LARGE DEVIATIONS FOR THE LARGEST EIGENVALUE OF SUB-GAUSSIAN MATRICES

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Abstract: We establish large deviations estimates for the largest eigenvalue of Wigner matrices with sub-Gaussian entries. Under technical assumptions, we show that the large deviation behavior of the largest eigenvalue is universal for small deviations, in the sense that the speed and the rate function are the same as in the case of the GOE. In contrast, in the regime of very large deviations, we obtain a non-universal rate function and we prove that the associated eigenvector is localized given the large deviation event, thus establishing the existence of a transition between two different large deviation mechanisms.

1. Introduction

In a breakthrough paper [25], Wigner showed that the empirical distribution of the eigenvalues of a Wigner matrix converges to the semi-circle law provided the off-diagonal entries have a finite second moment. Following the pioneering work of Kómlos and Fűredi [17], it was proved in [3] that assuming the Wigner matrix has centered entries, the largest eigenvalue converges to the right edge of the support of the semi-circle law if and only if the fourth moment of the off-diagonal entries is finite. But what is the probability that the empirical measure or the largest eigenvalue have a different behavior ? Analyzing the probability that the largest eigenvalue of a random matrix takes an unexpected value is a challenging question, with many applications in statistics [16], mobile communications systems [16, 12] or the energy landscape of disordered systems [8, 5]. This turns out to be a much more challenging question than to analyze the typical behavior which could be only answered so far for very specific models. It was first solved in the case of Gaussian ensembles, such as the Gaussian Unitary Ensemble (GUE) and Gaussian Orthogonal Ensemble (GOE), where the joint law of the eigenvalues is explicit. In these cases, large deviations principles were derived for the empirical distribution of the eigenvalues and the largest eigenvalue in [7] and [6] merely by Laplace's method, up to taking care of the singularity of the interaction. The question was revived in a breakthrough paper by Bordenave and Caputo [13] who considered Wigner matrices with entries with tails heavier than in the Gaussian case. They proved a large deviations principle for the law of the empirical measure by a completely different argument based on the fact that deviations are created by a relatively small number of large entries. These large deviations have a smaller speed than in the Gaussian case. This phenomenon was shown to hold as well for the largest eigenvalue by one of the authors [2].

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Yet, the case of sub-Gaussian entries remained open and the general mechanism which creates large deviations mysterious. Last year, two of the authors showed in [18] that if the Laplace transform of the entries is pointwise bounded from above by the one of the GUE or GOE, then a large deviations principle holds with the same rate function as in the Gaussian case. This special case of entries, which was said to have *sharp sub-Gaussian tails*, includes Rademacher variables and uniform variables. Yet, many entries with sub-Gaussian tails are not sharp, as for instance sparse entries which are obtained by multiplying a Gaussian variable with an independent Bernoulli random variable. In this article, we investigate this general setting. We derive large deviations estimates for the largest eigenvalue of Wigner matrices with sub-Gaussian entries. In particular, we show that the rate function of this large deviations estimates is different from the one of the GOE.

This article is restricted to matrices with entries which are centered and covariance of order of the inverse of the dimension. Non centered models, such as the adjacency matrix of Erdős-Rényi matrices, may have different deviations properties as the mean of the entries can be seen as a rank one deformation of the later. The large deviation behavior of the extreme eigenvalues of Erdős-Rényi graphs has attracted some attention recently. Inside the regime where the average degree goes to infinity, that is $np \gg 1$ where n is the number of vertices and p is the edge-probability, Cook and Dembo [14] have computed the tail distribution of the operator norm for $p \gg n^{-1/2}$, and proved that it is governed by a certain mean-field variational problem. Very recently, the joint large deviations of the extreme eigenvalues were established in [11] for the "localized" regime where $1 \ll np \ll \sqrt{\log n}$. The case were np is of order one is still open and might be studied by our techniques.

We will consider hereafter a $N \times N$ symmetric random matrix X_N with independent entries $(X_{ij})_{i\le j}$ above the diagonal so that $\sqrt{N}X_{ij}$ has law μ for all $i<j$ and $\sqrt{N/2}X_{ii}$ has law μ for all i. In particular, the variance profile is the same as the one of the GOE. We assume that μ is centered and has a variance equal to 1. For a real number x, let

$$\psi(x) = \frac{1}{x^2}\log \int e^{xt} d\mu(t)\,.$$

ψ is a continuous function on the real line such that $\psi(0) = 1/2$. Assume that μ is sub-Gaussian so that

$$\frac{A}{2} := \sup_{x\in\mathbb{R}} \psi(x) < +\infty. \tag{1}$$

The case where $A = 1$ is the case of *sharp sub-Gaussian tails* studied in [18]. We investigate here the case where $A > 1$ and we show the following result.

Theorem 1.1. *Denote by λ_{X_N} the largest eigenvalue of X_N. Under some technical assumptions, there exist a good rate function $I_\mu : \mathbb{R} \to [0,+\infty]$ and a set $\mathcal{O}_\mu \subset \mathbb{R}$ such that $(-\infty, 2] \cup [x_\mu, +\infty) \subset \mathcal{O}_\mu$ for some $x_\mu \in (2,+\infty)$ and such that for any $x \in \mathcal{O}_\mu$,*

$$\lim_{\delta\to 0}\liminf_{N\to+\infty}\frac{1}{N}\log \mathbb{P}\left(|\lambda_{X_N} - x| \le \delta\right) = \lim_{\delta\to 0}\limsup_{N\to+\infty}\frac{1}{N}\log \mathbb{P}\left(|\lambda_{X_N} - x| \le \delta\right) = -I_\mu(x).$$

The rate function I_μ is infinite on $(-\infty, 2)$ and satisfies

$$I_\mu(x) \underset{x\to+\infty}{\sim} \frac{1}{4A}x^2.$$

If $A \in (1,2)$, then $[2, \sqrt{A-1} + 1/\sqrt{A-1}] \subset \mathcal{O}_\mu$ and I_μ coincides on this interval with the rate function of the GOE, that is,

$$I_\mu(x) = \frac{1}{2}\int_2^x \sqrt{y^2-4}dy =: I_{GOE}(x). \tag{2}$$

Moreover, for all $x \geq 2$, $I_\mu(x) \leq I_{GOE}(x)$.

The technical assumptions include the case where ψ is increasing (which holds in the case of sparse Gaussian entries) and the case where the maximum of ψ is achieved on $\mathbb{R}$ at a unique point in a neighborhood of which it is strictly concave. In the later case, $I_\mu(x)$ only depends on A for x large enough.

The method introduced in [18] is based on a tilt of the measure by spherical integrals and therefore the estimation of the annealed spherical integrals (given by the average of spherical integrals over the entries of the matrix X_N). In the case of sharp sub-Gaussian entries, the annealed spherical integrals are easy to estimate because they concentrate on delocalized vectors. In the general case we deal with in this article, estimating the annealed spherical integral becomes much more complicated and interesting because it can concentrate on localized vectors, at least in a regime corresponding to sufficiently large deviations. This new phenomenon comes with a "phase" transition at least when $A < 2$, since then for small deviations the annealed spherical integral concentrates on delocalized vectors whereas for large deviations it concentrates on more localized vectors. This fact is reflected in the eigenvector of the largest eigenvalue when the later is conditioned to be large, see section 7. We show that it is delocalized for sharp sub-Gaussian entries, whereas otherwise it localizes for large enough deviations. The existence of such a transition makes it difficult to compute large deviations on the whole real line, but in fact our formulas may just be wrong then. For instance, our formulas would predict a convex rate function, which may not always be the case.

1.1. **Assumptions.** We now describe more precisely our assumptions.

Assumption 1.1. *Let $\mu \in \mathcal{P}(\mathbb{R})$ be a symmetric probability measure with unit variance. We denote by L its log-Laplace transform,*

$$\forall x \in \mathbb{R},\ L(x) = \log \int e^{xt} d\mu(t),$$

and $\psi(x) = L(x)/x^2$. We assume that μ is sub-Gaussian in the sense that

$$\frac{A}{2} := \sup_{x\in\mathbb{R}} \psi(x) < +\infty,$$

and we define $B \geq 0$ by

$$\frac{B}{2} := \lim_{|x|\to+\infty} \psi(x).$$

We assume moreover that $L(\sqrt{.})$ is a Lipschitz function and that μ does not have sharp sub-Gaussian tails, meaning that $A > 1$.

We describe below a few examples of probability measures μ which satisfy the above assumptions. In each of these cases, the fact that $L(\sqrt{\cdot})$ is Lipschitz is clear and left to the reader.

Example 1.2.

- *(Combination of Gaussian and Rademacher laws). Let*
$$\mu(dx) = a\frac{e^{-\frac{1}{2B}x^2}}{\sqrt{2\pi B}}dx + (1-a)\frac{1}{2}(\delta_{-b}+\delta_{+b})$$
where a, b, B are non negative real numbers such that $a \in (0,1)$ and $aB+(1-a)b^2 = 1$. Then, for all $x \in \mathbb{R}$,
$$L_\mu(x) = \log\left(ae^{\frac{B}{2}x^2} + (1-a)\cosh(bx)\right).$$
If $B > 1$ and $b \in (0,1)$ we see that our conditions are fulfilled and $A = B$.
- *(Sparse Gaussian case). Let μ be the law of $\zeta\Gamma$ with ζ a Bernoulli variable of parameter $p \in (0,1)$ and Γ a centered Gaussian variable with variance $1/p$. For any $x \in \mathbb{R}$,*
$$L_\mu(x) = \log\left(pe^{\frac{x^2}{2p}} + 1 - p\right)$$
so that $A = B = \frac{1}{p}$.
- *(Combination of Rademacher laws). Let*
$$\mu = \sum_{i=1}^{p} \frac{\alpha_i}{2}(\delta_{\beta_i} + \delta_{-\beta_i})$$
with $\alpha_i \geq 0$, $\beta_i \in \mathbb{R}$ and $p \in \mathbb{N}$ so that $\sum \alpha_i = 1, \sum \alpha_i\beta_i^2 = 1$. Since μ is compactly supported $B = 0$. The fact that μ does not have sharp sub-Gaussian tails means that there exist some t and $A > 1$ such that
$$\sum_{i=1}^{p} \alpha_i \cosh(\beta_i t) \geq e^{A\frac{t^2}{2}}.$$
The latter is equivalent to
$$\sum_{i=1}^{p} \frac{\alpha_i}{2} e^{\frac{\beta_i^2}{2A}}\left(e^{-\frac{A}{2}(t-\frac{\beta_i}{A})^2} + e^{-\frac{A}{2}(t+\frac{\beta_i}{A})^2}\right) \geq 1.$$
This inequality holds as soon as $\alpha_i e^{\frac{\beta_i^2}{2A}} \geq 2$ for some $i \in \{1,\ldots,p\}$ by taking $t = \frac{\beta_i}{A}$. This can be fulfilled if β_i is large enough while $\alpha_i\beta_i^2 < 1$. We also see with this family of examples that A can be taken arbitrarily large even if $B = 0$ (take e.g $p = 2$, $A = \beta_1, t = 1, \alpha_1 = (2\beta_1^2)^{-1}$, $e^{\beta_1/2} \geq 4\beta_1^2, \beta_2^2 = (2-\beta_1^{-2})^{-1}, \alpha_2 = 1-\alpha_1$).

Let $\mathcal{H}_N$ be the set of real symmetric matrices of size N. We denote for any $A \in \mathcal{H}_N$ by λ_A its largest eigenvalue, $||A||$ is spectral radius and by $\hat{\mu}_A$ the empirical distribution of its eigenvalues, that is
$$\hat{\mu}_A = \frac{1}{N}\sum_{i=1}^{N} \delta_{\lambda_i},$$

where $\lambda_1, \dots, \lambda_N$ are the eigenvalues of A. We make the following assumption of concentration of the empirical distribution of the eigenvalues at the scale N.

Assumption 1.2. *The empirical distribution of the eigenvalues $\hat{\mu}_{X_N}$ concentrates at the scale N:*

$$\limsup_{N\to+\infty} \frac{1}{N} \log \mathbb{P}\left(d(\hat{\mu}_{X_N}, \sigma) > N^{-\kappa}\right) = -\infty, \tag{3}$$

for some $\kappa > 0$, where d is a distance compatible with the weak topology and σ is the semi-circle law, defined by

$$\sigma(dx) = \frac{1}{2\pi}\sqrt{4-x^2}\mathbb{1}_{|x|\leq 2}dx.$$

Remark 1.3. (1) *From* [18, Lemmas 1.8, 1.11]*, we know that Assumption 1.2 is fulfilled if μ is either compactly supported, or if μ satisfies a logarithmic Sobolev inequality in the sense that there exists $c > 0$ so that for any smooth function $f : \mathbb{R} \to \mathbb{R}$, such that $\int f^2 d\mu = 1$,*

$$\int f^2 \log f^2 d\mu \leq c \int \|\nabla f\|_2^2 d\mu \,.$$

(2) *If μ is a symmetric sub-Gaussian probability measure on $\mathbb{R}$ with log-concave tails in the sense that $t \mapsto \mu(|x| \geq t)$ is a log-concave function, then the Wigner matrix X_N satisfies Assumption 1.2. In particular, if B is a Wigner matrix with Bernoulli entries with parameter p and Γ is a GOE matrix, then the sparse Gaussian matrix $B \circ \Gamma/\sqrt{p}$, where $\circ$ is the Hadamard product, satisfies Assumption 1.2. We refer the reader to section 8.1 of the appendix for more details.*

Let us remark that when μ is sub-Gaussian, (1), the spectral radius of X_N is exponentially tight [18, Lemma 5.1] and that this fact remains true under any tilted measure

$$\mathbb{P}^{(e,\theta)} = \frac{e^{\theta N \langle e, X_N e\rangle}}{\mathbb{E}_X(e^{\theta N \langle e, X_N e\rangle})} d\mathbb{P}(X), \tag{4}$$

where $e \in \mathbb{S}^{N-1}$ and $\theta \geq 0$. More precisely, we can make the following remark.

Remark 1.4. *If* (1) *holds, then for any $\theta \geq 0$,*

$$\lim_{K\to+\infty} \limsup_{N\to+\infty} \sup_{e\in\mathbb{S}^{N-1}} \frac{1}{N} \log \mathbb{P}^{(e,\theta)}\big(||X_N|| \geq K\big) = -\infty.$$

1.2. **Statement of the results and scheme of the proof.** As in [18], our approach to derive large deviations estimates is based on a tilting of the law of the Wigner matrix X_N by spherical integrals. Let us recall the definition of spherical integrals. For any $\theta \geq 0$, we define

$$I_N(X_N, \theta) = \mathbb{E}_e[e^{\theta N \langle e, X_N e\rangle}]$$

where e is uniformly sampled on the sphere $\mathbb{S}^{N-1}$ with radius one. The asymptotics of

$$J_N(X_N, \theta) = \frac{1}{N} \log I_N(X_N, \theta)$$

were studied in [19] where the following result was proved.

Theorem 1.5. [19, Theorem 6] *Let $(E_N)_{N\in\mathbb{N}}$ be a sequence of $N \times N$ real symmetric matrices such that:*

- *The sequence of empirical measures $\hat{\mu}_{E_N}$ converges weakly to a compactly supported measure μ.*
- *There is a real number λ_E such that the sequence of the largest eigenvalues λ_{E_N} converges to λ_E.*
- $\sup_N ||E_N|| < +\infty$.

For any $\theta \geq 0$,

$$\lim_{N\to+\infty} J_N(E_N,\theta) = J(\mu, \lambda_E, \theta)$$

The limit J is defined as follows. For a compactly supported probability measure $\mu \in \mathcal{P}(\mathbb{R})$ we define its Stieltjes transform G_μ by

$$\forall z \notin \operatorname{supp}(\mu),\ G_\mu(z) := \int_{\mathbb{R}} \frac{1}{z-t} d\mu(t),$$

where $\operatorname{supp}(\mu)$ is the support of μ. Let r_μ denote the right edge of the support of μ. Then G_μ is a bijection from $(r_\mu, +\infty)$ to $\big(0, G_\mu(r_\mu)\big)$ where

$$G_\mu(r_\mu) = \lim_{t\downarrow r_\mu} G_\mu(t).$$

Let K_μ be the inverse of G_μ on $(0, G_\mu(r_\mu))$ and let

$$\forall z \in (0, G_\mu(r_\mu)),\ R_\mu(z) := K_\mu(z) - 1/z,$$

be the R-transform of μ as defined by Voiculescu in [24]. Then, the limit of spherical integrals is defined for any $\theta \geq 0$ and $x \geq r_\mu$ by,

$$J(\mu, x, \theta) := \theta v(\mu, x, \theta) - \frac{1}{2}\int \log\big(1 + 2\theta v(\mu, x, \theta) - 2\theta y\big) d\mu(y),$$

with

$$v(\mu, x, \theta) := \begin{cases} R_\mu(2\theta) & \text{if } 0 \leq 2\theta \leq G_\mu(x), \\ x - \frac{1}{2\theta} & \text{if } 2\theta > G_\mu(x). \end{cases}$$

In the case of the semi-circle law, we have

$$G_\sigma(x) = \frac{1}{2}(x - \sqrt{x^2-4}),\ R_\sigma(x) = x.$$

We denote by $J(x,\theta)$ as a short-hand for $J(\sigma, x, \theta)$. In the next lemma we compute explicitly $J(x,\theta)$, whose proof is left to the reader.

Lemma 1.6. *Let $\theta \geq 0$ and $x \geq 2$. For $\theta \leq \frac{1}{2}G_\sigma(x)$,*

$$J(x,\theta) = \theta^2.$$

Whereas for $\theta \geq \frac{1}{2}G_\sigma(x)$,

$$J(x,\theta) = \theta x - \frac{1}{2} - \frac{1}{2}\log 2\theta - \frac{1}{2}\int \log(x-y) d\sigma(y).$$

To derive large deviations estimates using a tilt by spherical integrals, it is central to obtain the asymptotics of the annealed spherical integral $F_N(\theta)$ defined as,

$$F_N(\theta) = \frac{1}{N} \log \mathbb{E}_{X_N} \mathbb{E}_e [\exp(N\theta \langle e, X_N e\rangle)] \,.$$

In the following lemma, we obtain the limit of F_N as the solution of a certain variational problem. We denote by $\overline{F}(\theta)$ and $\underline{F}(\theta)$ its upper and lower limits:

$$\overline{F}(\theta) = \limsup_{N\to+\infty} F_N(\theta),$$

$$\underline{F}(\theta) = \liminf_{N\to+\infty} F_N(\theta).$$

For any measurable subset $I \subset \mathbb{R}$, we denote by $\mathcal{M}(I)$ and $\mathcal{P}(I)$ respectively the set of measures and the set of probability measures supported on I.

Proposition 1.7. *Assume X_N satisfies Assumptions 1.1 and 1.2.*

$$\overline{F}(\theta) = \limsup_{\substack{\delta\to0, K\to+\infty\\ \delta K\to 0}} \sup_{\substack{\alpha_1+\alpha_2+\alpha_3=1\\ \alpha_i\geq 0}} \limsup_{N\to+\infty} \mathcal{F}^N_{\alpha_1,\alpha_2,\alpha_3}(\delta, K),$$

$$\underline{F}(\theta) = \sup_{\substack{\alpha_1+\alpha_2+\alpha_3=1\\ \alpha_i\geq 0}} \liminf_{\substack{\delta\to0, K\to+\infty\\ \delta K\to 0}} \limsup_{N\to+\infty} \mathcal{F}^N_{\alpha_1,\alpha_2,\alpha_3}(\delta, K)\,.$$

$\mathcal{F}^N_{\alpha_1,\alpha_2,\alpha_3}(\delta, K)$ is the function given by:

$$\begin{aligned}\mathcal{F}^N_{\alpha_1,\alpha_2,\alpha_3}(\delta, K) &= \theta^2\big(\alpha_1^2 + 2\alpha_1\alpha_2 + B\alpha_3^2\big) \\ &+ \sup_{\substack{t_i\in I_2, i\leq l\\ |\sum_i t_i^2 - N\alpha_2|\leq \delta N}} \sup_{\substack{s_i\in I_3, i\leq k\\ |\sum_i s_i^2 - N\alpha_3|\leq \delta N}} \Big\{ \frac{1}{N}\sum_{i=1}^k \sum_{j=1}^l L\Big(\frac{2\theta s_i t_j}{\sqrt{N}}\Big) + \frac{1}{2N}\sum_{i,j=1}^l L\Big(\frac{2\theta t_i t_j}{\sqrt{N}}\Big) \\ &+ \sup_{\substack{\nu_1\in\mathcal{P}(I_1)\\ \int x^2 d\nu_1(x)=\alpha_1}} \Big\{ \sum_{i=1}^k \int L\Big(\frac{2\theta s_i x}{\sqrt{N}}\Big) d\nu_1(x) - H(\nu_1)\Big\} - \frac{1}{2}\log(2\pi) - \frac{1}{2}\Big\},\end{aligned}$$

where $I_1 = \{x : |x| \leq \delta^{1/2} N^{1/4}\}$, $I_2 = \{x : \delta^{1/2}N^{1/4} < |x| \leq K^{1/2}N^{1/4}\}$, $I_3 = \{x : K^{1/2}N^{1/4} < |x| \leq \sqrt{N\alpha_3}\}$, and

$$H(\nu) = \int \log \frac{d\nu}{dx} d\nu(x),$$

if ν is absolutely continuous with respect to the Lebesgue measure, whereas $H(\nu)$ is infinite otherwise.

Remark 1.8. *Note that $\underline{F}$ and $\overline{F}$ are convex by Hölder inequality. Since the entries of X_N are sub-Gaussian, $\overline{F}(\theta) \leq A\theta^2$. In particular $\overline{F}$ and $\underline{F}$ are finite convex functions and therefore are continuous on $\mathbb{R}_+$.*

The above proposition gives quite an intricate definition for the limit of the annealed spherical integrals. Yet, for small enough θ it can be computed explicitly.

Lemma 1.9. *For any $\theta \leq \frac{1}{2\sqrt{A-1}}$,*

$$\underline{F}(\theta) = \overline{F}(\theta) = \theta^2.$$

Note that for large θ this formula is not valid anymore when $A > 1$ since $\underline{F}$ grows like $A\theta^2$ at infinity (see the proof of Proposition 1.11).

Proof. Using the bound $L(x) \leq Ax^2/2$ for any $x \geq 0$ and the notation of Proposition 1.7, we have

$$\overline{F}(\theta) \leq \sup_{\alpha_1+\alpha_2+\alpha_3=1} \left\{\theta^2\big(\alpha_1^2 + 2\alpha_1\alpha_2 + B\alpha_3^2 + 2A\alpha_3\alpha_2 + A\alpha_2^2 + 2A\alpha_1\alpha_3\big) + \frac{1}{2}\log\alpha_1\right\}.$$

Here we used the fact that

$$\inf\{H(\nu_1) : \int x^2 d\nu_1 = \alpha_1, \nu_1 \in \mathcal{P}(I_1)\} \geq \inf\{H(\nu_1) : \int x^2 d\nu_1 = \alpha_1, \nu_1 \in \mathcal{P}(\mathbb{R})\},$$

where the infimum in the RHS is achieved at $\nu_1(dx) = (2\pi\alpha_1)^{-1/2}e^{-\frac{x^2}{2\alpha_1}}dx$ and hence equals $-1/2(1+\log(2\pi\alpha_1))$.

As $A \geq 1$ and $B \leq A$, we deduce the upper bound,

$$\begin{aligned}\overline{F}(\theta) &\leq \sup_{\alpha\in[0,1]} \left\{\theta^2\big(\alpha^2 + 2A\alpha(1-\alpha) + A(1-\alpha)^2\big) + \frac{1}{2}\log\alpha\right\} \\ &= \sup_{\alpha\in[0,1]} \left\{\theta^2\big(A - (A-1)\alpha^2\big) + \frac{1}{2}\log\alpha\right\}.\end{aligned}$$

Hence for all $\theta \geq 0$, (and as we could have seen directly from the uniform upper bound $L(\theta) \leq \frac{A}{2}\theta^2$)

$$\overline{F}(\theta) \leq A\theta^2. \tag{5}$$

We see that if $2\theta\sqrt{A-1} \leq 1$ then the function

$$\alpha \mapsto \theta^2\big(A - (A-1)\alpha^2\big) + \frac{1}{2}\log\alpha,$$

is increasing on $[0,1]$. Thus the supremum is achieved at $\alpha = 1$, and $\overline{F}(\theta) \leq \theta^2$. Moreover, taking $\alpha_1 = 1, \alpha_2 = \alpha_3 = 0$, and ν_1 the standard Gaussian restricted to I_1, $\nu_1(dx) = \mathbb{1}_{I_1}e^{-\frac{x^2}{2}}dx/Z$, we find that

$$\underline{F}(\theta) \geq \theta^2. \tag{6}$$

Thus, if $2\theta\sqrt{A-1} \leq 1$, we get that $\overline{F}(\theta) = \underline{F}(\theta) = \theta^2$.

$\square$

Although the limit of the annealed spherical integrals may not be explicit for all θ, we can still use it to obtain large deviations upper bounds as we describe now in the following theorem.

Theorem 1.10. *Under Assumptions 1.1 and 1.2, the law of the largest eigenvalue λ_{X_N} satisfies a large deviation upper bound with good rate function $\bar{I}$ which is infinite on $(-\infty, 2)$ and otherwise given by:*

$$\forall y \geq 2,\ \bar{I}(y) = \sup_{\theta\geq 0}\{J(y,\theta) - \overline{F}(\theta)\}. \tag{7}$$

Moreover, $\bar{I}(y) \leq I_{GOE}(y)$ for all $y \geq 2$.

Proof. From Remark 1.4, we know that the law of the largest eigenvalue is exponentially tight at the scale N. Therefore, it is sufficient to prove a weak large deviations upper bound by [15, Lemma 1.2.18]. Let $\delta > 0$. We have,

$$\mathbb{P}(\lambda_{X_N} < 2-\delta) \leq \mathbb{P}(\hat{\mu}_{X_N}(f) = 0),$$

where f is a smooth compactly supported function with support in $(2-\delta, 2)$. Since $\mathrm{supp}(\sigma) = [-2, 2]$, we deduce that,

$$\mathbb{P}(\lambda_{X_N} < 2-\delta) \leq \mathbb{P}(d(\hat{\mu}_{X_N}, \sigma) > \varepsilon),$$

for some $\varepsilon > 0$. As the empirical distribution of the eigenvalues concentrates at the scale N according to (3), we conclude that

$$\lim_{N\to+\infty} \frac{1}{N} \log \mathbb{P}(\lambda_{X_N} < 2-\delta) = -\infty.$$

Let now $x \geq 2$ and $\delta > 0$. Recall from (6) that $\underline{F}(\theta) \geq \theta^2$ for any $\theta \geq 0$. Therefore,

$$\bar{I}(x) \leq \sup_{\theta\geq 0}\{J(x,\theta) - \theta^2\}.$$

From [18, Section 4.1], we know that

$$\sup_{\theta\geq 0}\{J(x,\theta) - \theta^2\} = I_{GOE}(x),$$

where I_{GOE} is the rate function of the largest eigenvalue of a GOE matrix. Therefore we have proved that

$$\bar{I}(x) \leq I_{GOE}(x), \forall x \geq 2\,.$$

In particular $\bar{I}(2) = 0$ since $I_{GOE}(2) = 0$. Therefore we only need to estimate small ball probabilities around $x \neq 2$. As $\hat{\mu}_{X_N}$ concentrates at the scale N by Assumption 1.2, and $||X_N||$ is exponentially tight at the scale N by Remark 1.4 it is enough to show that for any $K > 0$,

$$\limsup_{\delta\to 0} \limsup_{N\to+\infty} \frac{1}{N} \log \mathbb{P}(X_N \in V^K_{\delta,x}) \leq -\bar{I}(x),$$

where $V^K_{\delta,x} = \{Y \in \mathcal{H}_N : |\lambda_Y - x| < \delta, d(\hat{\mu}_Y, \sigma) < N^{-\kappa}, ||Y|| \leq K\}$, for some $\kappa > 0$. Let $\theta \geq 0$. From [22, Proposition 2.1], we know that the spherical integral is continuous, more precisely, for N large enough and any $X_N \in V^K_{\delta,x}$,

$$|J_N(X_N,\theta) - J(x,\theta)| < g(\delta),$$

for some function $g(\delta)$ going to 0 as $\delta \to 0$. Therefore,

$$\mathbb{P}(X_N \in V^K_{\delta,x}) = \mathbb{E}\Big(\mathbb{1}_{X_N\in V^K_{\delta,x}} \frac{I_N(X_N,\theta)}{I_N(X_N,\theta)}\Big) \leq \mathbb{E}[I_N(X_N,\theta)] e^{-NJ(x,\theta)+Ng(\delta)}.$$

Taking the limsup as $N \to 0$ and $\delta \to 0$ at the logarithmic scale, we deduce

$$\limsup_{\delta\to 0} \limsup_{N\to+\infty} \frac{1}{N} \log \mathbb{P}(X_N \in V^K_{\delta,x}) \leq \overline{F}(\theta) - J(x,\theta).$$

Opimizing over $\theta \geq 0$, we get the claim. $\square$

Proposition 1.11. *Under Assumption 1.1, the rate function $\bar{I}$ defined in Theorem 1.10 is lower semi-continuous, and growing at infinity like $x^2/4A$. In particular, $\bar{I}$ is a good rate function.*

Proof. $\bar{I}$ is lower semi-continuous as a supremum of continuous functions (recall here that $J(\theta, .)$ is continuous by Lemma 1.6 and $\overline{F}$ is continuous by Remark 1.6). It remains to show that its level sets are compact, for which it is sufficient to prove that $\bar{I}$ goes to infinity at infinity. Let $x > 2$. Let $C > 0$ be a constant to be chosen later such that $Cx \geq 1/2$. We have by taking $\theta = Cx$ and using (5), that

$$\begin{aligned}\bar{I}(x) &\geq J(x, Cx) - \overline{F}(Cx) \\ &\geq Cx^2 - \frac{1}{2} - \frac{1}{2}\log(2Cx) - \frac{1}{2}\log x - AC^2x^2. \end{aligned} \tag{8}$$

Taking $C = 1/2A$, and assuming that $x > A$, we obtain that

$$\bar{I}(x) \geq \frac{x^2}{4A} - o(x^2). \tag{9}$$

To get the converse bound, we show that as θ goes to infinity, $\overline{F}$ goes to infinity like $A\theta^2$. We distinguish two cases. First, we consider the case $A = B$. Using Proposition 1.7, we get the lower bound for $\theta \geq 1$,

$$\underline{F}(\theta) \geq A\theta^2\Big(1 - \frac{1}{\theta^2}\Big) - \frac{1}{4}\log\theta,$$

by taking $\alpha_2 = 0$, $\alpha_3 = 1 - \theta^{-2}$, $\alpha_1 = \theta^{-2}$ and ν_1 the Gaussian law restricted to I_1 with variance α_1. In the case $A > B$, we define m_* such that $\psi(m_*) = A/2$. Taking $\alpha_3 = 0$, $\alpha_2 = 1 - \theta^{-2}$, $\alpha_1 = \theta^{-2}$, $t_i = \frac{\sqrt{m_*}N^{1/4}}{\sqrt{2\theta}}$, $l = \lfloor \frac{2\theta\alpha_2\sqrt{N}}{m_*} \rfloor$, and ν_1 the Gaussian law restricted to I_1 with variance α_1, we obtain,

$$\underline{F}(\theta) \geq A\theta^2\Big(1 - \frac{1}{\theta^2}\Big) - \frac{1}{4}\log\theta. \tag{10}$$

It follows that for any $\varepsilon > 0$, there exists $M < \infty$ such that for $\theta \geq M$,

$$\underline{F}(\theta) \geq (1 - \varepsilon)A\theta^2.$$

Therefore

$$\bar{I}(x) \leq \max\left\{\sup_{\theta \geq M}\{J(x,\theta) - (1-\varepsilon)A\theta^2\}, \sup_{\theta \leq M}\{J(x,\theta) - \overline{F}(\theta)\}\right\}.$$

But from Lemma 1.6 one can see that the second term in the above right-hand side is bounded by $Mx + C$ where C is a numerical constant. Besides, using the same argument as in (8), we get

$$\sup_{\theta \geq M}\{J(x,\theta) - (1-\varepsilon)A\theta^2\} \geq \frac{x^2}{4(1-\varepsilon)A} - o(x^2).$$

Hence, for x large enough,

$$\bar{I}(x) \leq \sup_{\theta \geq M}\{J(x,\theta) - (1-\varepsilon)A\theta^2\}.$$

But, for x large enough and $\theta \geq 1/2$, $J(\theta, x) \leq \theta x$. Thus,

$$\sup_{\theta \geq M}\{J(x,\theta) - (1-\varepsilon)A\theta^2\} \leq \sup_{\theta \geq 0}\{\theta x - (1-\varepsilon)A\theta^2\} = \frac{x^2}{4(1-\varepsilon)A},$$

which ends the proof.

□

Proposition 1.12. *For any $\theta \geq 0$, $J(.,\theta)$ is a convex function. Therefore, $\bar{I}$ is also convex.*

Proof. Let $x, y \geq 2$ and $t \in (0,1)$. Let E_N be a sequence of diagonal matrices such that $||E_N|| \leq 2$ and such that $\hat{\mu}_{E_N}$ converges weakly to σ. Let E_N^x and E_N^y be such that $(E_N^x)_{i,i} = (E_N^y)_{i,i} = (E_N)_{i,i}$ for any $i \in \{1, \dots, N-1\}$, and

$$(E_N^x)_{N,N} = x \quad (E_N^y)_{N,N} = y.$$

We have $\lambda_{E_N^x} = x$ and $\lambda_{E_N^y} = y$. Then, $H_N = tE_N^x + (1-t)E_N^y$ is such that its empirical distribution of eigenvalues converges to σ, and $\lambda_{H_N} = tx + (1-t)y$. By Hölder's inequality we have,

$$\log I_N(H_N, \theta) \leq t \log I_N(E_N, \theta) + (1-t)\log I_N(D_N, \theta).$$

Taking the limit as $N \to +\infty$, we get,

$$J(tx + (1-t)y, \theta) \leq tJ(x,\theta) + (1-t)J(y,\theta).$$

Therefore, $J(\theta, .)$ is convex and I is convex as a supremum of convex functions. □

To derive the large deviation lower bound, we denote by $\mathcal{C}_\mu$ the set of $\theta \in \mathbb{R}^+$ such that

$$\underline{F}(\theta) = \overline{F}(\theta) =: F(\theta)\,.$$

By Lemma 1.9, $\mathcal{C}_\mu$ is not empty. We observe also that by continuity of both $\underline{F}$ and $\overline{F}$ (see Remark 1.8), $\mathcal{C}_\mu$ is closed. Let

$$\forall x \geq 2,\ I(x) = \sup_{\theta \in \mathcal{C}_\mu}\{J(x,\theta) - F(\theta)\}.$$

Theorem 1.13. *For any $x \geq 2$, denote by*

$$\Theta_x = \{\theta \geq 0 : \bar{I}(x) = J(x,\theta) - F(\theta)\},$$

where $\bar{I}$ is defined in (7). *Let $x \geq 2$ such that there exists $\theta \in \Theta_x \cap \mathcal{C}_\mu$ and $\theta \notin \Theta_y$ for any $y \neq x$. Then, $I(x) = \bar{I}(x)$ and*

$$\lim_{\delta \to 0}\liminf_{N \to +\infty}\frac{1}{N}\log \mathbb{P}\left(|\lambda_{X_N} - x| \leq \delta\right) \geq -I(x).$$

We apply this general theorem in two cases. We first investigate the case where the function ψ is increasing, case for which we can check that our hypotheses on the sets Θ_x holds for x large enough. This includes the case where μ is the sparse Gaussian law, see Example 1.2.

Proposition 1.14. *Suppose that Assumptions 1.1 and 1.2 hold. If ψ is increasing on $\mathbb{R}_+$, then $\mathcal{C}_\mu = \mathbb{R}^+$. Moreover, there exists $x_\mu \geq 2$ such that for any $x \geq x_\mu$, the large deviation lower bound holds with rate function I.*

We then consider the case where μ is such that $B < A$. This includes any compactly supported measure μ since then $B = 0$. We prove in this case the following result.

Proposition 1.15. *Suppose that Assumptions 1.1 and 1.2 hold. If μ is such that $B < A$ and such that the maximum of ψ is attained on $\mathbb{R}^+$ for a unique m_* such that $\psi''(m_*) < 0$, then there exists a positive finite real number θ_0 such that $[\theta_0, +\infty[\subset \mathcal{C}_\mu$. Therefore, there exists a finite constant x_μ such that for $x \geq x_\mu$, the large deviation lower bound holds with rate function I. Furthermore, on the interval $[x_\mu, +\infty)$ the rate function I depends only on A.*

In the case where A is sufficiently small, we can show without any additional assumption that the large deviation lower bound holds in a vicinity of 2 and the rate function I is equal to the one of the GOE. This contrasts with Proposition 1.11 which shows that the rate function $\bar{I}$ goes to infinity like $x^2/4A$ at infinity and therefore depends on A. In other words the “heavy tails” only kicks in above a certain threshold.

Proposition 1.16. *Assume $A < 2$. The large deviation lower bound holds with rate function $\bar{I}$ on $[2, 1/\sqrt{A-1} + \sqrt{A-1}]$. Moreover, $\bar{I}$ coincides on this interval with the rate function in the GOE case I_{GOE}, defined in (2). As a consequence, for all $x \in [2, 1/\sqrt{A-1} + \sqrt{A-1}]$,*

$$\lim_{\delta\to 0}\liminf_{N\to+\infty}\frac{1}{N}\log\mathbb{P}\left(|\lambda_{X_N} - x| \leq \delta\right) = \lim_{\delta\to 0}\limsup_{N\to+\infty}\frac{1}{N}\log\mathbb{P}\left(|\lambda_{X_N} - x| \leq \delta\right) = -I_{GOE}(x).$$

Organization of the paper. In the next section 2, we detail our approach to prove large deviations lower bounds. Since Proposition 1.7 is crucial to all our results, we prove it in the next section 3. Then, we will apply these results to prove the large deviations lower bounds close to the bulk in section 4, that is, we give a proof of Proposition 1.16. To prove the large deviations lower bounds for large x, we consider first the case of increasing ψ in section 5 and then the case of $B < A$ in section 6. Indeed, the variational formulas for the limiting annealed spherical integrals differ in these two cases, as $B = A$ in the first case whereas $B < A$ in the second.

2. A general large deviation lower bound

We first prove Theorem 1.13 and will then give more practical descriptions of the sets Θ_x in order to apply it.

Proof of Theorem 1.13. By assumption, there exists $\theta \in \Theta_x \cap \mathcal{C}_\mu$ such that $\theta \notin \Theta_y$ for $y \neq x$. In particular, it entails that $I(x) = \bar{I}(x)$. Introducing the spherical integral with parameter $\theta \geq 0$, we have

$$\mathbb{P}\left(|\lambda_{X_N} - x| \leq \delta\right) \geq \mathbb{E}\Big(\mathbb{1}_{X_N \in V_{\delta,x}^K}\frac{I_N(X_N,\theta)}{I_N(X_N,\theta)}\Big),$$

where $V_{\delta,x}^K = \{Y \in \mathcal{H}_N : |\lambda_Y - x| \leq \delta, d(\hat{\mu}_Y, \sigma) < N^{-\kappa}, ||X_N|| \leq K\}$ for some $K > 0$ and $\kappa > 0$. Using the continuity of the spherical integral (see [22, Proposition 2.1]), we get

$$\mathbb{P}\left(|\lambda_{X_N} - x| \leq \delta\right) \geq \frac{\mathbb{E}\big(\mathbb{1}_{X_N \in V_{\delta,x}^K} I_N(X_N,\theta)\big)}{\mathbb{E} I_N(X_N,\theta)} e^{N\underline{F}(\theta) - NJ(x,\theta) - Ng(\delta) - o(N)}, \tag{11}$$

where g is a function such that $g(\delta) \to 0$ as $\delta \to 0$. We claim that

$$\liminf_{N\to+\infty} \frac{1}{N} \log \frac{\mathbb{E}\big(\mathbb{1}_{X_N \in V^K_{\delta,x}} I_N(X_N, \theta_x)\big)}{\mathbb{E} I_N(X_N, \theta_x)} \geq 0.$$

To this end we will use our large deviation upper bound. Since $\hat{\mu}_{X_N}$ concentrates at scales faster than N by Assumption 1.2, and by Remark 1.4, $||X_N||$ and $\hat{\mu}_{X_N}$ are exponentially tight uniformly under any tilted measures $\mathbb{P}^{(e,\theta)}$, as defined in (4), and therefore under the measure tilted by spherical integrals. Hence, it suffices to prove that for all $y \neq x$, for δ small enough, and K large enough,

$$\limsup_{N} \frac{1}{N} \log \frac{\mathbb{E}[\mathbb{1}_{X_N \in V^K_{\delta,y}} I_N(X_N, \theta)]}{\mathbb{E} I_N(X_N, \theta)} < 0.$$

By assumption, there exists $\theta \in \Theta_x \cap \mathcal{C}_\mu$ such that $\theta \notin \Theta_y$ for $y \neq x$. We introduce a new spherical integral with argument θ' and use again the continuity of J_N to show that:

$$\begin{aligned}
\frac{\mathbb{E}[\mathbb{1}_{X_N \in V^K_{\delta,y}} I_N(X_N, \theta)]}{\mathbb{E} I_N(X_N, \theta)} &= \frac{\mathbb{E}[\mathbb{1}_{X_N \in V^K_{\delta,y}} \frac{I_N(X_N,\theta')}{I_N(X_N,\theta')} I_N(X_N, \theta)]}{\mathbb{E} I_N(X_N, \theta)} \\
&\leq e^{-NJ(y,\theta') - N\underline{F}(\theta) + NJ(y,\theta) + N\varepsilon(\delta)} \mathbb{E}[1_{X_N \in V^K_{\delta,y}} I_N(X_N, \theta')] \\
&\leq e^{-NJ(y,\theta') - N\underline{F}(\theta) + NJ(y,\theta) + N\overline{F}(\theta') + N\varepsilon(\delta)},
\end{aligned}$$

where $\varepsilon(\delta) \to 0$ as $\delta \to 0$. We can conclude that

$$\begin{aligned}
\limsup_{\substack{N\to+\infty \\ \delta\to 0}} \frac{1}{N} \log \frac{\mathbb{E}[\mathbb{1}_{X_N \in V^K_{\delta,y}} I_N(X_N, \theta)]}{\mathbb{E} I_N(X_N, \theta)} &\leq -\sup_{\theta'} \{J(y, \theta') - \overline{F}(\theta')\} + J(y, \theta) - \underline{F}(\theta) \\
&= -\overline{I}(y) + J(y, \theta) - \underline{F}(\theta) \qquad (12)
\end{aligned}$$

By assumption, $\theta \notin \Theta_y$, and $\theta \in \mathcal{C}_\mu$ so that $\underline{F}(\theta) = \overline{F}(\theta)$. Hence,

$$-\overline{I}(y) + J(y, \theta) - \overline{F}(\theta) < 0$$

and the conclusion follows from (12). Therefore, coming back to (11), we obtain since $\theta_x \in \Theta_x$ and $I(x) = \bar{I}(x)$,

$$\liminf_{N\to+\infty} \frac{1}{N} \log \mathbb{P}\left(|\lambda_{X_N} - x| \leq \delta\right) \geq -I(x).$$

□

In a first step, we identify a subset defined in terms of the subdifferential sets of F at the points of non-differentiability where the large deviation lower bound holds. Let $\mathcal{D}$ be the set of $\theta \geq 0$ such that $\overline{F}$ is differentiable at θ.

Lemma 2.1. *The lower bound holds for any $x > 2$ such that $I(x) = \bar{I}(x) > 0$ and*

$$x \notin E := \bigcup_{\theta \in \mathcal{D}^c} \Big(\frac{1}{2\theta} + \partial\overline{F}(\theta)\Big), \qquad (13)$$

where $\partial\overline{F}(\theta)$ denotes the subdifferential of $\overline{F}$ at θ.

Note that since $\overline{F}$ is a convex function, its subdifferentials are well defined. Moreover, by Lemma 1.9, $\mathcal{D}^c \subset [\frac{1}{2\sqrt{A-1}}, +\infty)$.

Proof. Let $x > 2$ such that $I(x) = \bar{I}(x) > 0$ and $x \notin E$. Since $\overline{F}(\theta) \geq \theta^2$ for any $\theta \geq 0$ by (6) and $\overline{F}$ is continuous by Remark 1.8, we deduce from Lemma 1.6 that $\theta \mapsto J(x,\theta) - \overline{F}(\theta)$ is continuous and goes to $-\infty$ as θ goes to $+\infty$. Since $\mathcal{C}_\mu$ is closed, the supremum

$$\sup_{\theta \in \mathcal{C}_\mu} \left\{ J(\theta, x) - \overline{F}(\theta) \right\} \tag{14}$$

is achieved at some $\theta \in \mathcal{C}_\mu$. We will show that $\theta \in \mathcal{D}$.

As $\bar{I}(x) \neq 0$ we must have $\theta > \frac{1}{2} G_\sigma(x)$. Indeed, $\overline{F}(\theta) \geq \theta^2$ and $J(x,\theta) = \theta^2$ for $\theta \leq \frac{1}{2} G_\sigma(x)$ by Lemma 1.6 so that

$$\sup_{\theta \leq \frac{1}{2} G_\sigma(x)} \{ J(x,\theta) - \overline{F}(\theta) \} = 0.$$

Since $\bar{I}(x) = I(x)$, we deduce by Fermat's rule that θ is a critical point of $J(x,.) - \overline{F}$ and therefore satisfies the condition:

$$0 \in \frac{\partial J}{\partial \theta}(x,\theta) - \partial \overline{F}(\theta) = x - \frac{1}{2\theta} - \partial \overline{F}(\theta)\,.$$

Since $x \notin E$, we deduce that $\overline{F}$ is differentiable at θ.

According to Theorem 1.13, to prove that the lower bound holds at x, it suffices to show that $\theta \notin \Theta_y$ for any $y \neq x$. Let us proceed by contradiction and assume that there exists $y \geq 2$, $y \neq x$, such that $\theta \in \Theta_y$. As $\overline{F}$ is differentiable at θ, it should be a critical point of both $J(y,.) - \overline{F}$ and $J(x,.) - \overline{F}$. Therefore, we should have

$$\frac{\partial}{\partial \theta} J(y,\theta) = \frac{\partial}{\partial \theta} J(x,\theta).$$

If $G_\sigma(y) < 2\theta$, then we obtain by Lemma 1.6 and the fact that $G_\sigma(x) < 2\theta$ that $x = y$. If $G_\sigma(y) \geq 2\theta$, then we have

$$x - \frac{1}{2\theta} = 2\theta.$$

On the other hand, $2\theta \leq G_\sigma(y) \leq 1$ and therefore we get the unique solution $2\theta = G_\sigma(x)$. As we assumed that $2\theta > G_\sigma(x)$, we get a contradiction and conclude that $\theta \notin \Theta_y$ for any $y \neq x$ such that $G_\sigma(y) \geq 2\theta$, which completes the proof. $\square$

We are now ready to prove the following result:

Proposition 2.2. *Assume that there exists $\theta_0 > 0$ such that $[\theta_0, +\infty) \subset \mathcal{C}_\mu$ and such that $\overline{F}$ is differentiable on $(\theta_0, +\infty)$. There exists $x_\mu \in [2, +\infty)$ such that for any $x \geq x_\mu$, $I(x) = \bar{I}(x)$ and the large deviation lower bound holds for any $x \geq x_\mu$ with rate function $I(x)$.*

Proof. On one hand,

$$\sup_{\theta \leq \theta_0} \{ J(\theta, x) - \overline{F}(\theta) \} \leq \theta_0 x + C, \tag{15}$$

where C is some positive constant. Since $\bar{I}(x) \geq x^2/4A - o(x^2)$ by (9), we deduce that there exists $x_\mu \in [2, +\infty)$ such that for $x \geq x_\mu$, $\bar{I}(x) > 0$ and together with (15) that the supremum of $J(.,x) - \overline{F}$ is achieved in $[\theta_0, +\infty)$, and therefore in $\mathcal{C}_\mu$ by our assumption. By definition of $\mathcal{C}_\mu$, we deduce that for any $x \geq x_\mu$, $\bar{I}(x) = I(x) > 0$. In view of Lemma 2.1, it remains to show that E, defined in (13), is a bounded set. From our assumption that $\overline{F}$ is differentiable on $(\theta_0, +\infty)$ and Lemma 1.9, we deduce that

$$\mathcal{D}^c \subset \Big[\frac{1}{2\sqrt{A-1}}, \theta_0\Big].$$

We observe that since $0 \leq \overline{F}(\theta) \leq A\theta^2$, we have for any $\zeta \in \partial \overline{F}(\theta)$,

$$\zeta\theta \leq \overline{F}(2\theta) - \overline{F}(\theta) \leq 4A\theta^2,$$

and thus $\zeta \leq 4A\theta$. Therefore, the set E defined in Lemma 2.1 is bounded, which ends the proof.

□

3. Asymptotics of the annealed spherical integral

In this section we prove Proposition 1.7. Taking the expectation first with respect to X_N, the annealed spherical integral is given by

$$\begin{aligned} F_N(\theta) &= \frac{1}{N} \log \mathbb{E}_{X_N} \mathbb{E}_e[\exp(N\theta\langle e, X_N e\rangle)] \\ &= \frac{1}{N} \log \mathbb{E}_e \exp\big(f(e)\big), \end{aligned}$$

where

$$f(e) = \sum_{i<j} L\big(2\sqrt{N}\theta e_i e_j\big) + \sum_{i=1}^{N} L(\sqrt{2N}\theta e_i^2\big).$$

In a first step, we will prove the following variational representation of the upper and lower limits of $F_N(\theta)$.

Lemma 3.1. *Let X_N be a Wigner matrix satisfying Assumptions 1.1 and 1.2. Then for any $\theta > 0$,*

$$\underline{F}(\theta) \leq \liminf_{N\to+\infty} F_N(\theta) \leq \limsup_{N\to+\infty} F_N(\theta) \leq \overline{F}(\theta)$$

with

$$\overline{F}(\theta) = \limsup_{\substack{\delta\to 0, K\to+\infty \\ \delta K \to 0}} \sup_{\substack{c = c_1+c_2+c_3 \\ c_i \geq 0}} \limsup_{N\to+\infty} F^N_{c_1,c_2,c_3}(\delta, K),$$

$$\underline{F}(\theta) = \sup_{\substack{c = c_1+c_2+c_3 \\ c_i \geq 0}} \liminf_{\substack{\delta\to 0, K\to+\infty \\ \delta K \to 0}} \liminf_{N\to+\infty} F^N_{c_1,c_2,c_3}(\delta, K),$$

where

$$\begin{aligned} F^N_{c_1,c_2,c_3}(\delta, K) = \sup_{\substack{s_i \geq \sqrt{cK} N^{1/4} \\ |\sum s_i^2 - c_3 N| \leq \delta N}} \sup_{\substack{\sqrt{c\delta} \leq t_i N^{-1/4} \leq \sqrt{cK} \\ |\sum t_i^2 - N c_2| \leq \delta N}} &\Big\{ \frac{\theta^2}{c^2}\big(c_1^2 + 2c_1c_2 + Bc_3^2\big) - \frac{1}{2}\big(c_2 + c_3\big) \\ &+ \frac{1}{N}\sum_{i,j} L\Big(\frac{2\theta s_i t_j}{\sqrt{N}c}\Big) + \frac{1}{2N}\sum_{i,j} L\Big(\frac{2\theta t_i t_j}{\sqrt{N}c}\Big) + \sup_{\substack{\nu\in\mathcal{P}(I_1) \\ \int x^2 d\nu(x) = c_1}} \big\{\Phi(\nu, s) - H(\nu|\gamma)\big\}\Big\}, \end{aligned}$$

and

$$\Phi(\nu, s) = \sum_i \int L\Big(\frac{2\theta x s_i}{\sqrt{N}c}\Big) d\nu(x),$$

with $I_1 = [-\sqrt{c\delta}N^{1/4}, \sqrt{c\delta}N^{1/4}]$. *Here, we have set* γ *to be the standard Gaussian law and*

$$H(\nu|\gamma) = \int \log \frac{d\nu}{d\gamma}(x) d\nu(x)\,.$$

Hereafter, $o_\delta(1)$ is a function which goes to zero as δ goes to zero (or infinity depending on the context). $\varepsilon_K(\delta)$ denotes a function which goes to zero as δ goes to zero or infinity while K is fixed. $O(\delta)$ is a function such that there exists a finite constant C such that the modulus of $O(\delta)$ is bounded by $C\delta$. These functions may change from line to line.

Proof. We use the representation of the law of the vector e uniformly distributed on the sphere as a renormalized Gaussian vector $g/\|g\|_2$ where g is a standard Gaussian vector in $\mathbb{R}^N$, to write

$$\mathbb{E}_e \exp\big(f(e)\big) = \mathbb{E}\left[\exp(\Sigma(g))\right],$$

where $g = (g_1, \ldots, g_N)$ and

$$\Sigma(g) = \sum_{i<j} L\Big(2\sqrt{N}\theta \frac{g_i g_j}{\sum_{i=1}^N g_i^2}\Big) + \sum_i L\Big(\sqrt{2N}\theta \frac{g_i^2}{\sum_{i=1}^N g_i^2}\Big).$$

To study the large deviation of $\Sigma(g)$, we split the entries of g into three possible regime: the regime where $g_i \ll N^{1/4}$, an intermediate regime where $g_i \simeq N^{1/4}$ and finally $g_i \gg N^{1/4}$. Fix some $K \geq 1, \delta > 0$ such that $0 < 2\delta < K^{-1}$. Let $c_1, c_2, c_3 > 0$ and $c = c_1 + c_2 + c_3$. We assume that $0 < K^{-1} \leq c_1 \leq c \leq K$. Define I_1, I_2, I_3 as

$$\begin{aligned} I_1 &= [0, \sqrt{c\delta}N^{\frac14}] \\ I_2 &= (\sqrt{c\delta}N^{\frac14}, \sqrt{cK}N^{\frac14}] \\ I_3 &= (\sqrt{cK}N^{\frac14}, \sqrt{N(c+3\delta)}]. \end{aligned}$$

Let for $i = 1, 2, 3,\ J_i = \{j : |g_j| \in I_i\}$ and $\hat{c}_i^N = \sum_{j \in J_i} g_j^2/N$. In a first step, we will fix the empirical variances $\hat{c}_i^N$ and compute the asymptotics of

$$F^N_{c_1,c_2,c_3}(\theta, \delta) = \mathbb{E}[\exp(\Sigma(g))\mathbb{1}_{\mathcal{A}^\delta_{c_1,c_2,c_3}}].$$

where

$$\mathcal{A}^\delta_{c_1,c_2,c_3} := \bigcap_{1 \leq i \leq 3} \{|\hat{c}_i^N - c_i| \leq \delta\}.$$

Let

$$\Sigma_c(g) = \sum_{i<j} L\Big(2\theta \frac{g_i g_j}{\sqrt{N}c}\Big) + \sum_i L\Big(\sqrt{2}\theta \frac{g_i^2}{\sqrt{N}c}\Big).$$

Using the fact the $L(\sqrt{.})$ is Lipschitz, we prove in the next lemma that on the event $\mathcal{A}^\delta_{c_1,c_2,c_3}$, $\Sigma_c(g)$ is a good approximation of $\Sigma(g)$.

Lemma 3.2. *On the event* $\mathcal{A}^{\delta}_{c_1,c_2,c_3}$,

$$\Sigma(g) - \Sigma_c(g) = N o_{\delta K}(1), \quad \text{as } \delta K \to 0. \tag{16}$$

Moreover

$$|J_3| \leq \frac{3\sqrt{N}}{K}, \qquad |J_2 \cup J_3| \leq \frac{3\sqrt{N}}{\delta}.$$

Proof. Note that since μ is symmetric, $L(x) = L(|x|)$ and since we assumed $L(\sqrt{.})$ Lipschitz, for any $x, y \in \mathbb{R}$, $|L(x) - L(y)| \leq L|x^2 - y^2|$ for some finite constant L. Therefore,

$$\sum_{1 \leq i \neq j \leq N} \big| L\Big(2\sqrt{N}\theta \frac{g_i g_j}{\sum_{i=1}^N g_i^2}\Big) - L\Big(2\theta \frac{g_i g_j}{\sqrt{N} c}\Big)\big| \leq \frac{4L\theta^2}{N} \sum_{i,j} g_i^2 g_j^2 \left| \frac{1}{(\hat{c}_1^N + \hat{c}_2^N + \hat{c}_3^N)^2} - \frac{1}{c^2} \right|$$

$$\leq CNL\theta^2 (c + \hat{c}_1^N + \hat{c}_2^N + \hat{c}_3^N) \frac{\delta}{c^2} \leq C' N L \theta^2 \delta K,$$

where C, C' are numerical constants and we used $K^{-1} < c < K$, and $2\delta < K^{-1}$. We get a similar estimate for the diagonal terms. The estimates on $|J_3|$ and $|J_2|$ are straightforward consequences of Tchebychev's inequality.

∎

We next fix the set of indices J_1, J_2, J_3. Using the invariance under permutation of the distribution of g, we can write

$$F^N_{c_1,c_2,c_3}(\theta, \delta) = \sum_{\substack{0 \leq k \leq 3\sqrt{N\delta} \\ 0 \leq l \leq 3\sqrt{NK}}} \binom{N}{k} \binom{N-k}{l} F^{k,l}_{c_1,c_2,c_3},$$

where

$$F^{k,l}_{c_1,c_2,c_3} = \mathbb{E}\Big[\exp(\Sigma_c(g)) \mathbb{1}_{\mathcal{A}^{\delta}_{c_1,c_2,c_3} \cap \mathcal{I}_{k,l}} \Big]$$

and

$$\mathcal{I}_{k,l} = \Big\{ J_3 = \{1, \ldots, k\}, J_2 = \{k+1, \ldots, k+l\}, J_1 = \{k+l+1, \ldots, N\} \Big\}.$$

As the number of all the possible configurations of J_2 and J_3 are sub-exponential in N by Lemma 3.2, that is, for any $k \leq 3\sqrt{N}/K$ and $l \leq 3\sqrt{N}/\delta$,

$$\max\Big(\binom{N}{k}, \binom{N-k}{l} \Big) = e^{O(\frac{\sqrt{N}}{\delta} \log N)},$$

we are reduced to compute $F^{k,l}_{c_1,c_2,c_3}$ for fixed k, l. More precisely, we have the following result.

Lemma 3.3.

$$\log F^N_{c_1,c_2,c_3}(\theta, \delta) = \max_{\substack{k \leq 3\sqrt{N}/K \\ l \leq 3\sqrt{N}/\delta}} \log F^{k,l}_{c_1,c_2,c_3} + O\Big(\frac{\sqrt{N}}{\delta} \log N \Big).$$

To simplify the notations, we denote for any $a, b \in \{1, 2, 3\}$,

$$\forall x, y \in \mathbb{R}^N, \ \Sigma_{a,b}(x, y) = \frac{1}{2N} \sum_{i \in J_a, j \in J_b} L\left(2\theta \frac{x_i y_j}{\sqrt{N} c} \right),$$

if $a \neq b$, and

$$\forall x, y \in \mathbb{R}^N,\ \Sigma_{a,a}(x,y) = \frac{1}{2N} \sum_{i \neq j \in J_a} L\left(2\theta \frac{x_i y_j}{\sqrt{N}c}\right) + \frac{1}{N} \sum_{i \in J_a} L\Big(\sqrt{2}\theta \frac{x_i y_i}{\sqrt{N}c}\Big),$$

where now $J_3 = \{1, \ldots, k\}, J_2 = \{k+1, \ldots, k+l\}, J_1 = \{k+l+1, \ldots, N\}$.

Next, we single out the interaction terms which involves the quadratic behavior of L at 0 or at $+\infty$.

Lemma 3.4. *On the event $\mathcal{A}^\delta_{c_1,c_2,c_3}$,*

$$\Sigma_{1,1}(g,g) + 2\Sigma_{1,2}(g,g) + \Sigma_{3,3}(g,g) = \frac{\theta^2}{c^2}\big(c_1^2 + 2c_1c_2 + Bc_3^2\big) + o_{\delta K}(1) + o_K(1),$$

as $\delta K \to 0$ and $K \to +\infty$.

Proof. Observe that for $i \in J_1$, $j \in J_1 \cup J_2$, $|g_i g_j| \leq \sqrt{NK\delta}c$. Since $L(x) \sim_0 x^2/2$, we get,

$$\Sigma_{1,1}(g,g) + 2\Sigma_{1,2}(g,g) = \theta^2 \frac{(\hat{c}_1^N)^2}{c^2} + 2\theta^2 \frac{\hat{c}_1^N \hat{c}_2^N}{c^2} + o_{\delta K}(1), \tag{17}$$

as $\delta K \to 0$. On the event $\mathcal{A}^\delta_{c_1,c_2,c_3}$, we have $|(\hat{c}_i^N)^2 - c_i^2| = O(\delta c)$ for any $i \in \{1,2,3\}$. But $c \geq K^{-1}$, therefore

$$\Sigma_{1,1}(g,g) + 2\Sigma_{1,2}(g,g) = \theta^2 \frac{c_1^2}{c^2} + 2\theta^2 \frac{c_1 c_2}{c^2} + o_{\delta K}(1).$$

For $i,j \in J_3$, $|g_i g_j| \geq K\sqrt{cN}$. Since $L(x) \sim_{+\infty} \frac{B}{2}x^2$, we deduce similarly that

$$\Sigma_{3,3}(g,g) = \Big(\frac{B}{2} + o_K(1)\Big) \frac{2\theta^2}{N^2c^2} \Big(\sum_{i \in J_3} g_i^2\Big)^2 = B\theta^2 \frac{c_3^2}{c^2} + o_K(1), \tag{18}$$

as $K \to +\infty$, which gives the claim.

$\square$

From the Lemmas 3.2 and 3.4, we have on the event $\mathcal{A}^\delta_{c_1,c_2,c_3}$,

$$\Sigma(g) = \frac{\theta^2}{c^2}\big(c_1^2 + 2c_1c_2 + Bc_3^2\big) + \Sigma_{1,3}(g,g) + 2\Sigma_{2,3}(g,g) + \Sigma_{2,2}(g,g) + o_{\delta K}(1) + o_K(1). \tag{19}$$

We now show that the deviations of the variables $g_i, i \in J_2 \cup J_3$ do not lead to any entropic terms, which yields the following lemma.

Lemma 3.5. *Let $k, l \in \mathbb{N}$ such that $k + l \leq N$. Define*

$$S_{c_1,c_2,c_3}(\delta) = \max_{\substack{t_i \in I_2, i \leq l \\ |\sum_i t_i^2 - c_2 N| \leq \delta N}} \max_{\substack{s_i \in I_3, i \leq k \\ |\sum_i s_i^2 - c_3 N| \leq \delta N}} \log \mathbb{E}\Big(\exp\Big\{N(\Sigma_{1,3}(g,s) + 2\Sigma_{2,3}(t,s) + \Sigma_{2,2}(t,t))\Big\} \mathbb{1}_{\mathcal{A}^\delta_{c_1}}\Big),$$

where

$$\mathcal{A}^\delta_{c_1} = \big\{(g_i)_{i=k+l+1}^N \in I_1^{N-k-l} : |\sum_{i=k+l+1}^N g_i^2 - Nc_1| \leq N\delta\big\}. \tag{20}$$

Then,

$$S_{c_1,c_2,c_3}(\delta/2) - \frac{N}{2}(c_2+c_3) - o_\delta(1)N + O\Big(\frac{\sqrt{N}}{\delta}\log\frac{\delta}{\sqrt{NK}}\Big)$$
$$\leq \log \mathbb{E}\Big(\exp\Big\{ N(\Sigma_{1,3}(g,g) + 2\Sigma_{2,3}(g,g) + \Sigma_{2,2}(g,g)) \Big\} \mathbb{1}_{\mathcal{A}^\delta_{c_1,c_2c_3} \cap \mathcal{I}_{k,l}} \Big)$$
$$\leq S_{c_1,c_2,c_3}(\delta) - \frac{N}{2}(c_2+c_3) + o_\delta(1)N + O\Big(\frac{\sqrt{N}}{\delta^2}\log(1/\delta)\Big).$$

Proof. Integrating on $g_i, i \leq k+l$, we find

$$\mathbb{E}\Big(\exp\Big\{ N(\Sigma_{1,3}(g,g) + 2\Sigma_{2,3}(g,g) + \Sigma_{2,2}(g,g)) \Big\} \mathbb{1}_{\mathcal{A}^\delta_{c_1,c_2c_3} \cap \mathcal{I}_{k,l}} \Big)$$
$$\leq \frac{1}{(2\pi)^{\frac{k+l}{2}}} \int \mathbb{1}_{\{|\frac{1}{N}\sum_{i=1}^{k+l} g_i^2 - (c_2+c_3)| \leq 2\delta\}} e^{-\frac{1}{2}\sum_{i=1}^{k+l} g_i^2} \prod_{i=1}^{k+l} dg_i$$
$$\times \max_{\substack{t \in I_2^l \\ |\sum_i t_i^2 - c_2 N| \leq \delta N}} \max_{\substack{s \in I_3^k \\ |\sum_i s_i^2 - c_3 N| \leq \delta N}} \mathbb{E}\Big(\exp\Big\{ N(\Sigma_{1,3}(g,s) + 2\Sigma_{2,3}(t,s) + \Sigma_{2,2}(t,t)) \Big\} \mathbb{1}_{\mathcal{A}^\delta_{c_1}} \Big),$$

But, using the fact that $c \leq K$, we get

$$\int \mathbb{1}_{\{|\frac{1}{N}\sum_{i=1}^{k+l} g_i^2 - (c_2+c_3)| \leq 2\delta\}} e^{-\frac{1}{2}\sum_{i=1}^{k+l} g_i^2} \prod_{i=1}^{k+l} dg_i \leq e^{-\frac{1}{2}(1-\delta)(c_2+c_3-2\delta)N} \int e^{-\frac{\delta}{2}\sum_{i=1}^{k+l} g_i^2} \prod_{i=1}^{k+l} dg_i$$
$$\leq (2\pi)^{\frac{k+l}{2}} e^{-\frac{N}{2}(c_2+c_3) + (O(\delta)+O(\delta K))N} \delta^{-\frac{k+l}{2}}.$$

By Lemma 3.2 we have $k+l = O(\sqrt{N}/\delta)$.Therefore,

$$\log \mathbb{E}\Big(\exp\Big\{ N(\Sigma_{1,3}(g,g) + 2\Sigma_{2,3}(g,g) + \Sigma_{2,2}(g,g)) \Big\} \mathbb{1}_{\mathcal{A}^\delta_{c_1,c_2,c_3} \cap \mathcal{I}_{k,l}} \Big)$$
$$\leq -\frac{N}{2}(c_2+c_3) + O(\delta)N + O(\delta K)N + O\Big(\frac{\sqrt{N}}{\delta}\log(1/\delta)\Big)$$
$$+ \max_{\substack{t_i \in I_2, i \leq l \\ |\sum_i t_i^2 - c_2 N| \leq \delta N}} \max_{\substack{s_i \in I_3, i \leq k \\ |\sum_i s_i^2 - c_3 N| \leq \delta N}} \log \mathbb{E}\Big(\exp\Big\{ N(\Sigma_{1,3}(g,s) + 2\Sigma_{2,3}(t,s) + \Sigma_{2,2}(t,t)) \Big\} \mathbb{1}_{\mathcal{A}^\delta_{c_1}} \Big).$$

To get the converse bound, we take $t \in I_2^l, s \in I_3^k$, optimizing the above maximum where δ is replaced by $\delta/2$. We next localize the integral on the set $\mathcal{B}_\delta$ where $|g_i - s_i| \leq \delta/8\sqrt{NK}$, $1 \leq i \leq k$, $|g_i - t_i| \leq \delta/4$, $k+1 \leq i \leq k+l$. One can check that $\mathcal{B}_\delta \times \mathcal{A}^\delta_{c_1} \subset \mathcal{A}^\delta_{c_1,c_2c_3} \cap \mathcal{I}_{k,l}$, and because $L \circ \sqrt{.}$ is Lipschitz, we have on this event

$$|\Sigma_{1,3}(g,s) + 2\Sigma_{2,3}(t,s) + \Sigma_{2,2}(t,t) - (\Sigma_{1,3}(g,g) + 2\Sigma_{2,3}(g,g) + \Sigma_{2,2}(g,g))| \leq C\theta^2\delta,$$

where C is some positive constant. Hence

$$\begin{aligned}
&\mathbb{E}\Big(\exp\Big\{N(\Sigma_{1,3}(g,g)+2\Sigma_{2,3}(g,g)+\Sigma_{2,2}(g,g))\Big\}\mathbb{1}_{\mathcal{A}^{\delta}_{c_1,c_2c_3}\cap\mathcal{I}_{k,l}}\Big)\\
&\geq \frac{1}{(2\pi)^{\frac{k+l}{2}}}\prod_{1\leq i\leq k}\int_{g\in I_3}\mathbb{1}_{|g-s_i|\leq\delta/8\sqrt{NK}}e^{-\frac{1}{2}g^2}dg\prod_{1\leq i\leq l}\int_{g\in I_2}\mathbb{1}_{|g-t_j|\leq\delta/4}e^{-\frac{1}{2}g^2}dg\\
&\times e^{-CN\theta^2\delta}\mathbb{E}\Big(\exp\Big\{N(\Sigma_{1,3}(g,s)+2\Sigma_{2,3}(t,s)+\Sigma_{2,2}(t,t))\Big\}\mathbb{1}_{\mathcal{A}^{\delta}_{c_1}}\Big),\\
&\geq e^{-\frac{N}{2}(c_2+c_3)+O(\delta)N}\Big(\frac{\delta}{8\sqrt{2\pi NK}}\Big)^k\Big(\frac{\delta}{4\sqrt{2\pi}}\Big)^l\mathbb{E}\Big(\exp\Big\{N(\Sigma_{1,3}(g,s)+2\Sigma_{2,3}(t,s)+\Sigma_{2,2}(t,t))\Big\}\mathbb{1}_{\mathcal{A}^{\delta}_{c_1}}\Big)
\end{aligned}$$

which completes the proof of Lemma 3.4 as $k+l=O(\sqrt{N}/\delta)$ and $K^{-1}\geq\delta$.

□

Hence, we are left to estimate

$$\Lambda_1^N=\mathbb{E}\big[\exp\big(N\Sigma_{1,3}(g,s)\big)\mathbb{1}_{\mathcal{A}^{\delta}_{c_1}}\big],$$

where $s\in I_3^k$ satisfies $|\sum_i s_i^2-Nc_3|\leq\delta N$ and $\mathcal{A}^{\delta}_{c_1}$ is defined in (20). Let $\hat\mu_N=\frac{1}{|J_1|}\sum_{i\in J_1}\delta_{g_i}$. We can write

$$\Sigma_{1,3}(g,s)=\frac{|J_1|}{N}\sum_{i=1}^k\int L\left(\frac{2\theta x s_i}{\sqrt{N}c}\right)d\hat\mu_N(x).$$

The first difficulty in estimating Λ_1^N lies in the fact that the function

$$x\mapsto\sum_{i=1}^k L\Big(\frac{2\theta x s_i}{\sqrt{N}c}\Big),$$

is not bounded so that Varadhan's lemma (see [15, Theorem 4.3.1]) cannot be applied directly. The second issue is that we need a large deviation estimate which is uniform in the choice of $s\in I_3^k$ such that $|\sum_{i=1}^k s_i^2-Nc_3|\leq\delta N$. In the next lemma, we prove a uniform large deviation estimate of the type of Varadhan's lemma. The proof is postponed to the appendix 8.2.

Lemma 3.6. *Let $f:\mathbb{R}\to\mathbb{R}$ such that $f(0)=0$ and $f(\sqrt{.})$ is a L-Lipschitz function. Let M_N,m_N be sequences such that $M_N=o(\sqrt{N})$ and $m_N\sim N$. Let $g_1,\dots,g_{m_N}$ be independent Gaussian random variables conditioned to belong to $[-M_N,M_N]$. Let $\delta\in(0,1)$ and $c>0$ such that $K^{-1}<c<K$ and $2\delta<K^{-1}$. Then,*

$$\begin{aligned}
\Big|\frac{1}{N}\log\mathbb{E}e^{\sum_{i=1}^{m_N}f\left(\frac{g_i}{\sqrt{c}}\right)}\mathbb{1}_{|\sum_{i=1}^{m_N}g_i^2-cN|\leq\delta N}-\sup_{\substack{\nu\in\mathcal{P}([-M_N,M_N])\\ \int x^2d\nu=c}}\Big\{\int f\Big(\frac{x}{\sqrt{c}}\Big)d\nu(x)-H(\nu|\gamma)\Big\}\Big|\\
\leq\varepsilon_{L,K}(N)+\varepsilon_L(\delta K),
\end{aligned}$$

where $\varepsilon_{L,K}(N)$ (resp. $\varepsilon_L(x)$) goes to zero as $N\to+\infty$ (resp. as $x\to0$), while L,K are fixed.

Let $s \in I_3^l$ such that $|\sum_i s_i^2 - Nc_3| \leq \delta N$. We consider the function

$$f : x \mapsto \sum_{i=1}^{k} L\Big(\frac{2\theta x s_i \sqrt{c_1}}{\sqrt{N}c}\Big).$$

Using the fact that $\delta \leq c$, one can observe that $f(\sqrt{.})$ is $8\theta^2 L$-Lipschitz. Using the previous lemma with $M_N = \delta N^{1/4}$, we deduce that for any $c_1 \geq K^{-1}$,

$$\Big|\frac{1}{N}\log \Lambda_1^N - \sup_{\substack{\nu\in\mathcal{P}(I_1)\\ \int x^2 d\nu(x)=c_1}} \Big\{\sum_{i=1}^{k}\int L\Big(\frac{2\theta x s_i}{\sqrt{N}c}\Big)d\nu(x) - H(\nu|\gamma)\Big\}\Big| \leq \varepsilon_K(N) + o_{\delta K}(1), \quad (21)$$

where $\varepsilon_K(N) \to 0$ as $N \to +\infty$. Putting together (19), Lemma 3.5 and (21), we obtain

$$\frac{1}{N}\log \mathbb{E}\Big(\exp(\Sigma(g))\mathbb{1}_{\mathcal{A}_{c_1,c_2,c_3}\cap\mathcal{I}_{k,l}}\Big) - \Psi_{k,l}^{(\delta)}(c_1,c_2,c_3) \leq \varepsilon_{\delta,K}(N) + o_{\delta K}(1) + o_K(1) + o_\delta(1),$$

and similarly for the lower bound where where $\Psi_{k,l}^{(\delta)}(c_1,c_2,c_3)$ is replaced by $\Psi_{k,l}^{(\delta/2)}(c_1,c_2,c_3)$; with $\varepsilon_K(N) \to 0$ as $N \to +\infty$, whereas δ and K are fixed,

$$\Psi_{k,l}^{(\delta)}(c_1,c_2,c_3) = Q(c_1,c_2,c_3) - \frac{1}{2}(c_2+c_3)$$
$$+ \max_{\substack{t_i\in I_2, i\leq l\\ |\sum_i t_i^2 - c_2 N|\leq \delta N}} \max_{\substack{s_i\in I_3, i\leq k\\ |\sum_i s_i^2 - c_3 N|\leq \delta N}} \Big\{\Sigma_{2,3}(t,s) + \Sigma_{2,2}(t,t) + \sup_{\substack{\nu\in\mathcal{P}(I_1)\\ \int x^2 d\nu_1(x)=c_1}}\{\Phi(\nu,s) - H(\nu|\gamma_1)\}\Big\},$$

with

$$Q(x,y,z) = \theta^2\frac{x^2+2xy+Bz^2}{(x+y+z)^2}, \quad \text{and} \quad \Phi_c(\nu,s) = \sum_{i=1}^{k}\int L\Big(\frac{2\theta x s_i}{\sqrt{N}c}\Big)d\nu(x). \quad (22)$$

By Lemma 3.3, we obtain

$$\frac{1}{N}\log F_{c_1,c_2,c_3}^N(\theta,\delta) - \max_{\substack{k\leq 3\sqrt{N}/K\\ l\leq 3\sqrt{N}/\delta}} \Psi_{k,l}^{(\delta)}(c_1,c_2,c_3) \leq \varepsilon_{\delta,K}(N) + o_{\delta K}(1) + o_K(1) + o_\delta(1), \quad (23)$$

and similarly for the lower bound, where $\Psi_{k,l}^{(\delta)}(c_1,c_2,c_3)$ is replaced by $\Psi_{k,l}^{(\delta/2)}(c_1,c_2,c_3)$. To prove the lower bound of Lemma 3.1, we can first write

$$\liminf_{N\to+\infty} F_N(\theta) \geq \liminf_{N\to+\infty}\frac{1}{N}\log F_{c_1,c_2,c_3}^N(\theta,2\delta).$$

Using (23), we deduce

$$\liminf_{N\to+\infty} F_N(\theta) \geq \liminf_{N\to+\infty} \max_{\substack{k\leq 3\sqrt{N}/K\\ l\leq 3\sqrt{N}/\delta}} \Psi_{k,l}^{(\delta)}(c_1,c_2,c_3) - o_{\delta K}(1) - o_K(1) - o_\delta(1).$$

To complete the proof of the lower bound, one can observe that

$$\Sigma_{2,2}(t,t) = \frac{1}{N}\sum_{i,j} L\Big(2\theta\frac{t_i t_j}{\sqrt{N}c}\Big) + o_N(1),$$

uniformly in $t_i \in I_2$ such that $|\sum_i t_i^2 - c_2 N| \leq \delta N$. Indeed, the diagonal terms are negligible in this case since

$$\frac{1}{N} \sum_{t_i \in J_2} \frac{t_i^4}{Nc^2} \leq \frac{K}{c\sqrt{N}} \frac{1}{N} \sum_{i \in J_2} t_i^2 = \varepsilon_K(N).$$

To complete the proof of the upper bound, we will use the exponential tightness of $||g||^2$ and of $||g||_{J_1}^{-2}$. More precisely, we claim that

$$\lim_{K \to +\infty} \limsup_{N \to +\infty} \frac{1}{N} \mathbb{P}(||g||^2 \geq KN, ||g||_{J_1}^2 \leq K^{-1}N) = -\infty. \tag{24}$$

Indeed, it is clear by Chernoff's inequality that for N large enough

$$\mathbb{P}(||g||^2 \geq KN) \leq Ce^{-CNK},$$

where C is a positive numerical constant. Whereas, using Lemma 3.2 and a union bound,

$$\mathbb{P}(||g||_{J_1}^2 \leq K^{-1}N) \leq \sum_{m \leq 3\sqrt{N}/\delta} \binom{N}{m} \mathbb{P}\big(\sum_{i=1}^{N-m} g_i^2 \leq K^{-1}N\big).$$

By Chernoff's inequality, we have for any $m \leq N$,

$$\mathbb{P}\big(\sum_{i=1}^{N-m} g_i^2 \leq K^{-1}N\big) \leq e^{-(N-m)\Lambda^*(K^{-1})},$$

where $\Lambda^*(x) = \frac{x}{2} - \frac{1}{4} - \frac{1}{2}\log(2x)$ for any $x > 0$. Since $\Lambda^*(K^{-1}) \to +\infty$ as $K \to +\infty$ and for any $m \leq 2\sqrt{N}/\delta$, the binomial $\binom{N}{m}$ is negligible in the exponential scale, we deduce the claim (24).

Using that $L(x) \leq Ax^2/2$ for any $x \in \mathbb{R}$, we have

$$\Sigma(g) \leq A\theta^2 N.$$

From the exponential tightness (24), we deduce that for K large enough

$$\mathbb{E} \exp(\Sigma(g)) \mathbb{1}_{\{\frac{1}{N}||g||^2 \geq K, \frac{1}{N}||g||_{J_1}^2 \leq K^{-1}\}} \leq 1. \tag{25}$$

Let $\mathcal{E}_K = \{\frac{1}{N}||g||^2 \geq K, \frac{1}{N}||g||_{J_1}^2 \leq K^{-1}\}$. We have

$$0 \leq \limsup_{N \to +\infty} F_N(\theta) \leq \max\Big(\limsup_{N \to +\infty} \frac{1}{N} \log \mathbb{E}\big(e^{\Sigma(g)} \mathbb{1}_{\mathcal{E}_K}\big), \limsup_{N \to +\infty} \frac{1}{N} \log \mathbb{E}\big(e^{\Sigma(g)} \mathbb{1}_{\mathcal{E}_K^c}\big)\Big)$$

Since we took K so that (25) holds and $F_N(\theta) \geq 0$ as X_N is centered, we deduce that

$$\limsup_{N \to +\infty} F_N(\theta) \leq \limsup_{N \to +\infty} \frac{1}{N} \log \mathbb{E}\big(e^{\Sigma(g)} \mathbb{1}_{\mathcal{E}_K}\big). \tag{26}$$

Let now $\mathcal{C}_\delta$ be a δ-net for the ℓ^∞-norm of the set

$$\{(c_1, c_2, c_3) \in \mathbb{R}_+^3 : c_1 + c_2 + c_3 \leq K, c_1 \geq K^{-1}\}.$$

As $|\mathcal{C}_\delta| = O(K/\delta^3)$ is independent of N, we have

$$\limsup_{N \to +\infty} F_N(\theta) \leq \max_{(c_1,c_2,c_3) \in \mathcal{C}_\delta} \limsup_{N \to +\infty} \frac{1}{N} \log F_{c_1,c_2,c_3}^N(\theta, \delta).$$

From (23), we deduce

$$\limsup_{N\to+\infty} F_N(\theta) \le \max_{\substack{c=c_1+c_2+c_3\\ c_1\ge K^{-1}}} \limsup_{N\to+\infty} \max_{\substack{k\le 3\sqrt{N}/K\\ l\le 3\sqrt{N}/\delta}} \Psi_{k,l}^{(\delta)}(c_1,c_2,c_3) + o_{\delta K}(1) + o_K(1) + o_\delta(1).$$

Taking now the limit as $\delta\to 0$ and $K\to+\infty$ such that $\delta K\to 0$, we obtain the upper bound of Lemma 3.1. □

We are now ready to prove Proposition 1.7. Building on Lemma 3.1, we show that we can optimize on the total norm c in order to simplify the variational problem.

Proof of Proposition 1.7 . We use the notation of Lemma 3.1. Observe that $F^N_{0,c_2,c_3}(\delta,K) = -\infty$ for any $c_2,c_3\ge 0$. Therefore, we may assume that $c=c_1+c_2+c_3>0$. We make the following changes of variables. For $i=1,2,3$, we set $\alpha_i=\frac{c_i}{c}$, and if $h_c : x\mapsto x/\sqrt{c}$, and $\nu\in\mathcal{P}(\mathbb{R})$, we set $\nu_1 = h_c\#\nu$ to be the push-forward of ν by h_c, defined for any bounded continuous function f to satisfy

$$\int f(x)d\nu_1(x) = \int f\Big(\frac{x}{\sqrt{c}}\Big)d\nu(x). \tag{27}$$

For any $\nu\in\mathcal{P}(I_1)$ such that $\int x^2 d\nu(x)=c_1$, we have $\int x^2 d\nu_1(x)=\alpha_1$ and

$$H(\nu|\gamma) = H(\nu)+\frac12 c_1+\frac12\log(2\pi) \text{ and } H(\nu)=\int \log\frac{d\nu}{dx}d\nu(x) = H(\nu_1)+\frac12\log c. \tag{28}$$

We obtain

$$\overline{F}(\theta) = \limsup_{\substack{\delta\to0,K\to+\infty\\ \delta K\to 0}} \sup_{\substack{\alpha_1+\alpha_2+\alpha_3=1\\ \alpha_i\ge0}} \sup_{c\ge0} \limsup_{N\to+\infty} \mathcal{F}^N_{\alpha_1,\alpha_2,\alpha_3,c}(\delta,K),$$

and similarly for $\underline{F}(\theta)$, where

$$\mathcal{F}^N_{\alpha_1,\alpha_2,\alpha_3,c}(\delta,K) = \sup_{\substack{s_i\in I_3\\ |\sum s_i^2-N\alpha_3|\le N\delta}} \sup_{\substack{t_i\in I_2\\ |\sum t_i^2-N\alpha_2|\le N\delta}} \Big\{\theta^2\big(\alpha_1^2+2\alpha_1\alpha_2+B\alpha_3^2\big)-\frac12 c+\frac12\log c$$
$$+\frac1N\sum_{i,j} L\Big(\frac{2\theta s_i t_j}{\sqrt N}\Big)+\frac{1}{2N}\sum_{i,j} L\Big(\frac{2\theta t_i t_j}{\sqrt N}\Big)+\sup_{\substack{\nu_1\in\mathcal{P}(h_c(I_1))\\ \int x^2 d\nu_1(x)=\alpha_1}}\big\{\Phi_1(\nu_1,s)-H(\nu_1)\big\}\Big\}-\frac12\log(2\pi),$$

Note that $h_c(I_1)=[-\sqrt{\delta}N^{1/4},\sqrt{\delta}N^{1/4}]$ is independent of c. Optimizing over c, and see that the maximum is achieved at $c=1$ for N large enough.

□

4. The large deviations close to the bulk

We prove in this section Proposition 1.16. By Theorem 1.13, the large deviation lower bound holds at every $x>2$ such that $I(x)=\bar{I}(x)\neq 0$ so that there exists $\theta\in\Theta_x$ which does not belong to any Θ_y for $y\neq x$. In the following lemma, we prove that if $\overline{F}(\theta)=\underline{F}(\theta)=\theta^2$ on a interval $(0,b)$ with $b>1/2$, then the large deviation lower bound holds in a neighborhood of 2 and the rate function I is equal to the one of the GOE.

Lemma 4.1. *If for $\theta \in \left(0, \frac{1}{2\varepsilon}\right)$, for some $\varepsilon \in (0,1)$, $\overline{F}(\theta) = \underline{F}(\theta) = \theta^2$, then for any $x \in [2, \varepsilon + \frac{1}{\varepsilon})$,*

$$\bar{I}(x) = I(x) = I_{GOE}(x).$$

As a consequence, $\bar{I}(x) > 0$ for any $x > 2$. Moreover, for $x \in [2, \varepsilon + \frac{1}{\varepsilon})$ the optimizer in I is taken at $\theta_x = 1/2G(x)$ and $\theta_x \notin \Theta_y$ for all $y \in [2, +\infty) \backslash \{x\}$.

Proof. As $\underline{F}(\theta) \geq \theta^2$ for any $\theta \geq 0$, we have that

$$\sup_{\theta \in [0, 1/2\varepsilon)} \left\{ J(\theta, x) - \theta^2 \right\} \leq I(x) \leq \bar{I}(x) \leq \sup_{\theta \geq 0} \left\{ J(\theta, x) - \theta^2 \right\}.$$

But if $x \in [2, \varepsilon + \frac{1}{\varepsilon})$,

$$I_{GOE}(x) = \sup_{\theta \geq 0} \left\{ J(\theta, x) - \theta^2 \right\}$$

is achieved at $\theta = 1/2G(x) \in (0, 1/2\varepsilon)$ since $G^{-1}(\varepsilon) = \varepsilon + 1/\varepsilon$. Therefore, if $x \in [2, 2\varepsilon + \frac{1}{2\varepsilon})$, then we obtain

$$I(x) = \bar{I}(x) = I_{GOE}(x).$$

The consequences are obvious as G is invertible on $[2, +\infty)$.

□

The result of Proposition 1.16 then follows from Lemma 4.1 and Lemma 1.9. We now study the convergence of the annealed spherical integrals for large values of θ, for which we need to make additional assumptions on μ.

5. Case where ψ is an increasing function

In this section we make the additional assumption that ψ is non-decreasing. This assumption is in particular satisfied in the sparse Gaussian case below.

Example 5.1 (Sparse Gaussian distribution)**.** *Let μ be the law of $\xi\Gamma$ where ξ is a Bernoulli variable of parameter $p \in (0,1)$ and Γ is a standard Gaussian random variable. In that case we have for any $x \in \mathbb{R}$,*

$$\psi(x) = \frac{\log[(1-p) + p\exp(x^2/2p)]}{x^2} = \int_0^1 \frac{t}{(1-p)\exp(-(xt)^2/2p) + p} dt$$

is increasing in x as the integral of increasing functions on $\mathbb{R}^+$.

5.1. Simplification of the variational problem.

We prove in this section that when ψ is increasing on $\mathbb{R}^+$, $\mathcal{C}_\mu = \mathbb{R}^+$ and we can simplify the limit $F(\theta)$ as follows.

Proposition 5.2. *For any $\theta \geq 0$, $\overline{F}(\theta) = \underline{F}(\theta) = F(\theta)$ where*

$$F(\theta) = \sup_{\alpha \in [0,1]} \sup_{\substack{\nu \in \mathcal{P}(\mathbb{R}) \\ \int x^2 d\nu(x) = \alpha}} \left\{ \theta^2 \alpha^2 + B\theta^2 (1-\alpha)^2 + \int L(2\theta\sqrt{1-\alpha} x) d\nu(x) - H(\nu) - \frac{1}{2}\log(2\pi) - \frac{1}{2} \right\}$$

Proof. With the notation of Proposition 1.7, we need to bound, for any $\delta, K > 0$, and $\alpha_1 + \alpha_2 + \alpha_3 = 1$, the quantity $\mathcal{F}^N_{\alpha_1,\alpha_2,\alpha_3}(\delta, K)$. Since ψ is non-decreasing on $\mathbb{R}^+$ and symmetric, we have for any $s \in I_3^k$ such that $|\sum_i s_i^2 - \alpha_3 N| \leq N\delta$, that for any $i \in$

$\{1,\dots,k\}$, $s_i \le \sqrt{(\alpha_3+\delta)N}$ so that for any $x\in\mathbb{R}$, $\psi(2\theta s_i x/\sqrt{N}) \le \psi(2\theta\sqrt{(\alpha_3+\delta)}x)$. Thus,

$$\sum_{i=1}^k \int L\Big(\frac{2\theta s_i x}{\sqrt{N}}\Big)d\nu_1(x) = \frac{4\theta^2}{N}\sum_{i=1}^k s_i^2 \int x^2\psi\Big(\frac{2\theta s_i x}{\sqrt{N}}\Big)d\nu_1(x)$$
$$\le \int L\Big(2\theta\sqrt{\alpha_3+\delta}x\Big)d\nu_1(x) = \int L\Big(2\theta\sqrt{\alpha_3}x\Big)d\nu_1(x) + o_\delta(1)$$

where we finally use that $L\circ\sqrt{.}$ is Lipschitz. On the other hand, since $L(x)\le Bx^2/2$ for any $x\ge 0$,

$$\frac{1}{N}\sum_{j=1}^l\sum_{i=1}^k L\Big(\frac{2\theta s_i t_j}{\sqrt{N}}\Big) + \frac{1}{2N}\sum_{i,j=1}^l L\Big(\frac{2\theta t_i t_j}{\sqrt{N}}\Big) \le \theta^2 B(\alpha_2^2 + 2\alpha_3\alpha_2) + o_\delta(1).$$

Therefore, we have the upper bound,

$$\overline{F}(\theta) \le \sup_{\alpha_1+\alpha_2+\alpha_3=1}\ \sup_{\substack{\nu_1\in\mathcal{P}(\mathbb{R})\\ \int x^2 d\nu_1(x)=\alpha_1}} \Big\{\theta^2\big(\alpha_1^2+2\alpha_1\alpha_2\big) + \theta^2 B(\alpha_3+\alpha_2)^2$$
$$+\int L(2\theta\sqrt{\alpha_3}x)d\nu_1(x) - H(\nu_1) - \frac{1}{2}\log(2\pi) - \frac{1}{2}\Big\}.$$

We can further simplify this optimization problem by showing that the assumption on the monotonicity of ψ entails that we can take $\alpha_2=0$ in the above RHS. Indeed, note that $\psi(0)=1/2$. Therefore, ψ is bounded below by $1/2$ everywhere. Hence, we deduce that

$$2\theta^2\alpha_1\alpha_2 + \int L(2\theta\sqrt{\alpha_3}x)d\nu_1(x) = 2\theta^2\alpha_1\alpha_2 + 4\theta^2\alpha_3\int x^2\psi(2\theta\sqrt{\alpha_3}x)d\nu_1(x)$$
$$\le 4\theta^2(\alpha_2+\alpha_3)\int x^2\psi(2\theta\sqrt{\alpha_2+\alpha_3}x)d\nu_1(x).$$

Thus, with the change of variables $\alpha_3+\alpha_2 \to \alpha_3$, we conclude that

$$\overline{F}(\theta) \le \sup_{\alpha_1+\alpha_3=1}\ \sup_{\substack{\nu_1\in\mathcal{P}(\mathbb{R})\\ \int x^2 d\nu_1(x)=\alpha_1}} \Big\{\theta^2\alpha_1^2 + \theta^2 B\alpha_3^2 + \int L(2\theta\sqrt{\alpha_3}x)d\nu_1(x) - H(\nu_1) - \frac{1}{2}\log(2\pi) - \frac{1}{2}\Big\}.$$

To prove that $\underline{F}(\theta)$ is bounded from below by the same quantity, we fix $\alpha_1,\alpha_2,\alpha_3$ such that $\alpha_1+\alpha_2+\alpha_3=1, \alpha_2=0$, and $\nu\in\mathcal{P}(\mathbb{R})$ such that $\int x^2 d\nu(x)=\alpha_1$. We take in the definition of $\mathcal{F}^N_{\alpha_1,\alpha_2,\alpha_3}(K,\delta)$, $k=1$, $s=\sqrt{\alpha_3 N}$, and $\nu_1 = h_\lambda \# \frac{1}{\nu(I_1)}\nu(.\cap I_1)$, with the notations of (27). We take λ such that $\int x^2 d\nu_1(x)=\alpha_1$, which goes to 1 as N goes to infinity. We have

$$\mathcal{F}^N_{\alpha_1,\alpha_2,\alpha_3}(K,\delta) \ge \theta^2\big(\alpha_1^2 + B\alpha_3^2\big) + \int L(2\theta\sqrt{\alpha_3}x)d\nu_1(x) - H(\nu_1) - \frac{1}{2}\log(2\pi) - \frac{1}{2}.$$

We deduce by monotone convergence and the fact that $L(\sqrt{.})$ is Lipschitz that

$$\underline{F}(\theta) \ge \theta^2\big(\alpha_1^2 + B\alpha_3\big) + \int L(2\theta\sqrt{\alpha_3}x)d\nu(x) - H(\nu) - \frac{1}{2}\log(2\pi) - \frac{1}{2}.$$

$\square$

We next compute the supremum over ν in the definition of F in Proposition 5.2. To this end, we denote by $G : [B/2, +\infty) \to \mathbb{R} \cup \{+\infty\}$ the function given by

$$\forall \zeta \in [B/2, +\infty),\ G(\zeta) = \log \int \exp(L(x) - \zeta x^2) dx\,. \tag{29}$$

Lemma 5.3. *Let*

$$l = - \lim_{\zeta \to B/2} G'(\zeta) \in (0, +\infty]. \tag{30}$$

For any $C \in (0, l)$, there exists a unique $\zeta \in (B/2, +\infty)$ solution to the equation

$$G'(\zeta) = -C\,.$$

It is denoted by ζ_C. For $C \geq l$, we set $\zeta_C = B/2$. Then,

$$\begin{aligned}\sup_{\substack{\nu \in \mathcal{P}(\mathbb{R}) \\ \int x^2 d\nu(x) = C}} \Big[\int L(x) d\nu(x) - H(\nu) \Big] &= \sup_{\substack{\nu \in \mathcal{P}(\mathbb{R}) \\ \int x^2 d\nu(x) \leq C}} \Big[\frac{BC}{2} + \int \Big(L(x) - \frac{B}{2} x^2 \Big) d\nu(x) - H(\nu) \Big] \\ &= C\zeta_C + G(\zeta_C)\end{aligned}$$

Note that if l is finite, then so is $G(B/2)$ since G is a convex function.

Proof. Define the function

$$\forall \nu \in \mathcal{P}(\mathbb{R}),\ E(\nu) = H(\nu) + \int \Big(\frac{B}{2} x^2 - L(x) \Big) d\nu(x).$$

We first show that

$$\inf_{\substack{\nu \in \mathcal{P}(\mathbb{R}) \\ \int x^2 d\nu(x) = C}} E(\nu) = \inf_{\substack{\nu \in \mathcal{P}(\mathbb{R}) \\ \int x^2 d\nu(x) \leq C}} E(\nu). \tag{31}$$

Clearly the RHS is bounded above by the LHS. To prove the equality, we therefore need to show that for any $\nu \in \mathcal{P}(\mathbb{R})$ such that $\int x^2 d\nu(x) \leq C$, there exists ν_ε such that $\int x^2 d\nu_\varepsilon(x) = C$, and

$$\lim_{\varepsilon \to 0} \nu_\varepsilon = \nu, \quad \text{and} \quad \lim_{\varepsilon \to 0} E(\nu_\varepsilon) \geq E(\nu).$$

We set $\nu_\varepsilon = (1-\varepsilon)\nu + \varepsilon \gamma_\varepsilon$ where γ_ε is a Gaussian measure of variance 1 and mean m_ε, given, if $D = \int x^2 d\nu(x)$ by

$$m_\varepsilon = \sqrt{\frac{C - (1-\varepsilon)D}{\varepsilon} - 1}\,.$$

With this choice of m_ε, one can check that $\int x^2 d\nu_\varepsilon(x) = C$. Moreover, ν_ε converges weakly to ν as ε goes to zero. As $\int x^2 d\nu_\varepsilon(x) \leq C$, and $(\frac{B}{2} x^2 - L(x))/x^2$ goes to zero as x goes to infinity, we deduce that

$$\lim_{\varepsilon \to 0^+} \int \Big(\frac{B}{2} x^2 - L(x) \Big) d\nu_\varepsilon(x) = \int \Big(\frac{B}{2} x^2 - L(x) \Big) d\nu(x). \tag{32}$$

Besides, as H is lower semi-continuous,

$$\liminf_{\varepsilon \to 0^+} H(\nu_\varepsilon) \geq H(\nu).$$

We conclude together with (32) that, $\lim_{\varepsilon \to 0^+} E(\nu_\varepsilon) \geq E(\nu)$, which ends the proof of the claim (31). Observe that E is a lower semi-continuous function for the weak topology since H is lower semi-continuous and $x \mapsto Bx^2/2 - L(x)$ is non-negative and continuous. Moreover, the set

$$\Big\{\nu \in \mathcal{P}(\mathbb{R}) : \int x^2 d\nu(x) \leq C\Big\},$$

is a compact set. Thus, the supremum of E over the set above is achieved. We will identify the maximizer. For any $\zeta \in [B/2, +\infty)$ such that $G(\zeta) < +\infty$, we let ν_ζ be the probability measure given by

$$d\nu_\zeta = \frac{\exp(L(x) - \zeta x^2)}{\int \exp(L(y) - \zeta y^2) dy} dx.$$

We will show that

$$\inf_{\substack{\nu \in \mathcal{P}(\mathbb{R}) \\ \int x^2 d\nu(x) \leq C}} E(\nu) = E(\nu_{\zeta_C}). \tag{33}$$

Let μ be a probability measure such that $H(\mu) < +\infty$ and $\int x^2 d\mu(x) \leq C$. As $H(\mu) < +\infty$, we can write,

$$\mu = (1 + \varphi) d\nu_{\zeta_C},$$

where φ is some measurable function such that $\varphi \geq -1$ ν_{ζ_C}-a.s. and $\int \varphi d\nu_{\zeta_C} = 0$. We have

$$E(\mu) = E(\nu_{\zeta_C}) + \int \Big(\frac{B}{2}x^2 - L(x)\Big)\varphi(x) d\nu_{\zeta_C}(x)$$
$$+ \int (1+\varphi)\log(1+\varphi) d\nu_{\zeta_C} + \int \log \frac{d\nu_{\zeta_C}}{dx} \varphi d\nu_{\zeta_C}.$$

By convexity of $x \mapsto x \log x$, we know that

$$\int (1+\varphi)\log(1+\varphi) d\nu_{\zeta_C} \geq (1 + \int \varphi d\nu_{\zeta_C}) \log(1 + \int \varphi d\nu_{\zeta_C}) = 0.$$

Therefore, using again that $\int \varphi d\nu_{\zeta_C} = 0$ to cancel the contribution from the partition function of ν_{ζ_C}, we get

$$E(\mu) - E(\nu_{\zeta_C}) \geq \int \Big(\frac{B}{2}x^2 - L(x) + \log \frac{d\nu_{\zeta_C}}{dx}\Big) \varphi(x) d\nu_{\zeta_C}(x)$$
$$= \Big(\frac{B}{2} - \zeta_C\Big) \int x^2 \varphi(x) d\nu_{\zeta_C}(x) = \Big(\frac{B}{2} - \zeta_C\Big)\Big(\int x^2 d\mu(x) - \int x^2 d\nu_{\zeta_C}(x)\Big).$$

If $C < l$, then $\int x^2 d\nu_{\zeta_C}(x) = C$. Since $\zeta_C \leq B/2$ and $\int x^2 d\mu(x) \leq C$ so that the RHS is non negative. If $C \geq l$, then $\zeta_C = B/2$, and we also get $E(\mu) \geq E(\nu_{\zeta_C})$. This shows that ν_{ζ_C} achieves the infimum in (33), and ends the proof of Lemma 5.3. □

5.2. **Differentiability of the limit of the annealed spherical integral.** This section is devoted to the proof of the following proposition.

Proposition 5.4. *F is continuously differentiable on $(1/\sqrt{B-1}, +\infty)$ except possibly at the point θ_0 such that*

$$\theta_0 = \inf\big\{\theta : F(\theta) > \theta^2\big\}.$$

Moreover, for any $\theta \leq 1/2\sqrt{B-1}$,

$$F(\theta) = \theta^2. \tag{34}$$

The second part of the claim of the above proposition (34) is due to Proposition 1.9 and the fact that $A = B$. From now on, we assume that $\theta^2(B-1) > 1$ and wish to prove the first part of Proposition 5.4. We define for any $\alpha \in [0,1]$, and $\nu \in \mathcal{P}(\mathbb{R})$,

$$H_\theta(\alpha,\nu) = \theta^2\alpha^2 + \theta^2 B(1-\alpha)^2 + \int L(2\theta\sqrt{1-\alpha}x)d\nu(x) - H(\nu) - \frac{1}{2}\log(2\pi) - \frac{1}{2}. \quad (35)$$

By Proposition 5.2, we have

$$F(\theta) = \sup_{(\alpha,\nu)\in S} H_\theta(\alpha,\nu), \quad (36)$$

where $S = \{(\alpha,\nu) \in [0,1] \times \mathcal{P}(\mathbb{R}) : \int x^2 d\nu(x) = \alpha\}$. We first show that we can can restrict the parameter α to the set $[0, \frac{1}{2}] \cup \{1\}$, as described in the following lemma.

Lemma 5.5. *If* $\theta^2(B-1) \geq 1$*, then*

$$F(\theta) = \max\Big(\sup_{\substack{(\alpha,\nu)\in S \\ \alpha \leq \frac{1}{2}}} H_\theta(\alpha,\nu), \theta^2\Big).$$

Proof. Up to replace ν by $h_\alpha\#\nu$, where $h_\alpha : x \mapsto x/\sqrt{\alpha}$,

$$\sup_{(\alpha,\nu)\in S} H_\theta(\alpha,\nu) = \sup_{\substack{\alpha\in(0,1] \\ \int x^2 d\nu = 1}} \{\tilde{H}_\theta(\alpha,\nu) - H(\nu)\},$$

where for any $\alpha \in (0,1]$, and $\nu \in \mathcal{P}(\mathbb{R})$,

$$\tilde{H}_\theta(\alpha,\nu) = \theta^2\alpha^2 + \theta^2 B(1-\alpha)^2 + \frac{1}{2}\log\alpha + \int L(2\theta\sqrt{(1-\alpha)\alpha}x)d\nu(x) - \frac{1}{2}\log(2\pi) - \frac{1}{2}.$$

We claim that for any $\nu \in \mathcal{P}(\mathbb{R})$ such that $\int x^2 d\nu(x) = 1$,

$$\max_{\alpha\in[1/2,1]} \tilde{H}_\theta(\alpha,\nu) \leq \max\Big(\tilde{H}_\theta\Big(\frac{1}{2},\nu\Big), \tilde{H}_\theta(1,\nu)\Big). \quad (37)$$

Indeed, first notice that since ψ is increasing, for all $\alpha \in [0,1]$ we have,

$$\begin{aligned} \int L(2\theta\sqrt{(1-\alpha)\alpha}x)d\nu(x) &= 4\theta^2\alpha(1-\alpha)\int x^2\psi(2\theta\sqrt{\alpha(1-\alpha)}x)d\nu(x) \\ &\leq 4\theta^2\alpha(1-\alpha)\int x^2\psi(\theta x)d\nu(x)\,. \end{aligned}$$

Denote by $m = 2\int x^2\psi(\theta x)d\nu(x) \in [1,B]$. For any $\alpha \in (0,1]$,

$$\tilde{H}_\theta(\alpha,\nu) \leq \theta^2\alpha^2 + \theta^2 B(1-\alpha)^2 + \frac{1}{2}\log\alpha + 2\theta^2\alpha(1-\alpha)m - \frac{1}{2}\log(2\pi) - \frac{1}{2} =: f_{\theta,m}(\alpha)\,.$$

We find that

$$f'_{\theta,m}\Big(\frac{1}{2}\Big) = \theta^2(1-B) + 1, \quad f''_{\theta,m}(\alpha) = 2\theta^2(B+1-2m) - \frac{1}{2\alpha^2}.$$

Since $f''_{\theta,m}$ is increasing and $f''_{\theta,m}(0) = -\infty$, we deduce that $f'_{\theta,m}$ is either decreasing or decreasing and then increasing. Since $f'_{\theta,m}(1/2) \leq 0$, we conclude that $f_{\theta,m}$ is either decreasing or decreasing and then increasing on $[1/2,1]$. Therefore,

$$\max_{\alpha\in[\frac{1}{2},1]} f_{\theta,m}(\alpha) = \max\Big(f_{\theta,m}\Big(\frac{1}{2}\Big), f_{\theta,m}(1)\Big)\,,$$

which yields the claim (37) since $f_{\theta,m}(\alpha) = \tilde{H}_\theta(\alpha,\nu)$ at the two points $\alpha = 1/2$ and 1. To conclude the proof we observe that since $\tilde{H}_\theta(1,\nu) = \theta^2$ for any $\nu \in \mathcal{P}(\mathbb{R})$, we have

$$\sup_{\int x^2 d\nu(x)=1} \{\tilde{H}_\theta(1,\nu) - H(\nu)\} = \theta^2 + \frac{1}{2} + \frac{1}{2}\log(2\pi) - \inf_{\int x^2 d\nu(x)=1} H(\nu) = \theta^2.$$

□

Due to Lemma 5.3, we can further simplify the optimization problem defining $F(\theta)$ in (36) by optimizing on $\nu \in \mathcal{P}(\mathbb{R})$ such that $\int x^2 d\nu(x) = \alpha$, given $\alpha \in (0,1)$.

Corollary 5.6. *Let R be the function*

$$R : C \in (0,+\infty) \mapsto C\zeta_C + G(\zeta_C),$$

where ζ_C is defined as in Lemma 5.3. Denote for any $\alpha \in (0,1)$,

$$K_\theta(\alpha) = \theta^2(\alpha^2 + B(1-\alpha)^2) + R(4\theta^2\alpha(1-\alpha)) - \frac{1}{2}\log(1-\alpha) - \log(2\theta) - \frac{1}{2}\log(2\pi) - \frac{1}{2},$$

and $K_\theta(1) = \theta^2$. Then, for any $\theta \geq 1/\sqrt{B-1}$,

$$F(\theta) = \sup_{\alpha\in(0,1]} K_\theta(\alpha)\,.$$

Proof. When $\alpha < 1$, we make the following change of variables which consists in replacing ν by its pushforward by $x \mapsto 2\theta\sqrt{1-\alpha}x$. Using (28), we find that

$$H(\nu) = \int \log\frac{d\nu}{dx} d\nu = H(\nu_1) - \frac{1}{2}\log(1-\alpha) - \log(2\theta)$$

and $\int x^2 d\nu(x) = 4\alpha(1-\alpha)\theta^2$. Thus,

$$F(\theta) = \max\Big(\theta^2, \sup_{(\alpha,\nu)\in S'} K_\theta(\alpha,\nu)\Big), \tag{38}$$

where $S' = \big\{(\alpha,\nu) \in (0,1)\times\mathcal{P}(\mathbb{R}) : \int x^2 d\nu(x) = 4\alpha(1-\alpha)\theta^2\big\}$, and for any $\alpha \in (0,1), \nu \in \mathcal{P}(\mathbb{R})$,

$$K_\theta(\alpha,\nu) = \theta^2(\alpha^2 + B(1-\alpha)^2) + \int L(x)d\nu(x) - H(\nu) - \frac{1}{2}\log(1-\alpha) - \log(2\theta) - \frac{1}{2}\log(2\pi e).$$

By Lemma 5.3, we obtain for any $\alpha \in (0,1)$,

$$\begin{aligned}\sup_{\int x^2 d\nu(x)=4\alpha(1-\alpha)\theta^2} K_\theta(\alpha,\nu) &= \theta^2(\alpha^2 + B(1-\alpha)^2) + 4\theta^2\alpha(1-\alpha)\zeta_{\alpha,\theta} + G(\zeta_{\alpha,\theta}) \\ &\quad - \frac{1}{2}\log(1-\alpha) - \log(2\theta) - \frac{1}{2}\log(2\pi) - \frac{1}{2},\end{aligned}$$

where $\zeta_{\alpha,\theta} = \zeta_{4\theta^2\alpha(1-\alpha)}$. Hence, if we set, for $\alpha \in (0,1)$,

$$K_\theta(\alpha) = \theta^2(\alpha^2 + B(1-\alpha)^2) + R(4\theta^2\alpha(1-\alpha)) - \frac{1}{2}\log(1-\alpha) - \log(2\theta) - \frac{1}{2}\log(2\pi) - \frac{1}{2},$$

we deduce from (38) that

$$F(\theta) = \max\Big(\theta^2, \sup_{\alpha\in(0,1)} K_\theta(\alpha)\Big). \tag{39}$$

To study the supremum of K_θ, we will need the following result on the limit of R at 0, which will allow us to compute the limit of K_θ at 1.

Lemma 5.7. *When $C \to 0^+$,*

$$R(C) = \frac{1}{2} + \frac{1}{2}\log(2\pi C) + o(1).$$

Proof. For $C < l$, we have

$$G'(\zeta_C) = -C. \tag{40}$$

Since $G'(C)$ goes to zero when C goes to infinity, and G' is invertible as $G''(x) > 0$ on $(-\infty, l)$, we find that

$$\lim_{C\to 0} \zeta_C = +\infty.$$

From the inequalities, $L(x)/x^2 \in [0, B/2]$, we deduce that by (29) of G we have the bounds

$$\frac{1}{2}\log\frac{\pi}{\zeta} \le G(\zeta) \le \frac{1}{2}\log\frac{\pi}{\zeta - \frac{B}{2}},$$

which implies that

$$G(\zeta) \sim_{+\infty} \frac{1}{2}\log\frac{\pi}{\zeta}. \tag{41}$$

On the other hand, inserting the bound $L(x)/x^2 \in [0, B/2]$ and (41) in the numerator and the denominator of the derivative, we obtain

$$\sqrt{\frac{\zeta - \frac{B}{2}}{\zeta}}\frac{1}{2\zeta} \le -G'(\zeta) \le \sqrt{\frac{\zeta}{\zeta - \frac{B}{2}}}\frac{1}{2(\zeta - \frac{B}{2})}.$$

We deduce, since $\zeta_C \to +\infty$ as $C \to 0$, that

$$G'(\zeta_C) = -\frac{1}{2\zeta_C} + o\Big(\frac{1}{\zeta_C}\Big).$$

Therefore, we get from the definition of ζ_C (40) that ζ_C is equivalent to $\frac{1}{2C}$ when C goes to zero. Using (41), we can conclude that

$$R(C) = \frac{1}{2} + o(1) + \frac{1}{2}\log\frac{\pi}{\frac{1}{2C} + o\Big(\frac{1}{2C}\Big)} = \frac{1}{2} + \frac{1}{2}\log(2\pi C) + o(1).$$

$\square$

From Lemma 5.7, we deduce that

$$\lim_{\alpha\to 1} K_\theta(\alpha) = \theta^2, \tag{42}$$

so that we can continuously extend K_θ to 1. Therefore, (38) gives

$$F(\theta) = \sup_{\alpha\in(0,1]} K_\theta(\alpha).$$

$\square$

We now study K_θ and show that it is continuously differentiable on $(0,1)$. This amounts to prove that R is continuously differentiable on $(0,1)$. On $(0,l)$, it is clear that R is continuously differentiable due to the implicit function theorem. Indeed, ζ_C is by definition the unique solution of the equation

$$G'(\zeta) = -C,$$

and G is strictly convex. On $(l,+\infty)$, $R(C) = BC/2 + G(B/2)$ is an affine function, therefore it is sufficient to prove that

$$\lim_{C\to l^-} R'(C) = \frac{B}{2}. \tag{43}$$

This is a consequence of the fact that for any $C < l$,

$$R'(C) = C\partial\zeta_C + \zeta_C + \partial\zeta_C G'(\zeta_C) = \zeta_C,$$

which gives (43). We deduce that K_θ is continuously differentiable on $(0,1)$ and

$$\forall \alpha \in (0,1),\ K'_\theta(\alpha) = 2\theta^2(\alpha + B(\alpha-1)) + 4\theta^2\zeta_{\alpha,\theta}(1-2\alpha) + \frac{1}{2(1-\alpha)}.$$

From Lemma 5.7, we know that K_θ goes to $-\infty$ when α goes to zero so that the supremum of K_θ on $(0,1]$ is achieved either at 1 or on $(0,1)$. From Lemma 5.5, we find for $\theta^2(B-1) \geq 1$,

$$F(\theta) = \max\big(K_\theta(1), \sup_{\alpha\leq\frac{1}{2}} K_\theta(\alpha)\big).$$

Let us assume that the maximum of K_θ is achieved on $(0,1)$. We deduce that the maximum of K_θ is achieved on $(0,\frac{1}{2}]$ at a critical point since K_θ is differentiable. The critical points α of K_θ satisfy the equation,

$$2\theta^2(\alpha + B(\alpha-1)) + 4\theta^2\zeta_{\alpha,\theta}(1-2\alpha) + \frac{1}{2(1-\alpha)} = 0, \tag{44}$$

As $\theta^2(B-1) > 1$, $1/2$ does not satisfy the above equation so that the critical points of K_θ are the $\alpha \neq 1/2$, such that

$$\zeta_{\alpha,\theta} = \frac{2\theta^2(\alpha + B(\alpha-1)) + \frac{1}{2(1-\alpha)}}{4\theta^2(2\alpha-1)} := \varphi(\alpha). \tag{45}$$

We find that

$$\varphi(\alpha) \geq \frac{B}{2} \iff \begin{cases} P(\alpha) \geq 0 & \text{if } \alpha \geq \frac{1}{2} \\ P(\alpha) \leq 0 & \text{if } \alpha \leq \frac{1}{2}, \end{cases}$$

with $P(\alpha) = 4\theta^2(B-1)\alpha^2 - 4\theta^2(B-1)\alpha + 1$. As $\zeta_{\alpha,\theta} \geq B/2$, we obtain that the maximum of K_θ is achieved at $\alpha \in (0,1/2)$ such that $P(\alpha) \leq 0$. The roots of P are

$$\alpha_\pm = \frac{1 \pm \sqrt{1 - [\theta^2(B-1)]^{-1}}}{2}. \tag{46}$$

Thus, the maximum of K_θ is achieved on $[\alpha_-, 1/2]$. We will show that K_θ is strictly concave on $(0,\frac{1}{2})$. Note that,

$$4\theta^2\alpha(1-\alpha) \geq l \iff \alpha \in [\beta_-, \beta_+],$$

with

$$\beta_\pm = \frac{1 \pm \sqrt{1 - l\theta^{-2}}}{2}.$$

For any $\alpha \in (\beta_-, \frac{1}{2})$, we must have $\zeta_{\alpha,\theta} = B/2$ and therefore

$$K_\theta(\alpha) = \theta^2(\alpha^2 + B(1-\alpha)^2) + 2B\theta^2\alpha(1-\alpha) + G\Big(\frac{B}{2}\Big) - \frac{1}{2}\log(1-\alpha) + C_\theta,$$

where C_θ is some constant depending on θ. Thus, for $\alpha \in (\beta_-, \frac{1}{2})$,

$$K''_\theta(\alpha) = 2\theta^2(1-B) + \frac{1}{2(1-\alpha)^2} < 2\theta^2(1-B) + 2 < 0.$$

For any $\alpha \in (0, \beta_-)$, we have

$$\begin{aligned} K_\theta(\alpha) &= \theta^2(\alpha^2 + B(1-\alpha)^2) + 4\theta^2\alpha(1-\alpha)\zeta_{\alpha,\theta} + G(\zeta_{\alpha,\theta}) \\ &\quad - \frac{1}{2}\log(1-\alpha) + C_\theta, \end{aligned}$$

where $\zeta_{\alpha,\theta}$ is such that

$$G'(\zeta_{\alpha,\theta}) = -4\theta^2\alpha(1-\alpha).$$

As G is strictly convex, we deduce by the implicit function theorem that $\alpha \in (0, \beta_-) \mapsto \zeta_{\alpha,\theta}$ is differentiable, and we have

$$\partial_\alpha\zeta_{\alpha,\theta}G''(\zeta_{\alpha,\theta}) = -4\theta^2(1-2\alpha).$$

We deduce that $\partial_\alpha\zeta_{\alpha,\theta} < 0$, for any $\alpha \in (0, \beta_-)$. Therefore, for $\alpha \in (0, \beta_-)$, we obtain

$$K''_\theta(\alpha) = 2\theta^2(B+1) - 8\theta^2\zeta_{\alpha,\theta} + 4\theta^2\partial_\alpha\zeta_{\alpha,\theta}(1-2\alpha) + \frac{1}{2(1-\alpha)^2}.$$

Using that $\zeta_{\alpha,\theta} > B/2$ and that $\partial_\alpha\zeta_{\alpha,\theta} < 0$ for $\alpha \in (0, \beta_-)$, we find that

$$\forall \alpha \in (0, \beta_-),\ K''_\theta(\alpha) \le 2\theta^2(B+1) - 4\theta^2 B + \frac{1}{2(1-\alpha)^2}, 2\theta^2(1-B) + 2 \le 0.$$

Thus, K'_θ is decreasing on $(0, \beta_-)$ and $(\beta_-, \frac{1}{2})$. Since K'_θ is continuous, we deduce that K'_θ is decreasing on $(0, \frac{1}{2})$ and K_θ is strictly concave on $(0, \frac{1}{2})$. Therefore, the maximum is achieved at the unique critical point of K_θ on $(0, \frac{1}{2})$ which we denote by α_θ. We distinguish two cases.

1^{st} *case:* $l \le \frac{1}{B-1}$. We then have $\beta_- \le \alpha_- \le \alpha_+ \le \beta_+$. We know that on one hand $P(\alpha_-) = 0$, so that $\varphi(\alpha_-) = \frac{B}{2}$. On the other hand $\zeta_{\alpha_-,\theta} = B/2$ since $\alpha_- \in [\beta_-, \beta_+]$. We deduce by (45) that α_- is a critical point of K_θ which lies in $(0, \frac{1}{2})$. Therefore $\alpha_\theta = \alpha_-$.

2^{nd} *case:* $l > \frac{1}{B-1}$. We have, $\alpha_- < \beta_- < \beta_+ < \alpha_+$. Note that $0 \leq \alpha_- < \frac{1}{2} < \alpha_+ \leq 1$. Since $\varphi(\alpha) \neq B/2$ for any $\alpha \in [\beta_-, \beta_+]^c$, we deduce that $\alpha_\theta \in [\alpha_-, \beta_-)$, and in particular $K''_\theta(\alpha_\theta) < 0$. We deduce by the implicit function theorem that $\theta \mapsto \alpha_\theta$ is C^1, and therefore $\theta \mapsto K_\theta(\alpha_\theta)$ is continuously differentiable on $(1/\sqrt{B-1}, +\infty)$.

In conclusion, we have shown that for any $\theta^2(B-1) \geq 1$, if $l \leq \frac{1}{B-1}$,

$$F(\theta) = \max\big(\theta^2, K_\theta(\alpha_-)\big),$$

where α_- is defined in (46), whereas if $l \geq \frac{1}{B-1}$,

$$F(\theta) = \max\big(\theta^2, K_\theta(\alpha_\theta)\big),$$

where α_θ is the unique solution in $(0, \beta_-)$ such that $G'(\zeta_{\alpha,\theta}) = -4\theta^2\alpha(1-\alpha)$.

To conclude that F is continuously differentiable on $(1/\sqrt{B-1}, +\infty)$ except at most at one point, we show that there exists θ_0 such that

$$\forall \theta \leq \theta_0,\ F(\theta) = \theta^2, \text{ and } \forall \theta > \theta_0,\ F(\theta) > \theta^2.$$

Since $F(\theta) \geq \theta^2$ for any $\theta \geq 0$, it suffices to prove that $\theta \mapsto F(\theta) - \theta^2$ is non-decreasing. Recall that

$$F(\theta) = \lim_{N \to +\infty} F_N(\theta),$$

where

$$F_N(\theta) = \frac{1}{N} \log \mathbb{E}_e \exp\Big(\sum_i L\big(\sqrt{2N}\theta e_i^2\big) + \sum_{i<j} L\big(2\sqrt{N}\theta e_i e_j\big)\Big),$$

and e is uniformly sampled on $\mathbb{S}^{N-1}$. Therefore,

$$F_N(\theta) - \theta^2 = \frac{1}{N} \log \mathbb{E}_e \exp\Big(\sum_i 2N\theta^2\Big(\psi\big(\sqrt{2N}\theta e_i^2\big) - \frac{1}{2}\Big)e_i^4 + \sum_{i<j} 4N\theta^2\Big(\psi\big(2\sqrt{N}\theta e_i e_j\big) - \frac{1}{2}\Big)e_i^2 e_j^2\Big).$$

As ψ is increasing and $\psi(0) = 1/2$, $\theta \mapsto F_N(\theta) - \theta^2$ is non-decreasing, and therefore $\theta \mapsto F(\theta) - \theta^2$ is non-decreasing as well.

For the sake of completeness, we show the following Proposition which indicates that it is unlikely we could prove the large deviation principle for all values of x by following our strategy because F is in general not differentiable everywhere.

Proposition 5.8. *Assume* $\theta_0 = \inf\{\theta \in \mathbb{R}^+ : F(\theta) > \theta^2\} > 1/\sqrt{B-1}$. *Then,* F *is not differentiable at* θ_0.

Proof. Let $\theta > \theta_0$. Lemma 5.5 shows that with H_θ defined in (35) , we have

$$F(\theta) = \max_{\substack{\int x^2 d\nu(x) \leq \alpha \\ \alpha \leq \frac{1}{2}}} H_\theta(\alpha, \nu).$$

Since $\theta_0 \geq 1/\sqrt{B-1}$, we know from the proof of Proposition 5.4 that there exists $\alpha_0 \leq 1/2$ and $\nu_0 \in \mathcal{P}(\mathbb{R})$ such that

$$H_{\theta_0}(\alpha_0, \nu_0) = F(\theta_0).$$

Define $g(\theta) = H_\theta(\alpha_0, \nu_0)$ for any $\theta \geq \theta_0$. Let F'_+ denote the right derivative of F. We have as $F \geq g$ and $F(\theta_0) = g(\theta_0)$,

$$F'_+(\theta_0) \geq g'(\theta_0).$$

We find

$$g'(\theta_0) = 2\theta_0(\alpha_0^2 + B(1-\alpha_0)^2) + 2\sqrt{1-\alpha_0}\int xL'(2\theta_0\sqrt{1-\alpha_0}x)d\nu_0(x).$$

Since ψ is increasing, $xL'(x) \geq 2L(x)$, and $L(x) \geq x^2/2$ for any $x \geq 0$. Therefore, $xL'(x) \geq x^2$ and we deduce

$$\begin{aligned} g'(\theta_0) &\geq 2\theta_0(\alpha_0^2 + B(1-\alpha_0)^2) + 4\theta_0\alpha_0(1-\alpha_0) \\ &\geq 2\theta_0 + 2\theta_0(1-\alpha)^2(B-1). \end{aligned}$$

This shows that $g'(\theta_0) > 2\theta_0$ and therefore $F'_+(\theta) > 2\theta_0$. It yields that F is not differentiable at θ_0.

□

5.3. **Proof of Proposition 1.14.** By Proposition 5.4, we know that F is differentiable on $(1/\sqrt{B-1}, +\infty)$ except possibly at θ_0. Using Proposition 2.2 we deduce that there exists x_μ finite such that the lower large deviation lower bound holds with rate function $I(x) = \overline{I}(x)$ for any $x \geq x_\mu$.

6. The case $B < A$

We consider in this section the case where the following assumption holds.

Assumption 6.1. *B exists and is strictly smaller than A. Moreover, we assume that ψ achieves its maximum A at a unique point m_* such that $\psi''(m_*) < 0$.*

The first condition includes in particular the case where the law of the entries have a compact support (since in this case $B = 0$) and we believe the second condition is true quite generically, as we check in the following example.

Example 6.1. *Let*

$$\mu = \frac{p}{2}(\delta_{-1/\sqrt{p}} + \delta_{1/\sqrt{p}}) + (1-p)\delta_0, \qquad \psi(x) = \frac{1}{x^2}\log(p(\cosh(\frac{x}{\sqrt{p}}) - 1) + 1)\,.$$

Then, for $p < 1/3$, μ satisfies Assumption 6.1 (but for $p > 1/3$ μ has a sharp sub-Gaussian tail). Indeed, we have

$$\forall x \geq 0,\ \psi'(x) = \frac{L'(x)}{x^2} - \frac{2L(x)}{x^3},\ \psi''(x) = \frac{L''(x)}{x^2} - \frac{4L'(x)}{x^3} + \frac{6L(x)}{x^4}.$$

We claim that $h : x \mapsto xL'(x) - 2L(x)$ is increasing and then decreasing on $\mathbb{R}_+$. Indeed,

$$\forall x \geq 0,\ h'(x) = xL''(x) - L'(x),\ h''(x) = xL^{(3)}(x),$$

and we have,

$$L(x) = \log\left(p\cosh\left(\frac{x}{\sqrt{p}}\right) + 1 - p\right),\ L'(x) = \frac{\sqrt{p}\sinh\left(\frac{x}{\sqrt{p}}\right)}{p\cosh\left(\frac{x}{\sqrt{p}}\right) + 1 - p}.$$

$$L''(x) = \frac{p + (1-p)\cosh\left(\frac{x}{\sqrt{p}}\right)}{(p\cosh\left(\frac{x}{\sqrt{p}}\right) + 1 - p)^2},\ L^{(3)}(x) = \frac{\frac{(1-p)^2}{\sqrt{p}} - 2p\sqrt{p} - \sqrt{p}(1-p)\cosh\left(\frac{x}{\sqrt{p}}\right)}{(p\cosh\left(\frac{x}{\sqrt{p}}\right) + 1 - p)^3}\sinh\left(\frac{x}{\sqrt{p}}\right).$$

We have, for $p > p_* = 1/3$

$$\frac{(1-p)^2}{\sqrt{p}} - 2p\sqrt{p} < \sqrt{p}(1-p).$$

Therefore, $L^{(3)}$ is negative and therefore h' is decreasing. Since $h'(0) = 0$, we deduce that h' is negative and ψ is decreasing. If $p > p_*$, we have that h'' is positive and then negative. Therefore, h' is increasing on $[0, x_0]$ and then decreasing on $[x_0, +\infty)$, with $x_0 = \sqrt{p}\cosh^{-1}(\frac{1-2p-p^2}{p(1-p)})$. But,

$$h'(0) = 0, \ \lim_{x \to +\infty} h'(x) = -\frac{1}{\sqrt{p}},$$

as $L'(x) \sim_{+\infty} 1/\sqrt{p}$ and $L''(x) \sim_{+\infty} 2(1-1/p)e^{-x/\sqrt{p}}$. Therefore, there exists $m_* > x_0$ such that h' is positive on $(0, m_*)$ and negative on $(m_*, +\infty)$. We deduce that ψ is increasing on $(0, m_*)$ and decreasing on $(m_*, +\infty)$ so that ψ achieves its unique maximum at m_*. Moreover, $\psi''(m_*) < 0$. Indeed, otherwise we have

$$\begin{aligned}\psi'(m_*) = 0, \psi''(m_*) = 0 &\iff m_* L'(m_*) = 2L(m_*), \ m_*^2 L''(m_*) = 4m_* L'(m_*) - 6L(m_*) \\ &\iff L'(m_*) = m_* L''(m_*), \ m_* L'(m_*) = 2L(m_*).\end{aligned}$$

This implies that $h'(m_*) = 0$ which contradicts $m_* > x_0$ (and $L^{(3)} \neq 0$ on $[x_0, m_*]$. As $m_* > x_0$, we have that $h'(m_*) < 0$ and therefore $\psi''(m_*) < 0$.

Studying the variational problem arising from the limit of the annealed spherical integral $\overline{F}(\theta)$ and $\underline{F}(\theta)$ defined in Proposition 1.7, we will show that for θ large enough we can give an explicit formula as stated in the following proposition.

Proposition 6.2. *There exists $\theta_0 > 1/\sqrt{A-1}$ such that for any $\theta \geq \theta_0$, $\overline{F}(\theta) = \underline{F}(\theta) = F(\theta)$ where*

$$F(\theta) = \sup_{\alpha \in (0,1]} V(\alpha),$$

with

$$\forall \alpha > 0, \ V(\alpha) = \theta^2(A-1)\alpha^2 + \theta^2 + \frac{1}{2}\log(1-\alpha).$$

More explicitly,

$$F(\theta) = \frac{\theta^2}{4}(A-1)\Big(1 + \sqrt{1 - \frac{1}{\theta^2(A-1)}}\Big)^2 + \theta^2 + \frac{1}{2}\log\Big(1 - \sqrt{1 - \frac{1}{\theta^2(A-1)}}\Big) - \frac{1}{2}\log 2.$$

Given the above proposition is true, the result of Proposition 1.15 immediately follows from Proposition 2.2.

We prove this proposition by first showing that $\underline{F}(\theta) \geq F(\theta)$ for all θ and then that, for large θ, $\overline{F}(\theta) \leq F(\theta)$.

6.1. **Proof of the lower bound.** Recall that by Proposition 1.7, we have the following formulation of the limit $\underline{F}(\theta)$.

$$\underline{F}(\theta) = \sup_{\substack{\alpha_1+\alpha_2+\alpha_3=1\\ \alpha_i\geq 0}} \liminf_{\substack{\delta\to 0, K\to+\infty\\ \delta K\to 0}} \limsup_{N\to+\infty} \mathcal{F}^N_{\alpha_1,\alpha_2,\alpha_3}(\delta,K)\,,$$

where

$$\begin{aligned}\mathcal{F}^N_{\alpha_1,\alpha_2,\alpha_3}(\delta,K) &= \theta^2\big(\alpha_1^2+2\alpha_1\alpha_2+B\alpha_3^2\big)\\ &\quad+\sup_{\substack{t_i\in I_2, i\leq l\\ |\sum_i t_i^2-N\alpha_2|\leq\delta N}} \sup_{\substack{s_i\in I_3, i\leq k\\ |\sum_i s_i^2-N\alpha_3|\leq\delta N}} \Big\{\frac{1}{N}\sum_{i=1}^k\sum_{j=1}^l L\Big(\frac{2\theta s_i t_j}{\sqrt{N}}\Big)+\frac{1}{2N}\sum_{i,j=1}^l L\Big(\frac{2\theta t_i t_j}{\sqrt{N}}\Big)\\ &\quad+\sup_{\substack{\nu_1\in\mathcal{P}(I_1)\\ \int x^2 d\nu_1(x)=\alpha_1}}\Big\{\sum_{i=1}^k\int L\Big(\frac{2\theta s_i x}{\sqrt{N}}\Big)d\nu_1(x)-H(\nu_1)\Big\}-\frac{1}{2}\log(2\pi)-\frac{1}{2}\Big\},\end{aligned}$$

Our goal is to show that we can take $\alpha_3=0$ and in the supremum defining $\mathcal{F}^N_{\alpha_1,\alpha_2,\alpha_3}(\delta,K)$ we can take all the t_i's equal. In fact we first prove the lower bound:

Lemma 6.3. *For any $\theta\geq 0$,*

$$\underline{F}(\theta)\geq \sup_{\alpha\in(0,1]} V(\alpha),$$

where V is defined in Proposition 6.2.

Proof. Indeed, if we take $\alpha_3=0$ and $t_j=N^{1/4}\sqrt{\frac{m_*}{2\theta}}, 1\leq j\leq l$, $\alpha_2\in[lm_*/2\theta\sqrt{N}-\delta, lm_*/2\theta\sqrt{N}+\delta]$, $\alpha_1=1-\alpha_2$, ν_1 to be the Gaussian law restricted to I_1 with variance α_1, then we get the lower bound

$$\mathcal{F}^N_{\alpha_1,\alpha_2,0}(\delta,K)\geq\theta^2(\alpha_1^2+2\alpha_1\alpha_2+\alpha_2^2 A)+\frac{1}{2}\log\alpha_1=V(\alpha_2)\,.$$

Hence, to derive the lower bound it is enough to remark that we can achieve any possible value of α_2 in $[0,1]$ as some large N limit of $l_N m_*/2\theta\sqrt{N}$ for some sequence of integer numbers l_N, which is obvious. □

6.2. **Proof of the upper bound.** The rest of this section is devoted to prove that the previous lower bound is sharp when θ is large enough. To this end, recall that by Proposition 1.7, we have the following formulation of the limit $\overline{F}(\theta)$.

$$\overline{F}(\theta) = \sup_{\substack{\alpha_1+\alpha_2+\alpha_3=1\\ \alpha_i\geq 0}} \limsup_{\substack{\delta\to 0, K\to+\infty\\ \delta K\to 0}} \limsup_{N\to+\infty} \mathcal{F}^N_{\alpha_1,\alpha_2,\alpha_3}(\delta,K)\,.$$

We first reformulate the supremum in $\mathcal{F}^N_{\alpha_1,\alpha_2,\alpha_3}(\delta,K)$ by denoting for $t\in I_2^l$ so that $|\sum t_i^2-N\alpha_2|\leq\delta N$,

$$\mu_2=\frac{1}{\alpha_2 N}\sum_{i=1}^l t_i^2\delta_{\frac{\sqrt{2\theta}t_i}{N^{1/4}}}\,.$$

μ_2 is a positive measure on $S_2=\{x:\sqrt{2\delta\theta}\leq|x|\leq\sqrt{2K\theta}\}$ whose total mass belongs to $[1-\frac{\delta}{\alpha_2},1+\frac{\delta}{\alpha_2}]$. We also denote by $S_3=\{x:\sqrt{K}\leq|x|\leq N^{1/4}\sqrt{\alpha_3}\}$. Then it is not hard to see that for any $\theta\geq 0$,

$$\overline{F}(\theta)\leq\hat{F}(\theta), \tag{47}$$

where $\hat{F}(\theta)$ is defined by

$$\hat{F}(\theta) = \sup_{\substack{\alpha_1+\alpha_2+\alpha_3=1\\ \alpha_i\geq 0}} \limsup_{\substack{\delta\to 0, K\to+\infty\\ \delta K\to 0}} \limsup_{N\to+\infty} \sup_{\mu_2\in\mathcal{P}(S_2)} \sup_{s\in S_3} \mathcal{G}^N_{\alpha_1,\alpha_2,\alpha_3}(\delta, K, s, \mu_2)$$

if

$$\begin{aligned}\mathcal{G}^N_{\alpha_1,\alpha_2,\alpha_3}(\delta, K, s, \mu_2) &= \theta^2\big(\alpha_1^2 + 2\alpha_1\alpha_2 + B\alpha_3^2\big)\\ &\quad + 4\theta^2\alpha_3\alpha_2\int\psi\big(\sqrt{2\theta}sx\big)d\mu_2(x) + 2\theta^2\alpha_2^2\int\psi(xy)d\mu_2(x)d\mu_2(y)\\ &\quad + \sup_{\substack{\nu_1\in\mathcal{P}(I_1)\\ \int x^2 d\nu_1(x)=\alpha_1}}\Big\{4\theta^2\alpha_3\int x^2\psi\Big(\frac{2\theta sx}{N^{\frac14}}\Big)d\nu_1(x) - H(\nu_1)\Big\} - \frac12\log(2\pi) - \frac12.\end{aligned}$$

Indeed, the upper bound proceeds in two steps: first we take the supremum over all measures μ_2 on S_2 with mass in $[1-\frac{\delta}{\alpha_2}, 1+\frac{\delta}{\alpha_2}]$, and then restrict ourselves to probability measures as δ goes to zero (since ψ is bounded). Then, we observe that for any $\mu_2\in\mathcal{P}(S_2)$, $\nu_1\in\mathcal{P}(I_1)$, and $s\in S_3^k$ such that $|\sum_i s_i^2 - \alpha_3\sqrt{N}|\leq\delta\sqrt{N}$,

$$\begin{aligned}&\frac{\alpha_2}{\sqrt{N}}\sum_{i=1}^k s_i^2\int\psi\big(\sqrt{2\theta}s_i x\big)d\mu_2(x) + \frac{1}{\sqrt{N}}\sum_{i=1}^k s_i^2\int x^2\psi\Big(\frac{2\theta s_i x}{N^{\frac14}}\Big)d\nu_1(x)\\ &\qquad\leq \alpha_3\int\psi\big(\sqrt{2\theta}sx\big)d\mu_2(x) + \int x^2\psi\Big(\frac{2\theta sx}{N^{1/4}}\Big)d\nu_1(x) + o_\delta(1),\end{aligned}$$

where s is a maximizer of the function

$$s\in S_3\mapsto \alpha_2\int\psi\big(\sqrt{2\theta}sx\big)d\mu_2(x) + \int x^2\psi\Big(\frac{2\theta sx}{N^{1/4}}\Big)d\nu_1(x),$$

which ends the proof of the claim (47). We will see that under our assumptions that $B<A$ and that the maximum of ψ is uniquely achieved at m_* such that $\psi''(m_*)<0$, the upper bound $\hat{F}(\theta)$ is sharp when θ is large.

The starting point of our analysis of the variational problem defining $\hat{F}(\theta)$ in the regime where θ is large is the fact that $\underline{F}(\theta)$ and $\hat{F}(\theta)$ behave like $A\theta^2$. More precisely, we know from (10) that there exists $\theta_0>0$ (depending on A) such that for all $\theta\geq\theta_0$,

$$\hat{F}(\theta)\geq\underline{F}(\theta)\geq A\theta^2-\kappa\log\theta, \tag{48}$$

for some constant $\kappa>0$.

As a consequence, we can localize the suprema over $(\alpha_1,\alpha_2,\alpha_3)$ and μ_2 in the definitions of $\hat{F}(\theta)$ in some subset of the constraint set, denoted by $\mathcal{S}$, and defined as follow,

$$\mathcal{S} = \Big\{(\underline{\alpha},\mu_2)\in[0,1]^3\times\mathcal{P}(S_2) : \alpha_1+\alpha_2+\alpha_3=1\Big\}.$$

Lemma 6.4. *There exists a constant $\theta_0>0$ depending on A such that for any $\theta\geq\theta_0$, the suprema defining $\hat{F}(\theta)$ can be restricted to the set $\mathcal{A}_\theta\times\mathcal{B}_\theta\subset\mathcal{S}$ defined by,*

$$\underline{\alpha}\in\mathcal{A}_\theta \iff \alpha_2\geq 1-\frac{C\sqrt{\log\theta}}{\theta},\ \ \alpha_1\leq\frac{C\log\theta}{\theta^2},\ \ \alpha_3\leq\frac{C\sqrt{\log\theta}}{\theta},$$

and

$$\mu_2 \in \mathcal{B}_\theta \Longleftrightarrow \int \Big(\frac{A}{2} - \psi(xy)\Big) d\mu_2(x) d\mu_2(y) \le \frac{C\log\theta}{\theta^2}. \tag{49}$$

where C is a some positive constant depending also on A.

Proof. From (48) we deduce that we can restrict the suprema in the definitions of $\hat{F}(\theta)$ to the parameters $\underline{\alpha}, s, \nu_1, \mu_2$ with $\alpha_1+\alpha_2+\alpha_3 = 1$, $s \in S_3$, $\int x^2 d\mu_1(x) = \alpha_1$ such that,

$$\begin{aligned}
&(A-1)(\alpha_1^2 + 2\alpha_1\alpha_2) + (A-B)\alpha_3^2 + 4\alpha_2\alpha_3 \int \Big(\frac{A}{2} - \psi(2\theta sy)\Big) d\mu_2(y) \\
&+ 2\alpha_2^2 \int \Big(\frac{A}{2} - \psi(2\theta xy)\Big) d\mu_2(y) d\mu_2(x) + 4\alpha_3 \int y^2 \Big(\frac{A}{2} - \psi(2\theta sy)\Big) d\nu_1(y) \\
&+ \frac{1}{\theta^2}\big(H(\nu_1) + \log\sqrt{2\pi}\big) \le \frac{2\kappa\log\theta}{\theta^2}.
\end{aligned}$$

But

$$4\alpha_3 \int y^2 \Big(\frac{A}{2} - \psi(2\theta sy)\Big) d\nu_1(y) + \frac{1}{\theta^2}\big(H(\nu_1) + \log\sqrt{2\pi}\big) \ge \frac{1}{2}\log\frac{1}{\alpha_1} \ge 0.$$

Therefore,

$$\begin{aligned}
(A-1)(\alpha_1^2 + 2\alpha_1\alpha_2) + (A-B)\alpha_3^2 + 4\alpha_2\alpha_3 \int y^2 \Big(\frac{A}{2} - \psi(2\theta sy)\Big) d\mu_2(y)& \\
+ 2\alpha_2^2 \int \Big(\frac{A}{2} - \psi(xy)\Big) d\mu_2(y) d\mu_2(x) \le \frac{2\kappa\log\theta}{\theta^2}&.
\end{aligned}$$

Since each term is non-negative, they are all bounded by $2\kappa\log\theta/\theta^2$. Note that this already yields with $C = 4\kappa/\min\{(A-B), A-1\}$,

$$\alpha_1^2 \le \frac{C\log\theta}{\theta^2},\ \alpha_3^2 \le \frac{C\log\theta}{\theta^2},\ \alpha_1\alpha_2 \le \frac{C\log\theta}{\theta^2}. \tag{50}$$

The two first estimates imply since $\alpha_2 = 1 - \alpha_1 - \alpha_3$,

$$\alpha_2 \ge 1 - 2\frac{\sqrt{C\log\theta}}{\theta}.$$

We can finally plug back this estimate into the last inequality of (50) to improve the estimate on α_1 as announced. □

Next, note that because ψ is bounded continuous, the function $\mathcal{G}^N_{\alpha_1,\alpha_2,\alpha_3}(\delta, K, s, .)$ we are optimizing over μ_2, is bounded continuous in μ_2 and therefore it achieves its maximal value. We denote by μ_2 such an optimizer. In the next lemma, we prove that the optimizers of $\mathcal{G}^N_{\alpha_1,\alpha_2,\alpha_3}(\delta, K, s, .)$ are concentrated around $\sqrt{m_*}$ if ψ takes its maximum value at m_* only.

Lemma 6.5. *Assume that ψ achieves its maximum value at m_* only and that it is strictly concave in an open neighborhood of this point. Let μ_2 be an optimizer of $\mathcal{G}^N_{\alpha_1,\alpha_2,\alpha_3}(\delta, K, s, .)$. There exists $\varepsilon_0 > 0$ such that for any $\mu_2 \in \mathcal{B}_\theta$,*

$$\forall 0 < \varepsilon < \varepsilon_0,\ \mu_2\big(|x - \sqrt{m_*}| \ge \varepsilon\big) \le \frac{C\sqrt{\log\theta}}{\theta\varepsilon},$$

where C is a positive constant depending on ψ.

Proof. Let $\mu_2 \in \mathcal{B}_\theta$. By Lemma 6.4 we have,

$$\int \Big(\frac{A}{2} - \psi(xy)\Big) d\mu_2(x) d\mu_2(y) \leq \frac{C \log \theta}{\theta^2}. \tag{51}$$

Since ψ is strictly concave in a neighborhood of m_*, and m_* is its unique maximizer, we deduce that there exists $\eta_0 > 0$ such that for all $0 < \eta < \eta_0$,

$$\forall |x - m_*| \in [\sqrt{\eta}, \sqrt{\eta_0}], \quad \frac{A}{2} - \psi(x) \geq \eta/c,$$

for some constant $c > 0$. As ψ is analytic, it admits a finite number of local maxima. Therefore, we can find $\eta_0 > 0$ such that for all $0 < \eta < \eta_0$,

$$\forall |x - m_*| \geq \sqrt{\eta}, \quad \frac{A}{2} - \psi(x) \geq \eta/c,$$

Since $\frac{A}{2} - \psi$ is non-negative, we deduce from (51) that

$$\forall \eta < \eta_0, \ \mu_2^{\otimes 2}\big(|xy - m_*| \geq \sqrt{\eta}\big) \leq \frac{C' \log \theta}{\eta \theta^2},$$

where $C' \geq 1$ is a constant depending on ψ. But for ε small enough, we have

$\mu_2([0, \sqrt{m_*} - \varepsilon])^2 \leq \mu_2^{\otimes 2}\big(xy \leq m_* - \sqrt{m_*}\varepsilon\big)$ and $\mu_2\big([\sqrt{m_*} + \varepsilon, +\infty)\big)^2 \leq \mu_2^{\otimes 2}\big(xy \geq m_* + \sqrt{m_*}\varepsilon\big)$ from which the result follows by a union bound. $\square$

Using Lemma 6.5, we will show that the optimization problem over μ_2 is asymptotically solved by $\delta_{\sqrt{m_*}}$, with an error which vanishes when K, and therefore the lower boundary point of S_3, goes to $+\infty$.

Lemma 6.6. *There exists θ_0 depending on ψ such that for any $\theta \geq \theta_0$, $\underline{\alpha} \in \mathcal{A}_\theta$ and $s \in S_3$,*

$$\sup_{\mu_2 \in \mathcal{B}_\theta} \Big\{ 2\alpha_3 \int \psi\big(\sqrt{2\theta} s x\big) d\mu_2(x) + \alpha_2 \int \psi(xy) d\mu_2(x) d\mu_2(y) \Big\} = \frac{A\alpha_2}{2} + \frac{B\alpha_3}{2} + o_K(1).$$

Proof. Letting $\overline{\psi}(x) = \psi(x) - \frac{B}{2}$, it is equivalent to show that:

$$\sup_{\mu_2 \in \mathcal{B}_\theta} \Big\{ 2\alpha_3 \int \overline{\psi}\big(\sqrt{2\theta} s x\big) d\mu_2(x) + \alpha_2 \int \overline{\psi}(xy) d\mu_2(x) d\mu_2(y) \Big\} = \frac{(A-B)\alpha_2}{2} + o_K(1).$$

Let us fix $\theta \geq \theta_0$ where θ_0 is given by Lemma 6.4. Observe that since ψ is bounded continuous,

$$Z : \mu \in \mathcal{P}(S_2) \mapsto 2\alpha_3 \int \overline{\psi}\big(\sqrt{2\theta} s x\big) d\mu(x) + \alpha_2 \int \overline{\psi}(xy) d\mu(x) d\mu(y)$$

achieves its maximum value in the closed set $\mathcal{B}_\theta$. Let μ_2 be an optimizer, and therefore a critical point of this function. Writing that $Z(\mu_2) \geq Z(\mu_2 + \varepsilon\nu)$ for all signed measures ν on S_2 such that $\mu_2 + \varepsilon\nu$ is a probability measure for small ε, we deduce that there exists a constant $C > 0$ such that,

$$\forall x \in S_2, \ \alpha_3 \overline{\psi}(\sqrt{2\theta} s x) + \alpha_2 \int \overline{\psi}(xy) d\mu_2(y) \leq C, \tag{52}$$

with equality μ_2-almost surely. Using Lemma 6.5, we get for any ε small enough,

$$\int \overline{\psi}(xy)d\mu_2(y) = \overline{\psi}(\sqrt{m_*}x) + \int_{[\sqrt{m_*}-\varepsilon, \sqrt{m_*}+\varepsilon]} \big(\overline{\psi}(xy) - \overline{\psi}(\sqrt{m_*}x)\big)d\mu_2(y) + O\Big(\frac{\sqrt{\log\theta}}{\theta\varepsilon}\Big).$$

Where we notice that our $O(\sqrt{\log\theta}/\theta\varepsilon)$ is a function that does not depend on δ, K or N. As L is the log-Laplace transform of a sub-Gaussian distribution, we have that $x \mapsto |L'(x)/x|$ is bounded. In particular, $|\psi'|$ is bounded and thus ψ is Lipschitz. Therefore, for any $x \leq M$,

$$\int \overline{\psi}(xy)d\mu_2(y) = \overline{\psi}(\sqrt{m_*}x) + O\Big(\varepsilon M + \frac{\sqrt{\log\theta}}{\theta\varepsilon}\Big).$$

Again, $O\Big(\varepsilon M + \frac{\sqrt{\log\theta}}{\theta\varepsilon}\Big)$ does not depend on δ, K or N. We choose $\varepsilon = \theta^{-1/2}$ and $M = \theta^{1/4}$ so that the two error term above goes to zero when θ goes to ∞, so that we have for any $x \geq 0$,

$$\int \overline{\psi}(xy)d\mu_2(y) = \overline{\psi}(\sqrt{m_*}x) + o_\theta(1). \tag{53}$$

In particular,

$$\alpha_3\overline{\psi}(\sqrt{2\theta}sx) + \alpha_2 \int \overline{\psi}(xy)d\mu_2(y) = \overline{\psi}(\sqrt{m_*}x) + o_\theta(1).$$

Taking $x = \sqrt{m_*}$ in (52), we get

$$C \geq \frac{A-B}{2} - o_\theta(1), \tag{54}$$

since $s \geq K$ and $1 - \alpha_2 \leq O(\frac{\sqrt{\log\theta}}{\theta})$. The term $o_\theta(1)$ above do not depend on K, δ or N. We claim that there exists θ_0 such that for any $\theta \geq \theta_0$,

$$\mu_2([0, \sqrt{m_*}/2]) = 0.$$

Indeed, if $x \leq \sqrt{m_*}/2$, we have by (53) and the fact that α_2 goes to 1 as θ goes to infinity,

$$\alpha_3\overline{\psi}(\sqrt{2\theta}sx) + \alpha_2 \int \overline{\psi}(xy)d\mu(y) \leq \sup_{t \leq \sqrt{m_*}/2} \overline{\psi}(\sqrt{m_*}t) + o_\theta(1),$$

with $\sup_{t\leq\sqrt{m_*}/2} \overline{\psi}(\sqrt{m_*}t) < (A-B)/2$ since the maximum of ψ is uniquely achieved at m_*. From (54) and the fact that equality in (52) holds μ_2-a.s, we deduce that for θ large enough (and not depending on δ, K or N) $[0, \sqrt{m_*}/2] \cap \operatorname{supp}(\mu_2) = \emptyset$. Therefore,

$$\begin{aligned} 2\alpha_3 \int \overline{\psi}\big(\sqrt{2\theta}sx\big)d\mu_2(x) + \alpha_2 \int \overline{\psi}(xy)d\mu_2(x)d\mu_2(y) &\leq \frac{(A-B)\alpha_2}{2} + 2 \sup_{y \geq K\frac{\sqrt{m_*\theta}}{2}} \overline{\psi}(y) \\ &= \frac{(A-B)\alpha_2}{2} + o_K(1). \end{aligned}$$

Thus,

$$\sup_{\mu_2 \in \mathcal{B}_\theta} \Big\{ 2\alpha_3 \int \overline{\psi}\big(\sqrt{2\theta}sx\big)d\mu_2(x) + \alpha_2 \int \overline{\psi}(xy)d\mu_2(x)d\mu_2(y) \Big\} \leq \frac{(A-B)\alpha_2}{2} + o_K(1).$$

The reverse inequality is achieved by taking $\mu_2 = \delta_{\sqrt{m_*}}$, which completes the proof. □

We deduce that taking δ to 0 and K to $+\infty$, we can simplify the expression of $\hat{F}(\theta)$.

Proposition 6.7. *There exists θ_0 depending on ψ such that for any $\theta \geq \theta_0$, $\overline{F}(\theta) \leq \hat{F}(\theta)$, where*

$$\hat{F}(\theta) = \sup_{(\underline{\alpha}, s, \nu) \in \mathcal{S}'} \mathcal{F}(\underline{\alpha}, s, \nu),$$

with

$$\mathcal{F}(\underline{\alpha}, s, \nu) = \theta^2 \big(\alpha_1^2 + 2\alpha_1\alpha_2\big) + \theta^2 A \alpha_2^2 + \theta^2 B(\alpha_3^2 + 2\alpha_3\alpha_2) \\ + 4\theta^2 \alpha_3 \int x^2 \psi(2\theta s \sqrt{\alpha_3} x) d\nu(x) - H(\nu) - \frac{1}{2}\log(2\pi) - \frac{1}{2},$$

and

$$\mathcal{S}' = \Big\{ (\underline{\alpha}, s, \nu) \in [0,1]^3 \times [0,1] \times \mathcal{P}(\mathbb{R}) : \alpha_1 + \alpha_2 + \alpha_3 = 1, \int x^2 d\nu(x) = \alpha_1 \Big\}.$$

Proof. By Lemmas 6.4 and 6.6, we know that

$$\hat{F}(\theta) = \sup_{\underline{\alpha} \in \mathcal{A}_\theta} \limsup_{\substack{\delta \to 0, K \to +\infty \\ \delta K \to 0}} \limsup_{N \to +\infty} \hat{\mathcal{F}}^N_{\alpha_1,\alpha_2,\alpha_3}(\delta, K),$$

where

$$\hat{\mathcal{F}}^N_{\alpha_1,\alpha_2,\alpha_3}(\delta, K) = \sup_{s \in S_3} \sup_{\substack{\nu_1 \in \mathcal{P}(I_1) \\ \int x^2 d\nu_1(x) = \alpha_1}} \Big\{ 4\theta^2 \alpha_3 \int x^2 \psi\Big(\frac{2\theta s x}{N^{\frac{1}{4}}}\Big) d\nu_1(x) - H(\nu_1) \Big\} \\ + \theta^2 \big(\alpha_1^2 + 2\alpha_1\alpha_2\big) + A\alpha_2^2 + B(\alpha_3^2 + 2\alpha_2\alpha_3) - \frac{1}{2}\log(2\pi) - \frac{1}{2},$$

$S_3 = [K, N^{1/4}\sqrt{\alpha_3}]$. Using the change of variable $s \mapsto sN^{-1/4}$ we have the upper bound,

$$\hat{F}(\theta) \leq \sup_{(\underline{\alpha}, s, \nu) \in \mathcal{S}'} \mathcal{F}(\underline{\alpha}, s, \nu).$$

□

We finally prove that the supremum is taken at $\alpha_3 = 0$.

Proposition 6.8. *There exists θ_0 depending on A such that for any $\theta \geq \theta_0$,*

$$\sup_{(\underline{\alpha}, s, \nu) \in \mathcal{S}'} \mathcal{F}(\underline{\alpha}, s, \nu) = \sup_{(\alpha_1, \alpha_2, 0, s, \nu) \in \mathcal{S}'} \mathcal{F}((\alpha_1, \alpha_2, 0), s, \nu).$$

Proof. We claim that for any $((\alpha_1, \alpha_2, \alpha_3), s, \nu) \in \mathcal{S}'$ such that $\alpha_2 \geq \frac{A-1}{2A-B-1}$, we have

$$\mathcal{F}((\alpha_1, \alpha_2, \alpha_3), s, \nu) \leq \sup_{\nu \in \mathcal{P}(\mathbb{R})} \mathcal{F}((\alpha_1, \alpha_2 + \alpha_3, 0), \nu). \tag{55}$$

Note that

$$\sup_{\nu \in \mathcal{P}(\mathbb{R})} \mathcal{F}((\alpha_1, \alpha_2 + \alpha_3, 0), \nu) = \theta^2(\alpha_1 + 2\alpha_1\alpha_2 + 2\alpha_1\alpha_3) + \theta^2 A(\alpha_2 + \alpha_3)^2 + \frac{1}{2}\log \alpha_1.$$

Now, for any $((\alpha_1,\alpha_2,\alpha_3),s,\nu)\in\mathcal{S}'$, using the fact that $\psi(x)\le A/2$ for any $x\in\mathbb{R}$, we have

$$\mathcal{F}((\alpha_1,\alpha_2,\alpha_3),s,\nu)\le\theta^2(\alpha_1+2\alpha_1\alpha_2)+\theta^2A\alpha_2^2+2\theta^2A\alpha_1\alpha_3+\theta^2B(\alpha_3^2+2\alpha_2\alpha_3)+\frac{1}{2}\log\alpha_1.$$

Therefore, it suffices to prove that for α_2 sufficiently near 1:

$$(A-B)(2\alpha_2\alpha_3+\alpha_3^2)\ge 2(A-1)\alpha_1\alpha_3$$

This is true if

$$2(A-1)\alpha_1\le(A-B)(\alpha_3+2\alpha_2).$$

A sufficient condition for the inequality to be true is that $(A-1)(1-\alpha_2)\le(A-B)\alpha_2$, which ends the proof of the claim (55). By Lemma 6.4, we know that for $\theta\ge\theta_0$,

$$\sup_{(\underline{\alpha},s,\nu)\in\mathcal{S}'}\mathcal{F}(\underline{\alpha},s,\nu)=\sup_{\substack{(\underline{\alpha},s,\nu)\in\mathcal{S}'\\ \alpha_1,\alpha_3\le C\sqrt{\log\theta}/\theta}}\mathcal{F}(\underline{\alpha},s,\nu).$$

Hence, for θ such that

$$1-2C\frac{\sqrt{\log\theta}}{\theta}\ge\frac{A-1}{2A-B-1},$$

we obtain (55). □

We can now conclude from the last two Propositions 6.7 and 6.8, that for $\theta\ge\theta_0$

$$\overline{F}(\theta)\le\sup_{((\alpha_1,\alpha_2,0),s,\nu)\in S}\mathcal{F}(\alpha_1,\alpha_2,0,s,\nu)=\sup_{\alpha\in[0,1)}V(\alpha)$$

where we optimized over ν (at the centered Gaussian law with covariance α_1). This completes the proof of the proof of Proposition 6.2 with Lemma 6.3.

7. Delocalization and localization of the eigenvector of the largest eigenvalue

In this section we consider a unit eigenvector u_{X_N} associated to the largest eigenvalue of X_N, conditioned to deviate towards a large value. We assume hereafter that μ is compactly supported, allowing us to use the sub-Gaussian concentration property of the titled measure $\mathbb{P}^{(e,\theta)}$, as defined in (4).

We first show when X_N has sharp sub-Gaussian tails, u_{X_N} stays close to the set of delocalized vectors. Then, we show that in the case where μ is not sharp sub-Gaussian, u_{X_N} is close to a set of localized vectors in the sense that it contains about $\sqrt{N}$ entries of order $N^{-1/4}$, the other being much smaller. It should be possible to consider as well the case where ψ is increasing, and we then expect that the eigenvector would localize over one entry. However, this would require more effort to obtain the required exponential estimates and we postpone this research to further investigations.

We denote by d_2 the Euclidean distance in $\mathbb{R}^N$: for a subset A of $\mathbb{R}^N$ and $u\in\mathbb{R}^N$ we set

$$d_2(u,A)=\inf\{\|u-v\|_2 : v\in A\}\,.$$

Proposition 7.1. *Assume that μ has sharp sub-Gaussian tail and is compactly supported. Let $\varepsilon > 0$ and define the set of delocalized vectors D_ε by:*

$$D_\varepsilon := \{e \in \mathbb{S}^{N-1} : \forall i \in \{1,\dots,N\},\ |e_i| \le \varepsilon N^{1/4}\}.$$

There exists a function $\eta(x)$ that goes to zero when x goes to $+\infty$ such that

$$\lim_{\varepsilon\to 0}\lim_{\delta\to 0}\lim_{N\to\infty} \mathbb{P}\left(d_2(u_{X_N}, B_\varepsilon) \le \eta(x) \big| |\lambda_{X_N} - x| \le \delta\right) = 1\,.$$

For any symmetric matrix X, we denote by u_X a unit eigenvector associated to the largest eigenvalue. For any $\chi \in (0,1)$, we set:

$$A_\chi = \{\sup_{e\in D_\varepsilon} |\langle u_{X_N}, e\rangle| \le 1-\chi\}\,.$$

Let $x > 2$ and $\theta_x \ge 1/2$ such that $x = 2\theta_x + 1/2\theta_x$. We know from [18, Section 5] that under the measure $\mathbb{P}_{\theta_x,N}$ defined by,

$$d\mathbb{P}_{\theta_x,N} = \frac{I_N(\theta_x, X_N)}{\mathbb{E} I_N(\theta_x, X_N)} d\mathbb{P}(X),$$

for $\delta, \gamma > 0$, N, M large enough, if $V^M_{\delta,x} = \{|\lambda_{X_N} - x| < \delta, d(\hat{\mu}_{X_N}, \sigma) < N^{-\gamma}, ||X_N|| \le M\}$

$$\mathbb{P}_{\theta_x,N}(V^M_{\delta,x}) \ge \frac{1}{2}. \tag{56}$$

By (11) we know that

$$\mathbb{P}\big(|\lambda_{X_N} - x| \le \delta\big) \ge e^{-N(J(\theta_x,x) - F(\theta_x) + o_\delta(1))}\mathbb{P}_{\theta_x,N}(V^M_{\delta,x}).$$

Similarly, we have

$$\mathbb{P}\big(V^M_{\delta,x} \cap A_\chi\big) \le e^{-N(J(\theta_x,x) - F(\theta_x) - o_\delta(1))}\mathbb{P}_{\theta_x,N}(A_\chi \cap V^M_{\delta,x}).$$

Using (56) and assumption 1.2 we find $f(M) \to +\infty$ when $M \to +\infty$ so that

$$\mathbb{P}\left(A_\chi \big| |\lambda_{X_N} - x| \le \delta\right) \le 2\mathbb{P}_{\theta_x,N}\big(A_\chi \cap V^M_{\delta,x}\big) + e^{-Nf(M)}.$$

Using the lower bound $\log \mathbb{E} I_N(X_N, \theta_x) \ge N\theta_x^2 - o(N)$, we deduce

$$\mathbb{P}\left(A_\chi \big| |\lambda_{X_N} - x| \le \delta\right) \le 2e^{-N\theta_x^2}\mathbb{E}_e[\mathbb{E}_X[\mathbb{1}_{\{A_\chi \cap V^M_{\delta,x}\}} e^{N\theta_x\langle e, X_N e\rangle}]] + e^{-Nf(M)}, \tag{57}$$

Let $\kappa \in (0,1)$ and define the set $D_{\varepsilon,\kappa} = \{e \in \mathbb{S}^{n-1} : \sum_{i,j} e_i^2 e_j^2 \mathbb{1}_{\sqrt{N}|e_ie_j| > \varepsilon^2/4} > \kappa\}$. Since we assumed that X_N has sharp sub-Gaussian tails, we have that $r_\varepsilon = \inf_{y\ge\varepsilon^2/4}(\psi(0) - \psi(y)) > 0$. Therefore, for any $e \in D_{\varepsilon,\kappa}$,

$$\sum_{i=1}^N L(\sqrt{2N}\theta_x e_i^2) + \sum_{i<j} L(2\sqrt{N}\theta_x e_i e_j) - N\theta_x^2 \le -\theta_x^2 r_\varepsilon \kappa N.$$

We deduce that

$$e^{-\theta_x^2 N}\mathbb{E}_e[\mathbb{1}_{e\in D_{\varepsilon,\kappa}}\mathbb{E}_X[e^{N\theta_x\langle e, X_N e\rangle}]] \le e^{-\theta_x^2 r_\varepsilon \kappa N}\,. \tag{58}$$

On the other hand, observe that for $e \in D^c_{\varepsilon,\kappa}$,

$$\Big(\sum_{i=1}^N e_i^2 1_{|e_i| \ge \varepsilon N^{-\frac{1}{4}}/2}\Big)^2 \le \sum_{i,j=1}^N e_i^2 e_j^2 1_{\sqrt{N}|e_ie_j| > \varepsilon^2/4} \le \kappa\,.$$

Therefore if we let $\bar{e}_i = \mathrm{sgn}(e_i)\min\{|e_i|, \varepsilon N^{-1/4}/2\}$, we have that

$$|\langle e, Xe\rangle - \langle \bar{e}, X\bar{e}\rangle| \leq 2\|X\|\sqrt{\kappa}\,.$$

Thus, we can write

$$\mathbb{E}_e[\mathbb{1}_{e\in D^c_{\varepsilon,\kappa}}\mathbb{E}_X[\mathbb{1}_{A_\chi\cap V^M_{\delta,x}}e^{N\theta_x\langle e,X_N e\rangle}]] \leq e^{2\theta_x M\sqrt{\kappa}N}\mathbb{E}_e[\mathbb{1}_{e\in D^c_{\varepsilon,\kappa}}\mathbb{E}_X[\mathbb{1}_{\{A_\chi\cap V^M_{\delta,x}\}}e^{N\theta_x\langle \bar{e},X\bar{e}\rangle}]].$$

But for $e \in D^c_{\varepsilon,\kappa}$,

$$\mathbb{E}_X[e^{N\theta_x\langle \bar{e},X\bar{e}\rangle}] \leq e^{\theta_x^2 N\|\bar{e}\|_2^2} \leq e^{\theta_x^2 N},$$

which implies that

$$\mathbb{E}_e[\mathbb{1}_{e\in D^c_{\varepsilon,\kappa}}\mathbb{E}_X[\mathbb{1}_{A_\chi\cap V^M_{\delta,x}}e^{N\theta_x\langle e,X_N e\rangle}] \leq e^{(2M\sqrt{\kappa}-\theta_x^2)N}\mathbb{E}_e[\mathbb{1}_{e\in D^c_{\varepsilon,\kappa}}\mathbb{P}^{(\bar{e},\theta_x)}(A_\chi)] \tag{59}$$

where

$$\mathbb{P}^{(\bar{e},\theta_x)} = \frac{e^{N\theta_x\langle \bar{e},X\bar{e}\rangle}}{\mathbb{E}_X[e^{N\theta_x\langle \bar{e},X\bar{e}\rangle}]}d\mathbb{P}(X). \tag{60}$$

We can conclude from (57), (58) and (59) that

$$\begin{aligned}\mathbb{P}\left(A_\chi \big| |\lambda_{X_N} - x| \leq \delta\right) \leq{}& 2e^{2N\theta_x M\sqrt{\kappa}}\mathbb{E}_e[\mathbb{1}_{e\in D^c_{\varepsilon,\kappa}}\mathbb{P}^{(\bar{e},\theta_x)}(A_\chi)] \\ &+ 2e^{-N\theta_x^2 r_\varepsilon \kappa} + e^{-Nf(M)}.\end{aligned}$$

Hence, it is sufficient to complete the proof of Proposition 7.1 to prove the following:

Lemma 7.2. *There exists a numerical constant $C > 0$ and a positive function h, such that for κ small enough, N large enough, $\chi \geq C\varepsilon^2$ and any $e \in D^c_{\varepsilon,\kappa}$,*

$$\mathbb{P}^{(\bar{e},\theta_x)}(A_\chi) \leq e^{-Nh(\chi)}.$$

Proof. We can proceed as in [18, section 5.1] and observe that under $\mathbb{P}^{(\bar{e},\theta_x)}$, X_N is symmetric and has independent entries with distribution

$$\bigotimes_{i\leq j} dP_N^{2^{i\neq j}\theta\sqrt{N}\bar{e}_i\bar{e}_j}(X_{ij})$$

where P_N^γ is the law of $x/\sqrt{N}$ under $e^{\gamma x}d\mu(x)/\int e^{\gamma y}d\mu(y)$. Using the fact that $\bar{e}_i = O(\varepsilon N^{-1/4})$, we see that we can write $X_N = W + M_{\bar{e},\theta_x}$ where $M_{\bar{e},\theta_x} = \mathbb{E}_{(\bar{e},\theta_x)}[X_N] = 2\theta_x\bar{e}\bar{e}^T + r_N$, with $\|r_N\| = O(\varepsilon^2)$. We denote $\hat{W} = W + r_N$. Provided that $\lambda_{X_N} > \lambda_{\hat{W}}$, a unit eigenvector u of X_N associated to λ_{X_N} satisfies the equation:

$$(\hat{W} + 2\theta_x\bar{e}\bar{e}^T)u = \lambda_{X_N}u \Rightarrow (\hat{W} - \lambda_{X_N})u = 2\theta_x\langle u, \bar{e}\rangle\bar{e} \Rightarrow u = \frac{(\hat{W} - \lambda_{X_N})^{-1}\bar{e}}{\|(\hat{W} - \lambda_{X_N})^{-1}\bar{e}\|_2}.$$

Therefore

$$|\langle u, \bar{e}\rangle|^2 = \frac{\langle \bar{e}, (\lambda_{X_N} - \hat{W})^{-1}\bar{e}\rangle^2}{\langle \bar{e}, (\lambda_{X_N} - \hat{W})^{-2}\bar{e}\rangle}\,.$$

Lemma 7.3. *For N large enough, and for any $e \in D^c_{\varepsilon,\kappa}$, $\delta > C\varepsilon^2$, and $K \geq C$ for some constant $C > 0$,*

$$\mathbb{P}^{(\bar{e},\theta_x)}\big(||\hat{W}|| > K\big) \leq e^{-c(K)N}, \qquad \mathbb{P}^{(\bar{e},\theta_x)}\big(|\lambda_{X_N} - x| \geq \delta\big) \leq e^{-c(\delta)N}, \tag{61}$$

where c is a positive function increasing to infinity. Furthermore, for any $\chi \geq C\varepsilon^2$,

$$\mathbb{P}^{(\bar{e},\theta_x)}\Big(|\langle u_{X_N}, \bar{e}\rangle|^2 \leq 1 - \chi\Big) \leq e^{-h(\chi)N},$$

where h is a positive function.

Proof. The first statement follows from Remark 1.4. The second claim is the consequence of Talagrand's concentration inequality for convex Lipschitz functions (see [21, Corollary 4.10]) and the fact that $\mathbb{E}^{(\bar{e},\theta_x)}\lambda_{X_N} = x + O(\varepsilon^2)$. Indeed, note that W is a centered random symmetric matrix with independent entries above the diagonal with variance close to $1/N$. It is known as the BBP transition, (see [4], [9] for example), that $\lambda_{W+2\theta_x\bar{e}\bar{e}^T}$ converges to x, almost surely and in expectation. Since $||r_N|| = O(\varepsilon^2)$, we deduce that for N large enough, $\mathbb{E}^{(\bar{e},\theta_x)}\lambda_{X_N} = x + O(\varepsilon^2)$.

Let now $x > 2$, $K \geq 1$ and $\delta \in (0,1)$ such that $x - \delta > 2K$. We have on the event $V = \{||\hat{W}|| \leq K,\ |\lambda_{X_N} - x| \leq \delta\}$, we have

$$\Big||\langle u_{X_N}, \bar{e}\rangle|^2 - v_N\Big| \leq C\delta, \text{ where } v_N = \frac{\langle \bar{e}, (x - \hat{W})^{-1}\bar{e}\rangle^2}{\langle \bar{e}, (x - \hat{W})^{-2}\bar{e}\rangle},$$

and C is a numerical constant. Moreover, one can check that on the event V,

$$|v_N - 1| \leq \frac{C'K}{x},$$

where $C' > 0$ is a numerical constant. Therefore, for $\chi \leq CK/x + C'\delta$,

$$\mathbb{P}^{(\bar{e},\theta_x)}\Big(\big||\langle u_{X_N}, \bar{e}\rangle|^2 - 1\big| > \chi\Big) \leq \mathbb{P}^{(\bar{e},\theta_x)}(W^c),$$

which gives the claim by an appropriate choice of K and δ. □

To conclude the proof of Lemma 7.2, note that for $e \in D_{\varepsilon,\kappa}$, we have $\bar{e}/||\bar{e}|| \in D_{\varepsilon/2\sqrt{1-\kappa}}$. For κ small enough, $\bar{e}/||\bar{e}|| \in D_\varepsilon$ and we have:

$$\mathbb{P}^{(\bar{e},\theta_x)}(A_\chi) \leq \mathbb{P}^{(\bar{e},\theta_x)}\Big(\big|\langle u_{X_N}, \bar{e}\rangle|^2 - 1\big| > \chi\Big),$$

which, using Lemma 7.3, ends the proof. □

We next consider what happens when μ is compactly supported and is not sharp sub-Gaussian. We shall prove that in this case, at least when we condition by deviations of the largest eigenvalue close to x large, the associated eigenvector becomes close to the set

$$\mathrm{Loc}_{r_1,r_2,\varepsilon} = \Big\{e \in \mathbb{S}^{N-1} : |I_{r_2,\varepsilon}(e)| \in [(r_1 - \varepsilon)\sqrt{N}, (r_1 + \varepsilon)\sqrt{N}],\ \forall i \notin I_{r_2,\varepsilon}(e),\ |e_i| \leq \varepsilon N^{-\frac{1}{4}}\Big\},$$

where for any $e \in \mathbb{S}^{N-1}$, $I_{r_2,\varepsilon}(e) = \big\{i : \frac{N^{-\frac{1}{4}}|e_i|}{\sqrt{r_2}} \in [1-\varepsilon, 1+\varepsilon]\big\}$.

Proposition 7.4. *Assume that μ is compactly supported and ψ achieves its maximum at a unique point m_* where it is strictly concave. For any $x \geq x_\mu$, let $v_x = 2\theta_x/m_*$, where $x = 1/2\theta_x + F'(\theta_x)$.*

$$\lim_{\varepsilon \to 0} \lim_{\delta \to 0} \lim_{N \to \infty} \mathbb{P}\big(d_2(u_{X_N}, Loc_{v_x, 1/v_x, \varepsilon}) \leq c(x) \big| |\lambda_{X_N} - x| \leq \delta\big) = 1\,,$$

where c is a function going to 0 as x goes to $+\infty$.

Proof. As in (50) we can replace the conditioning by the tilt by spherical integrals of parameter $\theta = \theta_x$ large. We then can use Lemmas 6.4 and 6.5 to see that up to exponentially small probability we can restrict the integration over a δ neighborhood of

$$\alpha_2^\theta \geq 1 - \frac{C\sqrt{\log \theta}}{\theta^2},\ \alpha_1^\theta \leq \frac{C \log \theta}{\theta^2},\ \alpha_3 = 0,$$

and $\mu_2 = \delta_{\sqrt{m_*}}$. Here, $\alpha_j = \sum_{i:\sqrt{N} e_i \in I_j} e_i^2$ and $\mu_2 = \frac{1}{\alpha_2 N} \sum_{\sqrt{N} e_i \in I_2}^{l} e_i^2 \delta_{\sqrt{2\theta} N^{1/4} e_i}$. Indeed, this is a simple consequence of the estimate of Lemma 6.4 which implies that we can restrict ourselves to the space $\mathcal{A}_\theta \times \mathcal{B}_\theta$ when we estimate the annealed free energy up to exponentially small errors. Note then that μ_2 is compactly supported and the rate function is smooth in μ_2 so that we can cover the integration over μ_2 by finitely many balls: this implies that we can restrict ourselves to a neighborhood of the minimizer $\mu_2 = \delta_{m_*}$ and $\alpha_3 = 0$. Note that this implies that with exponentially large probability, the uniform vector in the annealed spherical integral belongs to $\mathrm{Loc}_{r_1, r_2, \varepsilon}$ with $r_2 = m_*/2\theta, r_1 r_2 = \alpha_2$, hence $r_1 = 2\theta\alpha_2/m_*$. Since α_2 goes to one as θ (hence x) goes to infinity, we retrieve the fact that $r_1 r_2$ goes to one. Hence, as in the delocalized case we can write for every $x \geq x_0$ that for every event E

$$\mathbb{P}[E | |\lambda_{X_N} - x| \leq \delta] = \exp((o(\varepsilon) + o(\kappa) + o(\delta))N) \mathbb{E}_e[\mathbb{1}_{e \in \mathrm{Loc}_{r_1, r_2, \varepsilon}} \mathbb{P}^{(\bar{e}, \theta)}[E]]$$

where for $e \in \mathrm{Loc}_{r_1, r_2, \varepsilon}$, we denoted $\bar{e}$ the vector such that all the entries which are close to $r_2 N^{-1/4}$ are equal to this value and all the others are smaller than $\varepsilon N^{-1/4}$. Again, note that under $\mathbb{P}^{(\bar{e}, \theta)}$, we have

$$X_N = W + M_{\bar{e}, \theta}$$

where $M_{\bar{e}, \theta} = \mathbb{E}^{(\bar{e}, \theta)}[X_N] = 2\theta(\bar{e}\bar{e}^T - \bar{e}_1 \bar{e}_1^T) + L_N + R_N$. R_N is a matrix with negligible spectral radius, $\bar{e}_1$ is the restriction of $\bar{e}$ to the entries of order $\sqrt{r_2} N^{-1/4}$ and $L_N = (L'(\theta \bar{e}_1(i) \bar{e}_1(j))_{i,j}$. We may assume without loss of generality that they are the first $l = r_1 \sqrt{N}$ indices and then L_N is a $l \times l$ matrix with constant entry $\tilde{m}/\sqrt{N}$ where

$$\tilde{m} = \frac{\mathbb{E}[x e^{2\theta r_2 x}]}{\mathbb{E}[e^{2\theta r_2 x}]} = L'(2\theta r_2) = L'(m_*)\,.$$

L_N has rank one, with non-zero eigenvalue equal to $\gamma = \tilde{m} r_1 = 2\theta L'(m_*)\alpha_2/m_*$ and corresponding eigenvector $v = \frac{1}{\sqrt{\alpha_2}} \bar{e}_1$. Recalling that m_* is a critical point of ψ, we find that

$$\frac{L'(m_*)}{m_*^2} = \frac{2L(m_*)}{m_*^3} \Rightarrow \frac{L'(m_*)}{m_*} = 2\psi(m_*) = A\,,$$

so that $\gamma = 2\theta A\alpha_2$. Note that $\bar{e} = \sqrt{\alpha_2}v + \sqrt{1-\alpha_2}w$ with w a unit vector orthogonal to v. We then have up to a small error

$$M_{\bar{e},\theta} = 2\theta((1-\alpha_2)ww^T + \sqrt{\alpha_2(1-\alpha_2)}(vw^T + wv^T)) + 2\theta A\alpha_2 vv^T$$

We see that as x goes to infinity, α_2 goes to one and the largest eigenvalue of $M_{\bar{e},\theta}$ goes to $2\theta A$. More generally, the largest eigenvalue of $M_{\bar{e},\theta}$ is given by

$$\lambda_2^\theta = \theta\left((A+1)\alpha_2 + \sqrt{(A+1)^2\alpha_2^2 - 4(A-1)\alpha_2(1-\alpha_2)}\right)$$

and eigenvector v_2^θ converging to v when θ goes to infinity. Then, by the BBP transition [4, 10] the largest eigenvalue of X_N is given by $K_\sigma(\lambda_2^\theta)$ with $K_\sigma(x) = x + x^{-1}$. We therefore conclude that the optimal coefficients θ, α_2 must satisfy $x = K_\sigma(\lambda_2^\theta)$. At the same time, denoting by v_1^θ the second eigenvector of $M_{e,\theta}$, we find [10]:

$$M_{e,\theta} = \lambda_1^\theta v_1^\theta (v_1^\theta)^T + \lambda_2^\theta v_2^\theta (v_2^\theta)^T + o_N(1)$$

where $\lambda_1^\theta < \lambda_2^\theta$ and v_1^θ, v_2^θ are orthogonal unit vectors. Denoting u_{X_N} the eigenvector associated to the largest eigenvalue λ_{X_N} of X_N which is close to x, we deduce [10] that

$$u_{X_N} = C\left(\lambda_1^\theta \langle v_1^\theta, u\rangle (\lambda_{X_N} - W)^{-1} v_1^\theta + \lambda_2^\theta \langle v_2^\theta, u_{X_N}\rangle (\lambda_{X_N} - W)^{-1} v_2^\theta\right)$$

where C is the constant such that u is a unit vector. In expectation, the isotropic law shows that $\langle v_2^\theta, (\lambda_{X_N} - W)^{-1} v_1^\theta\rangle$ goes to zero. Hence, again by concentration of measure arguments we see that up to events with exponentially small probability

$$|\langle u, v_2^\theta\rangle|^2 = -\frac{G_\sigma^2(x)}{G'_\sigma(x)} + o(1)$$

Notice that the right hand side goes to one as x goes to infinity. Since when x goes to infinity, θ_x goes to infinity, α_2, v_2^θ goes to v which is the renormalized vector with $r_1\sqrt{N}$ non vanishing entries, and $r_1 = 2\theta\alpha_2/m_*$ the conclusion follows.

□

8. Appendix

8.1. Concentration for Wigner matrices with sub-Gaussian log-concave entries. In this section we show that Assumption 1.2 do not require to have compact support or log-Sobolev inequality as assumed in [18]. This hypothesis for instance would not include sparse Gaussian variables, whereas the following proposition handles this case.

Proposition 8.1. [20, 1] *Let μ be a symmetric probability measure on $\mathbb{R}$ which has log-concave tails in the sense that $t \mapsto \mu(x : |x| \geq t)$ is concave, and which is sub-Gaussian in the sense that* (1) *holds. Let X_N be a symmetric random matrix of size N such that $(X_{i,j})_{i\leq j}$ are independent random variables. Assume $\sqrt{N}X_{i,j}$ and $\sqrt{N/2}X_{i,i}$ have law μ for any $i \neq j$. There exists a numerical constant $\kappa > 0$ such that for any convex* 1*-Lipschitz function $f : \mathbb{R} \to \mathbb{R}$, and $t \geq 0$,*

$$\mathbb{P}\Big(\big|\frac{1}{n}\mathrm{Tr} f(X_N) - \frac{1}{n}\mathbb{E}\mathrm{Tr} f(X_N)\big| > t\Big) \leq 2e^{-\frac{\kappa}{A}N^2t^2}. \tag{62}$$

Moreover, for any $t > 0$,

$$\mathbb{P}\big(|\lambda_{X_N} - \mathbb{E}\lambda_{X_N}| > t\big) \leq 2e^{-\frac{\kappa}{A}Nt^2}. \tag{63}$$

One can take $\kappa = 1/8\beta^2$ with $\beta = 1680e$.

From these concentration inequalities, one can deduce as in the Appendix of [18] that a Wigner matrix with entries having sub-Gaussian and log-concave laws satisfy Assumptions 1.2.

Corollary 8.2. *Assume μ satisfies the assumptions of Proposition 8.1 and has variance 1. Then the matrix X_N satisfies Assumptions 1.2.*

We now prove Proposition 8.1. It will be a direct consequence of Klein's lemma (see [1, Lemma 4.4.12]) and the following concentration of convex Lipschitz functions under μ^n.

Proposition 8.3. *Let μ be a symmetric probability measure on $\mathbb{R}$ which has log-concave tails in the sense that $t \mapsto \mu(x : |x| \geq t)$ is concave, and which is sub-Gaussian in the sense that (1) holds. For any lower-bounded convex 1-Lipschitz function $f : \mathbb{R} \to \mathbb{R}$ such that $\int f d\mu^n = 0$ and any $t > 0$,*

$$\mu^n\big(x : |f(x)| > t\big) \leq 2e^{-\frac{t^2}{4\beta^2 A}},$$

where β is numerical constant. One can take $\beta = 1680e$.

Proof. By [23, Corollary 2.2], we know that there exists a numerical constant β such that μ^n satisfies a convex infimum convolution inequality with cost function $\Lambda^*(./\beta)$, where Λ^* is the Legendre transfom of Λ defined by,

$$\forall \theta \in \mathbb{R}^n,\ \Lambda(\theta) = \log \int e^{\langle \theta, x\rangle} d\mu^n(x).$$

Moreover, β can be taken to be $1680e$. More precisely, for any convex lower-bounded function $f : \mathbb{R} \to \mathbb{R}$,

$$\Big(\int e^{f \square \Lambda^*(./\beta)} d\mu^n\Big)\Big(\int e^{-f} d\mu^n\Big) \leq 1, \tag{64}$$

where $\square$ denotes the infimum convolution operator, defined by

$$f \square \Lambda^*(./\beta)(x) = \inf_{y \in \mathbb{R}^n} \big\{ f(y) + \Lambda^*\Big(\frac{y-x}{\beta}\Big)\big\}.$$

Since μ is sub-Gaussian in the sense of (1), for any $x \in \mathbb{R}^n$,

$$\Lambda^*(x) \geq \frac{1}{2A}||x||^2,$$

where $||\ ||$ denotes the Euclidean norm in $\mathbb{R}^n$. Therefore,

$$f \square \Lambda^*(./\beta)(x) \geq \inf_{y \in \mathbb{R}^n} \big\{ f(y) + \frac{1}{2\beta^2 A}||y-x||^2 \big\}.$$

Assume f is L-Lipschitz for some $L > 0$. Reproducing the arguments of [21, section 1.9, p19] we have for any $x \in \mathbb{R}^n$,

$$
\begin{aligned}
f\Box\Lambda^*(./\beta)(x) &\geq f(x) + \inf_{y\in\mathbb{R}^n} \Big\{ -L||y-x|| + \frac{1}{2\beta^2 A}||y-x||^2\Big\} \\
&\geq f(x) - \frac{1}{2}\beta^2 A L^2.
\end{aligned}
$$

Thus, by (64) we deduce that

$$
\Big(\int e^f d\mu^n\Big)\Big(\int e^{-f} d\mu^n\Big) \leq e^{\frac{1}{2}\beta^2 A L^2}. \tag{65}
$$

Assume now that f is 1-Lipschitz and $\int f d\mu^n = 0$. Using Jensen's inequality, we get for any $\lambda > 0$,

$$
\int e^{\lambda f} d\mu^n \leq e^{\frac{1}{2}\beta^2 A\lambda^2}.
$$

Using Chernoff inequality we deduce that for any $t > 0$,

$$
\mu^n\big(x : f(x) \geq t\big) \leq e^{-\frac{t^2}{4\beta^2 A}}.
$$

Using the symmetry in (65) between f and $-f$, we complete the proof. □

8.2. **A Uniform Varadhan's lemma.** We prove a quantitative version of Varadhan's lemma which is of independent interest.

Lemma 8.4. *Let $f : \mathbb{R} \to \mathbb{R}$ such that $f(0) = 0$ and $f(\sqrt{.})$ is L-Lipschitz for some $L > 0$. Let M_N, m_N be sequences such that $M_N = o(\sqrt{N})$ and $m_N = (1+o(1))N$. Let $g_1, \dots, g_{m_N}$ be independent Gaussian random variables conditioned to belong to $[-M_N, M_N]$. Let $\delta \in (0,1)$ and $c > 0$ such that $K^{-1} < c < K$ and $2\delta < K^{-1}$. Then,*

$$
\begin{aligned}
\Big|\frac{1}{N}\log \mathbb{E} e^{\sum_{i=1}^{m_N} f\big(\frac{g_i}{\sqrt{c}}\big)} \mathbb{1}_{|\sum_{i=1}^{m_N} g_i^2 - cN| \leq \delta N} - \sup_{\substack{\nu\in\mathcal{P}([-M_N,M_N]) \\ \int x^2 d\nu = c}} \Big\{\int f\Big(\frac{x}{\sqrt{c}}\Big) d\nu(x) - H(\nu|\gamma)\Big\}\Big| \\
\leq \varepsilon_{L,K}(N) + \varepsilon_L(\delta K),
\end{aligned}
$$

where $\varepsilon_{L,K}(N) \to +\infty$ as $N \to +\infty$ and $\varepsilon_L(x) \to 0$ as $x \to 0$.

Let $\varepsilon = 1/N$ and l_0 be the smallest integer such that $(1+\varepsilon)^{-l_0} \leq \varepsilon$. Define

$$
I_{l_0} = [-(1+\varepsilon)^{-l_0}, (1+\varepsilon)^{-l_0}], \text{ and } B_{l_0} = \{i : g_i \in I_{l_0}\}.
$$

For any $k > -l_0$, we set

$$
I_k = \{x \in \mathbb{R} : (1+\varepsilon)^{k-1} \leq |x| \leq (1+\varepsilon)^k\} \text{ and } B_k = \big\{i : g_i \in I_k\big\}. \tag{66}
$$

Let $\mu_k = |B_k|/m_N$. Let k_0 be the smallest integer such that $(1+\varepsilon)^{k_0} \geq M_N$. Since $g_i \in [-M_N, M_N]$ for all i, we obtain that for any $k > k_0$, $B_k = \emptyset$.

Lemma 8.5. *Let $\delta, \varepsilon = \frac{1}{N} \in [0,1]$ and $N/m_N \leq 2$. On the event $\big\{|\sum_{i=1}^{m_N} g_i^2 - cN| \leq \delta N\big\}$,*

$$
\Big|\frac{1}{m_N}\sum_{i=1}^{m_N} f\Big(\frac{g_i}{\sqrt{c}}\Big) - \sum_{k=-l_0}^{k_0} \mu_k f\Big(\frac{(1+\varepsilon)^k}{\sqrt{c}}\Big)\Big| \leq 30(c+1)CLK\varepsilon\,.
$$

Proof. As $f(\sqrt{.})$ is L-Lipschitz, we have

$$\Big|\frac{1}{m_N}\sum_{i=1}^{m_N} f\Big(\frac{g_i}{\sqrt{c}}\Big) - \sum_{k=-l_0}^{k_0}\mu_k f\Big(\frac{(1+\varepsilon)^k}{\sqrt{c}}\Big)\Big| \le \frac{1}{m_N}\sum_{k=-l_0}^{k_0}\sum_{i\in B_k}\Big|f\Big(\frac{g_i}{\sqrt{c}}\Big) - f\Big(\frac{(1+\varepsilon)^k}{\sqrt{c}}\Big)\Big|$$
$$\le \frac{L}{c}\sum_{k=-l_0+1}^{k_0}\mu_k(1+\varepsilon)^{2k}\big(1-(1+\varepsilon)^{-2}\big) + \frac{L}{c}\mu_{-l_0}(1+\varepsilon)^{-2l_0}\,.$$

Using the fact that $(1+\varepsilon)^{-l_0}\le\varepsilon$, we deduce

$$\Big|\frac{1}{m_N}\sum_{i=1}^{m_N} f\Big(\frac{g_i}{\sqrt{c}}\Big) - \sum_{k=-l_0}^{k_0}\mu_k f\Big(\frac{(1+\varepsilon)^k}{\sqrt{c}}\Big)\Big| \le \frac{3\varepsilon L}{c}\Big(\sum_{k=-l_0+1}^{k_0}\mu_k(1+\varepsilon)^{2k}+1\Big)\,.$$

But, on the other hand, on the event $\big\{|\sum_{i=1}^{m_N} g_i^2 - cN|\le\delta N\big\}$,

$$\sum_{k=-l_0+1}^{k_0}\mu_k(1+\varepsilon)^{2(k-1)} \le \frac{1}{m_N}\sum_{i=1}^{m_N} g_i^2 \le \frac{N}{m_N}(c+\delta)\le 2(c+1).$$

Thus, we conclude that

$$\Big|\frac{1}{m_N}\sum_{i=1}^{m_N} f\Big(\frac{g_i}{\sqrt{c}}\Big) - \sum_{k=-l_0}^{k_0}\mu_k f\Big(\frac{(1+\varepsilon)^k}{\sqrt{c}}\Big)\Big| \le \frac{3\varepsilon L}{c}\Big(2(1+\varepsilon)^2(c+1)+1\Big)\,.$$

□

Let $I=\{-l_0,\dots,k_0\}$ and $\mathcal{L}_N=\big\{y\in\mathbb{R}_+^I : \sum_{k\in I} y_k = 1,\ \forall k\in I, m_N y_k\in\mathbb{N}\big\}$. We know from [15, Lemma 2.1.6], that for any $y\in\mathcal{L}_N$,

$$(m_N+1)^{-n}e^{-m_N H(y|\gamma_{M_N})}\le\mathbb{P}(\mu_k=y_k,\ \forall k\in I)\le e^{-m_N H(y|\gamma_{M_N})}, \tag{67}$$

where $I_k=[(1+\varepsilon)^{k-1},(1+\varepsilon)^k]$, $n=|I|=l_0+k_0+1$, and with γ the standard Gaussian law

$$H(y|\gamma_{M_N})=\sum_{k\in I} y_k\log\frac{y_k}{\gamma_{M_N}(k)},\quad \text{with } \gamma_{M_N}(k)=\frac{\gamma(I_k)}{\gamma([-M_N,M_N])}.$$

Let $\mu=(\mu_k)_{k\in I}$ and denote

$$\mathcal{A}_{C_1,C_2}=\Big\{y\in\mathcal{L}_N : c-\delta+C_1\varepsilon\le\sum_{k=-l_0}^{k_0}(1+\varepsilon)^{2k}y_k\le c+\delta+C_2\varepsilon\Big\}$$

Then, by the previous lemma we see that there exists a finite constant $C=O(KL)$ such that if we denote $\mathcal{A}_\eta=\mathcal{A}_{-\eta C,\eta C}$, for N large enough,

$$\big\{\mu\in\mathcal{A}_-\big\}\subset\Big\{|\sum_{i=1}^{m_N} g_i^2-cN|\le\delta N\Big\}\subset\big\{\mu\in\mathcal{A}_+\big\}\,.$$

We used here the fact that the g_i belong to $[-M_N,M_N]$ and that $(I_k)_k$ is a partition of this set. By (67), we get the upper bound,

$$\mathbb{E}e^{m_N\sum_{k=-l_0}^{k_0}\mu_k f\left(\frac{(1+\varepsilon)^k}{\sqrt{c}}\right)}\mathbb{1}_{\mu\in\mathcal{A}_+}\le\sum_{y\in\mathcal{A}_+} e^{m_N\sum_{k=-l_0}^{k_0}y_k f\left(\frac{(1+\varepsilon)^k}{\sqrt{c}}\right)}e^{-m_N H(y|\gamma_{M_N})}, \tag{68}$$

whereas for the lower bound,

$$\mathbb{E}e^{m_N \sum_{k=-l_0}^{k_0} \mu_k f\left(\frac{(1+\varepsilon)^k}{\sqrt{c}}\right)} \mathbb{1}_{\mu\in\mathcal{A}_-} \geq (m_N+1)^{-n} \sum_{y\in\mathcal{A}_-} e^{m_N \sum_{k=-l_0}^{k_0} y_k f\left(\frac{(1+\varepsilon)^k}{\sqrt{c}}\right)} e^{-m_N H(y|\gamma_{M_N})}. \quad (69)$$

Let $y \in \mathcal{A}_+$ and define $\nu \in \mathcal{P}(\mathbb{R})$ by $d\nu(x) = \varphi(x)d\gamma(x)$, where

$$\varphi(x) = \sum_{k=-l_0}^{k_0} \mathbb{1}_{x\in I_k} \frac{y_k}{\gamma(I_k)}.$$

With this notation, we have

$$H(y|\gamma_{M_N}) = H(\nu|\gamma) - \log\gamma([-M_N, M_N]).$$

With the same argument as in Lemma 8.5, we also have for $y \in \mathcal{A}_+$,

$$\Big|\sum_{k=-l_0}^{k_0} y_k f\Big(\frac{(1+\varepsilon)^k}{\sqrt{c}}\Big) - \int f\Big(\frac{x}{\sqrt{c}}\Big)d\nu(x)\Big| \leq C\varepsilon, \Big|\int x^2 d\nu(x) - c\Big| \leq \delta + C\varepsilon \quad (70)$$

where C only depends on c. From (68) and Lemma 8.5, we deduce that

$$\frac{1}{N}\log\mathbb{E}e^{\sum_{i=1}^{m_N} f\left(\frac{g_i}{\sqrt{c}}\right)} \mathbb{1}_{|\sum_{i=1}^{m_N} g_i^2 - cN|\leq\delta N} \leq \frac{m_N}{N} \sup_{\substack{\nu\in\mathcal{P}([-M_N,M_N])\\ |\int x^2 d\nu(x) - c|\leq\delta+C\varepsilon}} \Big\{\int f\Big(\frac{x}{\sqrt{c}}\Big)d\nu(x) - H(\nu|\gamma)\Big\} + O(\varepsilon)$$

To complete the proof of the upper bound, we show the following result.

Lemma 8.6. *Let $K, L, \delta > 0$ such that $\delta < 2K^{-1}$ and $\varepsilon = \frac{1}{N}$. There exists a function $s_{L,K}$ depending on K and L such that for any function $f : \mathbb{R} \to \mathbb{R}$ such that $f(0) = 0$ and $f(\sqrt{\cdot})$ is L-Lipschitz, and any $K^{-1} < c < K$,*

$$\sup_{\substack{\nu\in\mathcal{P}([-M_N,M_N])\\ |\int x^2 d\nu(x) - c|\leq\delta+C\varepsilon}} \Big\{\int f\Big(\frac{x}{\sqrt{c}}\Big)d\nu(x) - H(\nu|\gamma)\Big\}$$

$$\leq \sup_{\substack{\nu\in\mathcal{P}([-M_N,M_N])\\ \int x^2 d\nu(x) = c}} \Big\{\int f\Big(\frac{x}{\sqrt{c}}\Big)d\nu(x) - H(\nu|\gamma)\Big\} + s_L((\delta+\varepsilon)K),$$

where $s_L(x) \to 0$ as $x \to 0$.

Proof. Let $\nu \in \mathcal{P}([-M_N, M_N])$ such that $|\int x^2 d\nu(x) - c| \leq \delta$. Let $\tilde{\nu} = h_\lambda \# \nu$ with the notations of (27).With the same arguments that below (27), we easily see that

$$H(\tilde{\nu}|\gamma) \leq H(\nu|\gamma) + \frac{1}{2}\delta + \frac{L}{2}\delta K,$$

which ends the proof. □

For the lower bound, fix ν a probability measure on $[-M_N, M_N]$ such that $\nu \ll \gamma$. We set $\varepsilon = \varepsilon_N$ such that $M_N^2/m_N\varepsilon_N \to 0$, and we define I_k and B_k as in (66). Define, for $k \in \{-l_0+1, \ldots, k_0\}$,

$$y_k = \frac{1}{m_N}\lfloor m_N \nu(I_k)\rfloor,$$

and $y_{-l_0} = 1 - \sum_{k=-l_0+1}^{k_0} y_k$. We claim that for N large enough and independent of ν,

$$\int x^2 d\nu(x) = c \Longrightarrow y \in \mathcal{A}_-.$$

Indeed, one can check that on one hand

$$\int x^2 d\nu(x) - \frac{(1+\varepsilon)^2 M_N^2}{m_N \varepsilon} \leq \sum_{k=-l_0}^{k_0} y_k (1+\varepsilon)^{2k} \leq (1+\varepsilon)^2 \int x^2 d\nu(x) + \varepsilon^2 \,.$$

We obtain from (69),

$$\log \mathbb{E} e^{m_N \sum_{k=-l_0}^{k_0} \mu_k f\left(\frac{(1+\varepsilon)^k}{\sqrt{c}}\right)} \mathbb{1}_{\mu \in \mathcal{A}_-} \geq (m_N+1)^{-n} e^{m_N \sum_{k=-l_0}^{k_0} y_k f\left(\frac{(1+\varepsilon)^k}{\sqrt{c}}\right)} e^{-m_N H(y|\gamma_{M_N})}. \quad (71)$$

In the next lemma we compare $H(y|\gamma_{M_N})$ and $H(\nu|\gamma)$.

Lemma 8.7.

$$H(y|\gamma_{M_N}) \leq H(\nu|\gamma) + o_N(1).$$

Proof. By definition we have,

$$H(y|\gamma_{M_N}) = \sum_{k=-l_0}^{k_0} y_k \log \frac{y_k}{\gamma(I_k)} + \log \gamma([-M_N, M_N]). \quad (72)$$

Let $f(x) = x \log x$ for $x > 0$ and $f(0) = 0$. We claim that

$$\forall 0 \leq x < y,\ f(x) \leq f(y) + (y - x). \quad (73)$$

Indeed, either $x > e^{-1}$ and $f(x) \leq f(y)$ since f is increasing on $[e^{-1}, +\infty)$. Or $x < e^{-1}$ and by convexity,

$$f(x) \leq f(y) + f'(x)(x - y).$$

Since $|f'(x)| \leq 1$ we get the claim. Note that we have for any $k > -l_0$,

$$\nu(I_k) - \frac{1}{m_N} < y_k \leq \nu(I_k), \text{ and } \nu(I_{-l_0}) \leq y_{-l_0} < \nu(I_{-l_0}) + \frac{k_0 + l_0}{m_N}.$$

Thus we deduce from (73) that

$$\begin{aligned} H(y|\gamma_{M_N}) &\leq \sum_{k=-l_0}^{k_0} \nu(I_k) \log \frac{\nu(I_k)}{\gamma(I_k)} + \sum_{k=-l_0+1}^{k_0} \gamma(I_k) \frac{1}{\gamma(I_k) m_N} + \gamma(I_{l_0}) \frac{k_0 + l_0}{\gamma(I_{-l_0}) m_N} + o_N(1) \\ &\leq \sum_{k=-l_0}^{k_0} \nu(I_k) \log \frac{\nu(I_k)}{\gamma(I_k)} + \frac{2(k_0 + l_0)}{m_N} + o_N(1). \end{aligned}$$

We have $k_0 = O(\log(M_N)/\varepsilon_N)$ and $l_0 = O(\log(1/\varepsilon_N)/\varepsilon_N)$. Since $M_N^2/m_N \varepsilon_N \to 0$, we get

$$H(y|\gamma_{M_N}) \leq \sum_{k=-l_0}^{k_0} \nu(I_k) \log \frac{\nu(I_k)}{\gamma(I_k)} + \frac{2(k_0 + l_0)}{m_N} + o_N(1).$$

Since $f : x \mapsto x \log x$ is convex, we complete the proof by using Jensen's inequality which yields

$$\sum_{k=-l_0}^{k_0} \nu(I_k) \log \frac{\nu(I_k)}{\gamma(I_k)} = \sum_{k=-l_0}^{k_0} \gamma(I_k) f\Big(\frac{1}{\gamma(I_k)} \int_{I_k} \frac{d\nu}{d\gamma} d\gamma\Big) \leq \sum_{k=-l_0}^{k_0} \int_{I_k} \frac{d\nu}{d\gamma} \log \frac{d\nu}{d\gamma} d\gamma.$$

□

Moreover, we can compare $\int f(x/\sqrt{c}) d\nu(x)$ and $\sum_{k=-l_0} y_k f((1+\varepsilon)^k/\sqrt{c})$.

Lemma 8.8.

$$\Big| \int f\Big(\frac{x}{\sqrt{c}}\Big) d\nu(x) - \sum_{k=-l_0}^{k_0} y_k f\Big(\frac{(1+\varepsilon)^k}{\sqrt{c}}\Big) \Big| \leq \varepsilon_{L,K}(N),$$

where $\varepsilon_{L,K}(N) \to 0$ *as* $N \to +\infty$.

Proof. As $f(\sqrt{.})$ is L-Lipschitz, we have on one hand using the same argument as in the proof of Lemma 8.5,

$$\Big| \int f\Big(\frac{x}{\sqrt{c}}\Big) d\nu(x) - \sum_{k=-l_0}^{k_0} \nu(I_k) f\Big(\frac{(1+\varepsilon)^k}{\sqrt{c}}\Big) \Big| \leq \frac{3L\varepsilon}{c} \Big(\int x^2 d\nu(x) + 1 \Big).$$

Therefore,

$$\Big| \int f\Big(\frac{x}{\sqrt{c}}\Big) d\nu(x) - \sum_{k=-l_0}^{k_0} \nu(I_k) f\Big(\frac{(1+\varepsilon)^k}{\sqrt{c}}\Big) \Big| \leq \varepsilon_{L,K}(N), \tag{74}$$

where $\varepsilon_{L,K}(N) \to 0$ as $N \to +\infty$. On the other hand, as $|\nu(I_k) - y_k| \leq 1/m_N$ for any $k > -l_0$ and $|\nu(I_{-l_0}) - y_{-l_0}| \leq (k_0 + l_0)/m_N$, we get since $f(0) = 0$ and $f(\sqrt{.})$ is L-Lipschitz

$$\begin{aligned} \sum_{k=-l_0}^{k_0} \Big| (y_k - \nu(I_k)) f\Big(\frac{(1+\varepsilon)^k}{\sqrt{c}}\Big) \Big| &\leq \frac{L}{c m_N} \sum_{k=-l_0+1}^{k_0} (1+\varepsilon)^{2k} + \frac{L(k_0+l_0)}{c m_N} (1+\varepsilon)^{-2l_0}, \\ &\leq \frac{\kappa L}{c m_N} \Big(\frac{M_N^2}{\varepsilon_N} + (k_0 + l_0) \varepsilon_N^2 \Big) \end{aligned}$$

As $M_N^2/m_N \to 0$ and $k_0 = O(\log(M_N)/\varepsilon_N)$ and $l_0 = O(\log(1/\varepsilon_N)/\varepsilon_N)$, we deduce that $\frac{(k_0+l_0)}{m_N} \varepsilon_N = o_N(1)$. Combining the above estimate with (70), we get the claim. □

Coming back to (71), using the results of Lemmas 8.7 and 8.8, we deduce

$$\mathbb{E} e^{m_N \sum_{k=-l_0}^{k_0} \mu_k f\left(\frac{(1+\varepsilon)^k}{\sqrt{c}}\right)} \mathbb{1}_{\mu \in \mathcal{A}_-} \geq (m_N + 1)^{-n} e^{m_N \int f(\frac{x}{\sqrt{c}}) d\nu(x) - m_N g_{L,\delta}(N)} e^{-m_N (H(\nu|\gamma) + o_N(1))},$$

which gives at the logarithmic scale,

$$\frac{1}{N} \log \mathbb{E} e^{m_N \sum_{k=-l_0}^{k_0} \mu_k f\left(\frac{(1+\varepsilon)^k}{\sqrt{c}}\right)} \mathbb{1}_{\mu \in \mathcal{A}_-} \geq \frac{m_N}{N} \Big(\int f\Big(\frac{x}{\sqrt{c}}\Big) d\nu(x) - H(\nu|\gamma) \Big) - \varepsilon_{L,K}(N),$$

We conclude by optimizing over the choice of $\nu \in \mathcal{P}([-M_N, M_N])$, such that $\int x^2 d\nu = c$.

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