frac{1}{\sqrt{2}}$, or $\Lambda \geq -\frac{1}{\sqrt{2}}$ and $x < \rho^\Lambda_{0,1/2}$. One can notice that

$$J'_\Lambda(x) = -\frac{1}{2}\sqrt{x^2 - 2} + \frac{x}{2} - \Lambda.$$

Moreover, by (3.3), we have

$$(I^\Lambda_{0,1/2})'(x) = \omega_{0,1/2}(x) - \Lambda = -\frac{1}{2}\sqrt{x^2 - 2} + \frac{x}{2} - \Lambda = J'_\Lambda(x).$$

Since $I^{\Lambda}_{0,1/2}(\rho^{\Lambda}_{0,1/2}) = J_\Lambda(\rho^{\Lambda}_{0,1/2})$, equality follows for all $x \le -\sqrt{2}$ if $\Lambda \le -\frac{1}{\sqrt{2}}$ and for all $x < \rho^{\Lambda}_{0,1/2}$ if $\Lambda \ge -\frac{1}{\sqrt{2}}$.

We finally consider the last case where $\Lambda \ge -\frac{1}{\sqrt{2}}$ and $x \in (\rho^{\Lambda}_{0,1/2}, -\sqrt{2})$. By Equation (3.4),

$$(I^{\Lambda}_{0,1/2})'(x) = \omega_{0,1/2}(x) - \omega^{*}_{0,1/2}(x) = -\sqrt{x^2-2} = J'_\Lambda(x).$$

Again, we conclude that $J_\Lambda = I^{\Lambda}_{0,1/2}$. □

4.2. The no outlier case

Let us now compare our result with [20] where no outlier is permitted. Again we take $\beta = 1$ in [20]. In the normalization of [20], the variance of the non-diagonal entry distribution is 1. In our setting (1.1), this corresponds to taking $t = 1$ (instead of $t = 1/2$ in the normalization of [19]).

We apply Theorem 1.2 to $\Lambda := \ell_\nu$. The rate function of the LDP of [20] (stated there for the largest eigenvalue) can be rewritten for the smallest eigenvalue by applying it to $-X_N$ and changing variables $\theta \mapsto -\theta$. It takes the form

quation1

$$J_\nu(x) = \sup_{\theta \le 0} J(x,\theta), \tag{4.1}$$

where for all $x \in \mathbb{R}$, $\theta \le 0$,

$$J(x,\theta) := J(\theta, \nu \boxplus \sigma_1, x) - J(\theta, \nu, x) - \theta^2,$$

and for all $x \in \mathbb{R}$, $\theta \le 0$ and probability measure μ on $\mathbb{R}$,

$$J(\theta,\mu,x) := \begin{cases} \frac{1}{2}\int_0^{2\theta} R_\mu(s)\mathrm{d}s & \text{if } G_\mu(x) \le 2\theta \le 0 \\ \theta x - \frac{1}{2}(1+\log(-2\theta)) - \frac{1}{2}\int \log|x-y|\mathrm{d}\mu(y) & \text{if } 2\theta < G_\mu(x), \end{cases}$$

where R_μ is the Voiculescu R-transform, which is given by $R_\mu : y \mapsto K_\mu(y) - \frac{1}{y}$ with K_μ as in (1.11).

Lemma 4.2. *We have*

$$J_\nu = I^{\ell_\nu}_{\nu,1}.$$

Proof. Let $\rho^{\ell_\nu}_{\nu,1} := \ell_\nu + G_\nu(\ell_\nu) \in \mathbb{R} \cup \{-\infty\}$. For all $x \in (\rho^{\ell_\nu}_{\nu,1}, \ell_{\nu,1})$, let $\gamma_1(x) \le \gamma_2(x)$ be the two solutions of the equation $H_{\nu,1}(\gamma) = x$ on $(-\infty, \ell_\nu)$.*** two solutions only on a certain range *** First observe that for all $x < \ell_{\nu,1}$,

eq:J'

$$J'_\nu(x) = \partial_x J(x,\theta(x)) + \partial_\theta J(x,\theta)\theta'(x) = \partial_x J(x,\theta(x)) = \theta(x) - \frac{1}{2}G_{\nu\boxplus\sigma_1}(x) = \theta(x) - \frac{1}{2}G_\nu(\gamma_1(x)). \tag{4.2}$$

where we have applied Lemma A.1 in the last equality.

Let $x \in (\rho^{\ell_\nu}_{\nu,1}, \ell_{\nu,1})$. As shown in [20, (2.14)], for $x \in (\rho^{\ell_\nu}_{\nu,1}, \ell_{\nu,1})$, the optimal $\theta(x)$ in (4.1) is the unique solution of the equation

$$2\theta(x) + K_\nu(2\theta(x)) = x, \quad \theta(x) \in \Big(\frac{1}{2}G_\nu(\ell_\nu), \frac{1}{2}G_{\nu\boxplus\sigma_1}(\ell_{\nu,1})\Big),$$

where K_ν is as in (1.11). One can easily show

$$\theta(x) = \frac{1}{2}G_\nu(\gamma_2(x)).$$

Substituting this into (4.2), we find that for all $x \in (\rho^{\ell_\nu}_{\nu,1}, \ell_{\nu,1})$,

$$J'_\nu(x) = \frac{1}{2}(G_\nu(\gamma_2(x)) - G_\nu(\gamma_1(x))) = \frac{1}{2}(\gamma_1(x) - \gamma_2(x)) = (I^{\ell_\nu}_{\nu,1})'(x),$$

where we have used (3.4) in the last equality.

If $x \le \rho_{\nu,1}^{\ell_\nu}$, then by [20],

$$\theta(x) = \frac{1}{2}(x - \ell_\nu),$$

which implies that for all $x < \rho_{\nu,1}^{\ell_\nu}$,

$$J_\nu'(x) = \frac{1}{2}(x - \ell_\nu - G_\nu(\gamma_1(x))) = \frac{1}{2}(\gamma_1(x) - \ell_\nu) = (I_{\nu,1}^{\ell_\nu})'(x).$$

Since J_ν and $I_{\nu,1}^{\ell_\nu}$ are $\mathcal{C}^1$ on $(-\infty, \ell_{\nu,1})$ and $\inf J_\nu = \inf I_{\nu,1}^{\ell_\nu} = 0$, this proves that $J_\nu = I_{\nu,1}^{\ell_\nu}$. □

5. Generalization to a deformed GUE matrix

sec:GUE

In this section we discuss why the proof of Theorem 1.3 is the same as for GOE (which is in general more complicated to study). It is enough to show that the volume of small balls satisfies the same functional equations and we discuss how the heuristics of Section 1.4 are modified. In fact, we have the same type of formula

$$\mathbb{P}(\lambda_1(X_N) \in (x - \varepsilon, x + \varepsilon)) = \frac{1}{Z_N^t} \int_{\lambda_1(X_N) \in (x-\varepsilon, x+\varepsilon)} e^{-\frac{N}{2t}\operatorname{Tr}((X_N - B_N)^2)} \, \mathrm{d}X_N,$$

except that now X_N is Hermitian, we have a term $\frac{1}{2t}$ instead of $\frac{1}{4t}$ to keep entries with variance t/N and the Vandermonde comes with a factor 2: so in the formulas t has to be divided by two. *** be more precise *** Equation (1.17) extends verbatim, with O now unitary. v_1 is now a vector in $\mathbb{C}^N$ and as a result Y_N will follow a Dirichlet distribution with parameters multiplied by 2, see [14, Theorem 3.3]. Moreover, because of the change in t, $L_{\nu,t}^{\Lambda}$ of (2.11) is multiplied by 2. Finally the constant C_t computed by Selberg integral is also multiplied by two. We therefore find that with the same notation as in (1.21), we have the following generalization of (1.22)

$$F_{\nu,t}^-(\Lambda, x) \lesssim \inf_{Y \in \Delta^{p+1}} \Big\{ 2L_{\nu,t}^{\Lambda}(x, Y) - \sum_{k=1}^{p} \alpha_k \log(y_k/\alpha_k) + C_t - 2\int \log|\lambda - x| \, d(\nu \boxplus \sigma_t)(\lambda) + \frac{x^2}{2t} + \inf_{y \ge x} F_{\nu,t}^-(\Phi(\Lambda, Y), y) \Big\}.$$

We therefore conclude that $F_{\nu,t}^-(\Lambda, x)/2$ satisfies the same inequality as $F_{\nu,t}^-(\Lambda, x)$ in the GOE case. Similarly $F_{\nu,t}^+(\Lambda, x)/2$ satisfies the same inequality as $F_{\nu,t}^+(\Lambda, x)$ in the GOE case. Thus, solving these functional inequalities exactly as before yields the weak large deviation principle with rate function $2I_{\nu,t}^{\Lambda}$.

Appendix A: Reminder on the free convolution

freeconv

We recall some classical results on free convolution from [7] that are useful throughout this article. Recall the definitions of $v_t(u)$ and $\Omega_{\mu,t}$ in (1.6). We have the following:

:free co

Lemma A.1 (Biane)**.** *Let μ be a probability measure on $\mathbb{R}$. Let $t > 0$. The function $H_{\mu,t}$ is a homeomorphism, which is conformal from $\Omega_{\mu,t}$ onto $\mathbb{C}^+$. We let $\omega_{\mu,t} : \mathbb{C}^+ \cup \mathbb{R} \to \overline{\Omega_{\mu,t}}$ be its inverse, called the subordination function. Moreover, for every $z \in \Omega_{\mu,t}$,*

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$$G_{\mu \boxplus \sigma_t}(z + tG_\mu(z)) = G_\mu(z). \tag{A.1}$$

Assume that the support of μ is bounded from below. The left edge of $\mu \boxplus \sigma_t$ is given by $\ell_{\mu,t} = H_{\mu,t}(\omega)$ where

eq:shock

$$\omega := \inf\Big\{ u \le \ell_\mu : \int \frac{\mathrm{d}\mu(\lambda)}{(\lambda - u)^2} \ge \frac{1}{t} \Big\}. \tag{A.2}$$

For every $x < \ell_{\mu,t}$, we have

egax exp

$$\omega_{\mu,t}(x) = x - tG_{\mu \boxplus \sigma_t}(x). \tag{A.3}$$

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Remark A.1. Lemma A.1 admits a reformulation in PDE terms. The Stieltjes transform of $\mu_t := \mu \boxplus \sigma_t$ indeed satisfies a complex Burgers equation: for any z outside the support of μ_t we have

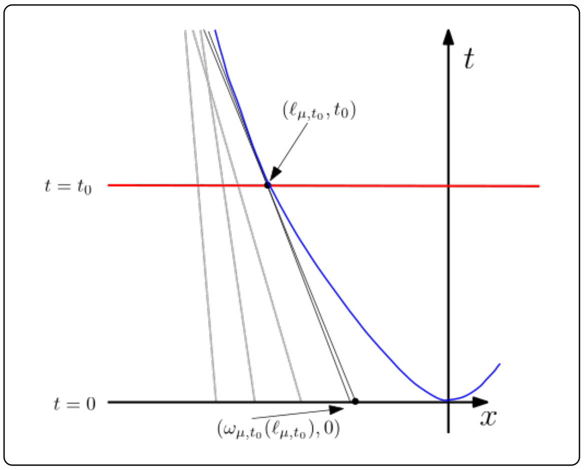
:Burgers

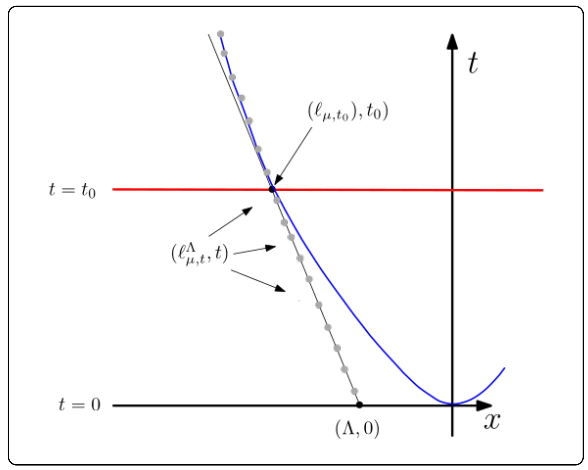
$$\partial_t G_{\mu_t}(z) + G_{\mu_t}(z) \partial_z G_{\mu_t}(z) = 0. \tag{A.4}$$

This equation can be solved via the method of characteristics. The characteristic starting from $z\in\mathbb{C}^+$ is given by the straight line $t\mapsto z+tG_\mu(z)=H_{\mu,t}(z)$. Equation (A.1) reflects the fact that the solution is constant along characteristics (after restriction to suitable z). Let $\omega\in(-\infty,\ell_\mu]$. One may note that $H_{\mu,t}$ is injective on $(-\infty,\omega)$ provided

$$\int\frac{\mathrm{d}\mu(\lambda)}{(\lambda-\omega)^2}<\frac{1}{t}.$$

From (A.2), the left edge of μ_t can be interpreted as the first value at which the solution develops a shock at time t.





e:caract (a) Characteristics and left edge of $\mu\boxplus\sigma_t$. (b) The BBP transition.

In Figure (a), the gray lines show characteristics $\{(H_{\mu,t}(x),t):t>0\}$ for fixed x (with x indicated on the horizontal axis). Characteristics that originate near $(\omega_{\mu,t}(\ell_{\mu,t}),t)$ coalesce at time t; see the discussion in Remark A.1. This coalescence corresponds to the point where the two solution branches merge. The blue curve is the locus of these merging points and traces the left edge $\ell_{\mu,t}$ of $\mu\boxplus\sigma_t$. Figure (b) illustrates the BBP transition. The x-coordinates of the gray dots are the limiting smallest eigenvalue $\ell^\Lambda_{\mu,t}$ of G_N+D_N, where G_N is a GOE matrix with variance t and D_N is as above; see Theorem 1.1. The BBP transition occurs at the point where the gray dots move from lying off the blue curve to lying on it.

k:twosol *Property* A.1. By Lemma A.1, the map $H_{\mu,t}|_{(-\infty,\ell_\mu)}$ is strictly concave, strictly increasing on the set $(-\infty,\omega_{\mu,t}(\ell_{\mu,t}))$ and strictly decreasing on the set $(\omega_{\mu,t}(\ell_{\mu,t}),\ell_\mu)$. In addition $H_{\mu,t}$ goes to $-\infty$ when x goes to $-\infty$ and to $H_{\mu,t}(\ell_\mu)\in\mathbb{R}\cup\{-\infty\}$ when x goes to ℓ_μ.

It follows that, for all $\lambda\in(H_{\mu,t}(\ell_\mu),\ell_{\mu,t})$, the equation

$$H_{\mu,t}(\gamma)=\lambda,\quad \gamma<\ell_\mu$$

admits two solutions $\gamma_1<\gamma_2$. Moreover if $\lambda\in(H_{\mu,t}(\ell_\mu),\ell_{\mu,t})$, we have $\gamma_1=\omega_{\mu,t}(\lambda)<\omega_{\mu,t}(\ell_{\mu,t})$ and $\gamma_2=\omega^*_{\mu,t}(\lambda)>\omega_{\mu,t}(\ell_{\mu,t})$.

Throughout this paper, we used several smoothness results that we summarize and prove here:

oothness **Lemma A.2.** *1. Let M be a positive finite real number. Then $\mu\mapsto\mu\boxplus\sigma_t$ is continuous for the weak topology on $\mathcal{P}([-M,M])$. In fact recalling the distance d from (1.15), we have*

$$d(\mu\boxplus\sigma_t,\mu'\boxplus\sigma_t)\le\left(\int_0^1|\eta_\mu(s)-\eta_{\mu'}(s)|^2\mathrm{d}s\right)^{\frac12}.$$

2. Moreover, for $\eta_1<\cdots<\eta_p$, denoting $\nu_\alpha=\sum\alpha_i\delta_{\eta_i}$, we have for every $t>0$ and $\alpha,\alpha'\in\Delta^p$,

$$d(\nu_\alpha\boxplus\sigma_t,\nu_{\alpha'}\boxplus\sigma_t)\le\left(\sum_{i=1}^p\eta_i^2|\alpha_i-\alpha_i'|\right)^{1/2}.$$

3. $\Lambda \in (-\infty, \ell_\nu) \mapsto \ell^\Lambda_{\nu,t}$ *is continuous.*
4. *For every probability measure* $\nu \in \mathcal{P}([-M,M])$*,* $x \mapsto \omega_{\nu,t}(x)$ *is continuous and increasing on* $(-\infty, \ell_{\nu,t}]$ *with values in* $(-\infty, \omega_{\nu,t}(\ell_{\nu,t})]$*, and* $x \mapsto \omega^*_{\nu,t}(x)$ *is continuous and decreasing on* $[H_{\nu,t}(\ell_\nu), \ell_{\nu,t})$ *with values in* $[\omega_{\nu,t}(\ell_{\nu,t}), \ell_\nu]$*.*
5. *For every probability measure* $\nu \in \mathcal{P}([-M,M])$*, let* η_ν *be such that* $\nu = \eta_\nu \# 1_{[0,1]}\mathrm{d}x$*. Then, for every* $\nu, \nu' \in \mathcal{P}([-M,M])$*,*

$$d(\nu \boxplus \sigma_t, \nu' \boxplus \sigma_t) \leq \|\eta_\nu - \eta_{\nu'}\|_\infty \text{ and } |\ell_{\nu,t} - \ell_{\nu',t}| \leq \|\eta_\nu - \eta_{\nu'}\|_\infty .$$

Moreover if $x < \ell_{\nu,t}$ *(resp.* $x \in [H_{\nu,t}(\ell_\nu), \ell_{\nu,t})$*) ,* $\omega_{\nu,t}(x) - \omega_{\nu',t}(x)$ *(resp.* $\omega^*_{\nu,t}(x) - \omega^*_{\nu',t}(x)$*) go to zero when* $\|\eta_\nu - \eta_{\nu'}\|_\infty$ *goes to zero.*

Proof. • The first point is a consequence of (2.29) by taking sequences of matrices B_N, B'_N so that the empirical measure of their eigenvalues converge towards μ and ν respectively. Similarly, the second point is an application of (2.30) by taking N going to infinity while M and M' are so that M_i/N (resp. M'_i/N) goes to α_i (resp. α'_i).

• To prove the third point, observe that the function $H_{\nu,t}$ is smooth on $(-\infty, \ell_\nu)$, strictly increasing on $(-\infty, \omega)$ and strictly decreasing on (ω, ℓ_ν) with $\omega \leq \ell_\nu$ the point where

$$\int \frac{\mathrm{d}\nu(x)}{(\omega - x)^2} = \frac{1}{t},$$

if it is strictly smaller than ℓ_ν, $\omega = \ell_\nu$ otherwise. The continuity of $\ell^\Lambda_{\nu,t}$ follows.
• Because the derivative of $H_{\nu,t}$ is strictly positive on $(-\infty, \omega)$, the implicit function theorem shows that $H_{\nu,t}$ has an inverse $\omega_{\nu,t}$ with values in $(-\infty, \omega)$ which is continuous. Similarly, because the derivative of $H_{\nu,t}$ is strictly negative on (ω, ℓ_ν), $H_{\nu,t}$ has a second continuous inverse with values in (ω, ℓ_ν). By [7], we know that $\ell_{\nu,t} = H_{\nu,t}(\omega)$ so $\omega = \omega_{\nu,t}(\ell_{\nu,t})$.

• For the last point, we can prove the bound on $d(\nu \boxplus \sigma_t, \nu' \boxplus \sigma_t)$ again by coupling them as the distribution of $\eta_\nu(\Lambda) + S_t$ and $\eta_{\nu'}(\Lambda) + S_t$ where Λ is distributed according to $1_{[0,1]}\mathrm{d}x$ and is free from S_t. The result follows. Moreover, $\ell_\nu = \inf\{\eta_\nu\}$ is continuous in η_ν. Furthermore $\ell_{\nu,t}$ is the smallest eigenvalue of $X_t(\nu) = \eta_\nu(\Lambda) + S_t$ and $\|X_t(\nu) - X_t(\nu')\|_{\mathrm{op}} \leq \|\eta_\nu - \eta_{\nu'}\|_\infty$ so that $|\ell_{\nu,t} - \ell_{\nu',t}| \leq \|\eta_\nu - \eta_{\nu'}\|_\infty$. The last statement regarding the continuity of $\omega_{\nu,t}(x)$ and $\omega^*_{\nu,t}(x)$ can be deduced from the following observations: First, we can assume $x < \min\{\ell_{\nu,t}, \ell_{\nu',t}\}$, which ensures that both $\omega_{\nu,t}(x)$ and $\omega^*_{\nu,t}(x)$ are away from ℓ_ν. Additionally, the inverse $H_{\nu,t}$ has a non vanishing derivative on this range. Therefore, by applying the implicit function theorem and considering that $H_{\nu,t} - H_{\nu',t}$ is small, we obtain the desired result. ☐

The PDE interpretation discussed in the Remark A.1 allows one to compute the logarithmic potential of the measure $\mu \boxplus \sigma_t$ through the Hopf-Lax formula (see also [**?** , Theorem 1.1])

:exp det **Lemma A.3.** *Let* μ *be a probability measure on* $\mathbb{R}$ *with support bounded from below and* $t > 0$*.*

1. *Let* $x \in \mathbb{R}$ *and* $\omega := \omega_{\mu,t}(x) \in \mathbb{C}^+ \cup \mathbb{R}$*. Let* $(z_s)_{s \in [0,t]}$ *given for all* $s \in [0,t]$ *by*

eq:zs (A.5)
$$z_s = \omega + sG_\mu(\omega).$$

If $x \leq \ell_{\mu,t}$*, then for all* $s \in [0,t]$*,*

eq:omegamu (A.6)
$$x \leq z_s \leq \ell_{\mu \boxplus \sigma_s}.$$

2. *Let* $x \in \mathbb{R}$*. Assume that* $x \leq \ell_{\mu,t}$*. We have*

eq:ch (A.7)
$$\int \log|\lambda - x| \mathrm{d}(\mu \boxplus \sigma_t)(\lambda) = \int \log|\lambda - \omega_{\mu,t}(x)| \mathrm{d}\mu(\lambda) + \frac{(\omega_{\mu,t}(x) - x)^2}{2t}.$$

Proof. Let $x \in \mathbb{R}$ and $\omega := \omega_{\mu,t}(x)$. Consider

$$z_s = \omega + sG_\mu(\omega), \quad 0 \le s \le t.$$

Assume first that $x \le \ell_{\mu \boxplus \sigma_t}$. First by [16, Lemma 11], we have

$$\omega_{\mu,t}((-\infty, \ell_{\mu \boxplus \sigma_t}]) \subset (-\infty, \ell_\mu]$$

and therefore $\omega \le \ell_\mu$. This proves that $z_0 \le \ell_\mu$. Let $s \in (0,t)$. Recall that $\mu \boxplus \sigma_t = (\mu \boxplus \sigma_s) \boxplus \sigma_{t-s}$. We have

$$\omega_{\mu \boxplus \sigma_s, t-s}(x) = z_s.$$

Indeed,

$$z_s + (t-s)G_{\mu \boxplus \sigma_s}(z_s) = z_s + (t-s)G_\mu(\omega) = \omega + sG_\mu(\omega) + (t-s)G_\mu(\omega) = x.$$

Therefore, applying the previous result to $t' = t - s$ and $\mu' = \mu \boxplus \sigma_s$, we obtain

$$z_s \le \ell_{\mu \boxplus \sigma_s}.$$

Because $\omega \le \ell_\mu$, $G_\mu(\omega)$ is non-positive and therefore z_s is decreasing which shows that $z_s \ge z_t = x$ for all $0 \le s \le t$. This completes the proof of (A.6).

Fix $x \in \mathbb{R}$ and let $\omega := \omega_{\mu,t}(x)$. For all $s \ge 0$, let $m_s := G_{\mu \boxplus \sigma_s}$. It is well-known, see Lemma A.1, that for all $z \in \mathbb{C}^+ \cup \mathbb{R}$ outside the support of $\mu \boxplus \sigma_s$,

$$\partial_s m_s(z) + m_s(z) \partial_z m_s(z) = 0$$

and that $m_s(z_s) = m_0(z_0) = G_\mu(\omega)$. Integrating this equality on $z \in (-\infty, \ell_{\mu,s})$ we deduce that there exists a constant C_s independent of z such that

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$$\partial_s \int \log|\lambda - z| \mathrm{d}(\mu \boxplus \sigma_s)(\lambda) + \frac{1}{2} m_s(z)^2 = C_s \,. \tag{A.8}$$

Furthermore, taking the limit z going to $-\infty$ shows that C_s must vanish. Define

$$F : (s,z) \in \bigcup_{s' \ge 0} \{(s', z') : z' \in \mathbb{C}^+ \cup (-\infty, \ell_{\mu,s'})\} \mapsto \int \log|\lambda - z| \mathrm{d}(\mu \boxplus \sigma_s)(\lambda) - \frac{(z-\omega)^2}{2s}.$$

Let $(z_s)_{s \in [0,t]}$ be as in (A.5). For every $z \in \mathbb{C}^+ \cup (-\infty, \ell_{\mu,t})$, we find by (A.8) that

$$\partial_s F(s,z) = -\frac{1}{2} m_s(z)^2 + \frac{(z-\omega)^2}{2s^2}.$$

As proved in item (1), for all $s \in [0,t)$, $z_s < \ell_{\mu,s}$. Hence, since $m_s(z_s) = m_0(z_0) = G_\mu(\omega)$ and $z_s - \omega = sG_\mu(\omega)$,

$$\partial_s F(s, z_s) = 0 \,.$$

It follows that

$$\frac{\mathrm{d}}{\mathrm{d}s} F(s, z_s) = 0,$$

which gives after integrating $s \in [0,t]$, $F(0,\omega) = F(0,z_0) = F(t,z_t) = F(t,x)$, or equivalently

$$\int \log|\lambda - x| \mathrm{d}(\mu \boxplus \sigma_t)(\lambda) = \int \log|\lambda - \omega_{\mu,t}(x)| \mathrm{d}\mu(\lambda) + \frac{(\omega_{\mu,t}(x) - x)^2}{2t},$$

which proves (A.7). $\square$

Appendix B: A priori properties of the rate function

monot **Lemma B.1.** *Let (B_N) satisfying Assumptions 1.1 with ν and Λ be as in (1.2) and (1.3). Let $\mathbb{P}$ be the law of the eigenvalues of $B_N + G_N$ where G_N is a $N \times N$ GOE matrix with variance t. Recall*

$$\ell_{\nu,t}^{\Lambda} := \begin{cases} H_{\nu,t}(\Lambda) & \text{if } \Lambda \le \omega_{\nu,t}(\ell_{\nu,t}) \\ \ell_{\nu,t} & \text{if } \Lambda \ge \omega_{\nu,t}(\ell_{\nu,t}). \end{cases}$$

Let $x < y < \ell_{\nu,t}^{\Lambda}$ or $\ell_{\nu,t}^{\Lambda} < y < x$. Then, for every $\delta \in (0, |\ell_{\nu,t}^{\Lambda} - y|)$:

$$\limsup_{N\to\infty} \frac{1}{N} \log \mathbb{P}(|\lambda_1 - x| \le \delta) \le \limsup_{N\to\infty} \frac{1}{N} \log \mathbb{P}(|\lambda_1 - y| \le \delta)\,.$$

Proof. We may and shall assume that $\limsup_{N\to\infty} \frac{1}{N} \log \mathbb{P}(|\lambda_1 - x| \le \delta)$ is finite since otherwise we are done. The proof follows the arguments of [11, Lemma 5.2] based on [11, Lemma 8.1] where a coupling is constructed. Because we restrict ourselves to Gaussian entries in this article, we propose a simpler version based on the symmetric Ornstein-Uhlenbeck

$$\mathrm{d}G_N(s) = \mathrm{d}H_N(s) - \frac{1}{2t} G_N(s)\mathrm{d}s$$

where H_N is a $N \times N$ symmetric Brownian motion and $G_0^N = G_N$ is a GOE with variance t. We also denote by $\bar{G}_N(s)$ the solution constructed with the same Brownian motion but starting from zero. It is well known that $G_N(s)$ has the law of a GOE with variance t at any time s, whereas $\bar{G}_N(s)$ has the law of a GOE with variance $t(1 - e^{-s/t})$. Moreover,

$$\mathrm{d}(G_N(s) - \bar{G}_N(s))^2 = -\frac{1}{t}(G_N(s) - \bar{G}_N(s))^2 \mathrm{d}s$$

implies that

dis (B.1) $$\|G_N(s) - \bar{G}_N(s)\|_{\mathrm{op}} \le \|G_N\|_{\mathrm{op}} e^{-s/2t}$$

where $\|.\|_{\mathrm{op}}$ denotes the operator norm. We will later take $s = N$. Moreover we work on the event where $\|G_N\|_{\mathrm{op}} \le N$ so that $\|G_N(s) - \bar{G}_N(s)\|_{\mathrm{op}}$ is of order $Ne^{-s/2t} \le \delta/2$ if s is of order N (here t is fixed, independently of N). We finally set $X_N(s) := B_N + G_N(s)$ and $\bar{X}_N(s) := B_N + \bar{G}_N(s)$. Note that for every integer p, $X_N(\frac{p}{N})$ has the same law as $X_N(0)$ and in particular its smallest eigenvalue converges towards $\ell_{\nu,t}^{\Lambda}$ almost surely and in expectation. Observe also that by (B.1), $\|\bar{X}_N(N) - X_N(N)\|_{\mathrm{op}}$ is bounded by $\|G_N\|_{\mathrm{op}} e^{-N/2t}$ so that

$$|\mathbb{E}[\lambda_1(\bar{X}_N(N))] - \mathbb{E}[\lambda_1(X_N(N))]| \le \mathbb{E}[\|G_N\|_{\mathrm{op}}] e^{-N/2t}$$

goes to zero as N goes to infinity. Hence, $\mathbb{E}[\lambda_1(\bar{X}_N(N))]$ converges as N goes to infinity towards $\ell_{\nu,t}^{\Lambda}$. By concentration of measure for $\lambda_1(\bar{X}_N(N))$ (see [1, Theorem 2.3.5]), we also have that for every $\eta > 0$,

conc (B.2) $$\mathbb{P}(|\mathbb{E}[\lambda_1(\bar{X}_N(N))] - \lambda_1(\bar{X}_N(N))| \ge \eta) \le 2e^{-\frac{N\eta^2}{4t}}\,.$$

We deduce that for $N \gg \log \delta^{-1}$ large enough,

$$Q_N := \mathbb{P}\left(|\lambda_1(X_N(N)) - \ell_{\nu,t}^{\Lambda}| \le \delta \big| |\lambda_1(X_N(0)) - x| \le \delta, \|G_N\|_{\mathrm{op}} \le N\right) \ge 1/2.$$

Indeed, (B.1) implies that $\|X_N(N) - \bar{X}_N(N)\|_{\mathrm{op}}$ is of order $Ne^{-N/2t}$ when the operator norm of G_N is bounded by N, for N large enough so that $Ne^{-N/2t} \le \delta/2$,

$$\begin{aligned} Q_N &\ge \frac{\mathbb{P}(\{|\lambda_1(\bar{X}_N(N)) - \ell_{\nu,t}^{\Lambda}| \le \delta/2\} \cap \{|\lambda_1(X_N(0)) - x| \le \delta, \|G_N\|_{\mathrm{op}} \le N\})}{\mathbb{P}(|\lambda_1(X_N(0)) - x| \le \delta, \|G_N\|_{\mathrm{op}} \le N\})} \\ &= \mathbb{P}(\{|\lambda_1(\bar{X}_N(N)) - \ell_{\nu,t}^{\Lambda}| \le \delta/2\}). \end{aligned}$$

where we finally used that $\bar{X}_N$ is independent from $X_N(0) = B_N + G_N$. Finally, we have seen that the above right-hand side goes to 1 as N goes to infinity. The fact that Q_N is bounded from below by $1/2$ implies that

$$\begin{aligned}
&\mathbb{P}(|\lambda_1(X_N(0)) - x| \le \delta, \|G_N\|_{\mathrm{op}} \le N) \\
&\le 2\mathbb{P}(\{|\lambda_1(X_N(N)) - \ell^{\Lambda}_{\nu,t}| \le \delta\} \cap \{|\lambda_1(X_N(0)) - x| \le \delta, \|G_N\|_{\mathrm{op}} \le N\}) \\
&\le 2\mathbb{P}(\{|\lambda_1(X_N(N)) - \ell^{\Lambda}_{\nu,t}| \le \delta\} \cap \{|\lambda_1(X_N(0)) - x| \le \delta, \|G_N\|_{\mathrm{op}} \le N\} \cap \Omega_N) + \mathbb{P}(\Omega_N^c),
\end{aligned}$$

where

$$\Omega_N = \bigcap_{0 \le p \le N^2 - 1} \Big\{ \Big\| X_N\Big(\frac{p+1}{N}\Big) - X_N\Big(\frac{p}{N}\Big) \Big\|_{\mathrm{op}} \le \delta \Big\}.$$

Since $X_N(\frac{p+1}{N}) - X_N(\frac{p}{N})$ is a GOE matrix with variance $2t(1 - e^{-1/(2tN)}) \sim \frac{1}{N}$, we find by (B.2) that $\mathbb{P}(\Omega_N^c)$ is bounded above by $2N^2 e^{-\frac{1}{4t} N^2 \delta^2}$. Moreover, since y lies in between x and $\ell^{\Lambda}_{\nu,t}$, we see that

$$\{|\lambda_1(X_N(N)) - \ell^{\Lambda}_{\nu,t}| \le \delta\} \cap \{|\lambda_1(X_N(0)) - x| \le \delta, \|G_N\|_{\mathrm{op}} \le N\} \cap \Omega_N \subset \bigcup_{0 \le p \le N^2} \Big\{ \Big| \lambda_1\Big(X_N\Big(\frac{p}{N}\Big)\Big) - y \Big| \le \delta \Big\}.$$

Hence, we deduce that

$$\begin{aligned}
\mathbb{P}(|\lambda_1(X_N(0)) - x| \le \delta, \|G_N\|_{\mathrm{op}} \le N) &\le 2\mathbb{P}\Big(\bigcup_{0 \le p \le N^2} \Big\{ \Big| \lambda_1\Big(X_N\Big(\frac{p}{N}\Big)\Big) - y \Big| \le \delta \Big\} \Big) + 4N^2 e^{-c(\delta)N^2} \\
&\le 2N^2 \mathbb{P}(\{|\lambda_1(X_N(0)) - y| \le \delta\}) + 4N^2 e^{-\frac{1}{4t} N^2 \delta^2},
\end{aligned}$$

where we used that $X_N(\frac{p}{N})$ has the same law as $X_N(0)$. Finally, since again by concentration of measure $\mathbb{P}(\|G_N\|_{\mathrm{op}} \ge N) \le e^{-cN^2}$, we conclude that for some $c'(\delta) > 0$ and all $N \in \mathbb{N}$

$$\mathbb{P}(|\lambda_1(X_N(0)) - x| \le \delta) \le 2N^2 \mathbb{P}(\{|\lambda_1(X_N(0)) - y| \le \delta\}) + 4N^2 e^{-c'(\delta)N^2}.$$

The result follows as long as $\mathbb{P}(|\lambda_1(X_N(0)) - x| \le \delta)$ is much larger than e^{-cN^2} which is clear since we assumed that $\limsup \frac{1}{N} \log \mathbb{P}(|\lambda_1(X_N(0)) - x| \le \delta)$ is finite.

□

Fprop

Lemma B.2. *The functions $\lambda \mapsto F^{\pm}_{\nu,t}(\Lambda, \lambda)$ are decreasing on $(-\infty, \ell^{\Lambda}_{\nu,t})$ and increasing on $(\ell^{\Lambda}_{\nu,t}, \ell_{\nu,t})$. Moreover, they vanish at $\ell^{\Lambda}_{\nu,t}$. These functions are bounded below by $c(\lambda - \ell^{\Lambda}_{\nu,t})^2$ for some constant $c > 0$ depending on t.*

Proof. The first point is a slight generalization of the previous proof where we take the supremum over vectors $\mathsf{M} \in \mathcal{B}^N_\delta(\alpha)$. Moreover, because $G_N \mapsto \lambda_1(B_N + G_N)$ is Lipschitz with Lipschitz constant 1, we deduce as in [1, Theorem 2.3.5] that since G_N has Gaussian entries with variance t/N, for every $\delta > 0$,

$$\mathbb{P}(|\lambda_1(B_N + G_N) - \mathbb{E}[\lambda_1(B_N + G_N)]| \ge \varepsilon) \le 2e^{-\frac{\varepsilon^2}{4t} N}.$$

By Theorem 1.1, we know that if N_i/N goes to α_i, $\mathbb{E}[\lambda_1(B_N + G_N)]$ converges towards $\ell^{\Lambda}_{\nu,t}$. Note that this limit is continuous in ν provided $\Lambda < \ell_\nu$. As a consequence, for $\varepsilon > 0$ and δ small enough so that

$$\sup_{\|\alpha - \alpha'\|_\infty \le \delta} \Big| \ell^{\Lambda}_{\sum \alpha_i \delta_{\eta_i}, t} - \ell^{\Lambda}_{\sum \alpha'_i \delta_{\eta_i}, t} \Big| \le \varepsilon,$$

we find that for $|x - \ell^{\Lambda}_{\nu,t}| \ge 2\varepsilon$,

$$\sup_{\mathsf{M} \in \mathcal{B}^N_\delta} \mathbb{P}(|\lambda_1(X_N) - x| \le \varepsilon) \le \sup_{\mathsf{M} \in \mathcal{B}^N_\delta} \mathbb{P}(|\lambda_1(X_N) - \mathbb{E}[\lambda_1(X_N)]| \ge \tfrac{1}{2}|x - \ell^{\Lambda}_{\nu,t}|) \le 2e^{-\frac{|x - \ell^{\Lambda}_{\nu,t}|^2}{16t} N}.$$

Since $\Lambda \mapsto \ell^{\Lambda}_{\nu,t}$ is continuous, it follows by first taking the limit when N goes to infinity and then the supremum over Λ in a small ball, that

$$F^{+}_{\nu,t}(\Lambda, \lambda) \ge \frac{|\lambda - \ell^{\Lambda}_{\nu,t}|^2}{16t}.$$

The same result holds for $F^-_{\nu,t}$. Finally, $F^{\pm}_{\nu,t}$ has to vanish somewhere. Indeed, by exponential tightness (2.3) we know that for some finite M, and N large enough, we must have

$$P(\lambda_1(X_N) \in [-M, M]) \geq \frac{1}{2}\,.$$

Moreover M is chosen independently of the choice of $\alpha \in \Delta^p$ governing the distribution of the eigenvalues of B_N. Covering $[-M, M]$ with balls of width δ and using the previous monotonicity, we deduce that $\inf F^{\pm}_{\nu,t}(\Lambda, \cdot)$ equals 0, with the infimum achieved at $\ell^{\Lambda}_{\nu,t}$. □

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