Asymptotics of Symmetric Polynomials: A Dynamical Pointview

Alice Guionnet*[1] and Jiaoyang Huang†[2]

[1]CNRS-ENS Lyon, France
[2]University of Pennsylvania, United States of America

Abstract

In this paper we study the asymptotic behavior of the (skew) Macdonald and Jack symmetric polynomials as the number of variables grows to infinity. We characterize their limits in terms of certain variational problems. As an intermediate step, we establish a large deviation principle for the θ analogue of non-intersecting Bernoulli random walks. When $\theta = 1$, these walks are equivalent to random Lozenges tilings of strip domains, where the variational principle (with general domains and boundary conditions) has been proven in the seminal work [15] by Cohn, Kenyon, and Propp. Our result gives a new proof of this variational principle, and also extends it to non-intersecting θ-Bernoulli random walks for any $\theta \in (0, \infty)$. Remarkably, the rate functions remain identical, differing only by a factor of $1/\theta$.

Contents

1 Introduction

Macdonald symmetric functions, initially discovered by Macdonald in the late 1980s [40], are central to a number of key developments in mathematics and mathematical physics. They are indexed by Young diagrams and implicitly depend on two parameters $q, t \in (0, 1)$. Their degeneracy, by taking $t = q^\theta$ and sending q to approach one, gives the Jack polynomials. This family of symmetric functions depends

*aguionne@ens-lyon.fr
†huangjy@wharton.upenn.edu

on the positive parameter $\theta > 0$, and was first introduced in [35, 34]. Notably, when $\theta = 1$, the Jack polynomials coincide with the well-known Schur polynomials, up to some multiplicative constants.

These symmetric polynomials appear in algebraic combinatorics as generating functions, and provide a systematic way to enumerate combinatorial objects. In representation theory, these symmetric polynomials offer a crucial bridge between algebraic structures, combinatorial objects and group representations. Recently, symmetric polynomials have become central tools in the development of integrable stochastic models. This theory has found numerous applications in areas such as random partitions, random matrix theory, and directed polymers. In this article, we study the asymptotic behaviors of symmetric polynomials, namely (skew) Macdonald and Jack polynomials, as the number of parameters going to infinite. In this introduction, we discuss the results for skew Jack polynomials. The results on skew Macdonald polynomials are more involved and are presented in Section 1.4. Recall that skew Jack polynomials $J_{\boldsymbol{\lambda}\setminus\boldsymbol{\mu}}(\boldsymbol{x};\theta)$ are symmetric functions in infinitely many variables $\boldsymbol{x} = (x_i)_{1\leqslant i\leqslant\infty}$ and parametrized by two Young diagrams $\boldsymbol{\mu} \subset \boldsymbol{\lambda}$, where $\boldsymbol{\mu} = (\mu_1 \geqslant \mu_2 \geqslant \cdots \geqslant \mu_{\ell(\boldsymbol{\mu})}), \boldsymbol{\lambda} = (\lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_{\ell(\boldsymbol{\lambda})})$ are arrays of integer numbers with length $\ell(\boldsymbol{\mu}), \ell(\boldsymbol{\lambda})$ respectively. We refer to Section 6.1 and Section 6.2 for more detailed discussion on (skew) Jack polynomials. We consider the limit of skew Jack polynomials along a sequence of Young diagrams $\boldsymbol{\lambda}^{(N)}, \boldsymbol{\mu}^{(N)}$ and parameters $\boldsymbol{b}^{(N)}$, where the column and row sizes of Young diagrams and the number of parameters grow linearly in N. We show the limit $N^{-2}\ln J_{\boldsymbol{\lambda}\setminus\boldsymbol{\mu}}(\boldsymbol{x};\theta)$ exists and give explicitly characterization of the limit in terms of a variational problem.

1.1 Nonintersecting θ-Bernoulli Walk Ensembles

Understanding the asymptotic behavior of symmetric polynomials is fundamental to modern combinatorics. However, there are few results in this direction, as symmetric polynomials tend to have complicated structures and thus are less approachable by algebraic combinatorics techniques. In this article, we will study the asymptotic behaviors of symmetric polynomials via a dynamical approach.

Many symmetric polynomials arise naturally as the partition functions of interacting particle systems. Some notable examples are

1. The partition function of nonintersecting Brownian bridges is given by the Harish-Chandra-Itzykson-Zuber integral formula [24]. It is closely related with Schur polynomials.
2. Macdonald processes as introduced by Borodin and Corwin [8], by definition have Macdonald polynomials as their partition functions. Degenerations of Macdonald processes include the Schur processes [46, 44], Jack processes [23, 32], Hall-Littlewood processes [7] and Whittaker processes [8].
3. Partition functions of vertex models give families of symmetric rational functions, which generalize Schur symmetric polynomials, as well as some of their variations [1, 2, 6, 10].

On one hand, the asymptotic behavior of symmetric polynomials provides a valuable tool for characterizing the dynamics of interacting particle systems, where these polynomials serve as partition functions. On the other hand, this also offers an alternative approach to investigating the asymptotic behaviors of symmetric polynomials by exploring the asymptotic behaviors of the associated interacting particle systems using dynamical approaches.

Fix large N, and take Young diagrams $\boldsymbol{\lambda} = (\lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_N), \boldsymbol{\mu} = (\mu_1 \geqslant \mu_2 \geqslant \cdots \geqslant \mu_N)$, and parameters $\boldsymbol{b} = (b_0, b_1, b_2, \cdots, b_{\mathsf{T}-1})$. The skew Jack symmetric polynomials $J_{\boldsymbol{\lambda}'\setminus\boldsymbol{\mu}'}(\boldsymbol{b};\theta^{-1})$ (where $\boldsymbol{\lambda}', \boldsymbol{\mu}'$ are the transposes of $\boldsymbol{\lambda}, \boldsymbol{\mu}$), can be interpreted as the partition functions of N-particle nonintersecting θ-Bernoulli walks. For any Young diagrams $\boldsymbol{\nu} = (\nu_1 \geqslant \nu_2 \geqslant \cdots \geqslant \nu_N)$ with at most N rows, we encode it by a particle configuration $\mathsf{x} = \mathsf{x}(\boldsymbol{\nu}) = (\mathsf{x}_1, \mathsf{x}_2, \cdots, \mathsf{x}_N)$, where

$$\mathsf{x}_i = \nu_i - (i-1)\theta, \quad 1 \leqslant i \leqslant N. \tag{1.1}$$ {{e:xtonu}}

From the construction, this particle configuration lives on the following θ-dependent lattice

$$\mathbb{W}_\theta^N := \big\{(\mathsf{x}_1, \dots, \mathsf{x}_N) \in \mathbb{R}^N \mid \mathsf{x}_1 \in \mathbb{Z}, \quad \mathsf{x}_i - \mathsf{x}_{i+1} \in \theta + \mathbb{Z}_{\geqslant 0}, \quad i = 1, 2, \dots, N-1\big\}. \tag{1.2}$$ {{e:defW}}

{d:bernoulliwal

Definition 1.1 (non-intersecting θ-Bernoulli walk ensembles)**.** *An n-particle non-intersecting θ-Bernoulli walk from time 0 to S is a sequence of particle configurations $\mathsf{p} = \big(\mathbf{x}(0), \mathbf{x}(1), \dots, \mathbf{x}(\mathsf{S})\big) \in (\mathbb{W}_\theta^n)^{\mathsf{S}}$ such that $\mathbf{x}(\mathsf{t}) = (\mathsf{x}_1(\mathsf{t}), \mathsf{x}_2(\mathsf{t}), \cdots, \mathsf{x}_n(\mathsf{t}))$, and $\mathsf{x}_i(\mathsf{t}+1) - \mathsf{x}_i(\mathsf{t}) \in \{0,1\}$ for each $\mathsf{t} \in [0, \mathsf{S}]$; viewing $\{\mathsf{x}_i(\mathsf{t})\}_{0 \leqslant \mathsf{t} \leqslant \mathsf{S}}$ as the space-time trajectory for the i-th particle, which may either jump to the right or not move at each step. From the construction of the lattice (1.2), the paths{e:path} are non-intersecting, since we have $\mathsf{x}_i(\mathsf{t}) > \mathsf{x}_j(\mathsf{t})$ whenever $1 \leqslant i < j \leqslant n$ and $0 \leqslant \mathsf{t} \leqslant \mathsf{S}$.*

Assume $\boldsymbol{\mu}, \boldsymbol{\lambda}$ are Young diagrams with at most N rows. We identify them as particle configurations $\mathbf{y} = (\mathsf{y}_1 > \mathsf{y}_2 > \cdots > \mathsf{y}_N), \mathbf{z} = (\mathsf{z}_1 > \mathsf{z}_2 > \cdots > \mathsf{z}_N) \in \mathbb{W}_\theta^N$

$$\mathsf{y}_i = \mu_i - \theta(i-1), \quad \mathsf{z}_i = \lambda_i - \theta(i-1), \quad 1 \leqslant i \leqslant N, \tag{1.3}$$ {{e:x2}}

We denote the set of non-intersecting θ-Bernoulli walks from $\mathbf{y} = (\mathsf{y}_1 > \mathsf{y}_2 > \cdots > \mathsf{y}_N) \in \mathbb{W}_\theta^N$ to $\mathbf{z} = (\mathsf{z}_1 > \mathsf{z}_2 > \cdots > \mathsf{z}_N) \in \mathbb{W}_\theta^N$ as

$$\mathcal{P}(\mathbf{y}, \mathbf{z}; \mathsf{T}) = \{\mathsf{p} = \{\mathbf{x}(\mathsf{t})\}_{0 \leqslant \mathsf{t} \leqslant \mathsf{T}} \in (\mathbb{W}_\theta^N)^{\mathsf{T}} : \mathbf{x}(0) = \mathbf{y}, \mathbf{x}(\mathsf{T}) = \mathbf{z}\}, \tag{1.4}$$ {{e:pathset}}

The set $\mathcal{P}(\mathbf{y}, \mathbf{z}; \mathsf{T})$ is nonempty if there exists a non-intersecting Bernoulli walk $\{\mathbf{x}(\mathsf{t})\}_{0 \leqslant \mathsf{t} \leqslant \mathsf{T}}$ from $\mathbf{x}(0) = \mathbf{y}$ to $\mathbf{x}(\mathsf{T}) = \mathbf{z}$, which is equivalent to

$$\mathsf{y}_i \leqslant \mathsf{z}_i \leqslant \mathsf{y}_i + \mathsf{T}, \quad 1 \leqslant i \leqslant N. \tag{1.5}$$ {{e:tilable}}

Indeed, if (1.5) holds{e:tilable} we can take

$$\mathsf{x}_i(\mathsf{t}) = \max\{\mathsf{y}_i, \mathsf{z}_i - (\mathsf{T} - \mathsf{t})\}, \quad 1 \leqslant i \leqslant N. \tag{1.6}$$

Then we can rewrite the skew Jack polynomial $J_{\boldsymbol{\lambda}'/\boldsymbol{\mu}'}(\cdot; \theta^{-1})$ evaluated at $(b_0, b_1, \cdots, b_{\mathsf{T}-1})$, in terms of the weights (1.8) as follows

$$J_{\boldsymbol{\lambda}'/\boldsymbol{\mu}'}(b_0, \cdots, b_{\mathsf{T}-1}; \theta^{-1}) = \frac{J_{\boldsymbol{\mu}}(1^N; \theta)}{J_{\boldsymbol{\lambda}}(1^N; \theta)} \sum_{\mathsf{p} \in \mathcal{P}(\mathbf{y}; \mathbf{z}; \mathsf{T})} \mathcal{W}(\mathsf{p}; \boldsymbol{b}), \tag{1.7}$$ {{e:Jackexp}}

where $\boldsymbol{\lambda}', \boldsymbol{\mu}'$ are the transposes of $\boldsymbol{\lambda}, \boldsymbol{\mu}$, and for non-intersecting θ-Bernoulli walk $\mathsf{p} = \{\mathbf{x}(t)\}_{0 \leqslant \mathsf{t} \leqslant \mathsf{T}}$ and $\boldsymbol{b} = (b_0, b_1, \cdots, b_{\mathsf{T}-1})$, we define their weights as

$$\mathcal{W}(\mathsf{p}; \boldsymbol{b}) = \prod_{0 \leqslant \mathsf{t} \leqslant \mathsf{T}-1} \prod_{1 \leqslant i < j \leqslant n} \frac{(\mathsf{x}_i(\mathsf{t}) + \theta e_i(\mathsf{t})) - (\mathsf{x}_j(\mathsf{t}) + \theta e_j(\mathsf{t}))}{\mathsf{x}_i(\mathsf{t}) - \mathsf{x}_j(\mathsf{t})} \prod_{1 \leqslant i \leqslant N} \mathsf{b}_{\mathsf{t}}^{e_i(\mathsf{t})} \tag{1.8}$$ {{e:weightpb}}

where $\boldsymbol{e}(\mathsf{t}) = (e_1(\mathsf{t}), e_2(\mathsf{t}), \cdots, e_N(\mathsf{t}))$ is the increment at time t,

$$\boldsymbol{e}(\mathsf{t}) = \mathbf{x}(\mathsf{t}+1) - \mathbf{x}(\mathsf{t}) \in \{0,1\}^N. \tag{1.9}$$

The formula (1.7) is derived from the construction of the Jack process, as explained in more detail in Section 6. On the left-hand side of (1.7), there are explicit formulas for the Jack symmetric polynomials evaluated at 1^N, i.e. $J_{\boldsymbol{\mu}}(1^N; \theta), J_\lambda(1^N; \theta)$, making them amenable for asymptotic analysis. The primary challenge lies in analyzing the sum of the weights $\mathcal{W}(\mathsf{p}; \boldsymbol{b})$. In the special case where $b_i = 1$, this sum of weights can be interpreted as the partition function of non-intersecting θ-Bernoulli random walks from $\mathbf{y}$ to $\mathbf{z}$, with the transition probability given by

$$\mathbb{P}(\mathbf{x}(\mathsf{t}+1) = \mathbf{x} + \boldsymbol{e} | \mathbf{x}(\mathsf{t}) = \mathbf{x}) = \frac{1}{2^N} \frac{V(\mathbf{x} + \theta \boldsymbol{e})}{V(\mathbf{x})} = \frac{1}{2^N} \prod_{1 \leqslant i < j \leqslant N} \frac{(\mathsf{x}_i + \theta e_i) - (\mathsf{x}_j + \theta e_j)}{\mathsf{x}_i - \mathsf{x}_j}, \tag{1.10}$$ {{e:mdensity2}}

for $0 \leqslant \mathsf{t} \leqslant \mathsf{T} - 1$, where V is the Vandermonde determinant, and $\boldsymbol{e} = (e_1, e_2, \cdots, e_N) \in \{0,1\}^N$, see Figure 1. We remark that the above transition probability is given by pairwise interactions, and the interactions between adjacent particles are singular to prevent colliding. More precisely, from the above

Figure 1: Shown above is the transition from the Young diagram $\boldsymbol{\mu} = (5, 3, 2, 1, 1, 1)$ to the Young diagram $\boldsymbol{\lambda} = (6, 4, 2, 2, 1, 1)$, which are encoded by the particle systems through the relation (1.1). {f:Young_diagra

Young_diagram.pdf

transition probability, if $\mathsf{x} \in \mathbb{W}_\theta^N$ and $\mathsf{x}_i - \mathsf{x}_{i+1} \geqslant \theta + 1$, then $(\mathsf{x}_i + e_i) - (\mathsf{x}_{i+1} + e_{i+1}) \in \theta + \mathbb{Z}_{\geqslant 0}$. If $\mathsf{x}_i - \mathsf{x}_{i+1} = \theta$, then (1.10) is nonzero only {nonzero:y2} if $(e_i, e_{i+1}) \in \{(0,0), (1,0), (1,1)\}$. In these cases, $(\mathsf{x}_i + e_i) - (\mathsf{x}_{i+1} + e_{i+1}) \in \theta + \mathbb{Z}_{\geqslant 0}$. Therefore the Markov process (1.10) stays in the lattice $\mathbb{W}_\theta^N$.

Our first main result establishes a large deviation principle for the non-intersecting θ-Bernoulli random walks (1.10). Notably, when $\theta = 1$, these walks reduce to non-intersecting Bernoulli random walks, which is equivalent to random Lozenge tiling of strip domains. Cohn, Kenyon, and Propp [15] previously proved a variational principle for random tiling models (domino tiling and lozenge tiling) with general boundary conditions. Our result extends this variational principle to non-intersecting θ-Bernoulli random walks for any $\theta \in (0, \infty)$. Remarkably, the rate functions remain identical, differing only by a factor of $1/\theta$. To state the large deviation principle, we need to introduce the notations of height function and surface tension.

1.2 Height Function and Surface Tension

Given any N-particle nonintersecting θ-Bernoulli walk ensemble $\mathsf{p} = \big(\mathsf{x}(0), \mathsf{x}(1), \dots, \mathsf{x}(\mathsf{T})\big) \in (\mathbb{W}_\theta^N)^{\mathsf{T}}$ from time 0 to time T, we denote the increment at time t as $\boldsymbol{e}(\mathsf{t}) = (e_1(\mathsf{t}), e_2(\mathsf{t}), \cdots, e_N(\mathsf{t}))$,

$$\boldsymbol{e}(\mathsf{t}) = \mathsf{x}(\mathsf{t}+1) - \mathsf{x}(\mathsf{t}) \in \{0,1\}^N. \tag{1.11}$$

We denote the rescaled particle configuration (by rescaling space and time by $1/N$),

$$\boldsymbol{x}(t) = (x_1(t), x_2(t), \cdots, x_N(t)) = \frac{\mathsf{x}(\mathsf{t})}{N}, \quad t = \frac{\mathsf{t}}{N}, \quad 0 \leqslant \mathsf{t} \leqslant \mathsf{T}. \tag{1.12}$$ {{e:scaling}}

In the rest of the paper, we will use the scaling above (we will use $\mathsf{x}(\mathsf{t})$ for particle configurations before scaling, and $\boldsymbol{x}(t)$ for after scaling). For $\mathsf{t}/N \leqslant t \leqslant (\mathsf{t}+1)/N$, we linearly interpolate the particle configurations

$$\boldsymbol{x}(t) = (\mathsf{t} + 1 - Nt)\, \boldsymbol{x}\left(\frac{\mathsf{t}}{N}\right) + (Nt - \mathsf{t})\, \boldsymbol{x}\left(\frac{(\mathsf{t}+1)}{N}\right) = \boldsymbol{x}(\mathsf{t}) + (Nt - \mathsf{t})\, \frac{\boldsymbol{e}(\mathsf{t})}{N}. \tag{1.13}$$

We encode the particle densities of $\boldsymbol{x}(t)$ as

$$\rho(x; \boldsymbol{x}(t)) = \sum_{i=1}^{N} \mathbf{1}(x \in [x_i(t), x_i(t) + \theta/N]), \quad 0 \leqslant t \leqslant T, \tag{1.14}$$ {{eq_rho_x_def

and define the associated height function

$$H(x,t) = \int_{-\infty}^{x} \rho(y; \boldsymbol{x}(t)) \mathrm{d}y, \quad (x,t) \in \mathbb{R} \times [0,T]. \tag{1.15}$$ {{e:Hxt}}

We notice that the total mass of ρ in (1.14) is θ for any $0 \leqslant t \leqslant \theta T$, and $H(x,t)$ is a non-decreasing function of x from 0 to θ. The particle configuration $\boldsymbol{x}(t)$ can be recovered easily by either the empirical particle density $\varrho(x; \boldsymbol{x}(t))$, or the height function $H(x,t)$. In the rest of the paper, we will not distinguish them.

Given an N-particle nonintersecting θ-Bernoulli walk ensemble $\mathsf{p} = \{\mathsf{x}(\mathsf{t})\}_{0 \leqslant \mathsf{t} \leqslant \mathsf{T}}$, we recall its weight $\mathcal{W}(\mathsf{p}; \boldsymbol{b})$ from (1.8). For $\boldsymbol{b} = 1^{\mathsf{T}}$, we simply denote the weight as $\mathcal{W}(\mathsf{p})$,

$$\mathcal{W}(\mathsf{p}) = \mathcal{W}(\mathsf{p}; 1^{\mathsf{T}}) = \prod_{0 \leqslant \mathsf{t} \leqslant \mathsf{T}} \prod_{1 \leqslant i < j \leqslant N} \frac{(\mathsf{x}_i(\mathsf{t}) + \theta e_i(\mathsf{t})) - (\mathsf{x}_j(\mathsf{t}) + \theta e_j(\mathsf{t}))}{\mathsf{x}_i(\mathsf{t}) - \mathsf{x}_j(\mathsf{t})}. \tag{1.16}$$ {{e:weightp}}

Under the transition probability (1.10), the probability of the walk p is given by $2^{-N\mathsf{T}}\mathcal{W}(\mathsf{p})$. We also define the weight of the height function H associated with $\mathsf{p} = \{\mathsf{x}(\mathsf{t})\}_{0\leqslant \mathsf{t}\leqslant \mathsf{T}}$ as $\mathcal{W}(H) = \mathcal{W}(\mathsf{p})$.

From the construction of the height function (1.15), $H(x,t)$ is 2-Lipschitz, and almost surely,

$$\nabla H(x,t) = (\partial_x H(x,t), \partial_t H(x,t)) \in \{(0,0),(1,0),(1,-1)\}. \tag{1.17}$$

To analyze the limits of height functions of these interacting particle systems, it will be useful to introduce continuum analogs of the height functions considered in (1.15). So, set

$$\mathcal{T} = \big\{(s,t) \in (0,1)\times \mathbb{R}_{<0} : s+t>0\big\} \subset \mathbb{R}^2, \tag{1.18}$$ {{t}}

and its closure $\overline{\mathcal{T}} = \big\{(s,t) \in [0,1]\times \mathbb{R}_{\leqslant 0} : s+t \geq 0\big\}$. We interpret $\overline{\mathcal{T}}$ as the set of possible gradients, also called *slopes*, for a continuum height function. The height functions $H(x,t)$ associated with particle configurations (from (1.15)) is 2-Lipschitz and satisfies $\nabla H(x,t) \in \overline{\mathcal{T}}$ for $(x,t) \in \mathfrak{R} = \mathbb{R}\times[0,T]$.

Definition 1.2. *We say that a function $H : \mathfrak{R} \to \mathbb{R}$ is* admissible *if H is 2-Lipschitz and $\nabla H(u) \in \overline{\mathcal{T}}$ for almost all $u \in \mathfrak{R}$. We further say that the boundary height function $h = (h(x,0), h(x,T)) : \partial\mathfrak{R} \to \mathbb{R}$* admits an admissible extension to $\mathfrak{R}$ *if $\mathrm{Adm}(\mathfrak{R}; h)$, the set of admissible functions $H : \mathfrak{R} \to \mathbb{R}$ with $H|_{\mathbb{R}\times\{0,T\}} = h$, is not empty.*

For any $x \in \mathbb{R}_{\geqslant 0}$ and $(s,t) \in \overline{\mathcal{T}}$ we denote the *Lobachevsky function* $L : \mathbb{R}_{\geqslant 0} \to \mathbb{R}$ and the *surface tension* $\sigma : \overline{\mathcal{T}} \to \mathbb{R}^2$ by

$$L(x) = -\int_0^x \log|2\sin z|\mathrm{d}z; \qquad \sigma(s,t) = \frac{1}{\pi}\Big(L\big(\pi(1-s)\big) + L(-\pi t) + L\big(\pi(s+t)\big)\Big). \tag{1.19}$$ {{sigmal}}

For any admissible height function H on $\mathfrak{R}$, we further denote the *entropy functional*

$$\mathcal{E}(H) = \int_{\mathfrak{R}} \sigma\big(\nabla H(z)\big)\mathrm{d}z. \tag{1.20}$$ {{efunctionh}}

The entropy functional (1.20) also governs the logarithm of the number of lozenge tilings for a given domain and boundary height function [15]. The surface tension σ is strictly concave within the interior of $\mathcal{T}$ and linear along the boundary of $\overline{\mathcal{T}}$. As a consequence, the maximizer of the energy functional (1.20) possesses what are known as "liquid regions", where the solution is real analytic, and "frozen regions", where the solution is piecewise linear. Variational problems associated with (1.20), and in more general setting, have been explored in previous studies [36, 52, 4].

1.3 Large Deviation Principle for non-intersecting θ-Bernoulli random walks

We prove a large deviation for non-intersecting θ-Bernoulli random walks (1.10) with given boundary condition at time 0 and time T. We make the following assumption on the height profile of the limiting boundary condition.

{a:boundaryheig

Definition 1.3. *Fix any $\theta > 0$. We denote by $\mathrm{Adm}^{\partial}_{\theta}(\mathfrak{R})$ the set of boundary height functions $h = (h(x,0), h(x,T)) : \partial\mathfrak{R} \mapsto \mathbb{R}$ which admit an admissible extension to $\mathfrak{R}$ so that $\partial_x h(x,0), \partial_x h(x,T)$ are two compactly supported positive measures with density bounded by 1 and total mass θ.*

We equip $\mathrm{Adm}(\mathfrak{R}, h)$ with the uniform topology. First, we remark that since we assumed H is Lipschitz, the uniform topology in the spatial variable is equivalent to the weak topology on its derivative, as can be easily checked by integration by parts. Moreover, it is evident that $\mathrm{Adm}(\mathfrak{R}, h)$ is a compact space. We also observe that $\mathrm{Adm}(\mathfrak{R}, h)$ depends on θ due to the fact that $h \in \mathrm{Adm}^{\partial}_{\theta}(\mathfrak{R})$. We want to stress this dependency because it has important consequences. Indeed, if $H \in \mathrm{Adm}(\mathfrak{R}, h)$, then for every $t \in [0,T]$, $\partial_x H(.,T)$ is compactly supported, bounded by one, and has total mass θ. Indeed, because $\nabla H \in \overline{\mathcal{T}}$, $\partial_x H(.,t)$ is a non-negative measure for all time t. Furthermore, since $\partial_t H(x,t)$ is non-positive, $H(x,T) \leqslant H(x,t) \leqslant H(x,0)$, and $H(x,t)$ is equal to θ for x sufficiently large x.

To state our main result, let us precise our running assumption concerning the particle configurations.

{assume}

Definition 1.4. *Let $\theta, T > 0$ and $h \in \mathrm{Adm}^{\partial}_{\theta}(\mathfrak{R})$. Given two sequences of particle configurations (y, z),*

$$\mathbf{y}^{(N)} = (\mathsf{y}_1^{(N)} \geqslant \mathsf{y}_2^{(N)} \geqslant \cdots \geqslant \mathsf{y}_N^{(N)}) \in \mathbb{W}_\theta^N, \quad \mathbf{z}^{(N)} = (\mathsf{z}_1^{(N)} \geqslant \mathsf{z}_2^{(N)} \geqslant \cdots \geqslant \mathsf{z}_N^{(N)}) \in \mathbb{W}_\theta^N, \quad N \geqslant 1\,. \tag{1.21}$$

we say that (y, z) are (h, θ, T)-admissible iff

1. *There exists a constant $C > 0$, $|\mathsf{y}_i^{(N)}|, |\mathsf{z}_i^{(N)}| \leqslant CN$ for $1 \leqslant i \leqslant N$.*
2. *The empirical density (recall from (1.14)) of $\mathbf{y}^{(N)}, \mathbf{z}^{(N)}$ converges, namely when $N \to \infty$*

$$\varrho(x; \mathbf{y}^{(N)}/N) \to \partial_x h(x, 0), \quad \varrho(x; \mathbf{z}^{(N)}/N) \to \partial_x h(x, T), \tag{1.22}$$ {{e:rhoconverge

in distribution.

Moreover, given $T > 0$ and T so that T/N goes to T as N goes to infinity, we further assume that for N large enough the set of non-intersecting θ-Bernoulli walks of length T from $\mathbf{y}^{(N)}$ to $\mathbf{z}^{(N)}$ is nonempty, namely $\mathcal{P}(\mathbf{y}^{(N)}, \mathbf{z}^{(N)}; \mathsf{T}) \neq \emptyset$ (see (1.4)).{e:rhoconverge}

We can now state the main result of this article concerning the large deviations for the distribution of the height function under the law $\mathbb{P}$ defined in (1.10).

{t:main1}

Theorem 1.5. *Let $\theta, T > 0$ and $h \in \mathrm{Adm}^{\partial}_{\theta}(\mathfrak{R})$. Consider two sequences of particle configurations (y, z) which are (h, θ, T)-admissible.*

1. *Then, for any $H \in \mathrm{Adm}(\mathfrak{R}, h)$, the following holds*

$$\lim_{\varepsilon \to 0} \limsup_{N \to \infty} \frac{1}{N^2} \ln \mathbb{P}(\{G \in \mathcal{P}(\mathbf{y}^{(N)}, \mathbf{z}^{(N)}; \mathsf{T}) : \|G - H\|_\infty \leqslant \varepsilon\}) = -\frac{1}{\theta} \mathcal{J}_h(H) + T \ln 2 \tag{1.23}$$ {{limsupwldp}}

where $\mathcal{J}_h$ is the functional defined for $H \in \mathrm{Adm}(\mathfrak{R}, h)$ by

$$\mathcal{J}_h(H) = -\int \sigma(\nabla H(x,t)) \mathrm{d}x\mathrm{d}t - \frac{1}{2} \int \ln|x - y| \mathrm{d}h(x,t) \mathrm{d}h(x,t) \Big|_0^T . \tag{1.24}$$ {{GRF}}

Moreover, (1.23) holds if we replace the $\limsup$ by a $\liminf$. {GRF}

2. *$\mathcal{J}_h$ has compact level sets in $\mathrm{Adm}(\mathfrak{R}, h)$.*
3. *The law of H conditioned to remain in $\mathcal{P}(\mathbf{y}^{(N)}, \mathbf{z}^{(N)}; \mathsf{T})$ satisfies a large deviation principle with speed N^2 and good rate function $\mathcal{I}_h(H)$ which is infinite outside $\mathrm{Adm}(\mathfrak{R}, h)$ and otherwise given by*

$$\mathcal{I}_h(H) = \frac{1}{\theta} (\mathcal{J}_h(H) - \inf_{G \in \mathrm{Adm}(\mathfrak{R}, h)} \mathcal{J}_h(G))\,. \tag{1.25}$$ {{e:Ih}}

Moreover, $\mathcal{I}_h$ is minimized at a unique minimizer $H^h \in \mathrm{Adm}(\mathfrak{R}, h)$.

The surface tension σ is the same as that of lozenge tiling. It has been proven in [18, Theorem 7.5], that $\mathcal{J}_h$ is lower semicontinuous over $\mathrm{Adm}(\mathfrak{R}, h)$ with the uniform topology. Since $\mathrm{Adm}(\mathfrak{R}, h)$ is compact, it follows that $\mathcal{J}_h$ has compact level sets in $\mathrm{Adm}(\mathfrak{R}, h)$. The uniqueness of the minimizer for the variational principle (1.25) has been proven in [52, Proposition 4.5]. {e:Ih}

Our second main result derives the large deviation asymptotics of skew Jack symmetric polynomials as the number of variables goes to infinity. As discussed after (1.7), the main challenge lies in analyzing the sum of the weights $\mathcal{W}(\mathsf{p}; \boldsymbol{b})$ from (1.8). It can be interpreted as the partition function of non-intersecting θ-Bernoulli random walks from $\mathbf{y}$ to $\mathbf{z}$, with time dependent drift. Explicitly, the transition probability is given by

$$\mathbb{P}^{\mathsf{b}}(\mathbf{x}(\mathsf{t}+1) = \mathbf{x} + \boldsymbol{e} | \mathbf{x}(\mathsf{t}) = \mathbf{x}) = \frac{1}{(1+\mathsf{b}_\mathsf{t})^N} \prod_{1 \leqslant i < j \leqslant N} \frac{(\mathsf{x}_i + \theta e_i) - (\mathsf{x}_j + \theta e_j)}{\mathsf{x}_i - \mathsf{x}_j} \prod_{1 \leqslant i \leqslant N} \mathsf{b}_\mathsf{t}^{e_i(\mathsf{t})}, \tag{1.26}$$ {{e:mdensitydri

for $0 \leqslant \mathsf{t} \leqslant \mathsf{T}-1$ and $\boldsymbol{e} = (e_1, e_2, \cdots, e_N) \in \{0,1\}^N$. The large deviation of the aforementioned non-intersecting θ-Bernoulli random walks with time-dependent drift can be deduced from Theorem 1.5 through the application of Varadhan's lemma. An immediate consequence of this is the following result concerning the asymptotics of skew Jack polynomials following from the formula (1.7). The proof for this {e:proofsforthis} result is provided in Section 6.4. We next state our large deviation results under the law $\mathbb{P}^{\mathsf{b}}$ and deduce the asymptotics of skew Jack polynomials in the related scaling.

{t:main2}

Theorem 1.6. *Let $\theta, T > 0$ and $h \in \mathrm{Adm}_\theta^\partial(\mathfrak{R})$. We consider a sequence of Young diagrams*

$$\boldsymbol{\lambda}^{(N)} = (\lambda_1^{(N)} \geqslant \lambda_2^{(N)} \geqslant \cdots \geqslant \lambda_N^{(N)}), \quad \boldsymbol{\mu}^{(N)} = (\mu_1^{(N)} \geqslant \mu_2^{(N)} \geqslant \cdots \geqslant \mu_N^{(N)}), \quad N \geqslant 1 \tag{1.27}$$

such that (y, z) given by (1.3) are (h, θ, T)-admissible and take $\mathsf{b}_\mathsf{t} = e^{f(\mathsf{t}/N)}$ for a continuously differentiable function $f(t)$.

1. *The asymptotics of the skew Jack polynomials are given by*

$$\begin{aligned}&\lim_{N\to\infty} \frac{1}{N^2} \ln J_{(\boldsymbol{\lambda}^{(N)})' \setminus (\boldsymbol{\mu}^{(N)})'}(b_0, b_1, \cdots, b_{\mathsf{T}-1}; \theta^{-1}) \\ &= \mathcal{J}_h^f + \frac{1}{2\theta}\left(\int_{\mathbb{R}^2} \ln|x-y| \mathrm{d}h(x,0)\mathrm{d}h(y,0) - \int_{\mathbb{R}^2} \ln|x-y|\mathrm{d}h(x,T)\mathrm{d}h(y,T)\right) + T\ln 2\end{aligned}$$

   where

$$\mathcal{J}_h^f := \frac{1}{\theta} \sup_{H \in \mathrm{Adm}(\mathfrak{R},h)} \left\{ \int \sigma(\nabla H(x,t))\mathrm{d}x\mathrm{d}t + \mathcal{F}^f(H) \right\} \tag{1.28}$$ {{lim1}}

   and the linear functional $\mathcal{F}^f(H)$ is given by

$$\mathcal{F}^f(H) := -\int_0^T \int_{\mathbb{R}} f(s)\partial_s H(y,s)\mathrm{d}y\mathrm{d}s. \tag{1.29}$$ {{lim2}}

2. *The law of H under $\mathbb{P}^{\mathsf{b}}$ defined in (1.26) conditioned {lim2} to remain in $\mathcal{P}(\mathbf{y}^{(N)}, \mathbf{z}^{(N)}; \mathsf{T})$ satisfies a large deviation principle with speed N^2 and good rate function $\mathcal{I}_h^f(H)$ which is infinite outside $\mathrm{Adm}(\mathfrak{R}, h)$ and otherwise given by*

$$\mathcal{I}_h^f(H) = -\frac{1}{\theta}\left(\int \sigma(\nabla H(x,t))\mathrm{d}x\mathrm{d}t + \mathcal{F}^f(H)\right) + \mathcal{J}_h^f\,. \tag{1.30}$$ {{ratef}}

   Moreover, $\mathcal{I}_h^f$ achieves its minimal value at a unique height function H_h^f.

1.4 Asymptotics for Skew Macdonald Polynomials

{s:Macdonaldint

The goal of this section is to study the asymptotics of the skew Macdonald polynomials $P_{\boldsymbol{\lambda}\setminus\boldsymbol{\mu}}(\boldsymbol{x}; q, t)$, which are symmetric functions in infinitely many variables $\boldsymbol{x} = (x_i)_{1\leqslant i \leqslant \infty}$ and parametrized by two Young diagrams $\boldsymbol{\mu} \subset \boldsymbol{\lambda}$. We refer to Appendix C for more detailed discussion on (skew) Macdonald polynomials.

Fix $\theta > 0$, and take $t = q^\theta$. Assume $\boldsymbol{\lambda}, \boldsymbol{\mu}$ are Young diagrams with at most N rows, and take any $\boldsymbol{b} = (b_0, b_1, \cdots, b_{\mathsf{T}-1})$. Then the same as in (1.3), we identify {ratef} $\boldsymbol{\mu}, \boldsymbol{\lambda}$ as particle configurations $\mathbf{y}, \mathbf{z}$. For non-intersecting Bernoulli walks $\mathsf{p} = \{\mathbf{x}(t)\}_{0\leqslant \mathsf{t} \leqslant \mathsf{T}}$ from $\mathbf{x}(0) = \mathbf{y}$ to $\mathbf{x}(\mathsf{T}) = \mathbf{z}$, we define their weights as

$$\widetilde{\mathcal{W}}(\mathsf{p}; \boldsymbol{b}) = \prod_{0\leqslant \mathsf{t} \leqslant \mathsf{T}-1} \prod_{1\leqslant i<j\leqslant n} \frac{q^{\mathsf{x}_i(\mathsf{t})+\theta e_i(\mathsf{t})} - q^{\mathsf{x}_j(\mathsf{t})+\theta e_j(\mathsf{t})}}{q^{\mathsf{x}_i(\mathsf{t})} - q^{\mathsf{x}_j(\mathsf{t})}} \prod_{1\leqslant i\leqslant N} \mathsf{b}_\mathsf{t}^{e_i(\mathsf{t})} \tag{1.31}$$ {{e:weightpbmac

Then we can rewrite the skew Macdonald polynomial evaluated at $(b_0, b_1, \cdots, b_{\mathsf{T}-1})$ as follows

$$P_{\boldsymbol{\lambda}'/\boldsymbol{\mu}'}(b_0, \cdots, b_{\mathsf{T}-1}; t, q) = \frac{P_{\boldsymbol{\mu}}(1, t, t^2, \cdots, t^{N-1}; q, t)}{P_{\boldsymbol{\lambda}}(1, t, t^2, \cdots, t^{N-1}; q, t)} \sum_{\mathsf{p}\in\mathcal{P}(\mathbf{y};\mathbf{z};\mathsf{T})} \widetilde{\mathcal{W}}(\mathsf{p}; \boldsymbol{b}), \tag{1.32}$$

where $\boldsymbol{\lambda}', \boldsymbol{\mu}'$ are the transposes of $\boldsymbol{\lambda}, \boldsymbol{\mu}$.

Similarly to the skew Jack polynomials, there are explicit formulas for the Macdonald symmetric polynomials evaluated at the principal specialization, i.e. $P_{\boldsymbol{\mu}}(1, t, t^2, \cdots, t^{N-1}; q, t), P_{\boldsymbol{\lambda}}(1, t, t^2, \cdots, t^{N-1}; q, t)$, making them amenable for asymptotic analysis. The sum of the weights $\widetilde{\mathcal{W}}(\mathsf{p}; \boldsymbol{b})$ can be interpreted as the partition function of non-intersecting θ-Bernoulli random walks from $\mathbf{y}$ to $\mathbf{z}$, with the transition probability given by

$$\mathbb{P}^{\mathsf{b},q}(\mathbf{x}(\mathsf{t}+1) = \mathbf{x} + \boldsymbol{e} | \mathbf{x}(\mathsf{t}) = \mathbf{x}) \propto \prod_{1 \leqslant i < j \leqslant N} \frac{q^{\mathsf{x}_i + \theta e_i} - q^{\mathsf{x}_j + \theta e_j}}{q^{\mathsf{x}_i} - q^{\mathsf{x}_j}} \prod_{1 \leqslant i \leqslant N} b_{\mathsf{t}}^{e_i}. \tag{1.33}$$ {{e:Macdonaldp}

for $0 \leqslant \mathsf{t} \leqslant \mathsf{T} - 1$, and $\boldsymbol{e} = (e_1, e_2, \cdots, e_N) \in \{0, 1\}^N$. The above Markov process (1.33) is a special case {e:Macdonaldp} of ascending Macdonald process as constructed in [8].

The Macdonald processes (1.33) are different from non-intersecting θ-Bernoulli random walks (1.10), and also have singular interactions among adjacent particles. However, by taking their ratios, we observe the cancellation of singularities. This observation allows us to establish the large deviation principle for the Macdonald process through the application of Varadhan's lemma. The following result on the asymptotics of skew Macdonald polynomials is a consequence of the large deviation principle of the Macdonald process (1.33). The proof is given in Section 6.4.

{t:main3}

Theorem 1.7. *Let $\theta, T > 0$ and $h \in \mathrm{Adm}_\theta^\partial(\mathfrak{R})$. We consider a sequence of Young diagrams $(\boldsymbol{\lambda}, \boldsymbol{\mu})$ such that (y, z) given by (1.3) are (h, θ, T)-admissible. We take $\mathsf{b}_\mathsf{t} = e^{f(\mathsf{t}/N)}$ for a continuously differentiable function f. Moreover, we take $q = e^{\kappa/N}$ for some $\kappa < 0$.*

1. *The asymptotics of the skew Macdonald polynomials are given by*

$$\begin{aligned} &\lim_{N \to \infty} \frac{1}{N^2} \ln P_{(\boldsymbol{\lambda}^{(N)})' \setminus (\boldsymbol{\mu}^{(N)})'}(b_0, b_1, \cdots, b_{\mathsf{T}-1}; q, t) \\ &= \mathcal{J}_h^f + \frac{1}{2\theta} \left(\int_{\mathbb{R}^2} \ln|x-y| \mathrm{d}h(x,0) \mathrm{d}h(y,0) - \int_{\mathbb{R}^2} \ln|x-y| \mathrm{d}h(x,T) \mathrm{d}h(y,T) \right) + T \ln 2 \end{aligned}$$

   where $\mathcal{J}_h^f$ is defined in (1.28) and $\mathcal{F}^f(H)$ in (1.29).

2. *The law of H under $\mathbb{P}^{\mathsf{b},q}$ defined in (1.33) conditioned to remain in $\mathcal{P}(\mathbf{y}^{(N)}, \mathbf{z}^{(N)}; \mathsf{T})$ satisfies a large deviation principle with speed N^2 and good rate function $\mathcal{I}_h^f(H)$ which is infinite outside $\mathrm{Adm}(\mathfrak{R}, h)$ and otherwise given by (1.30). Moreover, $\mathcal{I}_h^f$ achieves its minimal value at a unique height function H_h^f.*

We remark that the rate function surprisingly does not depend on κ. This indicates that q is converging toward one too rapidly to induce changes at the large N limit. However, other scalings seem difficult to deduce by Varadhan's lemma and require other kinds of large deviations estimates.

1.5 Related works on the Asymptotics of Symmetric Polynomials

The asymptotic behavior of symmetric polynomials can be studied under other scaling regimes. One intensively studied regime is to take limit along a sequence of Vershik-Kerov partitions, where the size and rows of the partitions all grow linearly in N. In this way, the normalized Schur functions approximate a character of the infinite unitary group. The asymptotics of normalized Schur functions in this scaling regime were originally derived by Vershik and Kerov in [55]. Generalizations of these results to the asymptotics of Jack and Macdonald polynomials under the same scaling can be found in [45, 16, 17].

While the Vershik-Kerov partitions involve thin partitions, this paper explores a different scaling regime. Specifically, we consider the limit along a sequence of partitions where the number of rows and columns grows linearly in N, but the size grows quadratically in N^2. Gorin and Panova's work in [21] presents an example within this regime. They obtained the asymptotics of Schur polynomials with all but finitely many parameters set to one, see also [28, 27]. These asymptotic results played an important role in the study of extreme characters of the infinite-dimensional unitary group [21, 11], as well as applications in random lozenge tilings [12, 13] and the six-vertex model [21, 20].

The structure constants of symmetric polynomials, such as Kostka number, Kronecker coefficients and Littlewood-Richardson coefficients are fundamental quantities in algebraic combinatorics. Despite the absence of explicit formulas, the asymptotics of specific extreme Kronecker and Littlewood-Richardson coefficients have been derived in [49, 54, 47, 48]. Using the asymptotics of Schur symmetric polynomials, the large deviation asymptotics of Kostka numbers and the large deviation upper bound for the Littlewood-Richardson coefficients have been obtained in [5]. These results were generalized in [33] to the asymptotic behavior of weight multiplicities of irreducible representations of compact or complex simple Lie algebras in the limit of large rank. The asymptotics of Jack polynomials and Macdonald polynomials should allow to address similar questions for the large deviations of their structure constants.

Any other things we want to add here?

1.6 Ideas of the proofs

In the continuous setting, the asymptotics of the Harish-Chandra-Itzykson-Zuber integral has been discovered through studying the asymptotics of Dyson's Brownian motion by the first named author and Zeitouni [29, 30, 25]. Subsequently, analogous findings were extended to the rectangular spherical integral and Generalized Bessel Functions in [33, 26]. However, these outcomes rely on stochastic calculus and specific structures of Dyson's Brownian motion.

In this paper, we focus on the discrete setting, where tools from stochastic calculus are not available. We summarize the main ideas of the proof of our main Theorems, Theorem 1.5. Theorem 1.6 and Theorem 1.7, can then be deduced by Varadhan's lemma. In the special case when $\theta = 1$, the non-intersecting θ-Bernoulli random walks (1.10) are equivalent to random Lozenge tiling of strip domains, where the variational principle of the height function has been proven in [15]. In fact their results apply to random tilings of more general domains. The proof of the variational principle for tilings in [15] relies on the exact computation of the partition function for tilings on a large torus, utilizing Kasteleyn's matrix. Unfortunately, this method is not applicable in our context when $\theta \neq 1$. Instead, our proof relies on the recently introduced dynamical loop equations by the second author and Gorin [19]. These equations have proven effective for analyzing large families of two-dimensional interacting particle systems in both discrete and continuous settings. Examples include nonintersecting Bernoulli/Poisson random walks [22, 39, 31], β-corner processes [23, 9], measures on Gelfand-Tsetlin patterns [14, 51, 50], and Macdonald processes [8]. In particular, a version of dynamical loop equations has been derived for the non-intersecting θ-Bernoulli random walks.

We establish the large deviation principle using Cramér's method, by tilting the measure of nonintersecting θ-Bernoulli walk using exponential martingales. Our proof is divided into an upper bound and a lower bound. The normalization constant for these exponential martingales can be determined using dynamical loop equations. Subsequently, the large deviation upper bound can be obtained straightforwardly by applying Markov's inequality. The measure tilted by the exponential martingale coincides with the drifted non-intersecting θ-Bernoulli random walk. Utilizing the dynamical loop equation once again, we establish a limit shape theorem for the drifted non-intersecting θ-Bernoulli random walk. As a consequence, for any targeting measure process, we can construct an exponential martingale such that the tilted measure concentrates around it. This provides the large deviation lower bound.

In the above outlined proof, we utilize dynamical loop equations to investigate nonintersecting θ-Bernoulli walks with drift. However, two challenges hinder their direct application. Firstly, dynamical loop equations require that the drift is analytic. To overcome this, we convolve the targeting measure process with a small Cauchy distribution (which is reminiscent of the strategy followed in [29]). This convolution enables the analytical extension of the density to a strip region around the real axis, allowing for a similar extension of the drift terms. As a trade-off, we are required to analyze measure-valued processes supported on the entire real axis. This challenge is addressed through precise truncations and the utilization of the explicit form of the rate function.

Secondly and more critically, dynamical loop equations require certain non-criticality conditions. Specifically, the particle system cannot be too dense, meaning the density cannot be too close to one. To tackle this challenge, rather than studying the nonintersecting θ-Bernoulli walks as a whole, we partition space-time into small regions of parallelogram shape. Within each region, we ensure that both the density and velocity remain nearly constant. In each region, the system resembles nonintersecting θ-Bernoulli walks but with a reduced number of particles and possibly left and right boundaries. If the density and velocity are non-extremal, we can establish a large deviation principle using the dynamical loop equation as outlined above. However, if the density or velocity are extremal, we prove a large deviation upper bound by directly analyzing the walk. Importantly, it is observed that the contribution from these regions is negligible. The original nonintersecting θ-Bernoulli walks have long-range pairwise interactions. After partitioning, particles from different regions also interact with each other. Fortunately, these interactions cccur between particles which are far from each other and are not singular, which can be approximated as smooth weights. In this manner, we can integrate all regions together and derive the variational principle for the original nonintersecting θ-Bernoulli walks.

add a paragraph on loop equations, and its role in computing partition function?

1.7 Organization of the paper

In Section 2, we collect various facts about height functions, Vandermonde determinants and the free entropy. They will be used repeatedly in the rest of this paper. Section 3 provides an overview of the proof for Theorem 1.5, which relies on both a large deviation upper bound (Proposition 3.3) and a corresponding lower bound (Proposition 3.6) for non-intersecting θ-Bernoulli random walks with a constant slope. The proof for Proposition 3.3 is detailed in Section 4, while the proof for Proposition 3.6 is presented in Section 5. Moving forward, Section 6 contains the proof for Theorem 1.6, on the asymptotics of skew Jack polynomials. Similarly, the proof for Theorem 1.7, which explores the asymptotics of skew Macdonald polynomials, is outlined in Appendix C. Finally in Appendix A, we collect the results on dynamical loop equations from [19].

1.8 Notations

We use $\boldsymbol{\lambda}, \boldsymbol{\mu}$ to represent Young diagrams. For macro particle locations we use mathsf letters: $\mathsf{x}_i(\mathsf{t}), \mathbf{\mathsf{x}}(\mathsf{t}), \mathbf{\mathsf{y}}, \mathbf{\mathsf{x}}$, where time t ranges from 0 to T. For micro particle locations we use standard letters $x_i(t), \boldsymbol{x}(t), \boldsymbol{y}, \boldsymbol{z}$, where the time t ranges from 0 to T. Height functions $H(x,t), \widetilde{H}(x,t), \ldots$ are all height functions associated with the micro particle locations. We use H^* for an asymptotic height function, that is an element of $\mathrm{Adm}(\mathfrak{R}, h)$ that arises as the limit of height functions, H is an arbitrary Height function, $\widetilde{H}$ stands for the smoothed version of H. And we use $\mathcal{H}$ for the discretization of H^* or H.

2 Setup and Preliminary Results

{s:setup}

In this section we collect various facts about height function, Vandermonde determinant and free entropy. They will be used repeatedly in the rest of this paper.

2.1 Approximating Height Function

For any height function $H^* \in \mathrm{Adm}(\mathfrak{R}; h)$, the following lemma states that if $\mathcal{P}(\mathbf{\mathsf{y}}^{(N)}, \mathbf{\mathsf{z}}^{(N)}; \mathsf{T}) \neq \emptyset$, then there exists a non-interesecting θ-Bernoulli walk from $\mathbf{\mathsf{y}}^{(N)}$ to $\mathbf{\mathsf{z}}^{(N)}$, and its height function is close to H^*. The proof follows from a careful discretization of H^*, and we postpone it to Appendix B.

{e:constructH}

Lemma 2.1. *Let θ, T be positive real numbers and $h \in \mathrm{Adm}_\theta^\partial(\mathfrak{R})$. Consider two sequences of particle configurations (y, z) which are (h, θ, T)-admissible. For any $\varepsilon > 0$, and any $H^* \in \mathrm{Adm}(\mathfrak{R}; h)$, there exists a non-intersecting Bernoulli paths $\{\boldsymbol{x}(t)\}_{0 \leqslant t \leqslant T}$ from $\boldsymbol{x}(0) = \mathbf{\mathsf{y}}^{(N)}/N$ to $\boldsymbol{x}(T) = \mathbf{\mathsf{z}}^{(N)}/N$ with height function $\mathcal{H}$, such that*

$$\|H^* - \mathcal{H}\|_\infty \leqslant \varepsilon, \tag{2.1}$$ {{e:heightclose

provided N is large enough.

We introduce the following ℓ-mesh, and Lemma 2.3 states that on most of the ℓ-mesh, H^* has an approximate linear approximation.

{d:lemesh}

Definition 2.2 (ℓ-mesh)**.** *Adopt Definition 1.3, there exists a large constant $A > 0$, such that for any $H \in \mathrm{Adm}(\mathfrak{R}; h)$,*

$$H(x,t) = 0, \quad x \leqslant -A, \quad H(x,t) = \theta, \quad x \geqslant A. \tag{2.2}$$

Take a small $\ell > 0$ such that $A/\ell, T/\ell \in \mathbb{Z}$. We cover the region $[-A, A] \times [0, T]$ by rectangles of size $6\ell \times \ell$:

$$[-A, A] \times [0, T] = \cup \mathfrak{R}(\alpha, \beta), \quad \mathfrak{R}(\alpha, \beta) = [(\alpha - 3)\ell, (\alpha + 3)\ell] \times [\beta\ell, (\beta + 1)\ell], \tag{2.3}$$

where α, β are integer numbers so that $-A/\ell \leqslant \alpha \leqslant A/\ell, \quad 0 \leqslant \beta \leqslant T/\ell$.

{l:Lipschitz}

Lemma 2.3. *Let H^* be an asymptotic height function, namely a given element of Adm$(\mathfrak{R}, h)$, and let $\varepsilon > 0$. If ℓ is sufficiently small then on at least a $1 - \varepsilon$ fraction of the rectangles $\mathfrak{R}(\alpha, \beta)$ from the ℓ-mesh (in Definition 2.2) we have the following two properties*

1. *On $\mathfrak{R}(\alpha, \beta)$, H has a linear approximation within an error bounded by $\ell\varepsilon$, namely there exists $(\varrho, -\varrho v) \in \overline{\mathcal{T}}$ such that for $(x,t) \in \mathfrak{R}(\alpha, \beta)$*

$$|H(x,t) - (H(\alpha\ell, \beta\ell) + \varrho(x - \alpha\ell) - \varrho v(t - \beta\ell))| \leqslant \varepsilon. \tag{2.4}$$

2. *For at least a $1 - \varepsilon$ fraction (in measure) of the points x of $\mathfrak{R}(\alpha, \beta)$, the gradient $\nabla H(x)$ exists and is within ε of $(\varrho, -\varrho v)$.*

Proof. The proof is the same as in [15, Lemma 2.2], so we omit. ☐

2.2 Smoothed Height Function and Complex Slope

{s:heightslope}

Given any height function $H \in \mathrm{Adm}(\mathfrak{R}; h)$, it is Lipschitz. We convolve it with a Cauchy distribution. Take small $\delta > 0$, denote

$$\widetilde{H}(x,t) = \frac{1}{\pi} \int \frac{\delta H(y,t)\mathrm{d}y}{(y - x)^2 + \delta^2}. \tag{2.5}$$ {{e:deftH0}}

In this section, we collect some estimates of the smoothed height function $\widetilde{H}(x,t)$ (as constructed in (2.5)) in {e:deftH0} Lemma 2.4 and Lemma 2.5. Their proofs follow from explicit computations, so we postpone them to Appendix B. We denote for two functions f, g, $f \asymp g$ iff there exists C, D finite so that, for every t, $C|g(t)| \leq |f(t)| \leq D|g(t)|$.

{c:nabHbound}

Lemma 2.4. *Fix any $0 \leqslant t \leqslant \ell$, denote*

$$\kappa_t(x) = \frac{1}{\pi} \int_{tv}^{\ell + tv} \frac{\delta \mathrm{d}y}{(y - x)^2 + \delta^2} \tag{2.6}$$ {{e:defkappat}}

then

$$\partial_x \widetilde{H}(x,t) = \varrho \kappa_t(x), \quad \partial_t \widetilde{H}(x,t) = -\varrho v \kappa_t(x). \tag{2.7}$$ {{e:dtH}}

The function $\kappa_t(x) \in [0, 1]$ and for $x \in [tv, \ell + tv]$, it holds

$$\kappa_t(x) \asymp 1, \quad 1 - \kappa_t(x) \asymp \frac{C\delta}{\delta + \mathrm{dist}(x, \{tv, \ell + tv\})}, \tag{2.8}$$

for $x \notin [tv, \ell + tv]$, it holds

$$1 - \kappa_t(x) \asymp 1, \quad \kappa_t(x) \asymp \frac{C\delta}{\delta + \mathrm{dist}(x, \{tv, \ell + tv\}) + \mathrm{dist}(x, \{tv, \ell + tv\})^2}. \tag{2.9}$$ {{e:ktxbound}}

We recall that the Hilbert transform $\mathrm{Hib}(u)$ of a function u is given by $\mathrm{Hib}(u)(x) = PV\int \frac{u(x)}{t-x}dx$ where PV denotes the principal value.

{l:kappaHib}

Lemma 2.5. *There exists a large constant $C>0$, for any small $\delta>0$, recall $\kappa_t(x)$ from (2.6), the following holds* {e:ktxbound}

1. *$\kappa_t(x)$ can be extended analytically to any $z = x + \mathrm{i}\eta$ with $\mathrm{Im}[\eta] \leqslant \delta/2$,*

$$|\kappa_t(z) - \kappa_t(x)| \leqslant \frac{C\eta}{\delta}\min\{\kappa_t(x), 1-\kappa_t(x)\}. \tag{2.10}$$ {{e:ktdiff}}

2. *The Hilbert transform $\mathrm{Hib}(\kappa_t)$ extends analytically to any $z = x+\mathrm{i}\eta$ with $\mathrm{Im}[\eta]\leqslant \delta/2$,*

$$|\mathrm{Hib}(\kappa_t)(z)| \leqslant \begin{cases} \frac{C\delta}{\mathrm{dist}(z,\{tv,\ell+tv\})} & \text{if } \mathrm{dist}(z,\{tv,\ell+tv\})\geqslant \ell \\ \ln(1/\delta)+C & \text{for all } z \end{cases}, \quad |\mathrm{Im}\,\mathrm{Hib}(\kappa_t)(z)| \leqslant \frac{C\eta}{\delta}. \tag{2.11}$$ {{e:Hkappa}}

If $\mathrm{dist}(z,\{tv,\ell+tv\}) \gtrsim \ell$, we have

$$\partial_z\kappa_t(z) \lesssim \frac{\delta\ell}{\mathrm{dist}(z,\{tv,\ell+tv\})^3}, \quad \mathrm{Hib}(\kappa_t(z) \lesssim \frac{\delta}{\mathrm{dist}(z,\{tv,\ell+tv\})} \tag{2.12}$$ {{e:largezbound

2.3 Vandermonde Determinant and Free Entropy

Lemma 2.6 and Lemma 2.7 concern certain useful identity and estimates of Vandermonde determinants.

{l:sumV}

Lemma 2.6. *For any particle configuration $\mathsf{x}\in\mathbb{W}_\theta^n$, and $0\leqslant k\leqslant n$, the following holds*

$$\sum_{e_1+e_2+\cdots+e_n=k}\frac{V(\mathsf{x}+\theta\boldsymbol{e})}{V(\mathsf{x})} = \binom{n}{k}. \tag{2.13}$$

{l:VVbb}

Lemma 2.7. *There exists a constant $C>0$. For any particle configuration $\mathsf{x}\in\mathbb{W}_\theta^n$, the following hold*

$$C^{-n}\leqslant \frac{V(\mathsf{x}+\theta\boldsymbol{e})}{V(\mathsf{x})}\leqslant 2^n. \tag{2.14}$$ {{e:VVbb}}

Proof. The upper bound in (2.14) follows from Lemma 2.6 which implies {l:VVbb}

$$\sum_{\boldsymbol{e}}\frac{V(\mathsf{x}+\theta\boldsymbol{e})}{V(\mathsf{x})} = 2^n \tag{2.15}$$ {{e:ubb}}

and the fact that all terms in the left hand side are non-negative. For the lower bound, we first show that there exists a constant $C=C(\theta)$ such that for every configuration x,

$$\left|\ln\frac{(\mathsf{x}_i+\theta e_i)-(\mathsf{x}_j+\theta e_j)}{\mathsf{x}_i-\mathsf{x}_j} - \theta\ln\frac{(\mathsf{x}_i+e_i)-(\mathsf{x}_j+e_j)}{\mathsf{x}_i-\mathsf{x}_j}\right| \leqslant C\frac{(e_i-e_j)^2}{(\mathsf{x}_i-\mathsf{x}_j)^2} \tag{2.16}$$ {{e:localeq1}}

If $e_i - e_j = 0$, there is nothing to prove, the lefthand side is 0. If $\mathsf{x}_i - \mathsf{x}_j = (j-i)\theta$, i.e. the particle configuration on $\{\mathsf{x}_i, \mathsf{x}_{i+1},\cdots,\mathsf{x}_j\}$ is tightly packed. The lefthand side of (2.16) is nonzero only if $e_i - e_j \in \{0,1\}$. Thus, {e:localeq1}

$$\frac{e_i-e_j}{\mathsf{x}_i-\mathsf{x}_j}\in\left\{0,\frac{1}{\theta(j-i)}\right\}. \tag{2.17}$$

Otherwise, $\mathsf{x}_i - \mathsf{x}_j \geqslant 1+\theta$, and we have

$$\left|\frac{e_i-e_j}{\mathsf{x}_i-\mathsf{x}_j}\right| \leqslant \frac{1}{1+\theta}. \tag{2.18}$$

The statement (2.16) follows from the inequality below

$$|\ln(1+\theta x)-\theta\ln(1+x)|\leqslant Cx^2,\quad x\in[-(1+\theta)^{-1},1], \tag{2.19}$$

which follows from the fact that for $x\in[-(1+\theta)^{-1},1]$

$$|\partial_x^2(\ln(1+\theta x)-\theta\ln(1+x))|=\theta(1-\theta)\frac{|1-\theta x^2|}{(1+x)^2(1+\theta x)^2}\leqslant\frac{(1+\theta)^5(1-\theta)}{\theta}. \tag{2.20}$$

It follows by summing over i,j in (2.16), that if y is the configuration $\mathsf{y}=\mathsf{x}+\mathsf{e}$,

$$\begin{aligned}\ln\frac{V(\mathsf{x}+\theta\boldsymbol{e})}{V(\mathsf{x})}&\geqslant\theta\ln\left(\frac{V(\mathsf{y})}{V(\mathsf{x})}\right)-C\sum_{i<j}\frac{4}{(\mathsf{x}_i-\mathsf{x}_j)^2}\\&\geqslant\theta\ln\frac{V(\mathsf{y})}{V(\mathsf{x})}-C\sum_{1\leqslant i<j\leqslant n}\frac{4}{(\theta(i-j))^2}\geqslant\theta\ln\frac{V(\mathsf{y})}{V(\mathsf{x})}-C'n\end{aligned} \tag{2.21}$$

{{e:localeq2}}

By the same argument as for (2.21), we also have that {e:localeq2}

$$n\ln 2\geqslant\ln\frac{V(\mathsf{y}-\theta\boldsymbol{e})}{V(\mathsf{y})}\geqslant\theta\ln\left(\frac{V(\mathsf{y}-\mathsf{e})}{V(\mathsf{y})}\right)-C'n=\theta\ln\left(\frac{V(\mathsf{x})}{V(\mathsf{y})}\right)-C'n \tag{2.22}$$

{{e:localeq3}}

where the upper bound by $n\ln 2$ comes from (2.14). Combining (2.21) and (2.22), we obtain the lower bound in (2.14) {e:localeq3}

$$\ln\frac{V(\mathsf{x}+\theta\boldsymbol{e})}{V(\mathsf{x})}\geqslant\theta\ln\frac{V(\mathsf{y})}{V(\mathsf{x})}-C'n\geqslant -n\ln 2-2C'n=-(\ln 2+2C')n. \tag{2.23}$$

□

In the following lemma we collect certain estimates on the limit of Vandermonde determinant, and on the free entropy.

{l:freeentropy}

Lemma 2.8. *Fix $\kappa<0$. There exists a constant $C=C(\kappa)>0$, the following holds. For any particle configuration $\boldsymbol{x}\in\mathbb{W}_\theta^N$, let $\boldsymbol{x}\in\boldsymbol{x}/N$. Assume $\boldsymbol{x}$ is supported on $[-A,A]$, and we denote its empirical density as $\varrho(x;\boldsymbol{x})$, then*

$$\begin{aligned}\frac{\theta^2}{N^2}\sum_{i<j}\ln(x_i-x_j)&=\frac{1}{2}\iint\ln|x-y|\varrho(x;\boldsymbol{x})\varrho(y;\boldsymbol{x})\mathrm{d}x\mathrm{d}y+\mathrm{O}\left(\frac{\ln N}{N}\right)\\\frac{\theta^2}{N^2}\sum_{i<j}\ln(1-e^{\kappa(x_i-x_j)})&=\frac{1}{2}\iint\ln(1-e^{\kappa|x-y|})\varrho(x;\boldsymbol{x})\varrho(y;\boldsymbol{x})\mathrm{d}x\mathrm{d}y+\mathrm{O}\left(\frac{\ln N}{N}\right).\end{aligned} \tag{2.24}$$

{{e:vandlimit}}

For any two height functions h_1,h_2 such that $\|h_1-h_2\|_\infty\leqslant\varepsilon$ with $0\leqslant\partial_x h_1,\partial_x h_2\leqslant 1$ and their supports $\mathrm{supp}(\partial_x h_1),\mathrm{supp}(\partial_x h_2)\in[-A,A]$*, then*

$$\begin{aligned}&\left|\iint\ln|x-y|\mathrm{d}h_1(x)\mathrm{d}h_1(y)-\iint\ln|x-y|\mathrm{d}h_2(x)\mathrm{d}h_2(y)\right|\leqslant C\varepsilon,\\&\left|\iint\ln(1-e^{\kappa|x-y|})\mathrm{d}h_1(x)\mathrm{d}h_1(y)-\iint\ln(1-e^{\kappa|x-y|})\mathrm{d}h_2(x)\mathrm{d}h_2(y)\right|\leqslant C\varepsilon.\end{aligned} \tag{2.25}$$

{{e:Hdiffbound}

Proof. We will only prove the first statements in (2.24) and (2.25), and the second statements follow from the same arguments. For any $i<j-1$, denote $x_i-x_j=r\geqslant 2\theta/N$, we have {e:Hdiffbound}

$$\int_{x_i}^{x_i+\theta/N}\int_{x_j}^{x_j+\theta/N}\ln|x-y|\mathrm{d}x\mathrm{d}y=\int_0^{\theta/N}\int_0^{\theta/N}\ln|r+x-y|\mathrm{d}x\mathrm{d}y \tag{2.26}$$

$$=\int_0^{\theta/N}(\ln(r-\tau)+\ln(r+\tau))\left(\frac{\theta}{N}-\tau\right)\mathrm{d}\tau=\frac{\theta^2}{N^2}\ln r+\int_0^{\theta/N}\ln\left(1-\frac{\tau^2}{r^2}\right)\left(\frac{\theta}{N}-\tau\right)\mathrm{d}\tau \tag{2.27}$$

$$=\frac{\theta^2}{N^2}\ln r+\mathrm{O}\left(\frac{1}{N^4r^2}\right). \tag{2.28}$$

By summing over all the pairs $i < j - 1$, and bound $\ln(x_i - x_{i+1}) = \mathrm{O}(\ln N)$, we have

$$\frac{\theta^2}{N^2}\sum_{i<j}\ln(x_i - x_j) = \sum_{i,j}\frac{1}{2}\int_{x_i}^{x_i+\theta/N}\int_{x_j}^{x_j+\theta/N}\ln|x-y|\mathrm{d}x\mathrm{d}y + \mathrm{O}\left(\frac{\ln N}{N}\right) \tag{2.29}$$

$$= \frac{1}{2}\iint \ln|x-y|\varrho(x;\boldsymbol{x})\varrho(y;\boldsymbol{x})\mathrm{d}x\mathrm{d}y + \mathrm{O}\left(\frac{\ln N}{N}\right). \tag{2.30}$$

This gives (2.24). The statement (2.25) follows from

$$\mathcal{I} := \left|\iint \ln|x-y|\mathrm{d}(h_1(x) - h_2(x))\mathrm{d}h_1(y)\right| \leqslant C\varepsilon \tag{2.31}$$

Denote $\varrho_1(y) = \partial_y h_1(y)$, by an integration by part

$$\mathcal{I} = \left|\iint \frac{\varrho_1(y)}{x-y}\mathrm{d}y(h_1(x) - h_2(x))\mathrm{d}x\right| \lesssim \varepsilon\int_{-A}^{A}|\operatorname{Hib}(\varrho_1)(y)|\mathrm{d}y \tag{2.32}$$

$$\leqslant \varepsilon\left(2A\int_{-A}^{A}|\operatorname{Hib}(\varrho_1)(y)|^2\mathrm{d}y\right)^{1/2} \leqslant \varepsilon\left(2A\int_{\mathbb{R}}\varrho_1^2\mathrm{d}y\right)^{1/2} = \varepsilon\sqrt{2A}, \tag{2.33}$$

where we used that the Hilbert transform preserves L_2 norm, and $\varrho_1 \leqslant 1$. □

3 Proof Outline of Theorem 1.5

{s:outline}

We recall the boundary height function h from Definition 1.3, and the sequences of particle configurations $\mathsf{y}^{(N)}, \mathsf{z}^{(N)}$ from Theorem 1.5. We recall the set of non-interesecting θ-Bernoulli walks from $\mathsf{y}^{(N)}$ to $\mathsf{z}^{(N)}$ from (1.4)

$$\mathcal{P}(\mathsf{y}^{(N)}, \mathsf{z}^{(N)}; \mathsf{T}) = \{\mathsf{p} = \{\mathsf{x}(\mathsf{t})\}_{0\leqslant \mathsf{t}\leqslant \mathsf{T}} : \mathsf{x}_0 = \mathsf{y}^{(N)}, \mathsf{x}_\mathsf{T} = \mathsf{z}^{(N)}\}. \tag{3.1}$$

Fix an asymptotic height function $H^* \in \mathrm{Adm}(\mathfrak{R}, h)$, and a sufficiently small $\ell > 0$ such that $\mathsf{L} = \ell N \in \mathbb{N}$. We recall the ℓ-mesh from Definition 2.2. We denote $\mathbb{B}_\varepsilon(H^*)$ the set of non-intersecting θ-Bernoulli walks with height function close to H^*

$$\mathbb{B}_\varepsilon(H^*) = \{\mathsf{p} = \{\mathsf{x}(\mathsf{t})\}_{0\leqslant \mathsf{t}\leqslant \mathsf{T}} \in \mathcal{P}(\mathsf{y}^{(N)}, \mathsf{z}^{(N)}; \mathsf{T}) : \|H - H^*\|_\infty \leqslant \varepsilon\}. \tag{3.2}$$

3.1 Upper Bound

To prove the large deviation upper bound for $\mathbb{P}(\mathbb{B}_\varepsilon(H^*))$, we partition $\mathbb{B}_\varepsilon(H^*)$ according to the possible particle configurations at times $\mathbb{N}\mathsf{L}$:

$$\mathbb{B}_\varepsilon(H^* : \{\mathsf{y}(\mathsf{t})\}_{\mathsf{t}\in\{0,\mathsf{L},2\mathsf{L},\cdots,\mathsf{T}\}}) := \{\mathsf{p} = \{\mathsf{x}(\mathsf{t})\}_{0\leqslant \mathsf{t}\leqslant \mathsf{T}} \in \mathbb{B}_\varepsilon(H^*) : \mathsf{x}(\mathsf{t}) = \mathsf{y}(\mathsf{t}) \text{ for } \mathsf{t} \in \{0, \mathsf{L}, 2\mathsf{L}, \cdots, \mathsf{T}\}\} \tag{3.3}$$

By our assumption that $|\mathsf{y}_i^{(N)}|, |\mathsf{z}_i^{(N)}| \leqslant CN$, then there exists $A > 0$ (say $A = C + \mathsf{T}$) such that $\operatorname{supp}\mathsf{y}(\mathsf{t}) \in [-AN, AN]$ for all $0 \leqslant \mathsf{t} \leqslant \mathsf{T}$. The number of total choices of $\{\mathsf{y}(\mathsf{t})\}_{\mathsf{t}\in\{0,\mathsf{L},2\mathsf{L},\cdots,\mathsf{T}\}}$ is given by

$$\binom{2AN}{N}^{T/\ell} \leqslant 2^{(2TA/\ell)N} = e^{\mathrm{O}(N)}, \tag{3.4}$$

which is negligible. In the rest of this section, we will fix the particle configurations $\{\mathsf{y}(\mathsf{t})\}_{\mathsf{t}\in\{0,\mathsf{L},2\mathsf{L},\cdots,\mathsf{T}\}}$.

We recall the ℓ-mesh from Definition 2.2 and restrict ourselves to the configurations in $\mathfrak{R}(\alpha, \beta)$ for some fixed integer numbers α, β. We consider the restriction of the particle configuration $\boldsymbol{y}(\beta\ell)$ on the interval $[\alpha\ell, (\alpha+1)\ell)$

$$y_{i-1}(\beta\ell) < \alpha\ell \leqslant y_i(\beta\ell) < \cdots < y_j(\beta\ell) < (\alpha+1)\ell \leqslant y_{j+1}(\beta\ell). \tag{3.5}$$

Denote the set of indices of the configuration inside $\mathfrak{R}(\alpha, \beta)$ by $I(\alpha,\beta)=\{i,i+1,i+2,\cdots,j\}$ and denote $n := |I(\alpha,\beta)| = j-i+1$. For any $\mathsf{p} \in \mathbb{B}_\varepsilon(H^* : \{\mathbf{y}(\mathsf{t})\}_{\mathsf{t}\in\{0,\mathsf{L},2\mathsf{L},\cdots,\mathsf{T}\}})$, its restriction on the index set $I(\alpha,\beta)$ and time interval $[\beta\ell,(\beta+1)\ell]$, $\{\mathsf{x}_k(t)\}_{k\in I(\alpha,\beta),t\in[\beta\ell,(\beta+1)\ell]}$ forms an n-particle nonintersecting Bernoulli random walk. The next lemma states that its height function is close to a height function with constant slope.

{d:constantslope}

Definition 3.1. *Given a slope $(\varrho, -\varrho v) \in \overline{\mathcal{T}}$, we construct the constant slope height function $\mathcal{A}$ on $\mathbb{R}\times[0,\ell]$:*

$$\mathcal{A}(x,t) = (x-tv)\varrho, \quad (x,t)\in\mathfrak{P}; \quad \mathcal{A}(x,t)=0, \quad x\leqslant tv; \quad \mathcal{A}(x,t)=\ell\varrho, \quad x\geqslant \ell+tv. \tag{3.6}$$ {{e:defH*}}

where $\mathfrak{P}$ is the parallelogram region:

$$\mathfrak{P} := \{x,t\in\mathbb{R}^2 : 0\leqslant t\leqslant \ell, \quad tv\leqslant x\leqslant \ell+tv\}. \tag{3.7}$$ {{e:defp}}

We say $\mathcal{A}$ is the height function with slope given by $(\varrho,-\varrho v)$.

{l:slope}

Lemma 3.2. *There exists a large finite constant $C>1$ so that the following holds. Assume that on $\mathfrak{R}(\alpha,\beta)$, the height function H^* has a linear approximation with slope $(\varrho,-\varrho v)\in\mathcal{T}$ and error ε. For any $\{\mathsf{x}(t)\}_{0\leqslant t\leqslant\ell} \in \mathbb{B}_\varepsilon(H^* : \{\mathbf{y}(\mathsf{t})\}_{\mathsf{t}\in\{0,\mathsf{L},2\mathsf{L},\cdots,\mathsf{T}\}})$, we denote the shifted particle configuration $\{\mathsf{z}_k(t)\}_{1\leqslant k\leqslant n, 0\leqslant t\leqslant \ell}$*

$$\mathsf{z}_k(t) = \mathsf{x}_{k+i-1}(t+\beta\ell)-\alpha\ell, \quad 1\leqslant k\leqslant n. \tag{3.8}$$ {{e:zk}}

Then $\{\mathsf{z}_k(t)\}_{1\leqslant k\leqslant n,0\leqslant t\leqslant\ell}$ form an n-particle nonintersecting Bernoulli random walk, where

$$\left|\frac{\theta n}{N}-\ell\varrho\right|\leqslant C\varepsilon. \tag{3.9}$$ {{e:particlebound}}

and its height function H defined in (1.15) satisfies {e:particlebound}

$$|H(x,t)-\mathcal{A}(x,t)|\leqslant C\varepsilon, \tag{3.10}$$ {{e:heightclose}}

where $\mathcal{A}$ is the height function with slope given by $(\varrho,-\varrho v)$ from Definition 3.1.

Proof. We denote the height function of $\{\boldsymbol{x}(t)\}_{0\leqslant t\leqslant \ell}$ as $\mathcal{H}$, then from the construction (3.8) for any $0\leqslant t\leqslant\ell$, {e:heightclose}

$$H(x,t) = \mathcal{H}(x+\alpha\ell,t+\beta\ell) - \frac{(i-1)\theta}{N}, \quad z_1(t)\leqslant x\leqslant z_n(t)+\frac{\theta}{N}. \tag{3.11}$$ {{e:Hxtdiff}}

And for $x\leqslant z_1(t)$ we have $H(x,t)=0$; for $x\geqslant z_n(t)$ we have $H(x,t)=n\theta/N$. Next, we show (3.9). {e:Hxtdiff}
Since $\mathcal{H}\in\mathbb{B}_\varepsilon(H^*)$, we have

$$\frac{n\theta}{N}+\mathrm{O}\left(\frac{1}{N}\right) = \mathcal{H}((\alpha+1)\ell)-\mathcal{H}(\alpha\ell) = H^*((\alpha+1)\ell)-H^*(\alpha\ell)+\mathrm{O}(\varepsilon) = \ell\varrho+\mathrm{O}(\varepsilon) \tag{3.12}$$ {{e:ntheta}}

The claim (3.9) follows by rearranging. We will prove (3.10), by dividing it into different regions. {e:ntheta}

1. For $x\leqslant \min\{z_1(t),tv\}$, $H(x,t)=\mathcal{A}(x,t)=0$.
2. For $x\geqslant\min\{z_n(t)+\theta/N, tv+\ell\}$, $H(x,t)=\theta n/N$ and $\mathcal{A}(x,t)=\ell\varrho$, and the claim (3.10) follows from (3.9) .
3. For $z_1(t)\leqslant x\leqslant z_n(t)+\theta/N$, we have

$$\begin{aligned}
H(x,t) &= \mathcal{H}(x+\alpha\ell,t+\beta\ell)-\frac{(i-1)\theta}{N} = \mathcal{H}(x+\alpha\ell,t+\beta\ell)-\mathcal{H}(\alpha\ell,\beta\ell)\\
&= H^*(x+\alpha\ell,t+\beta\ell)-H^*(\alpha\ell,\beta\ell)+\mathrm{O}(\varepsilon)\\
&= x\varrho - t\varrho v+\mathrm{O}(\varepsilon).
\end{aligned} \tag{3.13}$$ {{e:Hxt2}}

   It follows that for $\max\{z_1(t),tv\}\leqslant x\leqslant\min\{z_n(t)+\theta/N,tv+\ell\}$, we have $|H(x,t)-\mathcal{A}(x,t)|=\mathrm{O}(\varepsilon)$.

4. If $tv \leqslant z_1(t)$, then for $tv \leqslant x \leqslant z_1(t)$ we have

$$0 = H(x,t) \geqslant \mathcal{H}(x+\alpha\ell, t+\beta\ell) - \mathcal{H}(\alpha\ell, \beta\ell) = x\varrho - t\varrho v + \mathrm{O}(\varepsilon) = \mathcal{A}(x,t) + \mathrm{O}(\varepsilon), \tag{3.14}$$

where in the second statement we used the construction of (3.13); in the third statement we used that H^* and $\mathcal{H}$ are close, and H^* has a linear approximation with slope $(\varrho, -\varrho v)$. The claim (3.10) follows.

5. If $z_1(t) \leqslant tv$, then for $z_1(t) \leqslant x \leqslant tv$ we have $\mathcal{A}(x,t) = 0$, and

$$0 \leqslant H(x,t) = x\varrho - t\varrho v + \mathrm{O}(\varepsilon) \leqslant \mathrm{O}(\varepsilon), \tag{3.15}$$

where in the second statement we used (3.13); in the last statement we used $x \leqslant tv$.

6. If $z_n(t) + \theta/N \leqslant tv + \ell$, then for $z_n(t) + \theta/N \leqslant x \leqslant tv + \ell$ we have

$$\frac{\theta n}{N} = H(x,t) \leqslant \mathcal{H}(x+\alpha\ell, t+\beta\ell) - \mathcal{H}(\alpha\ell, \beta\ell) = x\varrho - t\varrho v + \mathrm{O}(\varepsilon) = \mathcal{A}(x,t) + \mathrm{O}(\varepsilon), \tag{3.16}$$

where in the second statement we used the construction of (3.13); in the third statement we used that H^* and $\mathcal{H}$ are close, and H^* has a linear approximation with slope $(\varrho, -\varrho v)$. The claim (3.10) follows from the above estimate together with (3.12).

7. If $tv + \ell \leqslant z_n(t) + \theta/N$, then for $tv + \ell \leqslant x \leqslant z_n(t) + \theta/N$ we have $\mathcal{A}(x,t) = \varrho\ell$, and

$$\frac{\theta n}{N} \geqslant H(x,t) = x\varrho - t\varrho v + \mathrm{O}(\varepsilon) \geqslant \ell\varrho + \mathrm{O}(\varepsilon). \tag{3.17}$$ {{e:Hextreme}}

where in the second statement we used (3.13); in {e:Hextreme} the last statement we used $x \geqslant \ell + tv$. The claim (3.10) follows from combining (3.17) with (3.12).

□

With the above decomposition, we can rewrite $\mathbb{P}(\mathbb{B}_\varepsilon(H^* : \{\mathsf{y}(\mathsf{t})\}_{\mathsf{t}\in\{0,\mathsf{L},2\mathsf{L},\cdots,\mathsf{T}\}}))$ as

$$\begin{aligned}\frac{1}{2^{N\mathsf{T}}} \sum_{\mathbb{B}_\varepsilon(H^*:\{\mathsf{y}(\mathsf{t})\}_{\mathsf{t}\in\{0,\mathsf{L},2\mathsf{L},\cdots,\mathsf{T}\}})} \prod_\alpha \prod_\beta \prod_{\alpha\mathsf{L}\leqslant\mathsf{t}<(\alpha+1)\mathsf{L}} \prod_{i,j\in I(\alpha,\beta):i<j} \left(1 + \frac{\theta(e_i(\mathsf{t}) - e_j(\mathsf{t}))}{\mathsf{x}_i(\mathsf{t}) - \mathsf{x}_j(\mathsf{t})}\right) \\ \times \prod_\alpha \prod_{\beta<\beta'} \prod_{\alpha\mathsf{L}\leqslant\mathsf{t}<(\alpha+1)\mathsf{L}} \prod_{i\in I(\alpha,\beta), j\in I(\alpha,\beta')} \left(1 + \frac{\theta(e_i(\mathsf{t}) - e_j(\mathsf{t}))}{\mathsf{x}_i(\mathsf{t}) - \mathsf{x}_j(\mathsf{t})}\right)\end{aligned} \tag{3.18}$$ {{e:ldpup}}

{p:LDP}

Proposition 3.3. *Let θ, T be positive real numbers and $h \in \mathrm{Adm}_\theta^\partial(\mathfrak{R})$. Consider two sequences of particle configurations (y, z) which are (h, θ, T)-admissible as defined in 1.4. There exists a constant $C > 0$,*

$$\frac{1}{(\ell N)^2} \ln \sum \prod_{\alpha\mathsf{L}\leqslant\mathsf{t}<(\alpha+1)\mathsf{L}} \prod_{i,j\in I(\alpha,\beta):i<j} \left(1 + \frac{\theta(e_i(\mathsf{t}) - e_j(\mathsf{t}))}{\mathsf{x}_i(\mathsf{t}) - \mathsf{x}_j(\mathsf{t})}\right) \leqslant C, \tag{3.19}$$ {{e:trivialub}}

where the summation is over all the non-intersecting Bernoulli walk paths.

If we assume that on $\mathfrak{R}(\alpha, \beta)$, the height function H^* has a linear approximation with slope $(\varrho, -\varrho v) \in \mathcal{T}$ and error ε.

$$\begin{aligned}&\frac{1}{(\ell N)^2} \ln \sum \prod_{\alpha\mathsf{L}\leqslant\mathsf{t}<(\alpha+1)\mathsf{L}} \prod_{i,j\in I(\alpha,\beta):i<j} \left(1 + \frac{\theta(e_i(\mathsf{t}) - e_j(\mathsf{t}))}{\mathsf{x}_i(\mathsf{t}) - \mathsf{x}_j(\mathsf{t})}\right) \\ &= \frac{1}{\theta\ell^2} \iint_{[\alpha\ell,(\alpha+1)\ell]\times[\beta\ell,(\beta+1)\ell]} \sigma(\nabla H^*) \mathrm{d}x\mathrm{d}t + \mathrm{O}(\varepsilon^{1/2}\log(1/\varepsilon))\end{aligned} \tag{3.20}$$ {{e:linearapub}

where the summation is over $\{\mathsf{x}_k(t)\}_{k\in I(\alpha,\beta), t\in[\beta\ell,(\beta+1)\ell]}$ such that (3.10) holds {e:linearapub}

{l:offdiagonal}

Lemma 3.4. *Let θ, T be positive real numbers and $h \in \mathrm{Adm}_\theta^{\partial}(\mathfrak{R})$. Consider two sequences of particle configurations (y, z) which are (h, θ, T)-admissible as defined in 1.4. For any $\{\mathbf{x}(t)\}_{0\leqslant \mathsf{t}\leqslant \mathsf{T}} \in \mathbb{B}_\varepsilon(H^* : \{\mathbf{y}(\mathsf{t})\}_{\mathsf{t}\in\{0,\mathsf{L},2\mathsf{L},\cdots,\mathsf{T}\}})$, the following holds*

$$\sum_{0\leq \mathsf{t}\leq \mathsf{T}}\sum_{\alpha}\sum_{\beta\neq\beta'}\sum_{i\in I(\alpha,\beta), j\in I(\alpha,\beta')} \ln\left(1+\frac{\theta(e_i(\mathsf{t})-e_j(\mathsf{t}))}{\mathsf{x}_i(\mathsf{t})-\mathsf{x}_j(\mathsf{t})}\right)$$
$$=\sum_{1\leqslant i<j\leqslant N}\theta\ln\left(\frac{\mathsf{y}_i(\mathsf{T})-\mathsf{y}_j(\mathsf{T})}{\mathsf{y}_i(0)-\mathsf{y}_j(0)}\right)+\mathrm{O}(\varepsilon\ln(1/\ell)N^2)$$

Proof of Large Deviation Upper bound in Theorem 1.5. Thanks to Lemma 3.4 and Proposition 3.3, we can bound (3.18) as

$$\begin{aligned}\frac{1}{N^2}\ln(3.18) &\leqslant -T\ln(2)+\frac{1}{N^2}\ln\sum_{\mathbb{B}_\varepsilon(H^*:\{\mathbf{y}(\mathsf{t})\}_{\mathsf{t}\in\{0,\mathsf{L},2\mathsf{L},\cdots,\mathsf{T}\}})}\prod_{\alpha,\beta}\prod_{\alpha\mathsf{L}\leqslant\mathsf{t}<(\alpha+1)\mathsf{L}}\prod_{i,j\in I(\alpha,\beta):i<j}\left(1+\frac{\theta(e_i(\mathsf{t})-e_j(\mathsf{t}))}{\mathsf{x}_i(\mathsf{t})-\mathsf{x}_j(\mathsf{t})}\right)\\
&+\frac{1}{N^2}\sum_{1\leqslant i<j\leqslant N}\theta\ln\left(\frac{\mathsf{x}_i(\mathsf{T})-\mathsf{x}_j(\mathsf{T})}{\mathsf{x}_i(0)-\mathsf{x}_j(0)}\right)+\mathrm{O}(\varepsilon\ln(1/\ell))\\
&=-T\ln(2)+\frac{1}{\theta}\sum_{\alpha,\beta}\int_{[\alpha\ell,(\alpha+1)\ell]\times[\beta\ell,(\beta+1)\ell]}\sigma(\nabla H^*)\mathrm{d}x\mathrm{d}t+\mathrm{O}(\varepsilon)\\
&+\frac{1}{N^2}\sum_{1\leqslant i<j\leqslant N}\theta\ln\left(\frac{\mathsf{y}_i(\mathsf{T})-\mathsf{y}_j(\mathsf{T})}{\mathsf{y}_i(0)-\mathsf{y}_j(0)}\right)+\mathrm{O}(\varepsilon\ln(1/\ell))\\
&=\frac{1}{\theta}\iint\sigma(\nabla H^*)\mathrm{d}x\mathrm{d}t+\frac{1}{2\theta}\int\ln|x-y|\mathrm{d}h(x,t)\mathrm{d}h(x,t)\Big|_0^T-T\ln(2)+\mathrm{O}(\varepsilon\ln(1/\ell))\end{aligned}$$

where in the second equality, we used that the number of (α,β) such that on $\mathfrak{R}(\alpha,\beta)$, H^* does not have a linear approximation is bounded by ε/ℓ^2; in the last line we used Lemma 2.8. ☐

3.2 Lower Bound

{s:lowbb}

To prove the large deviation lower bound, we recall the height function $\mathcal{H}$ as constructed in (2.1)

$$\|\mathcal{H}-H^*\|_\infty\leqslant\varepsilon. \quad (3.21)$$ {{e:cHbound}}

We denote $\{\boldsymbol{y}(t)\}_{t\in\{0,\ell,2\ell,\cdots,T\}}$ the particle configurations corresponding to $\mathcal{H}$ at times $0,\ell,2\ell,\cdots,T$.

We recall the ℓ-mesh from Definition 2.2. Take any $\mathfrak{R}(\alpha,\beta)$, the same as in the upper bound, we consider the restriction of the particle configuration $\boldsymbol{y}(\beta\ell)$ on the interval $[\alpha\ell,(\alpha+1)\ell)$

$$y_{i-1}(\beta\ell)<\alpha\ell\leqslant y_i(\beta\ell)<\cdots<y_j(\beta\ell)<(\alpha+1)\ell\leqslant y_{j+1}(\beta\ell). \quad (3.22)$$

Denote the index set $I(\alpha,\beta)=\{i,i+1,i+2,\cdots,j\}$. For any $\{\boldsymbol{x}(t)\}_{0\leqslant t\leqslant T}$ in $\mathbb{B}_\varepsilon(H^*:\{\mathbf{y}(\mathsf{t})\}_{\mathsf{t}\in\{0,\mathsf{L},2\mathsf{L},\cdots,\mathsf{T}\}})$, its restriction on the index set $I(\alpha,\beta):=n$ and time interval $[\beta\ell,(\beta+1)\ell]$, $\{x_k(t)\}_{k\in I(\alpha,\beta),t\in[\beta\ell,(\beta+1)\ell]}$ form an n-particle nonintersecting Bernoulli random walk. The shifted particle configuration

$$y_{k+i-1}(t+\beta\ell)-\alpha\ell,\quad 1\leqslant k\leqslant n, \quad (3.23)$$ {{e:zk3}}

form an n-particle nonintersecting Bernoulli random walk on the parallelogram shaped region

$$\mathfrak{P}=\{x,t\in\mathbb{R}^2:0\leqslant t\leqslant\ell, y_i(t+\beta\ell)-\alpha\ell\leqslant x\leqslant y_{j+1}(t+\beta\ell)-\alpha\ell\}. \quad (3.24)$$ {{e:deffP}}

To obtain a large deviation lower bound, we will restrict to the following set of N-particle nonintersecting Bernoulli random walks $\{\boldsymbol{x}(t)\}_{0\leqslant t\leqslant T}$: for any α,β,

1. If on $\mathfrak{R}(\alpha,\beta)$, the height function H^* has a linear approximation with slope $(\varrho,-\varrho v)\in\mathcal{T}_\zeta$ and error ε. We denote the shifted particle configuration $\{z_k(t)\}_{1\leqslant k\leqslant n, 0\leqslant t\leqslant \ell}$

$$z_k(t)=y_{k+i-1}(t+\beta\ell)-\alpha\ell,\quad 1\leqslant k\leqslant n. \tag{3.25}$$ {{e:zk2}}

   Then $z_k(t)$ form an n-particle nonintersecting Bernoulli walk inside $\mathfrak{P}$ from (3.24), and its height function H has a linear approximation {e:zk2}

$$(z_k(t),t)\in\mathfrak{P},\quad |H(x,t)-\mathcal{A}(x,t)|\leqslant C\varepsilon, \tag{3.26}$$ {{e:heightclose

   where $\mathcal{A}$ is the height function with slope given by $(\varrho,-\varrho v)$ from Definition 3.1.

2. Otherwise, we take

$$\{x_k(t)\}_{k\in I(\alpha,\beta),t\in[\beta\ell,(\beta+1)\ell]}=\{y_k(t)\}_{k\in I(\alpha,\beta),t\in[\beta\ell,(\beta+1)\ell]} \tag{3.27}$$ {{e:x=y}}

Lemma 3.5. *For any N-particle nonintersecting Bernoulli random walks $\{\boldsymbol{x}(t)\}_{t\in N^{-1}[\![0,\mathsf{T}]\!]}$ satisfying (3.26) and (3.27), its height function H satisfies* {e:x=y}

$$\|H-H^*\|\leqslant C\varepsilon. \tag{3.28}$$ {{e:HHdiff}}

Proof. For any α,β, we recall the index set $I(\alpha,\beta)=\{i,i+1,i+2,\cdots,j\}$. There are two cases. If on $\mathfrak{R}(\alpha,\beta)$, H^* has a linear approximation with slope $(\varrho,-\varrho v)$, then for $\beta\ell\leqslant t\leqslant(\beta+1)\ell$ and $y_i(t)\leqslant x\leqslant y_j(t)$, we have

$$\begin{aligned}&\varrho(x-y_i(t))-\varrho v(t-\beta\ell)=H^*(x,t)-H^*(y_i(t),\beta\ell)\\&=H^*(x,t)-\mathcal{H}(y_i(t),\beta\ell)+\mathrm{O}(\varepsilon)=H^*(x,t)-i\theta/N+\mathrm{O}(\varepsilon)\,.\end{aligned} \tag{3.29}$$ {{e:bb1}}

And the same argument as in (3.13) {e:bb1}

$$H(x,t)-i\theta/N=H(x,t)-H(y_i(t),\beta\ell)=\varrho(x-y_i(t))-\varrho v(t-\beta\ell)+\mathrm{O}(\varepsilon) \tag{3.30}$$

The claim (3.28) follows from combining (3.29) and (3.13).

In the second case that on $\mathfrak{R}(\alpha,\beta)$, H^* does not have a linear approximation. Then our construction gives

$$H(x,t)=\mathcal{H}(x,t)=H^*(x,t)+\mathrm{O}(\varepsilon). \tag{3.31}$$

□ {p:LDPlow}

Proposition 3.6. *Let θ,T be positive real numbers and $h\in\mathrm{Adm}_\theta^\partial(\mathfrak{R})$. Consider two sequences of particle configurations (y,z) which are (h,θ,T)-admissible. There exists a constant $C>0$,*

$$\frac{1}{(\ell N)^2}\ln\sum\prod_{\alpha\mathsf{L}\leqslant\mathsf{t}<(\alpha+1)\mathsf{L}}\prod_{i,j\in I(\alpha,\beta):i\neq j}\left(1+\frac{\theta(e_i(\mathsf{t})-e_j(\mathsf{t}))}{\mathsf{x}_i(\mathsf{t})-\mathsf{x}_j(\mathsf{t})}\right)\geqslant -C, \tag{3.32}$$ {{e:triviallow}

where the summation is over $\{\mathsf{x}_k(t)\}_{k\in I(\alpha,\beta),t\in[\beta\ell,(\beta+1)\ell]}$ such that (3.26) holds. {e:triviallow}

If we assume that on $\mathfrak{R}(\alpha,\beta)$, the height function H^* has a linear approximation with slope $(\varrho,-\varrho v)\in\mathcal{T}_\zeta$ and error ε.

$$\begin{aligned}&\frac{1}{(\ell N)^2}\ln\sum\prod_{\alpha\mathsf{L}\leqslant\mathsf{t}<(\alpha+1)\mathsf{L}}\prod_{i,j\in I(\alpha,\beta):i\neq j}\left(1+\frac{\theta(e_i(\mathsf{t})-e_j(\mathsf{t}))}{\mathsf{x}_i(\mathsf{t})-\mathsf{x}_j(\mathsf{t})}\right)\\&\geqslant\frac{1}{\theta}\iint_{[\alpha\ell,(\alpha+1)\ell]\times[\beta\ell,(\beta+1)\ell]}\sigma(\nabla H^*)\mathrm{d}x\mathrm{d}t+\mathrm{O}(\varepsilon^{1/2}\log(1/\varepsilon)).\end{aligned} \tag{3.33}$$ {{e:linearaplow

where the summation is over $\{x_k(t)\}_{k\in I(\alpha,\beta),t\in[\beta\ell,(\beta+1)\ell]}$ such that (3.26) holds. {e:linearaplow}

Proof of Large Deviation Lower bound in Theorem 1.5. Thanks to Lemma 3.4 and Proposition 3.6, we can bound (3.18) as

$$
\begin{aligned}
\frac{1}{N^2}\ln(3.18) \geqslant{}& -T\ln(2) + \frac{1}{N^2}\ln \sum_{\mathbb{B}_\varepsilon(H^*:\{\mathbf{y}(\mathsf{t})\}_{\mathsf{t}\in\{0,\mathsf{L},2\mathsf{L},\cdots,\mathsf{T}\}})} \prod_{\alpha,\beta}\prod_{\alpha\mathsf{L}\leqslant\mathsf{t}<(\alpha+1)\mathsf{L}}\prod_{i,j\in I(\alpha,\beta):i<j}\left(1+\frac{\theta(e_i(\mathsf{t})-e_j(\mathsf{t}))}{\mathsf{x}_i(\mathsf{t})-\mathsf{x}_j(\mathsf{t})}\right)\\
&+\frac{1}{N^2}\sum_{1\leqslant i<j\leqslant N}\theta\ln\left(\frac{\mathsf{x}_i(\mathsf{T})-\mathsf{x}_j(\mathsf{T})}{\mathsf{x}_i(0)-\mathsf{x}_j(0)}\right)+\mathrm{O}(\varepsilon\ln(1/\ell))\\
={}& -T\ln(2)+\frac{1}{\theta}\sum_{\alpha,\beta}\int_{[\alpha\ell,(\alpha+1)\ell]\times[\beta\ell,(\beta+1)\ell]}\sigma(\nabla H^*)\mathrm{d}x\mathrm{d}t+\mathrm{O}(\varepsilon)\\
&+\frac{1}{N^2}\sum_{1\leqslant i<j\leqslant N}\theta\ln\left(\frac{\mathsf{x}_i(\mathsf{T})-\mathsf{x}_j(\mathsf{T})}{\mathsf{x}_i(0)-\mathsf{x}_j(0)}\right)+\mathrm{O}(\varepsilon\ln(1/\ell))\\
={}& \frac{1}{\theta}\iint\sigma(\nabla H^*)\mathrm{d}x\mathrm{d}t+\frac{1}{2\theta}\int\ln|x-y|\mathrm{d}h(x,t)\mathrm{d}h(x,t)\Big|_0^T-T\ln(2)+\mathrm{O}(\varepsilon\ln(1/\ell))
\end{aligned}
$$

where in the second equality, we used that the number of (α,β) such that on $\mathfrak{R}(\alpha,\beta)$, H^* does not have a linear approximation is bounded by ε/ℓ^2; in the last line we used Lemma 2.8.

□

4 Large Deviation Upper Bound: Constant Slope Case

{s:upB}

In this section we study the following n-particle non-intersecting θ-Bernoulli walk from time 0 to $\mathsf{L}=\ell N$ $\big(\mathbf{x}(0),\mathbf{x}(1),\dots,\mathbf{x}(\mathsf{L})\big)\in(\mathbb{W}_\theta^n)^{\mathsf{L}}$ (see Definition 1.1)

$$
\mathbb{P}(\mathbf{x}(\mathsf{t}+1)=\mathbf{x}+\boldsymbol{e}|\boldsymbol{x}(\mathsf{t})=\mathbf{x})=\frac{1}{2^n}\frac{V(\mathbf{x}+\theta\boldsymbol{e})}{V(\mathbf{x})},\quad 0\leqslant\mathsf{t}\leqslant\mathsf{L}. \tag{4.1}
$$
{{e:Mkcopy}}

We will prove Proposition 3.3, the large deviation upper bound for the non-intersecting θ-Bernoulli walks with height function approximately linear.

We recall the height function $\mathcal{A}(x,t)$ with constant slope $\nabla\mathcal{A}(x,t)=(\varrho,-\varrho v)\in\overline{\mathcal{T}}$ from Definition 3.1,

$$
\mathcal{A}(x,t)=(x-tv)\varrho,\quad (x,t)\in\mathfrak{P};\quad \mathcal{A}(x,t)=0,\quad x\leqslant tv;\quad \mathcal{A}(x,t)=\ell\varrho,\quad x\geqslant\ell+tv. \tag{4.2}
$$
{{e:defH*}}

where $\mathfrak{P}$ is the parallelogram region

$$
\mathfrak{P}:=\{x,t\in\mathbb{R}^2:0\leqslant t\leqslant\ell,\quad tv\leqslant x\leqslant\ell+tv\}, \tag{4.3}
$$
{{e:defp}}

We assume that the density satisfies (3.9) {e:defp}

$$
\left|\frac{\theta n}{N}-\ell\varrho\right|\leqslant C\varepsilon \tag{4.4}
$$
{{e:densityboun

Fix a small parameter $\zeta>0$, and denote

$$
\mathcal{T}_\zeta=\{(u,v)\in\mathcal{T}:\zeta<u,-v,u+v\leqslant 1-\zeta\} \tag{4.5}
$$

There are two cases,

1. Extreme slope case where $\nabla H^*=(\varrho,-\varrho v)\notin\mathcal{T}_\zeta$.
2. Interior slope case where $\nabla H^*=(\varrho,-\varrho v)\in\mathcal{T}_\zeta$.

{p:LDPconstant}

Proposition 4.1. *We recall the rate function σ from (1.19). Fix small $\zeta>0$, and $(\varrho,-\varrho v)\in\overline{\mathcal{T}}$ such that (4.4) holds. Take the height function $\mathcal{A}(x,t)$ with constant slope $\nabla\mathcal{A}=(\varrho,-\varrho v)$ on the parallelogram region $\mathfrak{P}$, as in (4.2).*

1. *If* $(\varrho, -\varrho v) \notin \mathcal{T}_\zeta$, *then for any* $\varepsilon \leqslant \zeta\ell^2$, *we have*

$$\frac{1}{(\ell N)^2} \ln \mathbb{P}(\|H - \mathcal{A}\|_\infty \leqslant \varepsilon) = -\ln(2) + \mathrm{O}(-\zeta^{1/2}\ln(\zeta)) \tag{4.6}$$

{{e:contributio

2. *If* $(\varrho, -\varrho v) \in \mathcal{T}_\zeta$, *then for any* $\varepsilon \leqslant \zeta\ell^2$, *we have*

$$\frac{1}{(\ell N)^2} \ln \mathbb{P}(\|H - \mathcal{A}\|_\infty \leqslant \varepsilon) \leqslant \frac{1}{\pi}\sigma(\varrho, -\varrho v) - \ln(2) + \mathrm{O}(\varepsilon^{1/2}\log(1/\varepsilon)). \tag{4.7}$$

Proof of Proposition 3.3. The first statement (3.19) in Proposition 3.3 follows from Lemma 2.7.

For the second statement (3.20), we recall from Lemma 2.3 that at least $1-\varepsilon$ fraction of the points x of $\mathfrak{R}(\alpha, \beta)$, the gradient ∇H exists and is within ε of $(\varrho, -\varrho v)$. As a consequence,

$$\frac{1}{\theta\ell^2} \iint_{[\alpha\ell,(\alpha+1)\ell]\times[\beta\ell,(\beta+1)\ell]} \sigma(\nabla H^*)\mathrm{d}x\mathrm{d}t = \sigma(\varrho, -\varrho v) + \mathrm{O}(\varepsilon) \tag{4.8}$$

{{e:integralfor

If $(\varrho, -\varrho v) \notin \mathcal{T}_\zeta$, then the claim (3.20) follows from (4.8) {e:integralform} and the second statement of Proposition 4.1. If $(\varrho, -\varrho v) \in \mathcal{T}_\zeta$, without loss of generality we assume that $\varrho \leqslant \zeta$. The Lobachevsky function (recall from (1.19)) is π-periodic, namely $L(x+\pi) = L(x)$ and $-L(\pi - x) = L(x)$. Moreover, because of the logarithmic singularity, the Lobachevsky function satisfies

$$|L(x) - L(y)| \lesssim |x-y|\log(1/|x-y|). \tag{4.9}$$

{{e:Lobdiff}}

and $L(\varepsilon) = \mathrm{O}(\varepsilon\log(1/\varepsilon))$ for $\varepsilon \to 0+$. Therefore

$$\sigma(\varrho, -\varrho v) = -L(\varrho) + L(\varrho v) + L((1-v)\varrho) = \mathrm{O}(\zeta\ln(1/\zeta)). \tag{4.10}$$

{{e:sigmasmall}

The claim (3.20) follows from (4.8) {e:sigmasmall}, (4.10) and the second statement of Proposition 4.1 ☐

4.1 Proof Outline of Proposition 4.1

For the extreme slope case, to estimate $\mathbb{P}(\|H - \mathcal{A}\|_\infty \leqslant \varepsilon)$, we will directly upper bound the number of height profiles satisfying $\|H - \mathcal{A}\|_\infty \leqslant \varepsilon$ by a combinatorics argument, and the probability $\mathbb{P}(H)$ for each of these height profile by a delicate analysis of Vandermonde determinants. It turns out both parts are negligible. The proof is given in Section 4.2.

For the interior slope case, we will first smooth the height function $\mathcal{A}$ by convolving it with a Cauchy distribution. Take small $\delta > 0$, denote

$$\widetilde{H}(x,t) = \frac{1}{\pi}\int \frac{\delta\mathcal{A}(y,t)\mathrm{d}y}{(y-x)^2+\delta^2}. \tag{4.11}$$

{{e:deftH}}

Since for $(x,t) \in \mathfrak{P}$, $\nabla\mathcal{A}(x,t) = (\varrho, -\varrho v) \in \mathcal{T}_\zeta$, namely, $\zeta \leqslant \varrho, \varrho v, \varrho(1-v) \leqslant 1-\zeta$, it follows that $\nabla\widetilde{H}(x,t) \in \mathcal{T}$ for $(x,t) \in \mathfrak{R}$. Properties of the smoothed height function $\widetilde{H}(x,t)$ are collected in Section 4.3.

We define the associated *complex slope* $\widetilde{f} : \mathfrak{R} \to \mathbb{H}^-$ by setting, for any $(x,t) \in \mathfrak{R}$, $\widetilde{f}_t(x) = \widetilde{f}(x,t) \in \mathbb{H}^-$ to be the unique complex number satisfying

$$\arg^* \widetilde{f}_t(x) = -\pi\partial_x\widetilde{H}(x,t); \qquad \arg^*\big(\widetilde{f}_t(x)+1\big) = \pi\partial_t\widetilde{H}(x,t), \tag{4.12}$$

{{fh}}

where for any $z \in \mathbb{H}^-$ we have set $\arg^* z = \theta \in (-\pi, 0)$ to be the unique number in $(-\pi, 0)$ satisfying $e^{-\mathrm{i}\theta}z \in \mathbb{R}$; see Figure 2 for a depiction. By the law of sines (4.12) {fh} implies that

$$\frac{|\widetilde{f}_t(x)|}{\sin(-\pi\partial_t\widetilde{H}(x,t))} = \frac{|1+\widetilde{f}_t(x)|}{\sin(\pi\partial_x\widetilde{H}(x,t))} = \frac{1}{\sin(\pi\partial_x\widetilde{H}(x,t)+\pi\partial_t\widetilde{H}(x,t))} \tag{4.13}$$

{{e:sinelaw}}

Figure 2: Shown above the complex slope $\widetilde{f}=\widetilde{f}(x,t)$. {slope1}

complex_slope.pdf

From (4.13), we can solve {e:sinelaw} the complex slope $\widetilde{f}_t(x)$,

$$\begin{aligned}\widetilde{f}_t(x)&=e^{-\mathrm{i}\pi\partial_x\widetilde{H}(x,t)}\exp(\ln\sin(-\pi\partial_t\widetilde{H}(x,t))-\ln\sin(\pi\partial_x\widetilde{H}(x,t)+\pi\partial_t\widetilde{H}(x,t))),\\ 1+\widetilde{f}_t(x)&=e^{\mathrm{i}\pi\partial_t\widetilde{H}(x,t)}\exp(\ln\sin(\pi\partial_x\widetilde{H}(x,t))-\ln\sin(\pi\partial_x\widetilde{H}(x,t)+\pi\partial_t\widetilde{H}(x,t))),\end{aligned}\tag{4.14}$$ {{e:ftx}}

We take a drift function $g_t(x)$ given by

$$g_t(x)=\ln\sin(-\pi\partial_t\widetilde{H}(x,t))-\ln\sin(\pi\partial_x\widetilde{H}(x,t)+\pi\partial_t\widetilde{H}(x,t))-\mathrm{Hib}(\partial_x\widetilde{H}(x,t)).\tag{4.15}$$ {{e:gtx}}

As we will show in Section 4.3, the complex slope $\widetilde{f}_t(x)$, and the drift function $g_t(x)$ can both be extended analytically to a strip neighborhood of the real axis, and on this strip region they are related by

$$\widetilde{f}_t(z)=e^{m_t(z)+g_t(z)},\quad m_t(z)=\int_{\mathbb{R}}\frac{\partial_x\widetilde{H}(x,t)\mathrm{d}x}{z-x}\tag{4.16}$$

To prove the large deviation upper bound for the interior case in Proposition 3.3, we tilt the law of the Markov process (4.1) by an exponential martingale. For any $0\leqslant \mathsf{t}\leqslant \mathsf{L}-1$, let

$$\Delta\mathcal{M}_t^g:=\frac{1}{N}\left(\sum_{i=1}^n e_i(\mathsf{t})g_t(x_i(t))-\mathbb{E}\left[\sum_{i=1}^n e_i(\mathsf{t})g_t(x_i(t))\middle|\boldsymbol{x}(t)\right]\right),\quad t=\mathsf{t}/N\tag{4.17}$$

and the following is a Martingle

$$\prod_{t\in N^{-1}[\![0,\mathsf{L}-1]\!]}\frac{e^{N\Delta\mathcal{M}_t^g}}{\mathbb{E}[e^{N\Delta\mathcal{M}_t^g}|\boldsymbol{x}(t)]}\tag{4.18}$$ {{e:Mt}}

In particular it gives the following identity

$$1=\mathbb{E}\left[\prod_{t\in[\![0,\mathsf{L}-1]\!]/N}\frac{e^{N\Delta\mathcal{M}_t^g}}{\mathbb{E}[e^{N\Delta\mathcal{M}_t^g}|\boldsymbol{x}(t)]}\right]=\mathbb{E}\left[\prod_{t\in[\![0,\mathsf{L}-1]\!]/N}\frac{e^{\sum_{i=1}^N e_i(Nt)g_t(x_i(t))}}{\mathbb{E}[e^{\sum_{i=1}^N e_i(Nt)g_t(x_i(t))}|\boldsymbol{x}(t)]}\right].\tag{4.19}$$ {{e:ub1}}

The large deviation upper bound follows from restricting the above identity to the event $\{\|H-\mathcal{A}\|_\infty\leqslant\varepsilon)\}$.

$$1\geqslant\mathbb{E}\left[\prod_{t\in[\![0,\mathsf{L}-1]\!]/N}\frac{e^{\sum_{i=1}^N e_i(Nt)g_t(x_i(t))}}{\mathbb{E}[e^{\sum_{i=1}^N e_i(Nt)g_t(x_i(t))}|\boldsymbol{x}(t)]}\mathbf{1}(\|H-\mathcal{A}\|_\infty\leqslant\varepsilon)\right].\tag{4.20}$$ {{e:Pg2}}

In Section 4.4, we will show that on the event, $\{\|H(x,t)-\mathcal{A}(x,t)\|_\infty\leqslant\varepsilon)\}$, up to small error, the numerator in the exponential martingale (4.18) can be {e:Pg2} written in terms of $\mathcal{A}(x,t)$. For the denominator, we will use the dynamical loop equation Theorem A.5 to compute its expectation. They together lead to the following large deviation upper bound.

{p:ubb}

Proposition 4.2. *Let θ,T be positive real numbers and $h\in\mathrm{Adm}_\theta^\partial(\mathfrak{R})$. Consider two sequences of particle configurations (y,z) which are (h,θ,T)-admissible. Take the height function $\mathcal{A}(x,t)$ with constant slope $\nabla\mathcal{A}=(\varrho,-\varrho v)$ on the parallelogram region $\mathfrak{P}$, as in (4.2). If $(\varrho,-\varrho v)\in\mathcal{T}_\zeta$, then for any $\varepsilon\leqslant\zeta\ell^2$, we have*

$$\limsup_{N\to\infty}\frac{1}{(\ell N)^2}\ln\mathbb{P}(\|H-\mathcal{A}\|_\infty\leqslant\varepsilon)\leqslant\frac{S(\mathcal{A};g)}{\theta}+\mathrm{O}\left(\frac{\varepsilon}{\delta\zeta^2}\right)\tag{4.21}$$

where, if m_t^* denotes the Stieltjes transform of $\partial_x\mathcal{A}$,

$$S(\mathcal{A};g)=-\int_0^\ell\int_{\mathbb{R}}\partial_t\mathcal{A}(x,t)g_t(x)\mathrm{d}x\mathrm{d}t-\frac{1}{2\pi\mathrm{i}}\int_0^\ell\oint\int_0^1\ln(1+e^{m_t^*(z)+\tau g_t(z)})g_t(z)\mathrm{d}z\mathrm{d}\tau\mathrm{d}t\tag{4.22}$$ {{e:rateS}}

We will then analyze the rate function $S(\mathcal{A};g)$ in (4.22) in {Sec:rates} 4.5. Miraculously, after simplifying the rate function, we recover these Lobachevsky functions (see (1.19)).

{p:entropy}

Proposition 4.3. *Adopt the assumptions in Proposition 4.2. The rate function satisfies*

$$S(\mathcal{A};g) = -\frac{1}{\pi}\sigma(\varrho,-\varrho v) - \ln(2) + \mathrm{O}(\delta\ln(1/\delta)^2\ell). \tag{4.23}$$ {{e:entropy}}

4.2 Proof of Proposition 3.3: Extreme Slope Case

{s:extremeslope

There are three cases for the extreme slope $(\varrho,-\varrho v)\notin\mathcal{T}_\zeta$: i) $-\varrho v\leqslant\zeta$ ii) $\varrho+\varrho v\geqslant 1-\zeta$ iii) $\varrho\geqslant 1-\zeta$.

We start with the first two cases. Their proofs are the same, so we will only give the proof of the first case.

Proof of the First and Second Cases. In the first case $-\zeta\leqslant\partial_t\mathcal{A}\leqslant 0$, and most of $e_i(\mathsf{t})$ are zero. In fact, we have by integration by parts

$$\begin{aligned}
\frac{\theta}{N^2}\sum_{1\leqslant i\leqslant n}\sum_{1\leqslant \mathsf{t}\leqslant \mathsf{L}} e_i(\mathsf{t}) &= \frac{\theta}{N^2}\left(\sum_{1\leqslant i\leqslant n}\mathsf{x}_i(\mathsf{L}) - \sum_{1\leqslant i\leqslant n}\mathsf{x}_i(0)\right) = \int_{\mathbb{R}} x\varrho_\ell(x;\boldsymbol{x}(\ell))\mathrm{d}x - \int_{\mathbb{R}} x\varrho_0(x;\boldsymbol{x}(0))\mathrm{d}x\\
&= \int_{\mathbb{R}}(H(x,0)-H(x,\ell))\mathrm{d}x = \int_{\mathbb{R}}(\mathcal{A}(x,0)-\mathcal{A}(x,\ell))\mathrm{d}x + \mathrm{O}(\varepsilon\ell)\\
&= \int_{\mathbb{R}}\int_0^\ell -\partial_t\mathcal{A}(x,t)\mathrm{d}t\mathrm{d}x + \mathrm{O}(\varepsilon) = \mathrm{O}(\zeta\ell^2+\varepsilon) = \mathrm{O}(\zeta\ell^2)\,.
\end{aligned} \tag{4.24}$$ {{e:sume}}

We denote

$$I = \{1\leqslant \mathsf{t}\leqslant \mathsf{L} : \sum_i e_i(\mathsf{t})\leqslant \zeta^{1/2}\ell N\},\quad I^{\complement} = \{1\leqslant \mathsf{t}\leqslant \mathsf{L} : \sum_i e_i(\mathsf{t}) > \zeta^{1/2}\ell N\}. \tag{4.25}$$ {{e:defI}}

Then (4.24) gives that {e:defI} $|I^{\complement}| = \mathrm{O}(\zeta^{1/2}\ell N)$. There are $\binom{\ell N}{\mathrm{O}(\zeta^{1/2}\ell N)} = e^{\mathrm{O}(-\zeta^{1/2}\ln(\zeta)(\ell N)^2)}$ ways to choose the set $I^{\complement}$. Once we fix the sets $I, I^{\complement}$, the total weights of non-intersecting Bernoulli walks satisfying (4.25) is

$$\begin{aligned}
\sum_{\mathsf{p}\text{ satisfies }(4.25)}\mathcal{W}(\mathsf{p}) &\leqslant \prod_{\mathsf{t}\in I}\sum_{\sum e_i(\mathsf{t})\leqslant\zeta^{1/2}\ell N}\frac{V(\mathbf{x}(\mathsf{t})+\boldsymbol{e}(\mathsf{t}))}{V(\boldsymbol{x}(\mathsf{t}))}\prod_{\mathsf{t}\in I^{\complement}}\sum_{\mathbf{e}(\mathsf{t})}\frac{V(\mathbf{x}(\mathsf{t})+\boldsymbol{e}(\mathsf{t}))}{V(\boldsymbol{x}(\mathsf{t}))}\\
&\leqslant \left(\sum_{k\leqslant\zeta^{1/2}\ell N}\binom{n}{k}\right)^{|I|} 2^{n|I^{\complement}|} = e^{\mathrm{O}(-\zeta^{1/2}\ln(\zeta)(\ell N)^2)}
\end{aligned} \tag{4.26}$$ {{e:Wbound}}

where we used that

$$\sum_{\sum_i e_i = k}\frac{V(\mathbf{x}+\boldsymbol{e})}{V(\mathbf{x})} = \binom{n}{k}\leqslant n^k, \tag{4.27}$$

from Lemma 2.6. It follows from (4.26) and the fact that there at most $e^{\mathrm{O}(-\zeta^{1/2}\ln(\zeta)(\ell N)^2)}$ ways to choose the set $I^{\complement}$, we conclude (4.6) for $-\varrho v\leqslant\zeta$.

In the second case that $1-\zeta\leqslant\partial_x\mathcal{A}+\partial_t\mathcal{A}\leqslant 1$, we have that most of $e_i(\mathsf{t})$ are one. By the same argument as the first case, we will have that (4.6) holds. ☐

For the last case, we have $1-\zeta\leqslant\partial_x\mathcal{A}\leqslant 1$, and $(1-\zeta)\ell N\leqslant n\leqslant \ell N$, we need to bound the total weight of the paths p corresponding to H,

$$\sum_{\mathsf{p}:\|H-\mathcal{A}\|\leqslant\varepsilon}\mathcal{W}(\mathsf{p}) = \sum_{\mathsf{p}:\|H-\mathcal{A}\|\leqslant\varepsilon}\prod_{0\leqslant\mathsf{t}\leqslant\mathsf{T}}\prod_{1\leqslant i<j\leqslant n}\frac{(\mathsf{x}_i(\mathsf{t})+\theta e_i(\mathsf{t}))-(\mathsf{x}_j(\mathsf{t})+\theta e_j(\mathsf{t}))}{\mathsf{x}_i(\mathsf{t})-\mathsf{x}_j(\mathsf{t})} \tag{4.28}$$ {{e:Wpsum}}

We denote the index sets

$$I_0 = [\![1, \varepsilon N]\!], \quad I_1 = [\![\varepsilon N, n - \varepsilon N]\!], \quad I_2 = [\![n - \varepsilon N, n]\!]. \tag{4.29}$$ {{e:defI012}}

For $\|H - \mathcal{A}\| \leqslant \varepsilon$, we have that for any index $i \in I_1$, $\boldsymbol{x}_i(t) \in [tv, \ell + tv]$. Fix any time $0 \leqslant \mathsf{t} \leqslant \mathsf{T}$. For $i \in I_0 \cup I_2$, there are $2^{|I_0|+|I_2|} = 2^{2\varepsilon N}$ possible choices for $e_i(\mathsf{t})$. For the particle configuration $\{\mathsf{x}_i(\mathsf{t}) + e_i(\mathsf{t})\}_{i \in I_1}$, the number of empty sites is bounded by

$$\sum_{i \in I_1} ((\mathsf{x}_{i+1}(\mathsf{t}) + e_{i+1}(\mathsf{t})) - (\mathsf{x}_i(\mathsf{t}) + e_i(\mathsf{t})) - \theta) = \mathrm{O}(\zeta \ell N + \varepsilon N) = \mathrm{O}(\zeta \ell N). \tag{4.30}$$

Therefore, the number of configurations p satisfying $\|H - \mathcal{A}\| \leqslant \varepsilon$ is at most

$$2^{2\varepsilon N} \binom{\ell N}{\mathrm{O}(\zeta \ell N)} = e^{\mathrm{O}(-\zeta \ln(\zeta) \ell N)}. \tag{4.31}$$ {{e:totalp}}

The following lemma bounds each summand in (4.28) {e:totalp}

Lemma 4.4. *For configurations* x *so that* $\|H - \mathcal{A}\| \leqslant \varepsilon$, *The following holds*

$$\prod_{1 \leqslant i < j \leqslant n} \frac{(\mathsf{x}_i(\mathsf{t}) + \theta e_i(\mathsf{t})) - (\mathsf{x}_j(\mathsf{t}) + \theta e_j(\mathsf{t}))}{\mathsf{x}_i(\mathsf{t}) - \mathsf{x}_j(\mathsf{t})} = e^{\mathrm{O}(\zeta)} \left(\prod_{1 \leqslant i < j \leqslant n} \frac{(\mathsf{x}_i(\mathsf{t}) + e_i(\mathsf{t})) - (\mathsf{x}_j(\mathsf{t}) + e_j(\mathsf{t}))}{\mathsf{x}_i(\mathsf{t}) - \mathsf{x}_j(\mathsf{t})} \right)^{\theta} \tag{4.32}$$ {{e:xi-xj}}

Proof. To simplify the notation, we will omit the time t dependence. We recall from (2.16) {e:xi-xj}

$$\left| \ln \frac{(\mathsf{x}_i + \theta e_i) - (\mathsf{x}_j + \theta e_j)}{\mathsf{x}_i - \mathsf{x}_j} - \theta \ln \frac{(\mathsf{x}_i + e_i) - (\mathsf{x}_j + e_j)}{\mathsf{x}_i - \mathsf{x}_j} \right| \leqslant C \frac{(e_i - e_j)^2}{(\mathsf{x}_i - \mathsf{x}_j)^2} \tag{4.33}$$ {{e:localeq}}

Next, we show that for $\|H - \mathcal{A}\| \leqslant \varepsilon$,

$$\sum_{1 \leqslant i < j \leqslant n} \frac{(e_i - e_j)^2}{(\mathsf{x}_i - \mathsf{x}_j)^2} = \sum_{i \in I, j \in J} \frac{1}{(\mathsf{x}_i - \mathsf{x}_j)^2} = \mathrm{O}(\zeta \ln(\ell/\zeta) \ell N) \tag{4.34}$$

Let $I = \{i \in [\![1, n]\!] : e_i = 1\}$ and $J = \{j \in [\![1, n]\!] : e_j = 0\}$. The particle configuration $\{\mathsf{x}_i\}_{1 \leqslant i \leqslant n}$ consists of at most $\mathrm{O}(\zeta \ell N)$ tightly packed pieces. Namely, there exists $1 = i_0 < i_1 < \cdots < i_K = n + 1$ with $K = \mathrm{O}(\zeta \ell N)$, such that for any $i_k \leqslant i < j < i_{k+1}$, $\mathsf{x}_i = \mathsf{x}_j + \theta(j - i)$. Denote the interval $T_k = [\![i_k, i_{k+1} - 1]\!]$. Then either $I \cap T_k = \emptyset$ or $I \cap T_k = \{i_k, i_k + 1, \cdots, j_k\}$, and

$$\begin{aligned} \sum_{i \in I \cap T_k, j \in J} \frac{1}{|\mathsf{x}_i - \mathsf{x}_j|^2} &\leqslant \sum_{i \in I \cap T_k} \sum_{d \geqslant 1 + \min\{|i - j_k|, |i - i_k|\}} \frac{1}{(d\theta)^2} \\ &\lesssim \sum_{i \in I \cap T_k} \frac{1}{\theta^2 (1 + \min\{|i - j_k|, |i - i_k|\})} \lesssim \frac{1}{\theta^2} (1 + \ln(1 + i_k - j_k)) \end{aligned} \tag{4.35}$$ {{e:sumxixj}}

We can then sum over all the intervals T_k,

$$\sum_{i \in I, j \in J} \frac{1}{|\mathsf{x}_i - \mathsf{x}_j|^2} = \sum_{k=0}^{K-1} \sum_{i \in I \cap T_k, j \in J} \frac{1}{|\mathsf{x}_i - \mathsf{x}_j|^2} \lesssim \sum_{k=0}^{K-1} \frac{1}{\theta^2} (1 + \ln(1 + i_k - j_k)) \tag{4.36}$$

$$\lesssim \frac{K}{\theta^2} \left(1 + \ln \left(\frac{\sum_k 1 + i_k - j_k}{K} \right) \right) \lesssim \frac{K \ln(\ell N / K)}{\theta^2} \lesssim \frac{\zeta \ell N \ln(\ell/\zeta)}{\theta^2} \tag{4.37}$$

where for the second statement we used (4.35); the third statement we used Jensen's inequality; the last two statements we used that $\sum_k (1 + i_k - j_k) \leqslant K + \ell N$. □

In the following we prove the third case, when $\varrho \geqslant 1 - \zeta$.

Proof of the Third Case. Plugging (4.32) into (4.28), we get

$$\begin{aligned}
&\prod_{0\leqslant \mathsf{t}\leqslant \mathsf{T}}\prod_{1\leqslant i<j\leqslant n}\frac{(\mathsf{x}_i(\mathsf{t})+\theta e_i(\mathsf{t}))-(\mathsf{x}_j(\mathsf{t})+\theta e_j(\mathsf{t}))}{\mathsf{x}_i(\mathsf{t})-\mathsf{x}_j(\mathsf{t})}\\
&=e^{\mathrm{O}(-\zeta\ln(\zeta)\mathsf{T}\ell N)}\prod_{0\leqslant \mathsf{t}\leqslant \mathsf{T}}\left(\prod_{1\leqslant i<j\leqslant n}\frac{(\mathsf{x}_i(\mathsf{t})+e_i(\mathsf{t}))-(\mathsf{x}_j(\mathsf{t})+e_j(\mathsf{t}))}{\mathsf{x}_i(\mathsf{t})-\mathsf{x}_j(\mathsf{t})}\right)^{\theta}\\
&=e^{\mathrm{O}(-\zeta\ln(\zeta)(\ell N)^2}\left(\frac{V(\mathsf{x}(\mathsf{T}))}{V(\mathsf{x}(0))}\right)^{\theta}
\end{aligned}\tag{4.38}$$

Finally we need to estimate the ratio of the Vandermonde determinant

$$\frac{V(\mathsf{x}(\mathsf{T}))}{V(\mathsf{x}(0))}\leqslant e^{\mathrm{O}(\zeta(\ell N)^2)}.\tag{4.39}$$ {{e:weightp2}}

The claim (4.6) follows from plugging (1.16), (4.39) into (4.28), and using (4.31). To prove (4.39), we recall the three index sets I_0, I_1, I_2 from (4.29). It follows from showing that

$$\frac{V(\mathsf{x}(\mathsf{T}))}{V(\mathsf{x}(0))}\leqslant e^{\mathrm{O}(\varepsilon(\ell N)^2)}\frac{V(\mathsf{x}(\mathsf{T})|_{I_1})}{V(\mathsf{x}(0)|_{I_1})},\tag{4.40}$$ {{e:farparticle

and

$$\frac{V(\mathsf{x}(\mathsf{T})|_{I_1})}{V(\theta,2\theta,\cdots,|I_1|\theta)}=e^{\mathrm{O}(\zeta(\ell N)^2)}\tag{4.41}$$ {{e:addempty}}

The claim (4.40) follows from the following one time estimate

$$\frac{V(\mathsf{x}(\mathsf{T}))}{V(\mathsf{x}(0))}\frac{V(\mathsf{x}(0)|_{I_1})}{V(\mathsf{x}(\mathsf{T})|_{I_1})}=\prod_{(i,j)\notin I_1\times I_1}\frac{\mathsf{x}_i(\mathsf{T})-\mathsf{x}_j(\mathsf{T})}{\mathsf{x}_i(0)-\mathsf{x}_j(0)}\leqslant\prod_{\substack{i\in I_0\cup I_2, j\in[\![1,n]\!]\\ i<j}}\left(1+\frac{2\mathsf{T}}{|\mathsf{x}_i(0)-\mathsf{x}_j(0)|}\right)\tag{4.42}$$ {{e:onestep}}

$$\leqslant\prod_{\substack{i\in I_0\cup I_2, j\in[\![1,n]\!]\\ i<j}}\left(1+\frac{2\mathsf{T}}{|i-j|\theta}\right)\leqslant\prod_{i\in I_0\cup I_2}\prod_{1\leqslant k\leqslant \ell N}\left(\frac{3\ell N}{k\theta}\right)^2=e^{\mathrm{O}((|I_0|+|I_2|)\ell N)}=e^{\mathrm{O}(\varepsilon\ell N^2)}\tag{4.43}$$ {e:onestep}

where we used that $|\mathsf{x}_i(0)-\mathsf{x}_j(0)|\geqslant|i-j|\theta$. Next we prove (4.41). Since $\|H-\mathcal{A}\|_\infty\leqslant\varepsilon$ and $1-\zeta\partial_x\mathcal{A}\leqslant 1$, we have that up to some shift $\mathsf{x}(\mathsf{T})|_{I_1}$ is obtained from the particle configuration $\theta, 2\theta,\cdots,|I_1|\theta$ by adding at most $\mathrm{O}(\zeta\ell N)$ empty sites. Next we show that adding an empty site increases the Vandermonde determinant by at most a factor $e^{\mathrm{O}(\ell N)}$, then (4.41) follows. Say from the configuration $x_1>x_2>\cdots>x_{|I_1|}$, we add an empty site between x_k and x_{k+1}, and get the particle configuration $x_1+1>x_2+1>\cdots>x_k+1>x_{k+1}>\cdots x_{|I_1|}$, then

$$\frac{V(x_1+1,\cdots,x_k+1,x_{k+1},\cdots x_{|I_1|})}{V(x_1,x_2,\cdots,x_{|I_1|})}=\prod_{i\leqslant k,j\geqslant k+1}\left(1+\frac{1}{x_i-x_j}\right)\leqslant\prod_{i\leqslant k,j\geqslant k+1}\left(1+\frac{1}{|i-j|\theta}\right)\tag{4.44}$$

$$\leqslant e^{\sum_{i\leqslant k,j\geqslant k+1}\frac{1}{|i-j|\theta}}\leqslant e^{\frac{1}{\theta}\left(\frac{1}{1}+\frac{2}{2}+\cdots+\frac{\ell N}{\ell N}\right)}=e^{\ell N/\theta},\tag{4.45}$$

where we used that the multiset $\{|i-j|:i\leqslant k,j\geqslant k+1\}$ contains at most m copies of m for any $1\leqslant m\leqslant\ell N$. □

4.3 Complex Slope Estimates

{s:heightslope}

In this section, we prove in Lemma 4.5 that the complex slope $\widetilde{f}_t(x)$ and the drift function $g_t(x)$ (as constructed in (4.12) and (4.15)) can be extended analytically to a strip neighborhood of the real axis.

We recall the complex slope $\widetilde{f}_t(x)$ and drift function $g_t(x)$ from (4.14) and (4.15)

$$\begin{aligned}\widetilde{f}_t(x) &= e^{-\mathrm{i}\pi\partial_x \widetilde{H}(x,t)}\exp(\ln\sin(-\pi\partial_t\widetilde{H}(x,t)) - \ln\sin(\pi\partial_x\widetilde{H}(x,t)+\pi\partial_t\widetilde{H}(x,t))),\\ g_t(x) &= \ln\sin(-\pi\partial_t\widetilde{H}(x,t)) - \ln\sin(\pi\partial_x\widetilde{H}(x,t)+\pi\partial_t\widetilde{H}(x,t)) - \mathrm{Hib}(\partial_x\widetilde{H}(x,t)).\end{aligned} \tag{4.46}$$ {{e:ftgtx}}

{l:gtestimate}

Lemma 4.5. *There exists a large constant $C>0$, for any $\delta>0$, $\widetilde{f}_t(x)$ and $g_t(x)$ as in (4.46) can be* {e:ftgtx} *extended analytically to the strip region $\mathcal{D}=\mathcal{D}(C)=\{x+\mathrm{i}\eta : |\eta|\leqslant C^{-1}\delta\}$, and for any $z\in\mathcal{D}$ the following holds*

1. *The norm of $\widetilde{f}_t(z)$ satisfies: $|\widetilde{f}_t(z)|\leqslant C/\zeta$.*
2. *The norm of $g_t(z)$ satisfies: $|g_t(z)|\leqslant \ln(1/\zeta)+\ln(1/\delta)+C$. If $\mathrm{dist}(z,\{tv,\ell+tv\})\geqslant \ell$, we have*

$$\ln\frac{\sin(\pi\varrho v\kappa_t(z))}{\sin(\pi\varrho(1-v)\kappa_t(z))} = \ln\frac{\sin(v)}{\sin(1-v)} + \mathrm{O}\left(\frac{\delta}{\mathrm{dist}(z,\{tv,\ell+tv\})}\right) \tag{4.47}$$ {{e:glargez}}

   The imaginary part of $g_t(z)$ satisfies: $|\operatorname{Im}[g_t(z)]|\leqslant C\operatorname{Im}[z]/\delta$.

3. *The derivatives of $g_t(x)$ satisfies: for any $x\in\mathbb{R}$, $|\partial_x g_t(x)|\leqslant C/\delta$.*

Proof of Lemma 4.5. We will first estimate

$$\ln\sin(-\pi\partial_t\widetilde{H}(x,t)) - \ln\sin(\pi\partial_x\widetilde{H}(x,t)+\pi\partial_t\widetilde{H}(x,t)) \tag{4.48}$$

which can be extended to $z\in\mathcal{D}$ as

$$\ln\sin(\pi\varrho v\kappa_t(z)) - \ln\sin(\pi\varrho(1-v)\kappa_t(z)) = \ln\frac{\sin(\pi\varrho v\kappa_t(z))}{\sin(\pi\varrho(1-v)\kappa_t(z))}\,. \tag{4.49}$$ {{e:logdiff}}

Recall that $\kappa_t(x)\in[0,1]$ and $\zeta\leqslant\varrho v\leqslant 1-\zeta$, we have that

$$\sin(\pi\varrho v\kappa_t(x)) \asymp \min\{\varrho v\kappa_t(x), 1-\varrho v\kappa_t(x)\} \tag{4.50}$$

Thanks to (2.10), we have that

$$\begin{aligned}|\sin(\pi\varrho v\kappa_t(z)) - \sin(\pi\varrho v\kappa_t(x))| &\lesssim \pi\varrho v|\kappa_t(z)-\kappa_t(x)|\\ &\lesssim \frac{|\operatorname{Im}[z]|}{\delta}\min\{\varrho v\kappa_t(x), \varrho v(1-\kappa_t(x))\}\\ &\lesssim \frac{|\operatorname{Im}[z]|}{\delta}\sin(\pi\varrho v\kappa_t(x)) \leqslant \frac{1}{2}\sin(\pi\varrho v\kappa_t(x)),\end{aligned} \tag{4.51}$$ {{e:sindiff1}}

provided that $\delta\geqslant C\operatorname{Im}[z]$ for $C>1$ large enough. Similarly, for the second term on the left of (4.49), {e:sindiff1} we have

$$\begin{aligned}|\sin(\pi\varrho(1-v)\kappa_t(z)) - \sin(\pi\varrho(1-v)\kappa_t(x))| &\lesssim \frac{|\operatorname{Im}[z]|}{\delta}\sin(\pi\varrho(1-v)\kappa_t(x))\\ &\leqslant \frac{1}{2}\sin(\pi\varrho(1-v)\kappa_t(x))\,.\end{aligned} \tag{4.52}$$ {{e:sindiff2}}

The two estimates (4.51) and (4.52) together gives that {e:sindiff2}

$$\left|\frac{\sin(\pi\varrho v\kappa_t(z))}{\sin(\pi\varrho(1-v)\kappa_t(z))}\right| \asymp \left|\frac{\sin(\pi\varrho v\kappa_t(x))}{\sin(\pi\varrho(1-v)\kappa_t(x))}\right| \tag{4.53}$$ {{e:t1}}

If $\kappa_t(x)\leqslant 1/2$, then

$$\left|\frac{\sin(\pi\varrho v\kappa_t(x))}{\sin(\pi\varrho(1-v)\kappa_t(x))}\right| \asymp \frac{\varrho v}{\varrho(1-v)} \in \left[\frac{\zeta}{1-\zeta}, \frac{1-\zeta}{\zeta}\right] \tag{4.54}$$ {{e:t2}}

If $\kappa_t(x) \geqslant 1/2$, then

$$\begin{aligned}\zeta \lesssim \min\{\varrho v\kappa_t(x), 1-\varrho v\kappa_t(x)\} &\lesssim \left|\frac{\sin(\pi\varrho v\kappa_t(x))}{\sin(\pi\varrho(1-v)\kappa_t(x))}\right| \\ &\lesssim \frac{1}{\min\{\varrho(1-v)\kappa_t(x), 1-\varrho(1-v)\kappa_t(x)\}} \lesssim \frac{1}{\zeta}\end{aligned} \tag{4.55}$$ {{e:t3}}

We conclude from plugging (4.53), (4.54) {e:t3} and (4.55) into (4.49)

$$|\ln\sin(\pi\varrho v\kappa_t(z)) - \ln\sin(\pi\varrho(1-v)\kappa_t(z))| \leqslant \ln(1/\zeta) + C. \tag{4.56}$$ {{e:logdiff2}}

The above estimate (4.56) and {e:logdiff2} the first statement of Lemma 2.5 together imply that we can extend $f_t(z)$ to the strip region, and

$$|f_t(z)| \leqslant \frac{C}{\zeta}. \tag{4.57}$$

The estimate (4.56) and the second statement of Lemma 2.5 together imply that we can extend $g_t(z)$ to the strip region, and

$$|g_t(z)| \leqslant \ln(1/\zeta) + \ln(1/\delta) + C. \tag{4.58}$$

For $\mathrm{dist}(z, \{tv, \ell+tv\}) \geqslant \ell$, it follows from (2.10) and Lemma 2.4

$$|\kappa_t(z)| \lesssim \frac{C\delta}{\mathrm{dist}(z, \{tv, \ell+tv\}) + \mathrm{dist}(z, \{tv, \ell+tv\})^2} \tag{4.59}$$

Then by Taylor expansion

$$\ln\frac{\sin(\pi\varrho v\kappa_t(z))}{\sin(\pi\varrho(1-v)\kappa_t(z))} = \ln\frac{\sin(v)}{\sin(1-v)} + \ln\frac{\sin(\pi\varrho v\kappa_t(z))}{\pi\varrho v\kappa_t(z)} + \ln\frac{\pi\varrho(1-v)\kappa_t(z)}{\sin(\pi\varrho(1-v)\kappa_t(z))} \tag{4.60}$$

$$= \ln\frac{\sin(v)}{\sin(1-v)} + \mathrm{O}\left(\frac{\delta}{\mathrm{dist}(z, \{tv, \ell+tv\})}\right) \tag{4.61}$$

This together with (2.11) gives the claim (4.47). Next, we estimate $\mathrm{Im}[g_t(z)]$ for $z = x + \mathrm{i}\eta$. By taking imaginary part of (4.49), and using (4.51) and (4.52)

$$\mathrm{Im}\ln\frac{\sin(\pi\varrho v\kappa_t(z))}{\sin(\pi\varrho(1-v)\kappa_t(z))} = \mathrm{Im}\ln\frac{\sin(\pi\varrho v\kappa_t(z))}{\sin(\pi\varrho v\kappa_t(x))} - \mathrm{Im}\ln\frac{\sin(\pi\varrho(1-v)\kappa_t(z))}{\sin(\pi\varrho(1-v)\kappa_t(x))} \tag{4.62}$$

$$= \arg\left(1 + \mathrm{O}\left(\frac{|\mathrm{Im}[z]|}{\delta}\right)\right) - \arg\left(1 + \mathrm{O}\left(\frac{|\mathrm{Im}[z]|}{\delta}\right)\right) = \mathrm{O}\left(\frac{|\mathrm{Im}[z]|}{\delta}\right) \tag{4.63}$$

The above estimate together with the second statement of Lemma 2.5 imply that

$$|\mathrm{Im}\, g_t(z)| \leqslant \frac{C\,\mathrm{Im}[z]}{\delta}. \tag{4.64}$$

$\mathrm{Hib}(\kappa_t)(z)$ can be computed explicitely, see (B.26), and its derivative at $z = x$ can be bounded as

$$|\partial_x \mathrm{Hib}(\kappa_t)(x)| = \left|\frac{x-\ell-tv}{(x-\ell-tv)^2+\delta^2} - \frac{x-tv}{(x-tv)^2+\delta^2}\right| \lesssim \frac{1}{\mathrm{dist}(x, \{tv, \ell+tv\}) + \delta}. \tag{4.65}$$ {{e:der2}}

Since $g_t(z)$ is analytic, we have $|\partial_x g_t(x)| = |\partial_\eta g_t(x+\mathrm{i}\eta)|_{\eta=0}|$. As a consequence of (4.51) and (4.52) {e:der2}, we have

$$\left|\partial_x \ln\frac{\sin(\pi\varrho v\kappa_t(x))}{\sin(\pi\varrho(1-v)\kappa_t(x))}\right| \lesssim \frac{1}{\delta} \tag{4.66}$$ {{e:der1}}

The claim $|\partial_x g_t(x)| \lesssim 1/\delta$ follows from (4.66) and (4.65) {e:der1}. □

4.4 Proof of Proposition 3.3: Interior Slope Case

In this section we prove Proposition 4.2, namely Proposition 3.3 in the interior slope case, which is a consequence of the following two lemmas. {s:interiorslop {l:Nterm}

Lemma 4.6. *Adopt the assumptions in Proposition 4.2. Let H be a height function with $\|H-\mathcal{A}\|_\infty \leqslant \varepsilon$, and denote $\{\boldsymbol{x}(t)\}_{0\leqslant t\leqslant \ell}$ the particle configuration associated with $H(x,t)$, then*

$$\sum_{\mathsf{t}=0}^{\mathsf{L}-1}\sum_{i=1}^{N} e_i(\mathsf{t})g_t(x_i(\mathsf{t}/N)) = -\frac{N^2}{\theta}\int_0^\ell\int_{\mathbb{R}}\partial_t H(x,t)g_t(x)\mathrm{d}x\mathrm{d}t + \mathrm{O}(N(\|g\|_\infty+\|\partial_t g\|_\infty))$$
$$= -\frac{N^2}{\theta}\int_0^\ell\int_{\mathbb{R}}\partial_t \mathcal{A}(x,t)g_t(x)\mathrm{d}x\mathrm{d}t + \mathrm{O}((\varepsilon/\delta)(\ell N)^2). \tag{4.67}$$ {{e:expt1}}

{l:DLE}

Lemma 4.7. *Adopt the assumptions in Proposition 4.2. Let $H(x,t)$ be a height function with $\|H-\mathcal{A}\|_\infty \leqslant \varepsilon$, and denote $\{\boldsymbol{x}(t)\}_{0\leqslant t\leqslant \ell}$ the particle configuration associated with $H(x,t)$, then for any $0\leqslant \mathsf{t} = tN \leqslant \mathsf{L}-1$,*

$$\ln \mathbb{E}[e^{\sum_i e_i(\mathsf{t})g_t(x_i(t)/N)}|\boldsymbol{x}(t)] = \frac{N}{2\pi\mathrm{i}\theta}\oint\int_0^1 \ln(1+e^{m_t(z)+\tau g_t(z)})g_t(z)\mathrm{d}z\mathrm{d}\tau + \mathrm{O}(1)$$
$$= \frac{N}{2\pi\mathrm{i}\theta}\oint\int_0^1 \ln(1+e^{m_t^*(z)+\tau g_t(z)})g_t(z)\mathrm{d}z\mathrm{d}\tau + \mathrm{O}\left(\frac{\varepsilon N}{\delta\zeta^2}\right) \tag{4.68}$$ {{e:DLE}}

where $m_t(z)$ is the Stieltjes transform of the empirical measure of $\boldsymbol{x}(t)$.

Proof of Lemma 4.6. Let $G_t(z) = \int_0^z g_t(u)du$. We first notice that, for $t=\mathsf{t}/N$,

$$\frac{N^2}{\theta}\int_{\mathbb{R}}(\rho(x;\boldsymbol{x}(t+1/N))-\rho(x;\boldsymbol{x}(t/N)))G_t(x)\mathrm{d}x$$
$$= \sum_i \frac{Ne_i(\mathsf{t})}{\theta}\int_{x_i(t)}^{x_i(t)+\theta/N}(G_t(x+1/N)-G_t(x))\mathrm{d}x = \sum_i e_i(\mathsf{t})g_t(x_i(t)) + \mathrm{O}\left(\|g_t\|_\infty+\|\partial_t g\|_\infty\right). \tag{4.69}$$

Thus for the the lefthand side of (4.67), we have

$$\sum_{\mathsf{t}=0}^{L-1}\sum_{i=1}^{N} e_i(\mathsf{t})g_t(x_i(\mathsf{t})/N)$$
$$= \sum_{\mathsf{t}=0}^{L-1}\frac{N^2}{\theta}\int(\partial_x H(x,t+1/N)-\partial_x H(x,t))G_t(x)\mathrm{d}x + \mathrm{O}\left(N(\|g_t\|_\infty+\|\partial_t g\|_\infty)\right)$$
$$= -\sum_{\mathsf{t}=0}^{L-1}\frac{N^2}{\theta}\int(H(x,t+1/N)-H(x,t))g_t(x)\mathrm{d}x + \mathrm{O}\left(N(\|g_t\|_\infty+\|\partial_t g\|_\infty)\right)$$
$$= -\sum_{\mathsf{t}=0}^{L-1}\frac{N^2}{\theta}\int_t^{t+1/N}\partial_s H(x,s)\mathrm{d}s g_t(x)\mathrm{d}x + \mathrm{O}\left(N(\|g_t\|_\infty+\|\partial_t g\|_\infty)\right)$$
$$= -\frac{N^2}{\theta}\int_0^T\int_{\mathbb{R}}\partial_t H(x,t)g_t(x)\mathrm{d}x\mathrm{d}t + \mathrm{O}(N(\|g_t\|_\infty+\|\partial_t g\|_\infty)). \tag{4.70}$$ {{e:replaceH}}

This gives the first statement in (4.67). Next we replace H by $\mathcal{A}$, we can do an integration by part, and use $\|H-\mathcal{A}\|\leqslant\varepsilon$, {e:replace}

$$\left|\frac{N^2}{\theta}\int_0^T\int_{\mathbb{R}}(\partial_t H(x,t)-\partial_t\mathcal{A}(x,t))g_t(x)\mathrm{d}x\mathrm{d}t\right|$$
$$\leqslant \frac{N^2}{\theta}\left|\int_{\mathbb{R}}(H(x,t)-\mathcal{A}(x,t))g_t(x)\mathrm{d}x\right|_0^\ell + \frac{N^2}{\theta}\left|\int_0^L\int_{\mathbb{R}}(H(x,t)-\mathcal{A}(x,t))\partial_t g_t(x)\mathrm{d}x\mathrm{d}t\right| \tag{4.71}$$ {{e:replace}}
$$\lesssim \varepsilon\|g\|_\infty \ell N^2 + \varepsilon\|\partial_t g\|_\infty(\ell N)^2$$

Thanks to Lemma 4.5, we have $\|g\|_\infty \leqslant \ln(1/\zeta)+\ln(1/\delta)+C$ and $|\partial_x g_t(x)| \leqslant C/\delta$. Recall our construction of g_t from (4.46), since {e:replace} $\widetilde{H}(x-tv,t)$ is constant for $0\leqslant t\leqslant \ell$, so is $g_t(x-tv)$. By taking derivative with respect to t we get

$$\partial_t g_t - v\partial_x g_t = 0. \tag{4.72}$$

Thus $|\partial_t g_t| \leqslant v|\partial_x g_t| \leqslant C/\delta$, and the second statement of (4.67) follows from (4.71).

□

In the following, we prove Lemma 4.7. For simplicity of notations, we will omit the dependence on time t. We can rewrite the lefthand side of (5.30) as

$$\frac{1}{N}\ln \mathbb{E}[e^{\sum_i e_i g(x_i/N)}|\boldsymbol{x}] = \frac{1}{N}\int_0^1 \partial_\tau \ln \mathbb{E}[e^{\sum_i e_i \tau g(x_i/N)}|\boldsymbol{x}]\mathrm{d}\tau \tag{4.73}$$

$$= \frac{1}{N}\int_0^1 \frac{\mathbb{E}[\sum_i e_i g(x_i/N) e^{\sum_i e_i\tau g(x_i/N)}|\boldsymbol{x}]}{\mathbb{E}[e^{\sum_i e_i\tau g(x_i/N)}|\boldsymbol{x}]}\mathrm{d}\tau, \tag{4.74}$$

which is the expectation of $\sum_i e_i g(x_i/N)$ under the following deformed measure

$$\frac{1}{Z_\tau}\frac{V(\boldsymbol{x}+\theta\boldsymbol{e}/N)}{V(\boldsymbol{x})} e^{\sum_i e_i \tau g(x_i/N)}, \tag{4.75}$$ {{e:Etheta}}

where Z_τ is the normalization factor. We denote $\mathbb{E}_\tau[\cdot]$ the expectation with respect to the measure (4.75). {e:Etheta}

We will use the dynamical loop equation Theorem A.3 to compute the expectation

$$\frac{\mathbb{E}[\sum_i e_i g(x_i/N) e^{\sum_i e_i\tau g(x_i/N)}|\boldsymbol{x}]}{\mathbb{E}[e^{\sum_i e_i\tau g(x_i/N)}|\boldsymbol{x}]} = \mathbb{E}_\tau\left[\sum_i e_i g(x_i/N)\right]. \tag{4.76}$$

Take a large constant $8C/(\delta\zeta) < K < \zeta/(2\varepsilon)$, and denote the region

$$\Lambda = \Lambda_K = \{z\in\mathbb{C} : (1/K) \leqslant \mathrm{dist}(z, [-3\ell, 3\ell]) \leqslant 2/K\}. \tag{4.77}$$ {{e:defLambda}}

We need to verify the conditon that on Λ, the following are well defined, and lower and upper bounded

$$\ln(1+\widetilde{f}(z)), \quad \widetilde{f}(z) := e^{m(z)+\tau g(z)}. \tag{4.78}$$

{l:Bzcondition}

Lemma 4.8. *Take $8C/\delta\zeta < K < \zeta/(2\varepsilon)$, then on Λ, we have*

$$\sin\left(\frac{\zeta\pi}{4}\right) \leqslant |1+\widetilde{f}(z)| \leqslant \frac{C\ell K}{\delta\zeta}, \quad \frac{|\widetilde{f}(z)|}{|1+\widetilde{f}(z)|} \lesssim \frac{1}{\zeta} \tag{4.79}$$ {{e:fbound}}

and $\ln(1+\widetilde{f}(z))$ is well defined.

Proof. We first show that for any $z\in\Lambda$

$$-\pi(1-\zeta/2) \leqslant \mathrm{Im}[m(z)] \leqslant \pi(1-\zeta/2). \tag{4.80}$$ {{e:Immbound}}

Since $\mathrm{Im}[m(z)] = -\mathrm{Im}[m(\overline{z})]$, we will only prove (4.80) for {e:Immbound} $z\in\Lambda\cap\overline{\mathbb{H}}$. Thanks to Lemma 2.4, $H(\pm 3\ell, t) = \mathcal{A}(\pm 3\ell, t)$. Therefore, for any $z\in\Lambda$, we find

$$\begin{aligned}|m(z)-m^*(z)| &= \left|\int_{\mathbb{R}} \frac{(\partial_x H(x,t) - \partial_x\mathcal{A}(x,t))\mathrm{d}x}{x-z}\right| \leqslant \int_{-3\ell}^{3\ell}\frac{|H(x,t)-\mathcal{A}(x,t)|\mathrm{d}x}{|x-z|^2} \\ &\leqslant \varepsilon\int_{-\infty}^{\infty}\frac{\mathrm{d}x}{x^2+(1/K)^2} = \pi K\varepsilon \leqslant \zeta\pi/2,\end{aligned} \tag{4.81}$$ {{e:mm*diff}}

where in the last inequality we used that $K \leqslant \zeta/(2\varepsilon)$. We can also compute $m^*(z)$ and $\mathrm{Im}[m^*(z)]$ explicitly

$$m^*(z) = \int_{\mathbb{R}} \frac{\partial_x \mathcal{A} \mathrm{d}x}{z-x} = \int_{tv \leqslant x \leqslant \ell + tv} \frac{\rho \mathrm{d}x}{z-x} = -\rho \ln\left(1 - \frac{\ell}{z - tv}\right). \tag{4.82}$$ {{e:Imtm}}

and

$$-\mathrm{Im}[m^*(z)] = \eta \int_{\mathbb{R}} \frac{\partial_x \mathcal{A} \mathrm{d}x}{|z-x|^2} = \eta \int_{tv \leqslant x \leqslant 1+tv} \frac{\rho \mathrm{d}x}{|z-x|^2} \in [0, \pi\rho] \subset [0, (1-\zeta)\pi]. \tag{4.83}$$ {{e:Imtm2}}

The claim (4.80) follows from combining (4.81) and (4.82). {e:Imtm2} From Lemma 4.5, $|\mathrm{Im}\, g(z)| \leqslant C\,\mathrm{Im}[z]/\delta \leqslant 2C/(\delta K)$. Using (4.80), for any $z \in \Lambda$, we have $|\mathrm{Im}\, g(z)| \leqslant 2C/(\delta K) \leqslant \zeta\pi/4$, it follows that

$$-(1-\zeta/4)\pi \leqslant \mathrm{Im}[m(z) + \tau g(z)] \leqslant (1-\zeta/4)\pi. \tag{4.84}$$

It follows that $\arg e^{m(z)+\tau g(z)} \in (-(1-\zeta/4)\pi, (1-\zeta/4)\pi)$. Thus $1 + e^{m(z)+\tau g(z)}$ takes value in the sector region given by $\{1 + re^{\mathrm{i}u} : u \in (-(1-\zeta/4)\pi, (1-\zeta/4)\pi)\}$, which does not include the negative axis. Therefore, we can take $\ln(\cdot)$ the branch defined on $\mathbb{C}\backslash\mathbb{R}_{\leqslant 0}$ and $\ln(1+e^{m(z)+\tau g(z)})$ is well defined. Moreover

$$\min_{u \in (-(1-\zeta/4)\pi, (1-\zeta/4)\pi)} |1 + re^{\mathrm{i}u}| \geqslant \sin\left(\frac{\zeta\pi}{4}\right), \tag{4.85}$$

we conclude that for $z \in \Lambda$, it holds $|1 + e^{m(z)+\tau g(z)}| \geqslant \sin(\zeta\pi/4) \gtrsim \zeta$. The last statement of (5.67) also follows. On Λ, since $\mathrm{dist}(z, [-3\ell, 3\ell]) \geqslant 1/K$, we have

$$|e^{m(z)+\tau g(z)}| \leqslant e^{\mathrm{Re}\, m^*(z) + |m^*(z) - m(z)| + \|g\|_\infty} \leqslant \left|1 - \frac{\ell}{z - tv}\right| e^{\pi K\varepsilon + \|g\|_\infty} \leqslant (1 + \ell K) e^{\pi K \varepsilon + \|g\|_\infty}, \tag{4.86}$$

where we used (4.82) to bound $|\mathrm{Re}\, m^*(z)|$ and (4.81) to bound $|m(z) - m^*(z)|$. Using (4.80), for any $z \in \Lambda$, we have $|g(z)| \leqslant \ln(1/\zeta) + \ln(1/\delta) + \mathrm{O}(1)$ thus $|1 + e^{m(z)+\tau g(z)}| \leqslant C\ell K/(\zeta\delta)$. ☐

Proof of Lemma 4.7. Lemma 4.8 verifies that the measure (4.75) satisfies the assumption in Theorem A.5 (by taking ε, n there to be $1/N, N$ here). Let $\boldsymbol{y} = \boldsymbol{x} + \boldsymbol{e}/N$. We conclude that

$$\frac{N}{\theta} \int_{\mathbb{R}} \frac{(\rho(s; \boldsymbol{y}) - \rho(s; \boldsymbol{x}))}{z - s} \mathrm{d}s = \Delta\mathcal{M}(z) + \frac{1}{2\pi\mathrm{i}} \oint_{\omega_-} \frac{\ln(1 + e^{m(w)+\tau g(w)})\mathrm{d}w}{(w-z)^2} + \mathrm{O}\left(\frac{1}{N}\right), \tag{4.87}$$ {{e:dmg2}}

where the contour $\omega_- \subset \Lambda$ encloses $[-3\ell, 3\ell]$, but not z, the implicit constant in the error term depends on K, δ, ℓ. Moreover, $\Delta\mathcal{M}(z)$ is a random variable with mean zero under $\mathbb{E}_\tau$. By a contour integral with respect to $G(z) := \int_0^z g(w)\mathrm{d}w$ on both sides and taking expectation we get

$$\frac{N}{\theta} \mathbb{E}_\tau \int (\rho(s; \boldsymbol{y}) - \rho(s; \boldsymbol{x})) G(s) \mathrm{d}s = \frac{1}{2\pi\mathrm{i}} \oint_\omega \ln(1 + e^{m(z)+\tau g(z)}) g(z) \mathrm{d}z + \mathrm{O}\left(\frac{1}{N}\right), \tag{4.88}$$ {{e:rhodiff}}

where ω is a contour inside Λ. For the lefthand side of (4.88), we have {e:rhodiff}

$$\frac{N}{\theta} \mathbb{E}_\tau \int (\rho(s; \boldsymbol{y}) - \rho(s; \boldsymbol{x})) G(s) \mathrm{d}s = \mathbb{E}_\tau \sum_i \frac{e_i}{\theta} \int_{x_i/N}^{(x_i+\theta)/N} (G(s + 1/N) - G(s)) \mathrm{d}s \tag{4.89}$$

$$= \frac{1}{N} \mathbb{E}_\tau \left[\sum_i e_i g(x_i/N)\right] + \mathrm{O}\left(\frac{1}{N}\right) \tag{4.90}$$

It follows that

$$\frac{1}{N} \mathbb{E}_\tau \left[\sum_i e_i g(x_i/N)\right] = \frac{1}{2\pi\mathrm{i}\theta} \oint \int_0^1 \ln(1 + e^{\theta m(z)+\tau g(z)}) g(z) \mathrm{d}z \mathrm{d}\tau + \mathrm{O}\left(\frac{1}{N}\right) \tag{4.91}$$ {{e:linearst}}

This gives the first statement in (5.30) by integrating {e:integrating} from $\tau = 0$ to $\tau = 1$.

Next we replace $m(z)$ in the integral (4.91) by $m^*(z)$. Thanks to (4.81), on the contour we have $|m(z) - m^*(z)| \leqslant \varepsilon K\pi \leqslant 1/2$, thus

$$|\ln(1 + e^{m^*(z)+\tau g(z)}) - \ln(1 + e^{m(z)+\tau g(z)}))| \leqslant \left|\ln\left(1 - \frac{(e^{m^*(z)-m(z)} - 1)\widetilde{f}(z)}{1 + e^{m^*(z)+\tau g(z)}}\right)\right| \tag{4.92}$$

$$\leqslant \left|\frac{(e^{m^*(z)-m(z)} - 1)\widetilde{f}(z)}{1 + e^{m^*(z)+\tau g(z)}}\right| \lesssim \frac{\varepsilon K}{\zeta}, \tag{4.93}$$

where we used Lemma 4.8 in the last inequality. To get the second statement in (5.30), we can take $K = C\varepsilon/\delta\zeta$.

□

4.5 Rate Function and Proof of Proposition 4.3

{s:ratef}

We recall the rate function $S(\mathcal{A}; g)$ from (4.22)

$$S(\mathcal{A}; g) = -\int_0^\ell \int_{\mathbb{R}} \partial_t \mathcal{A}(x, t) g_t(x) \mathrm{d}x\mathrm{d}t - \frac{1}{2\pi \mathrm{i}} \int_0^\ell \oint \int_0^1 \ln(1 + e^{m_t^*(z)+\tau g_t(z)}) g_t(z) \mathrm{d}z\mathrm{d}\tau\mathrm{d}t. \tag{4.94}$$

In this section, we simplify $S(\mathcal{A}; g)$ and prove Proposition 4.3.

{l:angleest}

Lemma 4.9. *We introduce a new complex slope as* $f_t(z) = e^{m_t^*(z)+g_t(z)} \in \mathbb{H}^-$. *Then for* $x \in [tv, tv + \ell]$,

$$\arg f_t(x) = -\varrho\pi, \quad |\arg(1 + f_t(x)) + \varrho v\pi| \lesssim \frac{\delta}{\delta + \mathrm{dist}(x, \{tv, \ell + tv\})} \tag{4.95}$$

{e:tmm*diff}

Lemma 4.10. *If* $\mathrm{dist}(z, \{tv, \ell + tv\}) \gtrsim \delta$, *then*

$$|\widetilde{m}_t(z) - m_t^*(z)| \lesssim \frac{\delta\varrho}{\mathrm{dist}(z, \{tv, \ell + tv\})}, \tag{4.96}$$

Proof. The expressions of $m_t^*(x)$ and $\widetilde{m}_t(x)$ are explicit,

$$|\widetilde{m}_t(z) - m_t^*(z)| = \varrho\left|\ln\left(1 + \frac{\ell}{tv - z}\right) - \ln\left(1 + \frac{\ell}{tv - z - \mathrm{i}\delta}\right)\right| \tag{4.97}$$

$$= \varrho\left|\ln\left(1 - \frac{\mathrm{i}\delta}{tv - z}\right) - \ln\left(1 - \frac{\mathrm{i}\delta}{\ell + tv - z}\right)\right|, \tag{4.98}$$

from which the result follows from $|\ln(1 + x)| \leq |x|$. □

Proof. We recall $\kappa_t(x)$ from (2.6), $g_t(x)$ from (4.15), and $\widetilde{f}_t(x) = e^{\widetilde{m}_t(x)+g_t(x)}$ from (4.14). The three vertices $\{0, -1, \widetilde{f}\}$ form a triangle, with three angles give by $\varrho\kappa_t(x)\pi, \varrho v\kappa_t(x)\pi, \varrho(1 - v)\kappa_t(x)\pi$. From Lemma 2.4, for $x \in [tv, \ell + tv]$,

$$\kappa_t(x) \asymp 1, \quad 1 - \kappa_t(x) \asymp \frac{C\delta}{\delta + \mathrm{dist}(x, \{tv, \ell + tv\})}, \tag{4.99}$$ {{e:kappabound}

It follows that $|1 + \widetilde{f}_t(x)| \gtrsim \varrho$. We can rewrite $f_t(x)$ in terms of $\widetilde{f}_t(x)$

$$f_t(x) = \widetilde{f}_t(x) e^{m_t^*(x) - \widetilde{m}_t(x)}. \tag{4.100}$$

$m_t^*(x)$ and $\widetilde{m}_t(x)$ are explicit,

$$|\widetilde{m}_t(z) - m_t^*(z)| = \varrho\left|\ln\left(1 + \frac{\ell}{tv - z}\right) - \ln\left(1 + \frac{\ell}{tv - z - \mathrm{i}\delta}\right)\right| \tag{4.101}$$

$$= \varrho\left|\ln\left(1 - \frac{\mathrm{i}\delta}{tv - z}\right) - \ln\left(1 - \frac{\mathrm{i}\delta}{\ell + tv - z}\right)\right|, \tag{4.102}$$

If $\mathrm{dist}(z,\{tv,\ell+tv\})\gtrsim\delta$, then

$$|\widetilde{m}_t(z)-m_t^*(z)|\lesssim\frac{\delta\varrho}{\mathrm{dist}(z,\{tv,\ell+tv\})}, \tag{4.103}$$

For $\mathrm{dist}(x,\{tv,\ell+tv\})\gtrsim\delta$ we have

$$\begin{aligned}\arg(1+f_t(x))&=\arg(1+\widetilde{f}_t(x)e^{m_t^*(x)-\widetilde{m}_t(x)}) &(4.104)\\&=\arg(1+\widetilde{f}_t(x))e^{m_t^*(x)-\widetilde{m}_t(x)}-(e^{m_t^*(x)-\widetilde{m}_t(x)}-1)) &(4.105)\\&=\arg(1+\widetilde{f}_t(x))+\arg\left(1-\frac{(e^{m_t^*(x)-\widetilde{m}_t(x)}-1)}{(1+\widetilde{f}_t(x))e^{m_t^*(x)-\widetilde{m}_t(x)}}\right) &(4.106)\\&=-\varrho v\kappa_t(x)\pi+\mathrm{O}\left(\frac{|m_t^*(x)-\widetilde{m}_t(x)|}{|1+\widetilde{f}_t(x)|}\right)=-\varrho v\pi+\mathrm{O}\left(\frac{\delta}{\mathrm{dist}(x,\{tv,\ell+tv\})}\right), &(4.107)\end{aligned}$$

where in the last equality we used (4.99). □

Proof of Proposition 4.3. We examine the second term on the righthand side of (4.22). We recall the dilogarithm function (see [56]),

$$\mathrm{Li}_2(z)=-\int_0^z\ln(1-u)\frac{\mathrm{d}u}{u}, \tag{4.108}$$

which can be analytically extended to the cut plane $\mathbb{C}\setminus[1,\infty]$. By a change of variable,

$$\begin{aligned}\mathrm{Li}_2(-e^w)&=-\int_0^{-e^w}\ln(1-u)\frac{\mathrm{d}u}{u}\\&=-\int_{-\infty}^{w}\ln(1+e^x)\mathrm{d}x,\end{aligned}$$

which is analytic on the strip $\{w\in\mathbb{C}:-\pi<\mathrm{Im}[w]<\pi\}$. Then, we can rewrite the last term in (4.22) as

$$\begin{aligned}-\int_0^1\ln(1+e^{m_t^*(z)+\tau g_t(z)})g_t(z)\mathrm{d}\tau&=\int_{m_t^*(z)}^{m_t^*(z)+g_t(z)}-\ln(1+e^x)\mathrm{d}x\\&=\mathrm{Li}_2(-e^{m_t^*(z)+g_t(z)})-\mathrm{Li}_2(-e^{m_t^*(z)})\end{aligned}$$

For the last term, since $m_t^*(z)=1/z+\mathrm{O}(1/z^2)$ as $z\to\infty$. Thus $-e^{m_t^*(z)}=-1-1/z+\mathrm{O}(1/z^2)$ when $z\to\infty$. Thus the contour integral is given by

$$\begin{aligned}-\frac{1}{2\pi\mathrm{i}}\oint\mathrm{Li}_2(-e^{m_t^*(z)})\mathrm{d}z&=-\frac{1}{2\pi\mathrm{i}}\oint\mathrm{Li}_2\left(-1-\frac{1}{z}+\mathrm{O}\left(\frac{1}{z^2}\right)\right)\mathrm{d}z &(4.109)\\&=-\frac{1}{2\pi\mathrm{i}}\oint\mathrm{Li}_2\left(-1-w+\mathrm{O}\left(w^2\right)\right)\frac{1}{w^2}\mathrm{d}w=\mathrm{Li}_2'(-1)=\ln(2) &(4.110)\end{aligned}$$

For the first term we need to evaluate the integral

$$\frac{1}{2\pi\mathrm{i}}\oint\mathrm{Li}_2(-e^{m_t^*(z)+g_t(z)})\mathrm{d}z. \tag{4.111}$$ {{e:Literm}}

The dilogarithmic function function $\mathrm{Li}_2(z)$ jumps by $2\pi\mathrm{i}\log|z|$, as z crosses the cut. And the function $\mathrm{Li}_2(z)+\mathrm{i}\arg(1-z)\log|z|$, where arg denotes the branch of the argument lying between $-\pi$ and π, is continuous. Its imaginary part gives the Bloch-Wigner function $D(z)$

$$D(z)=\mathrm{Im}[\mathrm{Li}_2(z)]+\arg(1-z)\log|z|. \tag{4.112}$$

Bloch-Wigner function $D(z)$ can be expressed as a single real variable

$$D(z) = \frac{1}{2}\left[D\left(\frac{z}{\overline{z}}\right) + D\left(\frac{1-1/z}{1-1/\overline{z}}\right) + D\left(\frac{1-\overline{z}}{1-z}\right)\right], \tag{4.113}$$

and

$$D(e^{\mathrm{i}\theta}) = \mathrm{Im}[\mathrm{Li}_2(e^{\mathrm{i}\theta})] = 2L(\theta/2), \tag{4.114}$$

where L is the Lobachevsky function from (1.19). We recall the complex slope $f_t(z) = e^{m_t^*(z)+g_t(z)} \in \mathbb{H}^-$ from Lemma 4.9, and denote the three angle of the triangle $\{0,-1,f_t(x)\}$ as $\pi p_1, \pi p_2, \pi p_3$. Then Lemma 4.9 implies that for $\mathrm{dist}(x,\{tv,\ell+tv\}) \gtrsim \delta$,

$$p_1 = \varrho, \quad p_2 = \varrho v + \mathrm{O}\left(\frac{\delta}{\mathrm{dist}(x,\{tv,\ell+tv\})}\right), \quad p_3 = \varrho(1-v) + \mathrm{O}\left(\frac{\delta}{\mathrm{dist}(x,\{tv,\ell+tv\})}\right) \tag{4.115}$$ {{e:p123}}

and

$$\begin{aligned} D(-f) &= \frac{1}{2}\left[D\left(\frac{f}{\overline{f}}\right) + D\left(\frac{(f+1)/f}{(\overline{f}+1)/\overline{f}}\right) + D\left(\frac{1+\overline{f}}{1+f}\right)\right] \\ &= \Lambda(\pi p_1) + \Lambda(\pi p_1) + \Lambda(\pi p_3) = \sigma(\varrho, -\varrho v) + \mathrm{O}\left(\frac{\delta \log(1/\zeta)}{\mathrm{dist}(x,\{tv,\ell+tv\})}\right), \end{aligned} \tag{4.116}$$ {{e:Lobachevsky}}

where in the last equality, we also used that $\zeta \leqslant \varrho, \varrho v, \varrho(1-v) \leqslant 1-\zeta$, (4.9) and (4.9) {e:Lobachevsky}

With the notation above, we can rewrite (4.111) as

$$\frac{1}{2\pi\mathrm{i}}\oint_\omega \mathrm{Li}_2(-f_t(z))\mathrm{d}z, \tag{4.117}$$ {{e:Literm2}}

where the contour $\omega \in \Lambda$ (from (4.77)) encloses $[tv, \ell+tv]$. {e:Literm2} We can deform ω to the real axis,

$$\begin{aligned} \frac{1}{2\pi\mathrm{i}}\oint_\omega \mathrm{Li}_2(-f_t(z))\mathrm{d}z &= -\frac{1}{\pi}\int_{\mathbb{R}} \mathrm{Im}\,\mathrm{Li}_2(-f_t(x))\mathrm{d}x \\ &= -\frac{1}{\pi}\int_{\mathbb{R}} (D(-f_t(x)) - \arg(1+f_t(x))\log|f_t(x)|)\mathrm{d}x \end{aligned} \tag{4.118}$$ {{e:Literm2}}

where we used that $f_t(\overline{z}) = \overline{f_t(z)}$, and $\mathrm{Li}_2(\overline{w}) = \overline{\mathrm{Li}_2(w)}$.

Thanks to (4.118), we can rewrite {e:Literm2} the rate $S(\mathcal{A};g)$ from (4.22) as

$$S(\mathcal{A};g) = \frac{1}{\pi}\int_0^\ell\int_{tv}^{tv+\ell} \arg(1+f)\log|f| - \partial_t\mathcal{A}(x,t)g_t(x)\mathrm{d}x\mathrm{d}t - \frac{1}{\pi}\int_0^\ell\int_{tv}^{tv+\ell}\int_{\mathbb{R}} D(-f(x))\mathrm{d}x\mathrm{d}t + \ln(2). \tag{4.119}$$ {{e:SHg}}

Recall that $\log|f| = \mathrm{Re}\,m_t^*(x) + g_t(x)$, and $\int_{tv}^{tv+\ell}\mathrm{Re}\,m_t^*(x)\mathrm{d}x = 0$. We can rewrite the first term on the righthand side of (4.119) as {e:SHg}

$$\int_0^\ell\int_{tv}^{tv+\ell}(\pi^{-1}\arg(1+f) - \partial_t\mathcal{A}(x,t))(\mathrm{Re}\,m_t^*(x) + g_t(x))\mathrm{d}x\mathrm{d}t \tag{4.120}$$ {{e:cancel}}

$$\lesssim \int_0^\ell\int_{tv}^{tv+\ell}\frac{\delta}{\delta + \mathrm{dist}(x,\{tv,\ell+tv\})}(\ln(1/\mathrm{dist}(x,\{tv,tv+\ell\})) + \ln(1/\delta) + \ln(1/\zeta) + C)\mathrm{d}x\mathrm{d}t. \tag{4.121}$$ {e:cancel}

where we used Lemma 4.5 and Lemma 4.9. We divide the inner integral into two case: i) $\mathrm{dist}(x,\{tv,\ell+tv\}) \geqslant \delta$ and ii) $\mathrm{dist}(x,\{tv,\ell+tv\}) \leqslant \delta$:

$$\int_0^\ell\int_{\mathrm{dist}(x,\{tv,\ell+tv\})\geqslant\delta}\frac{\delta\ln(1/\delta)}{\mathrm{dist}(x,\{tv,\ell+tv\})}\mathrm{d}x \lesssim \delta\ln(1/\delta)^2\ell \tag{4.122}$$

and

$$\int_0^\ell \int_{\mathrm{dist}(x,\{tv,\ell+tv\})\leqslant\delta} \frac{\delta\ln(1/\delta)}{\delta}\mathrm{d}x\mathrm{d}t \lesssim \delta\ln(1/\delta)\ell. \tag{4.123}$$

Combining the discussions above, we get that the rate function is given by

$$\begin{aligned}
S(\mathcal{A};g) &= -\frac{1}{\pi}\int_0^\ell\int_{\mathbb{R}} D(-f(x))\mathrm{d}x + \ln(2) + \mathrm{O}(\delta\ln(1/\delta)^2\ell)\\
&= -\frac{1}{\pi}\int_0^\ell\int_{tv}^{tv+\ell}\left(\sigma(\varrho,-\varrho v) + \mathrm{O}\left(\frac{\delta\log(1/\zeta)}{\mathrm{dist}(x,\{tv,\ell+tv\})}\right)\right)\mathrm{d}x\mathrm{d}t + \ln(2) + \mathrm{O}(\delta\ln(1/\delta)^2\ell)\\
&= -\frac{\ell^2}{\pi}\sigma(\varrho,-\varrho v) + \ln(2) + \mathrm{O}(\delta\ln(1/\delta)^2\ell)
\end{aligned}$$

where in the second equality we used (4.116). This finishes the proof of Proposition 4.3.

□

Proof of Lemma 3.4. For any $\alpha, \beta < \beta'$, denote $I = I(\alpha,\beta)$ and $J = I(\alpha,\beta')$. Next we compute the interaction between different blocks. Thanks to (4.32), for $\alpha\mathsf{L} \leqslant \mathsf{t} < (\alpha+1)\mathsf{L}$,

$$\begin{aligned}
\sum_{i\in I(\alpha,\beta),j\in I(\alpha,\beta')} \ln\left(1+\frac{\theta(e_i(\mathsf{t})-e_j(\mathsf{t}))}{\mathsf{x}_i(\mathsf{t})-\mathsf{x}_j(\mathsf{t})}\right) &= \sum_{i\in I(\alpha,\beta),j\in I(\alpha,\beta')} \theta\ln\left(\frac{\mathsf{x}_i(\mathsf{t}+1)-\mathsf{x}_j(\mathsf{t}+1)}{\mathsf{x}_i(\mathsf{t})-\mathsf{x}_j(\mathsf{t})}\right)\\
&\quad + \mathrm{O}\left(\sum_{i\in I(\alpha,\beta),j\in I(\alpha,\beta')}\frac{1}{(\mathsf{x}_i(\mathsf{t})-\mathsf{x}_j(\mathsf{t}))^2}\right)
\end{aligned} \tag{4.124}$$ {{e:sumblock}}

We can bound the total error as

$$\sum_\beta\sum_{i\in I(\alpha,\beta),j\notin I(\alpha,\beta)}\frac{1}{(\mathsf{x}_i(\mathsf{t})-\mathsf{x}_j(\mathsf{t}))^2} \lesssim \sum_\beta\sum_{i\in I(\alpha,\beta),j\notin I(\alpha,\beta)}\frac{1}{(i-j)^2} \tag{4.125}$$ {{e:totalsum}}

where we used that $|\mathsf{x}_i(\mathsf{t})-\mathsf{x}_j(\mathsf{t})| \geqslant \theta|i-j|$.

The upper bound on the righthand side of (4.125) is achieved when particles in $I(\alpha,\beta)$ are tightly packed at $\{-|I(\alpha,\beta)|,\cdots,-1\}$, and the other particles are also tightly packed on $\{0,1,2,\cdots,N-|I(\alpha,\beta)|-1\}$. Therefore we can bound it as

$$\begin{aligned}
\sum_\beta\sum_{i\in I(\alpha,\beta),j\notin I(\alpha,\beta)}\frac{1}{(i-j)^2} &\lesssim \sum_\beta\sum_{1\leqslant i\leqslant|I(\alpha,\beta)|}\left(\frac{1}{i^2}+\frac{1}{(i+1)^2}+\cdots+\frac{1}{N^2}\right)\\
&\lesssim \sum_\beta \ln|I(\alpha,\beta)| \lesssim \frac{1}{\ell}\ln(\ell N),
\end{aligned} \tag{4.126}$$ {{e:tsum}}

where in the last inequality, we used that $\sum_\beta |I(\alpha,\beta)| = N$ and Jessen's inequality.

Thanks to (4.124), (4.125) and (4.126), we can sum over $\alpha\mathsf{L} \leqslant t < (\alpha+1)\mathsf{L}$, and notice the cancellation of the first term on the righthand side of (4.124) when summing over t

$$\begin{aligned}
&\sum_{\alpha\mathsf{L}\leqslant t<(\alpha+1)\mathsf{L}}\sum_{\beta\neq\beta'}\sum_{i\in I(\alpha,\beta),j\in I(\alpha,\beta')}\ln\left(1+\frac{\theta(e_i(\mathsf{t})-e_j(\mathsf{t}))}{\mathsf{x}_i(\mathsf{t})-\mathsf{x}_j(\mathsf{t})}\right)\\
&= \sum_{\beta\neq\beta'}\sum_{i\in I(\alpha,\beta),j\in I(\alpha,\beta')}\theta\ln\left(\frac{\mathsf{x}_i((\alpha+1)\mathsf{L})-\mathsf{x}_j((\alpha+1)\mathsf{L})}{\mathsf{x}_i(\alpha\mathsf{L})-\mathsf{x}_j(\alpha\mathsf{L})}\right) + \mathrm{O}(N\ln(\ell N))\\
&= \left(\sum_{i\neq j} - \sum_\beta\sum_{i\neq j\in I(\alpha,\beta)}\right)\theta\ln\left(\frac{\mathsf{x}_i((\alpha+1)\mathsf{L})-\mathsf{x}_j((\alpha+1)\mathsf{L})}{\mathsf{x}_i(\alpha\mathsf{L})-\mathsf{x}_j(\alpha\mathsf{L})}\right) + \mathrm{O}(N\ln(\ell N))
\end{aligned} \tag{4.127}$$ {{e:sumbb}}

To bound the summation over $i \neq j \in I(\alpha, \beta)$ on the righthand side of (4.127), we recall {e:sumab} from Lemma 2.8

$$\begin{aligned} &\frac{\theta^2}{N^2} \sum_{i \neq j \in I(\alpha,\beta)} \ln\left(\mathsf{x}_i(\alpha \mathsf{L}) - \mathsf{x}_j(\alpha \mathsf{L})\right) \\ &= \frac{1}{2} \iint \ln|x-y| \varrho(x; \boldsymbol{x}_{I(\alpha,\beta)}(\alpha\ell) \varrho(y; \boldsymbol{x}_{I(\alpha,\beta)}(\alpha\ell) \mathrm{d}x\mathrm{d}y + \mathrm{O}\left(\frac{\ln N}{\ell N}\right) = \mathrm{O}(\ell^2 \ln(1/\ell)). \end{aligned} \tag{4.128}$$ {{e:entropyboun

There are two cases. If on $\mathfrak{R}(\alpha, \beta)$, $\mathcal{A}$ has a linear approximation, then the height functions corresponding to $\varrho(x; \boldsymbol{x}_{I(\alpha,\beta)}(\alpha\ell)$ and $\varrho(x; \boldsymbol{x}_{I(\alpha+1,\beta)}((\alpha+1)\ell)$ differs by at most $\mathrm{O}(\varepsilon)$. Thus Lemma 2.8 and (4.128) {e:entropybound together imply

$$\sum_{\beta} \sum_{i \neq j \in I(\alpha,\beta)} \theta \ln\left(\frac{\mathsf{x}_i((\alpha+1)\mathsf{L}) - \mathsf{x}_j((\alpha+1)\mathsf{L})}{\mathsf{x}_i(\alpha \mathsf{L}) - \mathsf{x}_j(\alpha \mathsf{L})}\right) = \mathrm{O}(\varepsilon(\ell N)^2) \tag{4.129}$$ {{e:badab}}

if on $\mathfrak{R}(\alpha, \beta)$, $\mathcal{A}$ does not have a linear approximation, then we simply bound

$$\sum_{\beta} \sum_{i \neq j \in I(\alpha,\beta)} \theta \ln\left(\frac{\mathsf{x}_i((\alpha+1)\mathsf{L}) - \mathsf{x}_j((\alpha+1)\mathsf{L})}{\mathsf{x}_i(\alpha \mathsf{L}) - \mathsf{x}_j(\alpha \mathsf{L})}\right) = \mathrm{O}((\ell N)^2 \ln(1/\ell)). \tag{4.130}$$ {{e:goodab}}

Recall from Lemma 2.3, the total number of such pair (α, β) such that on $\mathfrak{R}(\alpha, \beta)$, $\mathcal{A}$ does not have a linear approximation, is at most $\mathrm{O}(\varepsilon/\ell^2)$. By plugging (4.129), (4.130) {e:goodab} into (4.127), and summing over α, we get

$$\sum_{\alpha} \sum_{\beta \neq \beta'} \sum_{i \in I(\alpha,\beta), j \in I(\alpha,\beta')} \ln\left(1 + \frac{\theta(e_i(\mathsf{t}) - e_j(\mathsf{t}))}{\mathsf{x}_i(\mathsf{t}) - \mathsf{x}_j(\mathsf{t})}\right) = \sum_{i \neq j} \theta \ln\left(\frac{\mathsf{x}_i(\mathsf{T}) - \mathsf{x}_j(\mathsf{T})}{\mathsf{x}_i(0) - \mathsf{x}_j(0)}\right) + \mathrm{O}(\varepsilon \ln(1/\ell) N^2).$$

This finishes the proof of Lemma 3.4. □

5 Large Deviation Lower Bound: Constant Slope Case

{s:lowB}

We recall the parallelogram shaped region $\mathfrak{P}$ from (3.24):

$$\mathfrak{P} = \{x, t \in \mathbb{R}^2 : 0 \leqslant t \leqslant t, a(t) \leqslant x \leqslant b(t)\}. \tag{5.1}$$ {{defp2}}

where $\{a(t), b(t)\}_{0\leqslant t\leqslant \ell}$ are two Bernoulli walk paths such that $a(t) < b(t)$ for all $0 \leqslant t \leqslant \ell$(more specifically $a(t) = y_i(t+\beta\ell) - \alpha\ell, b(t) = y_{j+1}(t+\beta\ell) - \alpha\ell$). . We denote the parallelogram shaped region. From the construction in (3.2), the{defp2}on $\mathfrak{P}$ satisfies the following assumption

{a:Hset}

Assumption 5.1. *Fix any $(\varrho, -\varrho v) \in \mathcal{T}_\zeta$. Denote the height function $\mathcal{A}(x,t)$ with constant $(\varrho, -\varrho v)$ (see Definition 3.1). Take large integer $n \geqslant 1$ such that*

$$\left|\frac{n\theta}{N} - \varrho\right| \leqslant \varepsilon, \tag{5.2}$$

and an n-particle Bernoulli random walk $\{\boldsymbol{y}(t)\}_{0\leqslant t\leqslant \ell}$ such that $\boldsymbol{y}(t) \in [a(t), b(t) - \theta]$ and height function $\mathcal{H}(x,t)$ satisfying

$$\|\mathcal{H} - \mathcal{A}\|_\infty \leqslant \varepsilon. \tag{5.3}$$ {{e:cHcAdiff}}

We denote $\mathrm{Adm}(y(0), y(\ell), a, b)$ the set of n-particle nonintersecting Bernoulli random walks $\mathsf{p} = \{\boldsymbol{x}(t)\}_{0\leqslant t\leqslant \ell}$ such that

$$\boldsymbol{x}(0) = \boldsymbol{y}(0), \quad \boldsymbol{x}(\ell) = \boldsymbol{y}(\ell), \quad a(t) \leqslant x_n(t) \leqslant \cdots \leqslant x_1 \leqslant b(t) - \theta, \quad 0 \leqslant t \leqslant \ell. \tag{5.4}$$

{p:LDPlower}

Proposition 5.2. *Adopt Assumption 5.1, and we recall the rate function σ from (1.19). For any $\varepsilon > 0$, there exists an n-particle configuration $\boldsymbol{x}$, such that the Markov process (4.1) starting from $\boldsymbol{x}$ satisfies*

$$\frac{1}{(\ell N)^2} \ln \mathbb{P}(\{H : \|H - \mathcal{A}\|_\infty \leqslant \varepsilon\}) \geqslant -\frac{1}{\pi}\sigma(\varrho, -\varrho v) + \mathrm{O}((\varepsilon/\delta + \delta \ln(1/\delta)^2), \tag{5.5}$$

provided N is large enough.

The nonintersecting Bernoulli walk ensembles starting from $\boldsymbol{x}$ in Proposition 5.2 may not belong to $\mathrm{Adm}(y(0), y(\ell), a, b)$. The following proposition states that we can slightly modify them such that they belong to $\mathrm{Adm}(y(0), y(\ell), a, b)$.

{p:changebounda

Proposition 5.3. *Given a height function H with $\|H - \mathcal{A}\|_\infty \leqslant \varepsilon$ corresponding to the nonintersecting Bernoulli walk $\mathsf{p} = \{\boldsymbol{x}(t)\}_{0\leqslant t\leqslant \ell}$. There exists a modified nonintersecting Bernoulli walk $\widehat{\mathsf{p}} = \{\widehat{\boldsymbol{x}}(t)\}_{0\leqslant t\leqslant \ell} \in \mathrm{Adm}(y(0), y(\ell), a, b)$ such that its height function $\widehat{H}$ satisfies*

$$\|\widehat{H} - \mathcal{A}\|_\infty \lesssim \varepsilon/\zeta \tag{5.6}$$ {{e:heightdiffh

Moreover, the map from p to $\widehat{\mathsf{p}}$ is at most $e^{\mathrm{O}(\delta\ell N^2)}$ to one, and

$$\ln \mathbb{P}(\mathsf{p}) = \ln \mathbb{P}(\widehat{\mathsf{p}}) + \mathrm{O}(\delta \ln(\ell/\delta) N^2). \tag{5.7}$$

Proof of Proposition 3.6. The first statement (3.32) in Proposition 3.6 follows Lemma 2.7. For the second statement (3.33), notice that the sum is taken over configurations satisfying (3.26), namely which are restricted to leave in $\mathfrak{P}$ defined in (??), or in other words which belong to $\mathrm{Adm}(y(0), y(\ell), a, b)$. Thanks to Proposition 5.3, we have

$$\frac{1}{(\ell N)^2} \ln \mathbb{P}(\widehat{H} \in \mathrm{Adm}(y(0), y(\ell), a, b) : \|\widehat{H} - \mathcal{A}\|_\infty \leqslant \varepsilon) = \frac{1}{(\ell N)^2} \sum_{\substack{\widehat{\mathsf{p}}: \widehat{H} \in \mathrm{Adm}(y(0), y(\ell), a, b) \\ \|\widehat{H} - \mathcal{A}\|_\infty \leqslant \varepsilon}} \ln \mathbb{P}(\widehat{\mathsf{p}})$$

$$\geqslant \frac{1}{(\ell N)^2} \sum_{\mathsf{p}: H \in \|H - \mathcal{A}\|_\infty \leqslant \varepsilon} \ln \mathbb{P}(\mathsf{p}) + \mathrm{O}(\delta/\ell) = \frac{1}{(\ell N)^2} \ln \mathbb{P}(\|H - \mathcal{A}\|_\infty \leqslant \varepsilon) + \mathrm{O}(\delta/\ell)$$

$$\geqslant -\frac{1}{\pi}\sigma(\varrho, -\varrho v) + \mathrm{O}(\delta \ln(1/\delta)^2 \ell) = \frac{1}{\theta} \iint_{[\alpha\ell, (\alpha+1)\ell] \times [\beta\ell, (\beta+1)\ell]} \sigma(\nabla H^*) \mathrm{d}x \mathrm{d}t + \mathrm{O}(\delta \ln(1/\delta)^2 \ell),$$

where in the last equality we used the second statement in (2.3).

□

5.1 Proof of Proposition 5.3

Recall that $\xi \gg \varepsilon/\zeta$, and recall the index sets from (4.29)

$$I_0 = [\![1, \xi N]\!], \quad I_1 = [\![\xi N, n - \xi N]\!], \quad I_2 = [\![n - \xi N, n]\!]. \tag{5.8}$$ {{e:defI012b}}

and set

$$\widehat{x}_i(t) = x_i(t), \quad \xi \leqslant t \leqslant \ell - \xi, \quad \xi N \leqslant i \leqslant n - \xi N. \tag{5.9}$$ {{e:constructha

To construct $\widehat{\mathsf{p}}$ given (5.9), we need to construct {e:constructhat} the Bernoulli walk paths $\{\widehat{\mathsf{x}}_i(\mathsf{t})\}_{i\in I_0}, \{\widehat{\mathsf{x}}_i(\mathsf{t})\}_{i\in I_2}$, and also $\{\widehat{\mathsf{x}}_i(\mathsf{t})\}_{i\in I_1, \mathsf{t}\leqslant \xi N}, \{\widehat{\mathsf{x}}_i(\mathsf{t})\}_{i\in I_1,\mathsf{t}\geqslant(\ell-\xi)N}$.

For any $1 \leqslant i \leqslant \varrho N/\theta$, we denote the i-th level line of $\mathcal{A}$ as

$$\gamma_i(t) = \inf\{x : \mathcal{A}(x,t) > \theta(i-1)/N\} = \gamma_i(0) + tv = \frac{\theta(i-1)}{\varrho N} + tv, \tag{5.10}$$ {{e:defgamma}}

which are straight lines.

By our assumption $\|H - \mathcal{A}\|_\infty \leqslant \varepsilon$, we have for $\lceil \varepsilon N/\theta \rceil < i < n - \lceil \varepsilon N/\theta \rceil$,

$$\mathcal{A}(\gamma_{i+\lceil \varepsilon N/\theta\rceil}(t), t) \leqslant \frac{\theta(i-1)}{N} - \varepsilon \leqslant \mathcal{A}(x_i(t), t) \leqslant \frac{\theta(i-1)}{N} + \varepsilon \leqslant \mathcal{A}(\gamma_{i-\lceil \varepsilon N/\theta\rceil}(t), t) \tag{5.11}$$ {{e:Hbound}}

It follows from (5.10) and (5.11) {e:Hbound}, and noticing $\xi \geqslant 2\varepsilon/\theta$, that for $i \in I_1$.

$$\gamma_{i-\lceil \varepsilon N/\theta\rceil}(t) \leqslant x_i(t) \leqslant \gamma_{i-\lceil \varepsilon N/\theta\rceil}(t), \quad \Rightarrow \quad |x_i(t) - \gamma_i(t)| \leqslant 2\varepsilon. \tag{5.12}$$ {{e:xloc}}

By the same argument, our assumption (5.3) {e:xloc} also implies that for any $i \in I_1$,

$$|y_i(t) - \gamma_i(t)| \leqslant 2\varepsilon, \quad 0 \leqslant t \leqslant \ell. \tag{5.13}$$ {{e:xloc2}}

Moreover, since $\boldsymbol{y}(t) \in [a(t), b(t) - \theta]$, (5.3) {e:xloc2} also implies that either $a(t) \leqslant tv$, or $a(t) \geqslant tv$ and $\mathcal{A}(a(t), t) \leqslant \mathcal{H}(a(t), t) + \varepsilon = \varepsilon$. In both cases we have

$$\gamma_i(t) - a(t) \geqslant \frac{1}{\varrho}\left(\frac{(i-1)\theta}{N} - \varepsilon\right) \tag{5.14}$$ {{e:gammaiat}}

We will construct the Bernoulli walk paths $\{\widehat{\mathsf{x}}_i(\mathsf{t})\}_{i\in I_0}, \{\widehat{\mathsf{x}}_i(\mathsf{t})\}_{i\in I_1,\mathsf{t}\leqslant \xi N}$ with the smallest possible height function; and $\{\widehat{\mathsf{x}}_i(\mathsf{t})\}_{i\in I_2}, \{\widehat{\mathsf{x}}_i(\mathsf{t})\}_{i\in I_1,\mathsf{t}\geqslant(\ell-\xi)N}$ with the largest possible height function, in the following way. For $1 \leqslant i \leqslant \xi N, 0 \leqslant \mathsf{t} \leqslant \mathsf{L}$, add a picture

$$\begin{aligned} \widehat{\mathsf{x}}_i(\mathsf{t}) &= \max\{Na(\mathsf{t}) + (i-1)\theta, \mathsf{y}_i(0), \mathsf{y}_i(\mathsf{L}) - (\mathsf{L} - \mathsf{t})\} \\ \widehat{\mathsf{x}}_{n-i+1}(\mathsf{t}) &= \min\{Nb(\mathsf{t}) - i\theta, \mathsf{y}_{n-i+1}(0) + \mathsf{t}, \mathsf{y}_{n-i+1}(\mathsf{L})\} \end{aligned} \tag{5.15}$$ {{e:extremes}}

And for $i \in I_1, 0 \leqslant \mathsf{t} \leqslant \xi N$,

$$\begin{aligned} \widehat{\mathsf{x}}_i(\mathsf{t}) &= \max\{\mathsf{x} : \mathsf{x} - \mathsf{y}_i(0) \in \mathbb{Z}, \mathsf{x} \leqslant (1 - t/\xi)\mathsf{y}_i(0) + (t/\xi)\mathsf{x}_i(\xi N)\}, \\ \widehat{\mathsf{x}}_i(\mathsf{L} - \mathsf{t}) &= \max\{\mathsf{x} : \mathsf{y}_i(\mathsf{L}) - \mathsf{x} \in \mathbb{Z}, \mathsf{x} \leqslant (1 - t/\xi)\mathsf{x}_i(\mathsf{L} - \xi N) + (t/\xi)\mathsf{y}_i(\mathsf{L})\}, \end{aligned} \tag{5.16}$$ {{e:extremet}}

The maximum or minimum over Bernoulli paths is still a Bernoulli path. So $\{\widehat{\mathsf{x}}_i(\mathsf{t})\}_{0\leqslant \mathsf{t}\leqslant \mathsf{L}}$ is a Bernoulli path.

To show that $\{\widehat{\mathsf{x}}_i(\mathsf{t})\}_{1\leqslant i\leqslant n, 0\leqslant \mathsf{t}\leqslant \mathsf{L}} \in \mathrm{Adm}(y(0), y(\ell), a, b)$, we need to check the following conditions {e:it1}

1. $\{\widehat{\mathsf{x}}_i(\mathsf{t})\}_{1\leqslant i\leqslant \xi N, 0\leqslant \mathsf{t}\leqslant \mathsf{L}}$ are non-interesecting Bernoulli paths form $\mathsf{y}_i(0)$ to $\mathsf{y}_i(\mathsf{L})$; $\{\widehat{\mathsf{x}}_{n-i+1}(\mathsf{t})\}_{1\leqslant i\leqslant \xi N, 0\leqslant \mathsf{t}\leqslant \mathsf{L}}$ are non-intersecting Bernoulli paths form $\mathsf{x}_{n-i+1}(0)$ to $\mathsf{y}_{n-i+1}(\mathsf{L})$. {e:it2}
2. $\{\widehat{\mathsf{x}}_i(\mathsf{t})\}_{i\in I_1, 0\leqslant \mathsf{t}\leqslant \xi N}$ are non-interesecting Bernoulli paths form $\mathsf{y}_i(0)$ to $\mathsf{x}_i(\xi N)$; $\{\widehat{\mathsf{x}}_i(\mathsf{L} - \mathsf{t})\}_{i\in I_1, 0\leqslant \mathsf{t}\leqslant \mathsf{L}}$ are non-intersecting Bernoulli paths form $\mathsf{x}_i(\mathsf{L} - \mathsf{t})$ to $\mathsf{y}_i(\mathsf{L})$.

{e:it3}

3. $\widehat{\mathsf{x}}_{\xi N}(\mathsf{t}) < \widehat{\mathsf{x}}_{\xi N+1}(\mathsf{t})$ and $\widehat{\mathsf{x}}_{n-\xi N}(\mathsf{t}) < \widehat{\mathsf{x}}_{n-\xi N+1}(\mathsf{t})$.

From Assumption 5.1, $\mathrm{Adm}(y(0), y(\ell), a, b) \neq \emptyset$, so $\mathsf{y}_i(0) \geqslant Na(0) + (i-1)\theta$, $\mathsf{y}_{n-i+1}(\mathsf{L}) \leqslant Nb(\mathsf{L}) - i\theta$ and $\mathsf{y}_i(0) \geqslant \mathsf{y}_i(\mathsf{L}) - \mathsf{L}$. Thus at time $\mathsf{t} = 0$, $\widehat{\mathsf{x}}_i(0) = \mathsf{y}_i(0)$; at time $\mathsf{t} = \mathsf{L}$, $\widehat{\mathsf{x}}_i(\mathsf{L}) = \mathsf{y}_i(\mathsf{L})$. In particular $\widehat{\mathsf{x}}_i(\mathsf{t})$ is a Bernoulli path from $\mathsf{y}_i(0)$ to $\mathsf{y}_i(\mathsf{L})$. Also from the construction (5.16), we have $\widehat{\mathsf{x}}_i(\mathsf{L}) < \widehat{\mathsf{x}}_j(\mathsf{L})$ {e:extremet} for $1 \leqslant i < j \leqslant \xi N$. The same argument also implies that $\{\widehat{\mathsf{x}}_{n-i+1}(\mathsf{t})\}_{1\leqslant i\leqslant \xi N, 0\leqslant \mathsf{t}\leqslant \mathsf{L}}$ are non-intersecting Bernoulli paths form $\mathsf{y}_{n-i+1}(0)$ to $\mathsf{y}_{n-i+1}(\mathsf{L})$. This proves Item 1.

For Item 2, by our construction (5.16), we have that $\widehat{\mathsf{x}}_i(0) = \mathsf{y}_i(0)$ and $\widehat{\mathsf{x}}_i(\xi N) = \mathsf{x}_i(\xi N)$. It follows from (5.12) and (5.13), that $\mathsf{x}_i(\xi N) - \mathsf{y}_i(0) = N(\gamma_i(\xi N) - \gamma_i(0)) + \mathrm{O}(\varepsilon N) = N\xi v + \mathrm{O}(\varepsilon N)$, and

$$\left(1 - \frac{\mathsf{t}+1}{N\xi}\right)\mathsf{y}_i(0) + \left(\frac{\mathsf{t}+1}{N\xi}\right)\mathsf{x}_i(\xi N) - \left(1 - \frac{\mathsf{t}}{N\xi}\right)\mathsf{y}_i(0) - \left(\frac{\mathsf{t}}{N\xi}\right)\mathsf{x}_i(\xi N) = v + \mathrm{O}(\varepsilon/\xi) \in (0,1), \quad (5.17)$$

provided $v \geqslant \zeta \gg \varepsilon/\xi$.

Thus the construction (5.16) implies that $\widehat{\mathsf{x}}_i(\mathsf{t}+1) - \widehat{\mathsf{x}}_i(\mathsf{t}) \in \{0,1\}$. We conclude that $\{\widehat{\mathsf{x}}_i(\mathsf{t})\}_{i\in I_1, 0\leqslant \mathsf{t}\leqslant \xi N}$ are non-interesecting Bernoulli paths form $\mathsf{y}_i(0)$ to $\mathsf{x}_i(\xi N)$. The same argument also implies that $\{\widehat{\mathsf{x}}_i(\mathsf{L} - \mathsf{t})\}_{i\in I_1, 0\leqslant \mathsf{t}\leqslant \mathsf{L}}$ are non-intersecting Bernoulli paths form $\mathsf{x}_i(\mathsf{L}-\mathsf{t})$ to $\mathsf{y}_i(\mathsf{L})$.

For Item 3, by symmetry, only need to show that

$$\widehat{\mathsf{x}}_i(\mathsf{t}) < \widehat{\mathsf{x}}_{\xi N+1}(\mathsf{t}), \quad i \in I_0, \quad 0 \leqslant \mathsf{t} \leqslant \mathsf{L} - \xi N. \quad (5.18)$$ {{e:xibound}}

We will check the three relations: $\widehat{\mathsf{x}}_{\xi N+1}(\mathsf{t}) \geqslant Na(\mathsf{t}) + (\xi N + 1)\theta$, $\widehat{\mathsf{x}}_{\xi N+1}(\mathsf{t}) \geqslant \mathsf{y}_i(0)$ and $\widehat{\mathsf{x}}_{\xi N+1}(\mathsf{t}) \geqslant \mathsf{y}_i(\mathsf{L}) - (\mathsf{L} - \mathsf{t})$. For $\xi N \leqslant \mathsf{t} \leqslant \mathsf{L} - \xi N$, the first relation follows from (5.12) and (5.14) {e:xibound}, $\mathsf{x}_{\xi N+1}(\mathsf{t}) = N\gamma_{\xi N+1}(t) + \mathrm{O}(\varepsilon N) \geqslant Na(t) + \theta(\xi N)/\varrho + \mathrm{O}(\varepsilon N) > Na(\mathsf{t}) + \xi\theta N$. The second statement follows from $\mathsf{x}_{\xi N+1}(\mathsf{t}) = N\gamma_{\xi N+1}(t) + \mathrm{O}(\varepsilon N) = N\gamma_{\xi N+1}(0) + Ntv + \mathrm{O}(\varepsilon N) = \mathsf{y}_i(0) + Ntv + \mathrm{O}(\varepsilon N) > \mathsf{y}_i(0)$, where we used that $\gamma_{\xi N}(t)$ is linear in t, (5.13) and $t \geqslant \xi$. The last statement follows from similar argument $\mathsf{x}_{\xi N+1}(\mathsf{t}) = N\gamma_{\xi N+1}(t) + \mathrm{O}(\varepsilon N) = N\gamma_{\xi N+1}(\ell) + Nv(\ell - t) + \mathrm{O}(\varepsilon N) = \mathsf{y}_i(\mathsf{L}) + Nv(\ell - t) + \mathrm{O}(\varepsilon N) > \mathsf{y}_i(\mathsf{L})$, where we used that $\gamma_{\xi N}(t)$ is linear in t, (5.13) and $t \leqslant \ell - \xi$. This finishes the proof of (5.18) for $\xi N \leqslant \mathsf{t} \leqslant \mathsf{L} - \xi N$.

For $0 \leqslant \mathsf{t} \leqslant \xi N$, using (5.12) the first statment follows from:

$$\begin{aligned}\widehat{\mathsf{x}}_{\xi N+1}(\mathsf{t}) &\geqslant (1 - t/\xi)\mathsf{y}_{\xi N+1}(0) + (t/\xi)\mathsf{x}_{\xi N+1}(\xi N) - 1 = N\gamma_{\xi N+1}(t) - \mathrm{O}(\varepsilon N)\\ &= Na(t) + \xi\theta N/\varrho - \mathrm{O}(\varepsilon N) > Na(t) + \xi\theta N,\end{aligned}$$

where we used (5.12) and (5.13). The second statement that $\widehat{\mathsf{x}}_{\xi N+1}(\mathsf{t}) \geqslant \mathsf{y}_i(0)$ holds trivially; and the third statement follows from (5.13) that $\widehat{\mathsf{x}}_{\xi N+1}(\mathsf{t}) \geqslant N\gamma_{\xi N+1}(t) - \mathrm{O}(\varepsilon N) = N\gamma_{\xi N+1}(\mathsf{L}) - (\mathsf{L} - \mathsf{t})v - \mathrm{O}(\varepsilon N) \geqslant \mathsf{y}_i(\mathsf{L}) - (\mathsf{L} - \mathsf{t})$. This finishes the proof of (5.18) for $0 \leqslant \mathsf{t} \leqslant \xi N$. And we conclude that $\widehat{\mathsf{p}}$ constructed above belongs to $\mathrm{Adm}(y(0), y(\ell), a, b)$.

Next we show for any $i \in I_0$, we have

$$|\widehat{\boldsymbol{x}}_i(t) - \gamma_i(t)| \leqslant 3\varepsilon. \quad (5.19)$$ {{e:levelinebou

and the claim (5.6) follows. {e:levelinebound}

For $\xi \leqslant t \leqslant \ell - \xi$, (5.19) follows from (5.12). For $0 \leqslant t \leqslant \xi$, from the construction (5.16) and (5.12) and (5.13)

$$\begin{aligned}|\widehat{\boldsymbol{x}}_i(t) - \gamma_i(t)| &\leqslant |(1 - t/\xi)y_i(0) + (t/\xi)x_i(\xi N) - \gamma_i(t)| + 1/N\\ &\leqslant |(1 - t/\xi)\gamma_i(0) + (t/\xi)\gamma_i(\xi N) - \gamma_i(t)| + (\varepsilon + 1/N) = \varepsilon + 1/N.\end{aligned}$$

The case that $\ell - \xi \leqslant t \leqslant \ell$ follows from the same argument.

We can decompose the weight $\mathbb{P}(\mathsf{p})$

$$\begin{aligned}\ln \mathbb{P}(\mathsf{p}) = &\sum_{\mathsf{t}\leqslant \xi N \text{ or } \mathsf{t}\geqslant(\ell-\xi)N} \ln \frac{V(\mathbf{x}(\mathsf{t}) + \theta \boldsymbol{e}(\mathsf{t}))}{V(\mathbf{x}(\mathsf{t}))} + \sum_{\xi N\leqslant \mathsf{t}\leqslant(\ell-\xi)N} \left(\ln \frac{V(\mathbf{x}_{I_0\cup I_2}(\mathsf{t}) + \boldsymbol{e}_{I_0\cup I_2}(\mathsf{t}))}{V(\mathbf{x}_{I_0\cup I_2}(\mathsf{t}))} \right.\\ &\left. + \ln \frac{V(\mathbf{x}_{I_1}(\mathsf{t}) + \boldsymbol{e}_{I_1}(\mathsf{t}))}{V(\mathbf{x}_{I_1}(\mathsf{t}))} + \sum_{i\in I_0\cup I_2, j\in I_1} \ln\left(1 + \frac{\theta(e_i(\mathsf{t}) - e_j(\mathsf{t}))}{(\mathsf{x}_i(\mathsf{t}) - \mathsf{x}_j(\mathsf{t}))}\right)\right) - \ell n N.\end{aligned} \quad (5.20)$$ {{e:Wpdecompose

Thanks to Lemma 2.7, we can bound terms in (5.20) as{e:Wpdecompose}

$$\sum_{\mathsf{t}\leqslant \xi N \text{ or } \mathsf{t}\geqslant(\ell-\xi)N} \ln \frac{V(\mathbf{x}(\mathsf{t})+\theta \boldsymbol{e}(\mathsf{t}))}{V(\mathbf{x}(\mathsf{t}))} = \mathrm{O}(\xi n N) = \mathrm{O}(\xi \ell N^2)$$
$$\sum_{\xi N\leqslant \mathsf{t}\leqslant(\ell-\xi)N} \ln \frac{V(\mathbf{x}_{I_0\cup I_2}(\mathsf{t})+\boldsymbol{e}_{I_0\cup I_2}(\mathsf{t}))}{V(\mathbf{x}_{I_0\cup I_2}(\mathsf{t}))} = \mathrm{O}((|I_1|+|I_2|)\ell N) = \mathrm{O}(\xi \ell N^2) \tag{5.21}$$ {{e:Vterm}}

And for the last term in (5.20) {e:Vterm}

$$\sum_{i\in I_0\cup I_2, j\in I_1} \ln\left(1+\frac{\theta(e_i(\mathsf{t})-e_j(\mathsf{t}))}{(\mathsf{x}_i(\mathsf{t})-\mathsf{x}_j(\mathsf{t}))}\right) \leqslant 2\theta \sum_{i\in I_0\cup I_2, j\in I_1} \frac{1}{|\mathsf{x}_i(\mathsf{t})-\mathsf{x}_j(\mathsf{t})|} \leqslant 2 \sum_{i\in I_0\cup I_2, j\in I_1} \frac{1}{|i-j|}, \tag{5.22}$$ {{e:IJdiff}}

where in the last inequality we used $|\mathsf{x}_i(\mathsf{t})-\mathsf{x}_j(\mathsf{t})|\geqslant \theta|i-j|$. Recall the sets I_0, I_1, I_2 from (4.29), we can {e:IJdiff} further bound (5.22) as

$$\sum_{i\in I_0\cup I_2, j\in I_1} \frac{1}{|i-j|} \leqslant \sum_{i=1}^{\xi N}\sum_{j=i}^{\ell N}\frac{1}{j} \leqslant \xi N + \xi N \sum_{\xi N\leqslant j\leqslant \ell N}\frac{1}{j} \leqslant \xi N + \xi \ln(\ell/\xi) N = \mathrm{O}(\xi \ln(\ell/\xi) N). \tag{5.23}$$ {{e:sumb}}

Plugging (5.21), (5.22) {e:sumb} and (5.23) into (5.20), we get

$$\ln \mathbb{P}(\mathsf{p}) = \sum_{\xi N\leqslant \mathsf{t}\leqslant(\ell-\xi)N} \ln \frac{V(\mathbf{x}_{I_1}(\mathsf{t})+\boldsymbol{e}_{I_1}(\mathsf{t}))}{V(\mathbf{x}_{I_1}(\mathsf{t}))} - \ell n N + \mathrm{O}(\xi \ln(\ell/\xi) N^2) \tag{5.24}$$ {{e:Wpdecompose

The same statement (5.24) holds for $\widehat{\mathsf{p}}$ {e:Wpdecompose2} Moreover, from our construction (5.9), we conclude from (5.24)

$$\ln \mathbb{P}(\mathsf{p}) = \ln \mathbb{P}(\widehat{\mathsf{p}}) + \mathrm{O}(\xi \ln(\ell/\xi) N^2). \tag{5.25}$$

5.2 Proof of Proposition 5.2

We fix three small parameters $\omega \ll \varepsilon \ll \delta$, where ω and δ will be chosen later. We recall the smoothed height function $\widetilde{H}$ from (4.11), the associated complex slope $\widetilde{f}_t$ from (4.12), and the drift $g_t(z)$ from (4.15). We consider Markov process $\{\boldsymbol{x}(t)\}_{0\leqslant t\leqslant \ell}$ with height function $H(x,t)$, such that the initial data satisfies

$$\|H(x,0)-\widetilde{H}(x,0)\|_\infty \leqslant \omega. \tag{5.26}$$

From the construction of $\widetilde{H}$ in (4.11), thanks to Lemma 2.4, we have

$$|\mathcal{A}(x,t)-\widetilde{H}(x,t)| \lesssim \frac{\delta}{\ell} + \int_{-3\ell}^{x} \frac{\delta}{\delta+\mathrm{dist}(x,\{tv,\ell+tv\})} \lesssim \delta \ln(\ell/\delta), \tag{5.27}$$

and it follows

$$\frac{1}{(\ell N)^2}\ln \mathbb{P}(\|H(x,t)-\mathcal{A}(x,t)\|_\infty \leqslant \varepsilon+\delta) \geqslant \frac{1}{(\ell N)^2}\ln \mathbb{P}(\|H(x,t)-\widetilde{H}(x,t)\|_\infty \leqslant \varepsilon) \tag{5.28}$$ {{e:tHcenter}}

To prove (5.28), by the same {e:tHcenter} argument as in proof of the large deviation upper bound, we tilt the Markov chain by the exponential Martingale (4.18). The following lemma collects some estimates for the numerator and denominator of the exponential Martingale on the event $\|H(x,t)-\widetilde{H}(x,t)\|_\infty \leqslant \varepsilon$,

{l:Nterm2}

Lemma 5.4. *Take any large number $B>1$. Let $H(x,t)$ be the height function associated with the particle configuration $\{\boldsymbol{x}(t)\}_{0\leqslant t\leqslant \ell}$, with $\mathrm{supp}(\boldsymbol{x}(t))\in[-B,B]$ and $\|H(x,t)-\widetilde{H}(x,t)\|_\infty\leqslant\varepsilon$ then*

$$\sum_{\mathsf{t}=0}^{\mathsf{L}-1}\sum_{i=1}^{n} e_i(\mathsf{t}) g_t(x_i(\mathsf{t}/N)) = -\frac{N^2}{\theta}\int_0^\ell\int_{\mathbb{R}} \partial_t \mathcal{A}(x,t) g_t(x)\mathrm{d}x\mathrm{d}t + \mathrm{O}((\varepsilon/\delta+\delta\ln(1/\delta)^2)(\ell N)^2). \tag{5.29}$$ {{e:exp}}

Dynamical loop equation gives

$$\ln \mathbb{E}[e^{\sum_i e_i(\mathsf{t}) g_t(x_i(t)/N)}|\boldsymbol{x}(t)] = \frac{N}{2\pi \mathrm{i}\theta}\oint \int_0^1 \ln(1+e^{m_t^*(z)+\tau g_t(z)})g_t(z)\mathrm{d}z\mathrm{d}\tau + \mathrm{O}\left(\delta \ln^2(1/\delta)N\right) \tag{5.30}$$ {{e:DLE}}

where $m^*(z)$ is the Stieltjes transform of the empirical measure of $\partial_x \mathcal{A}(x,t)$.

With Lemma 5.4, we can lower bound the righthand side of (5.28) as{e:DLE}

$$\begin{aligned} &\frac{1}{(\ell N)^2}\ln \mathbb{P}(\|H(x,t)-\widetilde{H}(x,t)\|_\infty \leqslant \varepsilon) \geqslant S(\mathcal{A};g) + \mathrm{O}\left(\varepsilon/\delta + \delta\ln(1/\delta)^2\right) \\ &+\frac{1}{(\ell N)^2}\ln\mathbb{E}\left[\prod_{t\in[\![0,\mathsf{L}-1]\!]/N}\frac{e^{\sum_{i=1}^N e_i(Nt)g_t(x_i(t))}}{\mathbb{E}[e^{\sum_{i=1}^N e_i(Nt)g_t(x_i(t))}|\boldsymbol{x}(t)]}\mathbf{1}(\|H(x,t)-\widetilde{H}(x,t)\|_\infty\leqslant\varepsilon)\right]\end{aligned} \tag{5.31}$$ {{e:tilt}}

where $S(\mathcal{A};g)$ is from (4.22) and Proposition 4.3 gives{e:Prop}

$$\begin{aligned} S(\mathcal{A};g) &= -\int_0^\ell\int_{\mathbb{R}}\partial_t\mathcal{A}(x,t)g_t(x)\mathrm{d}x\mathrm{d}t - \frac{1}{2\pi\mathrm{i}}\int_0^\ell\oint\int_0^1\ln(1+e^{m_t^*(z)+\tau g_t(z)})g_t(z)\mathrm{d}z\mathrm{d}\tau\mathrm{d}t \\ &= -\frac{\ell^2}{\pi}\sigma(\varrho,-\varrho v) + \ln(2) + \mathrm{O}(\delta\ln(1/\delta)^2\ell). \end{aligned} \tag{5.32}$$

To lower bound the second term on the righthand side of (5.31), we introduce the tilded measure $\mathbb{P}^g(\cdot)$ from (5.33),

$$\mathbb{P}^g(\cdot) = \mathbb{E}\left[\prod_{t\in[\![0,\mathsf{L}-1]\!]/N}\frac{e^{\sum_{i=1}^N e_i(Nt)g_t(x_i(t))}}{\mathbb{E}[e^{\sum_{i=1}^N e_i(Nt)g_t(x_i(t))}|\boldsymbol{x}(t)]}(\cdot)\right]. \tag{5.33}$$ {{e:Pg}}

Under $\mathbb{P}^g(\cdot)$, the Markov process (5.34) becomes{e:Pg}

$$\mathbb{P}^g(\mathsf{x}(\mathsf{t}+1)=\mathsf{x}+\boldsymbol{e}|\mathsf{x}(\mathsf{t})=\mathsf{x}) \propto \frac{V(\mathsf{x}+\theta\boldsymbol{e})}{V(\mathsf{x})}\prod_{i=1}^N e^{e_i g_t(\mathsf{x}_i/N)}, \quad t=\mathsf{t}/N. \tag{5.34}$$ {{e:Mk}}

The large deviation lower bound follows from the following limit shape result

Proposition 5.5. *For any $\varepsilon>0$, there exists a small $c(\varepsilon)>0$, such that if the height function $H(x,0)$ of the initial data $\boldsymbol{x}(0)$ satisfies $\|H(x,t)-\widetilde{H}(x,t)\|\leqslant c(\varepsilon)$, the*

$$\mathbb{P}^g(\|H(x,t)-\widetilde{H}(x,t)\|\leqslant\varepsilon) = 1-\varepsilon, \tag{5.35}$$ {{e:Pg2}}

provided N is large enough.

5.3 Proof of Lemma 5.4

Proof. From the construction of $\widetilde{H}$ in (4.11), and thanks to Lemma 2.4{e:Pg2}

$$\widetilde{H}(-3\ell,t)\lesssim\frac{\delta}{\ell},\quad \theta-\widetilde{H}(3\ell,t)\lesssim\frac{\delta}{\ell}, \tag{5.36}$$

it follows that

$$H(-3\ell,t)\lesssim\frac{\delta}{\ell}+\varepsilon\lesssim\frac{\delta}{\ell},\quad \theta-\widetilde{H}(3\ell,t)\lesssim\frac{\delta}{\ell}+\varepsilon\lesssim\frac{\delta}{\ell}. \tag{5.37}$$

Using the above estimate we can rewrite the lefthand side of (5.29) as

$$\sum_{\mathsf{t}=0}^{\mathsf{L}-1}\sum_{i=1}^N e_i(\mathsf{t})g_t(x_i(t)) = \sum_{\mathsf{t}=0}^{\mathsf{L}-1}\sum_{i:|x_i(t)|\leqslant 3\ell}e_i(\mathsf{t})g_t(x_i(t)) + \mathrm{O}(\delta\ell N^2\|g_t\|_\infty). \tag{5.38}$$ {{e:replaceH2}}

By the same argument as in (4.70), (5.38) leads to {e:replaceH2}

$$\sum_{\mathsf{t}=0}^{\mathsf{L}-1}\sum_{i=1}^{N} e_i(\mathsf{t})g_t(x_i(\mathsf{t})/N) = -\frac{N^2}{\theta}\int_0^{\ell}\int_{-3\ell}^{3\ell}\partial_t H(x,t)g_t(x)\mathrm{d}x\mathrm{d}t + \mathrm{O}(\delta\ell N^2\|g_t\|_\infty). \tag{5.39}$$

Next we replace $H(x,t)$ by $\widetilde{H}(x,t)$. Using the second statement of Lemma 4.5,

$$\left|\frac{N^2}{\theta}\int_0^{\ell}\int_{-3\ell}^{3\ell}(\partial_t H(x,t)-\partial_t\widetilde{H}(x,t))g_t(x)\mathrm{d}x\mathrm{d}t\right| \tag{5.40}$$

$$\leqslant \frac{N^2}{\theta}\left|\int_{-3\ell}^{3\ell}(H(x,t)-\widetilde{H}(x,t))g_t(z)\mathrm{d}x\Big|_0^{\ell}\right| + \frac{N^2}{\theta}\left|\int_0^{\ell}\int_{-3\ell}^{3\ell}(H(x,t)-\widetilde{H}(x,t))\partial_t g_t(x)\mathrm{d}x\mathrm{d}t\right| \tag{5.41}$$

$$\lesssim \varepsilon\|g\|_\infty \ell N^2 + \varepsilon\|\partial_t g\|_\infty(\ell N)^2 \tag{5.42}$$

Finally, we replace $\widetilde{H}(x,t)$ by $\mathcal{A}(x,t)$

$$\begin{aligned}
&\left|\frac{N^2}{\theta}\int_0^{\ell}\int_{-3\ell}^{3\ell}(\partial_t\mathcal{A}(x,t)-\partial_t\widetilde{H}(x,t))g_t(x)\mathrm{d}x\mathrm{d}t\right| \\
&\leqslant \frac{N^2\|g\|_\infty}{\theta}\int_0^{\ell}\int_{-3\ell}^{3\ell}|\partial_t\mathcal{A}(x,t)-\partial_t\widetilde{H}(x,t)|\mathrm{d}x\mathrm{d}t \\
&\lesssim \frac{N^2\|g\|_\infty}{\theta}\int_0^{\ell}\int_{-3\ell}^{3\ell}\frac{\delta}{\delta+\mathrm{dist}(x,\{tv,\ell+tv\})+\mathrm{dist}(x,\{tv,\ell+tv\})^2}\mathrm{d}x\mathrm{d}t \\
&\lesssim (\delta\ln(\ell/\delta)\ell+\delta)\|g\|_\infty N^2
\end{aligned} \tag{5.43}$$

where we used Lemma 2.4 for the second inequality. The same as in Lemma 5.4, by using the dynamical loop equation, we have

$$\ln\mathbb{E}[e^{\sum_i e_i(\mathsf{t})g_t(x_i(t)/N)}|\boldsymbol{x}(t)] = \frac{N}{2\pi\mathrm{i}\theta}\oint_\omega\int_0^1 \ln(1+e^{m_t(z)+\tau g_t(z)})g_t(z)\mathrm{d}z\mathrm{d}\tau + \mathrm{O}(1), \tag{5.44}$$ {{e:DLE2}}

where $m_t(z)$ is the Stieltjes transform of the empirical measure of $\boldsymbol{x}(t)$, and the implicit constant in the error term $\mathrm{O}(1)$ depends on δ,ζ. We can deform the contour ω such that it consists of two parallel lines $\{z:\mathrm{Im}[z]=\pm 1/K\}$.

Recall from Lemma 4.5, if $\mathrm{dist}(z,\{tv,\ell+tv\})\geqslant \ell$, we have

$$\left|g_t(z)-\ln\frac{\sin(v)}{\sin(1-v)}\right| = \mathrm{O}\left(\frac{\delta}{\mathrm{dist}(z,\{tv,\ell+tv\})}\right) = \mathrm{O}\left(\frac{\delta}{|z|}\right) \tag{5.45}$$

We can replace $g_t(z)$ in (5.44) by $g_t(z)-\ln(\sin(v)/\sin(1-v))$ and contour integral does not change. To replace $m_t(z)$ in (5.44) by $m_t^*(z)$, the error is given by

$$\begin{aligned}
&\left|\oint\int_0^1 \ln(1+e^{m_t(z)+\tau g(z)})g(z)\mathrm{d}z\mathrm{d}\tau - \oint\int_0^1 \ln(1+e^{m_t^*(z)+\tau g(z)})g(z)\mathrm{d}z\mathrm{d}\tau\right| \\
&\leqslant \oint\int_0^1 |\ln(1+e^{m_t^*(z)+\tau g(z)})-\ln(1+e^{m_t(z)+\tau g(z)})|\left|g_t(z)-\ln\frac{\sin(v)}{\sin(1-v)}\right|\mathrm{d}z\mathrm{d}\tau
\end{aligned} \tag{5.46}$$ {{e:replacemm*2

For the difference $|\ln(1+e^{m_t^*(z)+\tau g(z)})-\ln(1+e^{m_t(z)+\tau g(z)})|$, we need to upper bound $|m_t^*(z)-m_t(z)|$. Recall from Lemma 4.10, for $\mathrm{dist}(z,\{tv,\ell+tv\})\geqslant\delta$

$$|m_t^*(z)-\widetilde{m}_t(z)| \lesssim \frac{\delta}{\delta+\mathrm{dist}(z,\{tv,\ell+tv\})}. \tag{5.47}$$ {{e:difln2}}

Next we bound $|m_t(z)-\widetilde m_t(z)|$. Since $\mathrm{Im}[z]\geqslant 1/K$, the same as in (4.81), we have {e:difln2} $|\widetilde m_t(z)-m_t^*(z)|\leqslant \varepsilon K$. They together give a simple bound $|m_t(z)-m_t^*(z)|\leqslant \delta/\operatorname{dist}(z,\{tv,\ell+tv\})+\varepsilon K\pi \ll 1$ for $\operatorname{dist}(z,\{tv,\ell+tv\})\gg\delta$. It follows that

$$\begin{aligned}&|\ln(1+e^{m_t^*(z)+\tau g_t(z)})-\ln(1+e^{m_t(z)+\tau g_t(z)}))|\leqslant\left|\ln\left(1+\frac{(e^{m_t(z)-m_t^*(z)}-1)e^{m_t^*(z)+\tau g_t(z)}}{1+e^{m_t^*(z)+\tau g_t(z)}}\right)\right|\\ \leqslant&\left|\frac{(e^{m_t(z)-m_t^*(z)}-1)e^{m_t^*(z)+\tau g_t(z)}}{1+e^{m_t^*(z)+\tau g_t(z)}}\right|\lesssim\frac{|m_t(z)-m_t^*(z)|}{\zeta}\lesssim\frac{|m_t(z)-\widetilde m_t(z)|+|\widetilde m_t(z)-m_t^*(z)|}{\zeta}\end{aligned}\tag{5.48}$$
{{e:difln}}

Next we will prove an improved estimate of the difference $|m_t(z)-\widetilde m_t(z)|$ when z is far away from 0. For $\operatorname{dist}(z,\{tv,\ell+tv\})\geqslant 3\ell$,

$$|m_t(z)-\widetilde m_t(z)|\lesssim\frac{\delta}{|z|}+\int_{|x|\geqslant|z|/2}\frac{\partial_x H(x,t)\mathrm{d}x}{|z-x|}\tag{5.49}$$
{{e:mtdiff}}

By symmetry, we will only prove (5.49) for {e:mtdiff} $\mathrm{Re}[z]\leqslant-\ell$, then $\operatorname{dist}(z,\{tv,\ell+tv\})\asymp|z|\geqslant\ell$.

$$\begin{aligned}|m_t(z)-\widetilde m_t(z)|&=\left|\int_{\mathbb R}\frac{\partial_x H(x,t)-\partial_x\widetilde H(x,t)}{z-x}\right|\\&\leqslant\left|\int_{x\leqslant\mathrm{Re}[z]/2}\frac{\partial_x H(x,t)-\partial_x\widetilde H(x,t)}{z-x}\right|+\left|\int_{x\geqslant\mathrm{Re}[z]/2}\frac{\partial_x H(x,t)-\partial_x\widetilde H(x,t)}{z-x}\right|\\&=\left|\int_{x\leqslant\mathrm{Re}[z]/2}\frac{\partial_x H(x,t)-\partial_x\widetilde H(x,t)}{z-x}\right|+\mathrm{O}\left(\frac{\varepsilon}{|z|}\right),\end{aligned}\tag{5.50}$$
{{e:diffmtm}}

where the last term is bounded the same as in (4.81). For {e:diffmtm} the first term on the righthand side of (5.50) and we have

$$\left|\int_{x\leqslant\mathrm{Re}[z]/2}\frac{\partial_x\widetilde H(x,t)}{z-x}\right|\leqslant\left|\int_{x\leqslant 2\,\mathrm{Re}[z]}\frac{\varrho\kappa_t(x)}{z-x}\right|+\left|\int_{2\,\mathrm{Re}[z]\leqslant x\leqslant\mathrm{Re}[z]/2}\frac{\varrho\kappa_t(x)}{z-x}\right|\lesssim\frac{\delta}{|z|}.\tag{5.51}$$
{{e:dtH}}

The claim (5.49) follows {e:dtH} from combining (5.50) and (5.51).

We decompose the contour integral on the righthand side of (5.46) into the following three parts

1. For $\operatorname{dist}(z,\{tv,\ell+tv\})\leqslant\delta$, using $|\ln(1+e^{m_t^*(z)+\tau g(z)})-\ln(1+e^{m_t(z)+\tau g(z)})|\lesssim\ln(1/\zeta)$, we can bound the integral as

$$\oint_{\operatorname{dist}(z,\{tv,\ell+tv\})\leqslant\delta}\ln(1/\zeta)(\ln(1/\zeta)+\ln(1/\delta)+C)|\mathrm{d}z|\lesssim\delta\ln^2(1/\delta).\tag{5.52}$$
{{e:case1}}

2. For $\delta\leqslant\operatorname{dist}(z,\{tv,\ell+tv\})\leqslant 10\ell$, using (5.48) and {e:case1} (5.47) we can bound the integral as

$$\oint_{\delta\leqslant\operatorname{dist}(z,\{tv,\ell+tv\})\leqslant 10\ell}\frac{\varepsilon K}{\zeta}(\ln(1/\zeta)+\ln(1/\delta)+C)|\mathrm{d}z|\lesssim\varepsilon\ln(1/\delta)\ell K/\zeta.\tag{5.53}$$
{{e:case2}}

3. For $\operatorname{dist}(z,\{tv,\ell+tv\})\geqslant 10\ell$, using (5.48), (5.47) {e:case2} (5.49), we can bound the integral as

$$\begin{aligned}&\oint_{\operatorname{dist}(z,\{tv,\ell+tv\})\geqslant 10\ell}\frac{1}{\zeta}\left(\frac{\delta}{|z|}+\int_{|x|\geqslant|z|/2}\frac{\partial_x H(x,t)\mathrm{d}x}{|z-x|}\right)\frac{\delta}{|z|}|\mathrm{d}z|\\ \lesssim&\frac{\delta^2}{\zeta\ell}+\int_{|x|\geqslant 3\ell}\partial_x H(x,t)\int_{6\ell\leqslant|z|\leqslant 2|x|}\frac{\delta}{\zeta|z-x||z|}|\mathrm{d}z|,\end{aligned}\tag{5.54}$$
{{dztv}}

where the inner integral is integrable

$$\int_{6\ell\leqslant|z|\leqslant 2|x|}\frac{\delta}{\zeta|z-x||z|}|\mathrm{d}z|\lesssim\frac{\delta}{\ell}, \tag{5.55}$$ {{e:inner}}

and it follows from combining (5.54) and (5.55) {e:inner}

$$\oint_{\mathrm{dist}(z,\{tv,\ell+tv\})\geqslant 3\ell}\frac{1}{\zeta}\left(\frac{\delta}{|z|}+\int_{|x|\geqslant|z|/2}\frac{\partial_x H(x,t)\mathrm{d}x}{|z-x|}\right)\frac{\delta}{|z|}|\mathrm{d}z|\lesssim\frac{\delta^2}{\ell^2} \tag{5.56}$$ {{e:case3}}

The claim (5.30) follows from combining (5.52), (5.53) and (5.56). {e:case3}

□

5.4 Limit Shape around a regular profile

In this section we study the Markov process

$$\mathbb{P}^g(\mathbf{x}(\mathrm{t}+1)=\mathbf{x}+\boldsymbol{e}|\mathbf{x}(\mathrm{t})=\mathbf{x})\propto\frac{V(\mathbf{x}+\theta\boldsymbol{e})}{V(\mathbf{x})}\prod_{i=1}^{N}e^{e_i g_{\mathrm{t}}(\mathrm{x}_i/N)}, \tag{5.57}$$ {{e:Mk2}}

where g_t is analytic in a small neighborhood of $[-A,A]$

{a:Mk}

Assumption 5.6. *We assume the Markov process (5.57) satisfies* {e:Mk2}

1. *There exists a large $A>0$, such that $\mathrm{supp}(\boldsymbol{x}(t))\subset[-A,A]$.*
2. *There exists a height function $H^*(x,t)\in\mathrm{Adm}(\mathfrak{R},h)$ such that for $x\in\mathbb{R}$, and a complex slope $f_t(x)\in\mathbb{H}^-$ which satisfies*

$$\begin{aligned}-\arg f_t(x)&=\pi\partial_x H^*(x,t)\\ \arg(1+f_t(x))&=\pi\partial_t H^*(x,t)\end{aligned} \tag{5.58}$$

3. *There exists a neighborhood Λ of $[-A,A]$, such that $f_t(x)$ extends continuously to $\Lambda\cap\overline{\mathbb{H}}$ as $f_t(z)=e^{m_t(z)+g_t(z)}$, where $m_t(z)$ is the Stieltjes transform of $\varrho_t(x)=\partial_x H^*(x,t)$,*

$$m_t(z)=\int_{\mathbb{R}}\frac{\varrho_t(x)\mathrm{d}x}{x-z}, \tag{5.59}$$

and $g_t(z)$ is analytic on Λ, and $g_t(\overline{z})=\overline{g_t(z)}$. analytically to $\Lambda\setminus[-A,A]$ is analytic in a small neighborhood of $[-A,A]$.

4. *We assume that $\ln(1+f_t(z))$ is well defined on $\Lambda\setminus[-A,A]$ and is uniformly bounded away from $[-A,A]$.*

{t:limitshape}

Theorem 5.7. *Adopt Assumption 5.6. We consider Markov process $\{\boldsymbol{x}(t)\}_{0\leqslant t\leqslant\ell}$, and we define the empirical height function*

$$H(x,t)=\int_{-\infty}^{x}\varrho_t(s;\boldsymbol{x}(t))\mathrm{d}x, 0\leqslant t\leqslant T. \tag{5.60}$$

For any $\varepsilon>0$, there exists a small $\delta=\delta(\varepsilon)>0$, the following holds, If $\|H(x,0)-H^*(x,0)\|_\infty\leqslant\delta(\varepsilon)$, then

$$\sup_t\|H(x,t)-H^*(x,t)\|_\infty\leqslant\varepsilon, \tag{5.61}$$

with probability $1-\varepsilon$ for N large enough.

5.5 Complex Burger Type Equation

From the Assumptions in (5.6), we have

$$\partial_t m_t(w) = \partial_t \int \frac{\varrho_t(s)\mathrm{d}s}{w-s} = \frac{1}{2\pi \mathrm{i}} \oint_{\omega_-} \frac{\ln(1+f_t(z))}{(w-z)^2}\mathrm{d}z, \tag{5.62}$$ {{e:Burger1}}

where the contour $\omega_- \in \Lambda$ encloses $[-A, A]$, but not w. We can deform the contour ω_- to enclose w, and

$$\begin{aligned}\partial_t m_t(w) = \partial_t \int \frac{\varrho_t(s)\mathrm{d}s}{w-s} &= -\partial_w \ln(1+f_t(w)) + \frac{1}{2\pi \mathrm{i}} \oint_{\omega} \frac{\ln(1+f_t(z))}{(w-z)^2}\mathrm{d}z \\ &= -\frac{(\partial_w m_t(w) + \partial_w g_t(w))f_t(w)}{1+f_t(w)} + \frac{1}{2\pi \mathrm{i}} \oint_{\omega} \frac{\ln(1+f_t(z))}{(w-z)^2}\mathrm{d}z \\ &= -\frac{\partial_w m_t(w) f_t(w)}{1+f_t(w)} + \frac{1}{2\pi \mathrm{i}} \oint_{\omega} \frac{\ln(1+f_t(z))}{(w-z)^2}\mathrm{d}z.\end{aligned} \tag{5.63}$$ {{e:Burger2}}

where the contour $\omega \in \Lambda$ encloses $[-A, A]$ and w.

5.6 Uniqueness of Solution and Characteristic Flow

In this section we prove the uniqueness of the solutions of the equation (5.62). {e:Burger1} Assume that there is another solution $\widehat{f}_t(z) = e^{\widehat{m}_t(z)+g_t(z)}$ satisfying (5.62)

$$\partial_t \widehat{m}_t(w) = \partial_t \int \frac{\widehat{\varrho}_t(s)\mathrm{d}s}{w-s} = \frac{1}{2\pi \mathrm{i}} \oint_{\omega_-} \frac{\ln(1+\widehat{f}_t(z))}{(w-z)^2}\mathrm{d}z, \tag{5.64}$$ {{e:Burger2}}

where $\widehat{m}_t(z)$ is the Stieltjes transform of $\widehat{\varrho}_t(s)$ supported inside $[-A, A]$.

{p:unique}

Proposition 5.8. *If $\widehat{m}_0(z) = m_0(z)$ for any $z \in \mathbb{C} \setminus [-A, A]$, then $\widehat{m}_t(z) = m_t(z)$ for any $t \geqslant 0$ and $z \in \mathbb{C} \setminus [-A, A]$.*

Local uniqueness implies the global uniqueness. We only need to show that the statement of Proposition 5.8 holds for $t \in [0, \delta]$ for some small δ. For the equation (5.62), {e:Burger1} we define the characteristic flow

$$\partial_t z_t = \frac{f_t(z_t)}{f_t(z_t)+1}. \tag{5.65}$$ {{e:ccff}}

By plugging the characteristic flow (5.65) {e:ccff} into (5.64), we have

$$\partial_t m_t(z_t) = -\frac{\partial_z g_t(z_t) f_t(z_t)}{1+f_t(z_t)} + \frac{1}{2\pi \mathrm{i}} \oint_{\omega} \frac{\ln(1+f_t(w))}{(z_t-w)^2}\mathrm{d}w. \tag{5.66}$$ {{e:mtzt}}

We take a contour $\mathcal{C}_0 \in \Lambda$ surrounding the interval $[-A, A]$, such that the characteristic flow maps $z_0 \in \mathcal{C}_0$ to $z_t \in \mathcal{C}_t$ for $t \geqslant 0$. For $t \leqslant \delta$, by the continuity of the characteristic flow, we have that $\mathcal{C}_t$ is inside an annulars region S, see Figure 3

Claim 5.9. *There exists some constant $0 < c < 1$, the following holds uniformly for $z \in S$ and $0 \leqslant t \leqslant \delta$,*

$$|\partial_z \widetilde{m}_t(z)| \leqslant 1/c, \quad |\partial_z g_t(z)| \leqslant 1/c, \quad |f_t(z)| \leqslant 1/c, \quad |1+f_t(z)| \geqslant c. \tag{5.67}$$ {{e:fbound}}

Denote the control parameter

$$\Gamma_t := \sup_{z \in \mathcal{C}_t} |\widetilde{m}_t(z) - m_t(z)| \tag{5.68}$$

In the following we assume $\Gamma_t < c^2/4$.

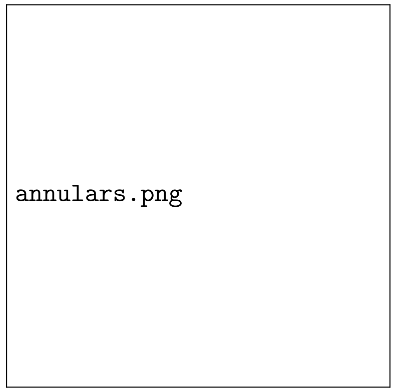

Figure 3: Shown above is the contour $\mathcal{C}_t$ inside the annulars region S. {f:annulars}

We notice that $\lim_{z\to\infty} |\widetilde{m}_t(z) - m_t(z)| = 0$. By maximal principle, $|\widetilde{m}_t(z) - m_t(z)| \leqslant \Gamma_t$ for any z outside the contour $\mathcal{C}_t$.

$$|\widetilde{f}_t(z) - f_t(z)| = |(e^{\widetilde{m}_t(z)-m_t(z)} - 1)f_t(z)| \leqslant (e^{\Gamma_t} - 1)|f_t(z)| \leqslant 2\Gamma_t/c \leqslant c/2, \tag{5.69}$$

where in the last two inequalities we used (5.67). It follows that $|1+\widetilde{f}_t(z)| \geqslant |1+f_t(z)| - |f_t(z) - \widetilde{f}_t(z)| \geqslant c/2$, and

$$\left|\frac{f_t(z)}{1+f_t(z)} - \frac{\widetilde{f}_t(z)}{1+\widetilde{f}_t(z)}\right| = \frac{|f_t(z) - \widetilde{f}_t(z)|}{|1+f_t(z))(1+\widetilde{f}_t(z))|} \leqslant \frac{2\Gamma_t/c}{(c/2)c} = \frac{4\Gamma_t}{c^3}. \tag{5.70}$$ {{e:fdiff2}}

And finally, we also have that

$$|\ln(1+\widetilde{f}_t(z)) - \ln(1+f_t(z))| = \left|1 + \frac{\widetilde{f}_t(z) - f_t(z)}{1+f_t(z)}\right| \leqslant \frac{|\widetilde{f}_t(z) - f_t(z)|}{|1+f_t(z)|} \leqslant \frac{2\Gamma_t/c}{c} = \frac{2\Gamma_t}{c^2} \tag{5.71}$$ {{e:fdiff3}}

By plugging the characteristic flow (5.65) into (5.64) {e:fdiff3}, we have

$$\partial_t \widetilde{m}_t(z_t) = -\partial_z \widetilde{m}_t(z_t)\left(\frac{\widetilde{f}_t(z_t)}{1+\widetilde{f}_t(z_t)} - \frac{f_t(z_t)}{1+f_t(z_t)}\right) - \frac{\partial_z g_t(z_t)\widetilde{f}_t(z_t)}{1+\widetilde{f}_t(z_t)} + \frac{1}{2\pi \mathrm{i}}\oint_\omega \frac{\ln(1+\widetilde{f}_t(w))}{(z_t - w)^2}\mathrm{d}w, \tag{5.72}$$ {{e:tmtzt}}

where the the contour is bounded away from z_t, namely $|w - z_t| \geqslant c$. Moreover, the length of the contour is bounded by $10A$.

By taking the difference of (5.66) and (5.72) {e:tmtzt}, we have

$$\partial_t(\widetilde{m}_t(z_t) - m_t(z_t)) = -(\partial_z \widetilde{m}_t(z_t) + \partial_z g_t(z_t))\left(\frac{\widetilde{f}_t(z_t)}{1+\widetilde{f}_t(z_t)} - \frac{f_t(z_t)}{1+f_t(z_t)}\right) \tag{5.73}$$ {{e:mdiff1}}

$$+ \frac{1}{2\pi \mathrm{i}}\oint_\omega \frac{\ln(1+\widetilde{f}_t(w)) - \ln(1+f_t(w))}{(z_t - w)^2}\mathrm{d}w. \tag{5.74}$$ {e:mdiff1}

Thanks to (5.67), (5.70) and (5.71), the righthand side of (5.73) is bounded by

$$\left|(\partial_z \widetilde{m}_t(z_t) + \partial_z g_t(z_t))\left(\frac{\widetilde{f}_t(z_t)}{1+\widetilde{f}_t(z_t)} - \frac{f_t(z_t)}{1+f_t(z_t)}\right)\right| \leqslant \frac{2}{c}\frac{2\Gamma_t}{c^2} = \frac{4\Gamma_t}{c^3}. \tag{5.75}$$

and

$$\left| \frac{1}{2\pi \mathrm{i}} \oint_\omega \frac{\ln(1+\widetilde{f}_t(w)) - \ln(1+f_t(w))}{(z_t - w)^2} \mathrm{d}w \right| \leqslant \frac{2\Gamma_t}{c^2} \frac{1}{2\pi} \oint_\omega \frac{|\mathrm{d}w|}{|z_t - w|^2} \leqslant \frac{2\Gamma_t}{c^2} \frac{1}{2\pi} \frac{10A}{c^2} = \frac{10A\Gamma_t}{\pi c^4}. \tag{5.76}$$

It then follows that

$$\partial_t \Gamma_t \leqslant \frac{4\Gamma_t}{c^3} + \frac{10A\Gamma_t}{\pi c^4} = \left(\frac{4}{c^3} + \frac{10A}{\pi c^4} \right) \Gamma_t. \tag{5.77}$$

Since $\Gamma_0 = 0$, Grönwell inequality implies that $\Gamma_t = 0$. Since $\widetilde{m}_t(z) - m_t(z)$ is analytic outside $[-A, A]$, it follows that $\widetilde{m}_t(z) = m_t(z)$ and the solution is unique.

5.7 Stochastic Differential Equation

We denote the empirical particle density $\widetilde{\rho}_t(x) = \varrho(x; \boldsymbol{x}(t))$ and its Stieltjes transform $\widetilde{m}_t(z)$, and the complex slope

$$\widetilde{\rho}_t(x) = \sum_{i=1}^n \mathbf{1}([x_i(t), x_i(t) + \theta/N])\mathrm{d}x, \quad \widetilde{m}_t(z) = \int \frac{\widetilde{\rho}_t(x)}{z - x} \mathrm{d}x, \quad \widetilde{f}_t(z) = e^{\widetilde{m}_t(z) + g_t(z)} \quad 0 \leqslant t \leqslant \ell. \tag{5.78}$$ {{e:rhomt}}

Transition probability (A.2) with parameters {e:rhomt} satisfying Assumptions A.1 and A.4 for all small enough ε. Thanks to Theorem A.5, the time difference of the Stieltjes transform for the empirical particle density (5.78) satisfy for for any $z \in \Lambda \setminus [-\boldsymbol{l}, \boldsymbol{r}]$, and $\mathsf{t} \in [0, \mathsf{L}]$, we have as $\varepsilon \to 0$:

$$N \left(\widetilde{m}_{t+1/N}(z) - \widetilde{m}_t(z)) \right) = \Delta \mathcal{M}_t(z) + \frac{1}{2\pi \mathrm{i}\theta} \oint_{\omega_-} \frac{\ln(1 + \widetilde{f}_t(z))\mathrm{d}w}{(w-z)^2} + \mathrm{O}\left(\varepsilon\right), \quad t = \mathsf{t}/N \tag{5.79}$$ {{e:dmg2}}

Moreover, $\Delta\mathcal{M}(z)$ are mean 0 random variables such that $\{\varepsilon^{-1/2}\Delta\mathcal{M}(z)\}_{z \in \Lambda \setminus [\boldsymbol{l}, \boldsymbol{r}]}$ are asymptotically Gaussian with covariance given by

$$N\mathbb{E}\left[\Delta\mathcal{M}(z_1), \Delta\mathcal{M}(z_2)\right] = \frac{1}{2\pi \mathrm{i}\theta} \oint_{\omega_-} \frac{\widetilde{f}_t(z)}{1 + \widetilde{f}_t(z)} \frac{1}{(w - z_1)^2} \frac{1}{(w - z_2)^2} \mathrm{d}w + \mathrm{o}(1), \tag{5.80}$$ {{e:covT}}

where the contour $\omega_- \subset \Lambda$ encloses $[-3\ell, 3\ell]$, but not z_1, z_2. We also have that the higher order joint moments of $\{\sqrt{N}\Delta\mathcal{M}(z)\}_{z \in \Lambda \setminus [\boldsymbol{l}, \boldsymbol{r}]}$ converge as $\varepsilon \to 0$ to the Gaussian joint moments.

5.8 Proof of Theorem 5.7

{Section_proof_

Let $C([0, \ell], \mathscr{M}_1(\mathbb{R})])$ denote the space of continuous functions from $[0, \ell]$ to the set of probability measures on $\mathbb{R}$ equipped with the weak topology of measures. Then $\rho(\cdot; \boldsymbol{x}(t))$ represents a random element of this space. In fact, there is a compact set $\mathcal{K} \subset C([0, \ell], \mathscr{M}_1(\mathbb{R})])$, such that the distribution of $\rho(\cdot; \boldsymbol{x}(t))_{0 \leqslant t \leqslant \ell}$ is supported on $\mathcal{K}$ — this is because measures $\rho(\cdot; \boldsymbol{x}(t))$ are supported inside $[\boldsymbol{l}, \boldsymbol{r}]$ and the dependence on t is Lipschitz (each component of $\boldsymbol{x}(t)$ jumps at most by $1/N$ when t grows by $1/N$), cf. [3, Lemma 4.3.13].

Since the space of probability measures on a compact set is compact, we conclude that the stochastic processes $\rho(\cdot; \boldsymbol{x}(t))_{0 \leqslant t \leqslant \ell}$ have subsequential limits in distribution as $N \to \infty$. We let $(\rho_t)_{0 \leqslant t \leqslant \ell}$ be one of the limiting points. Our task is to show that ρ_t are as described in Theorem 5.7 (Implying, in particular, that all the limiting points are the same.) Note that for each $0 \leqslant t \leqslant \ell$, ρ_t is an absolutely continuous measures of density at most 1, because so were the prelimit measures.

For any $t \in [\![0, \mathsf{L}]\!]/N$, using (A.9), we have {e:covT}

$$\widetilde{m}_t(z) = \widetilde{m}_0(z) + \frac{1}{N}\widetilde{\mathcal{M}}_t(z) + \sum_{\tau \in N^{-1}[\![0, \mathsf{t}-1]\!]} \frac{1}{2\pi \mathrm{i}\theta N} \oint_{\omega_-} \frac{\ln(1 + \widetilde{f}_t(z))\mathrm{d}w}{(w-z)^2} + \mathrm{O}\left(\frac{1}{N}\right), \tag{5.81}$$ {{e:martingale_

where $\widetilde{\mathcal{M}}_t(z)$ is a martingale given by

$$\widetilde{\mathcal{M}}_t(z) := \sum_{\tau \in N^{-1}[\![0, \mathsf{t}-1]\!]} \Delta \mathcal{M}_\tau(z). \tag{5.82}$$ {{e:deftMt}}

We can estimate the $\widetilde{\mathcal{M}}_t(z)$ term by Doob/Kolmogorov's inequality for martingales:

$$\operatorname{Prob}\left(\sup_{t \in N^{-1}[\![0, \mathsf{L}]\!]} |\widetilde{\mathcal{M}}_t(z)/N| > \lambda \right) \leqslant \frac{1}{\lambda^2 N^2} \sum_{\tau \in N^{-1}[\![0, \mathsf{T}-1]\!]} \mathbb{E}[\Delta \mathcal{M}_\tau(z)]^2, \qquad \lambda > 0.$$

Using (A.10), each term {e:deftMt} in the last sum is $\mathrm{O}(1)$. There are $\mathrm{O}(N)$ terms and therefore the probability decays as $\mathrm{O}(1/N\lambda^2)$. We conclude that the martingale part in (5.81) goes to 0 in probability as $N \to \infty$. Hence, $N \to \infty$ limit of (5.81) gives an integral equation for ρ_t:

$$\widehat{m}_t(z) = m_0(z) + \int_0^t \frac{1}{2\pi \mathrm{i}} \oint_{\omega_-} \frac{\ln(1 + \widehat{f}_s(z)) \mathrm{d}w}{(w-z)^2} \mathrm{d}s \tag{5.83}$$ {{e:LLN_integra

where $\widehat{f}_s(z) = e^{\widehat{m}_s(z) + g_s(z)}$ is the limit of $\widetilde{f}_s(z)$. We arrive at the same partial differential equation as (5.64). {e:LLN_integral_equation}

6 Asymptotics for Jack Polynomials

{s:Jprocess}

In this section we explain the correspondence between non-intersecting θ-Bernoulli walk ensembles and certain Jack ascending process (6.17). This correspondence will be used to derive large deviation asymptotics for (skew) Jack polynomials. We collect some basic properties of Jack symmetric functions in Section 6.1 and Section 6.2. We recall the Jack ascending process in Section 6.3. Our main references are [41] and [53]. In Section 6.4, we prove our main result Theorem 1.6.

6.1 Young Diagrams and Symmetric Functions

{s:YoungD}

Given a Young diagram $\boldsymbol{\lambda}$, a box $\square \in \boldsymbol{\lambda}$ is a pair of integers,

$$\square = (i,j) \in \boldsymbol{\lambda}, \text{ if and only if } 1 \leqslant i \leqslant \ell(\boldsymbol{\lambda}), 1 \leqslant j \leqslant \lambda_i.$$

We denote $\boldsymbol{\lambda}'$ the transposed diagram of $\boldsymbol{\lambda}$, defined by

$$\lambda'_j = |\{i : 1 \leqslant j \leqslant \lambda_i\}|, \quad 1 \leqslant j \leqslant \lambda_1.$$

For a box $\square = (i,j) \in \boldsymbol{\lambda}$, its arm $a_\square$ and leg $l_\square$ are

$$a_\square = \lambda_i - j, \quad l_\square = \lambda'_j - i.$$

Let $\boldsymbol{x} = (x_1, x_2, x_3, \cdots)$ be an infinite set of indeterminates, and Sym the graded algebra of symmetric functions in infinitely many variables over $\mathbb{R}$. We use the following notations for certain symmetric functions indexed by partitions $\boldsymbol{\lambda}$:

- We denote the monomial symmetric functions $m_{\boldsymbol{\lambda}}(\boldsymbol{x})$, which is defined as the sum of all monomials $\boldsymbol{x}^{\boldsymbol{\lambda}_\sigma}$ where $\boldsymbol{\lambda}_\sigma$ ranges over all distinct permutations of $\boldsymbol{\lambda}$.
- We denote the power sum symmetric functions $p_{\boldsymbol{\lambda}}(\boldsymbol{x})$, as

$$p_{\boldsymbol{\lambda}}(\boldsymbol{x}) = \prod_{1 \leqslant i \leqslant \ell(\boldsymbol{\lambda})} p_{\lambda_i}(\boldsymbol{x}), \quad p_n(\boldsymbol{x}) = x_1^n + x_2^n + x_3^n + \cdots.$$

Both the set of monomial symmetric functions $\{m_{\boldsymbol{\lambda}}(\boldsymbol{x}) : \boldsymbol{\lambda} \in \mathbb{Y}\}$, and the set of power sum symmetric functions $\{p_{\boldsymbol{\lambda}}(\boldsymbol{x}) : \boldsymbol{\lambda} \in \mathbb{Y}\}$ form homogeneous bases of Sym.

6.2 Jack Symmetric Polynomials

{s:Jack}

Let $\theta > 0$ be a parameter (indeterminante), and let $\mathbb{Q}(\theta)$ denote the field of all rational functions of θ with rational coefficients. Define a scalar product $\langle \cdot, \cdot \rangle$ on the vector space $\mathrm{Sym} \otimes \mathbb{Q}(\theta)$ of all symmetric functions over the field $\mathbb{Q}(\theta)$ by the condition

$$\langle p_{\boldsymbol{\lambda}}(\boldsymbol{x}), p_{\boldsymbol{\mu}}(\boldsymbol{x}) \rangle = \delta_{\boldsymbol{\lambda}\boldsymbol{\mu}} z_{\boldsymbol{\lambda}} \theta^{-\ell(\boldsymbol{\lambda})}. \tag{6.1}$$ {{e:scalarP}}

We recall that a partial order on Young diagrams is obtained by declaring $\boldsymbol{\lambda} \preceq \boldsymbol{\mu}$ if $|\boldsymbol{\lambda}| = |\boldsymbol{\mu}|$ and $\lambda_1 + \lambda_2 + \cdots + \lambda_i \leqslant \mu_1 + \mu_2 + \cdots + \mu_i$ for all i. The following fundamental result is due to Macdonald, which characterizes Jack symmetric functions.

Let $\theta > 0$ be a parameter (indeterminante). Jack symmetric functions $J_{\boldsymbol{\lambda}}(\boldsymbol{x}; \theta)$ are elements of the algebra Sym of the symmetric functions in infinitely many variables $(x_i)_{i=1}^{\infty}$ uniquely determined by the following two properties:

1. $J_{\boldsymbol{\lambda}}$, $|\boldsymbol{\lambda}| = m$, can be expressed in terms of the monomial symmetric functions via a strictly upper unitriangular transition matrix:

$$J_{\boldsymbol{\lambda}} = m_{\boldsymbol{\lambda}} + \sum_{\boldsymbol{\mu} < \boldsymbol{\lambda} \in \mathbb{Y}_m} R_{\boldsymbol{\lambda}\boldsymbol{\mu}} m_{\boldsymbol{\mu}},$$

   where $R_{\boldsymbol{\lambda}\boldsymbol{\mu}}$ are functions of θ and $\boldsymbol{\mu} < \boldsymbol{\lambda}$ is comparison in the dominance order on the set $\mathbb{Y}_m$ of all partitions of m (equivalently, Young diagrams with m boxes).

2. They are pairwise orthogonal with respect to the scalar product defined on the power sums via

$$\langle p_{\boldsymbol{\lambda}}, p_{\boldsymbol{\mu}} \rangle_\theta = \delta_{\boldsymbol{\lambda}\boldsymbol{\mu}} z_{\boldsymbol{\lambda}} \theta^{-\ell(\boldsymbol{\lambda})}, \qquad z_{\boldsymbol{\lambda}} = \prod_{1 \leqslant i \leqslant \lambda_1} i^{m_i(\boldsymbol{\lambda})} \prod_{1 \leqslant i \leqslant \lambda_1} m_i(\boldsymbol{\lambda})!, \tag{6.2}$$ {{e:scalarP}}

   where $\boldsymbol{\lambda} = 1^{m_1} 2^{m_2} \ldots$, i.e. m_i is the multiplicity of i in $\boldsymbol{\lambda}$, $\ell(\boldsymbol{\lambda})$ is the number of rows, and $p_k = (x_1)^k + (x_2)^k + \ldots$, $k \geqslant 1$.

We recall the scalar product on Sym as defined in (6.2), and introduce the following notation, {e:scalarP}

$$j_{\boldsymbol{\lambda}}(\theta) = \langle J_{\boldsymbol{\lambda}}(\boldsymbol{x}; \theta), J_{\boldsymbol{\lambda}}(\boldsymbol{x}; \theta) \rangle. \tag{6.3}$$ {{e:jl}}

The number $j_{\boldsymbol{\lambda}}(\theta)$ is explicitly given by

$$j_{\boldsymbol{\lambda}} = \prod_{\square \in \boldsymbol{\lambda}} \frac{a_\square + \theta l_\square + 1}{a_\square + \theta l_\square + \theta}. \tag{6.4}$$ {{def:jl2}}

See [53, Theorem 5.8] for a proof. We denote the dual polynomial $\widetilde{J}_{\boldsymbol{\lambda}}(\boldsymbol{x}; \theta) = J_{\boldsymbol{\lambda}}(\boldsymbol{x}; \theta)/j_{\boldsymbol{\lambda}}(\theta)$. Finally, the skew Jack polynomials $J_{\boldsymbol{\lambda}/\boldsymbol{\mu}}, \widetilde{J}_{\boldsymbol{\lambda}/\boldsymbol{\mu}}$ are defined through the expansions:

$$J_{\boldsymbol{\lambda}}(x_1, x_2, \ldots, y_1, y_2, \ldots; \theta) = \sum_{\boldsymbol{\mu}} J_{\boldsymbol{\lambda}/\boldsymbol{\mu}}(x_1, x_1, \ldots; \theta) J_{\boldsymbol{\mu}}(y_1, y_2, \ldots; \theta),$$
$$\widetilde{J}_{\boldsymbol{\lambda}}(x_1, x_2, \ldots, y_1, y_2, \ldots; \theta) = \sum_{\boldsymbol{\mu}} \widetilde{J}_{\boldsymbol{\lambda}/\boldsymbol{\mu}}(x_1, x_1, \ldots; \theta) \widetilde{J}_{\boldsymbol{\mu}}(y_1, y_2, \ldots; \theta).$$

{t:J1N}

Theorem 6.1. *We have*

$$J_{\boldsymbol{\lambda}}(1^N; \theta) = \prod_{\square \in \boldsymbol{\lambda}} \frac{N\theta + (j-1) - \theta(i-1)}{a_\square + \theta l_\square + \theta}. \tag{6.5}$$ {{e:Jlambda}}

Proof. See [53, Proposition 2.3]. □

The following lemma gives the asymptotics of the Jack symmetric polynomials at the principal specialization

Lemma 6.2. *Given a Young diagram $\boldsymbol{\lambda} \in \mathbb{Y}_N$, we identify it as a particle configuration*

$$\mathsf{x} = (\mathsf{x}_1, \mathsf{x}_2, \dots, \mathsf{x}_N) \in \mathbb{W}_\theta^N, \quad \mathsf{x}_i = \lambda_i - \theta(i-1) \quad 1 \leqslant i \leqslant N, \tag{6.6}$$ {{e:defxfromla}}

then

$$J_{\boldsymbol{\lambda}}(1^N; \theta) = \prod_{i<j} \frac{\Gamma(\mathsf{x}_i - \mathsf{x}_j + \theta)}{\Gamma(\mathsf{x}_i - \mathsf{x}_j)} \prod_{i=1}^{N} \frac{\Gamma(\theta)}{\Gamma(i\theta)}. \tag{6.7}$$ {{e:Jlambda2}}

Given a sequence of Young diagrams

$$\boldsymbol{\lambda}^{(N)} = (\lambda_1^{(N)} \geqslant \lambda_2^{(N)} \geqslant \cdots \geqslant \lambda_N^{(N)}) \in \mathbb{Y}_N, \quad N \geqslant 1 \tag{6.8}$$

such that

1. *There exists a constant $C > 0$, $\lambda_1^{(N)} \leqslant CN$*
2. *There exists a 1-Lipschitz nondecreasing function $h : \mathbb{R} \mapsto [0, \theta]$, and when $N \to \infty$*

$$\frac{\theta}{N} \sum_{i=1}^{N} \delta\left(\frac{\lambda_i^{(N)} - (i-1)\theta}{N} \right) \to \partial_x h(x), \tag{6.9}$$

in distribution. Then

$$\lim_{N\to\infty} \frac{1}{N^2} \ln J_{\boldsymbol{\lambda}^{(N)}}(1^N; \theta) = \frac{1}{2\theta} \int_{\mathbb{R}^2} \ln|x-y| \mathrm{d}h(x)\mathrm{d}h(y) - \frac{\theta \ln(\theta)}{2} + \frac{3\theta}{4}. \tag{6.10}$$ {{e:NlimitJ1}}

Proof. We can reorganize the numerator and denominator in (6.5), and rewrite it {e:NlimitJ1} in terms of the particle configuration x from (C.9).

$$\prod_{\square \in \boldsymbol{\lambda}} N\theta + (j-1) - \theta(i-1) = \prod_{i=1}^{N} \frac{\Gamma(N\theta + \mathsf{x}_i)}{\Gamma((N-i+1)\theta)} = \prod_{i=1}^{N} \frac{\Gamma(N\theta + \mathsf{x}_i)}{\Gamma(i\theta)} \tag{6.11}$$

and

$$\prod_{\square \in \boldsymbol{\lambda}} \frac{1}{a_\square + \theta l_\square + \theta} = \prod_{i \leqslant j} \frac{\Gamma(\lambda_i - \lambda_j + \theta(j-i) + \theta)}{\Gamma(\lambda_i - \lambda_{j+1} + \theta(j-i) + \theta)} = \Gamma(\theta)^N \prod_i \frac{1}{\Gamma(\mathsf{x}_i + N\theta)} \prod_{i<j} \frac{\Gamma(\mathsf{x}_i - \mathsf{x}_j + \theta)}{\Gamma(\mathsf{x}_i - \mathsf{x}_j)} \tag{6.12}$$

The claim (6.7) follows from combining the above two expressions.

For the asymptotics (6.10), simply write $\lambda_i^{(N)} - (i-1)\theta = \mathsf{x}_i$. We notice that there exists $C = C(\theta) > 0$, such that the log Gamma function satisfies $z - 1/2) \ln z - z - C \leqslant \ln \Gamma(z) \leqslant (z - 1/2) \ln z - z + C$ for $z \geqslant \theta$, and $|\partial_z \ln \Gamma(z) - \ln z| \leqslant C/z$ for $z \geqslant \theta$.

So we have

$$\begin{aligned}
\frac{1}{N^2} \sum_{i=1}^{N} (\ln \Gamma(\theta) - \ln \Gamma(i\theta)) &= -\frac{1}{N^2} \sum_{i=1}^{N} i\theta \ln(i\theta/e) + \mathrm{O}(N) \\
&= -\frac{\theta \ln N}{2} - \theta \int_0^1 x \ln x \mathrm{d}x - \frac{\theta}{2} \ln(\theta/e) + \mathrm{O}(N \ln N) \\
&= -\frac{\theta \ln N}{2} - \frac{\theta \ln(\theta)}{2} + \frac{3\theta}{4} + \mathrm{O}(N \ln N)
\end{aligned} \tag{6.13}$$ {{e:Jlimitt1}}

and

$$\sum_{i<j} \ln \frac{\Gamma(\mathsf{x}_i - \mathsf{x}_j + \theta)}{\Gamma(\mathsf{x}_i - \mathsf{x}_j)} = \sum_{i<j} \theta \ln(\mathsf{x}_i - \mathsf{x}_j) + \mathrm{O}\left(\sum_{i<j} \frac{1}{\mathsf{x}_i - \mathsf{x}_j}\right)$$
$$= \sum_{i<j} \theta \ln(\mathsf{x}_i - \mathsf{x}_j) + \mathrm{O}\left(N \ln N\right) = \frac{\theta \ln N}{2} + \frac{1}{2\theta} \int \ln|x-y| \mathrm{d}h(x)\mathrm{d}h(y) + \mathrm{O}(N \ln N). \tag{6.14}$$
{{e:Jlimitt2}}

where we used that $\mathsf{x}_i - \mathsf{x}_j \geqslant \theta(i-j)$ for the second inequality and in the last equality we used Lemma 2.8. The claim (6.10) follows from plugging (6.13) and (6.14) into (6.7). {e:J3limit2} □

6.3 Jack Process

{s:Jackprocess}

Computations in the algebra of symmetric functions Sym can be converted into numeric identities by means of *specializations*, which are algebra homomorphism from Sym to the set of complex numbers. A specialization ρ is uniquely determined by its values on any set of algebraic generators of Sym and we use $(p_k)_{k=1}^{\infty}$ as such generators. The value of ρ on a symmetric function f is denoted $f(\rho)$. Given two specializations ρ, ρ', we define their union (ρ, ρ') through the formula:

$$p_k(\rho, \rho') = p_k(\rho) + p_k(\rho'), \quad k \geqslant 1.$$

A specialization ρ is an algebra homomorphism from Sym to the set of complex numbers. A specialization ρ is called *Jack-positive* if its values on all (skew) Jack symmetric functions with a fixed θ are real and non-negative, i.e.,

$$J_{\boldsymbol{\lambda}}(\rho; \theta) = \rho(J_{\boldsymbol{\lambda}}(\boldsymbol{x}; \theta)) \geqslant 0,$$

for all $\boldsymbol{\lambda} \in \mathbb{Y}$. The set of Jack positive specializations are classified by Kerov, Okounkov and Olshanski in [37, Theorem A] (We slightly modify the statement (our β_i is $\theta\beta_i$ in [37, Theorem A]) such that it matches with Theorem C.1).

{t:Jackspeciali

Theorem 6.3. *For any fixed $\theta > 0$, Jack positive specializations can be parameterized by triplets $(\boldsymbol{\alpha}, \boldsymbol{\beta}, \gamma)$, where $\boldsymbol{\alpha}, \boldsymbol{\beta}$ are sequences of real numbers with*

$$\alpha_1 \geqslant \alpha_2 \geqslant \cdots \geqslant 0, \quad \beta_1 \geqslant \beta_2 \geqslant \cdots \geqslant 0, \quad \sum_i \alpha_i + \frac{\beta_i}{\theta} < \infty,$$

and γ is a non-negative real number. The specialization corresponding to a triplet $(\boldsymbol{\alpha}, \boldsymbol{\beta}, \gamma)$ is given by its values on Newton power sums p_k,

$$p_1(\rho) = \sum_{i=1}^{\infty} \alpha_i + \frac{1}{\theta}\left(\gamma + \sum_{i=1}^{\infty} \beta_i\right), \qquad p_k(\rho) = \sum_{i=1}^{\infty} (\alpha_i)^k + \frac{(-1)^{k-1}}{\theta} \sum_{i=1}^{\infty} (\beta_i)^k.$$

Following [], we create Markov chains out of the Jack-positive specializations:

{Definition_Jac

Definition 6.4. *Given two specializations ρ and ρ' we define the ascending transition through*

$$\mathbb{P}(\boldsymbol{\lambda} \mid \boldsymbol{\mu}) = \frac{1}{H_\theta(\rho; \rho')} \frac{J_{\boldsymbol{\lambda}}(\rho; \theta)}{J_{\boldsymbol{\mu}}(\rho; \theta)} \widetilde{J}_{\boldsymbol{\lambda}/\boldsymbol{\mu}}(\rho'; \theta), \tag{6.15}$$
{{eq_Jascending

where $\boldsymbol{\lambda}$ and $\boldsymbol{\mu}$ are partitions with $\boldsymbol{\mu} \subset \boldsymbol{\lambda}$ and the normalization constant $H_\theta(\rho; \rho')$ is given by

$$H_\theta(\rho, \rho') = \exp\left\{\sum_{k \geqslant 1} \frac{\theta}{k} p_k(\rho) p_k(\rho')\right\}.$$

We further use the w_β automorphism of the algebra of symmetric functions Sym defined on the power sums by:

$$w_\beta(p_k) = (-1)^{k-1}\beta p_k.$$

As shown in [40, Chapter VI, Section 10, (10.6)],

$$w_{\theta^{-1}}\left(J_{\boldsymbol{\lambda}'/\boldsymbol{\mu}'}(\cdot;\theta)\right) = \widetilde{J}_{\boldsymbol{\lambda}/\boldsymbol{\mu}}(\cdot;\theta^{-1}),$$

where $\boldsymbol{\lambda}'$ and $\boldsymbol{\mu}'$ are transposed Young diagrams $\boldsymbol{\lambda}$ and $\boldsymbol{\mu}$, respectively. Hence, with the specialization $\boldsymbol{\beta} = (b_0, b_1, \cdots, b_{\mathsf{T}-1})$, as in (6.15), we have {eq:Jascending_transition}

$$\widetilde{J}_{\boldsymbol{\lambda}/\boldsymbol{\mu}}(\boldsymbol{\beta} = (b_0, b_1, \cdots, b_{\mathsf{T}-1}); \theta) = J_{\boldsymbol{\lambda}'/\boldsymbol{\mu}'}(\boldsymbol{\alpha} = (b_0, b_1, \cdots, b_{\mathsf{T}-1}); \theta^{-1}).$$

To obtain the asymptotics of skew Jack polynomials, we study the following special Jack ascending process. We fix $N, \mathsf{T} \geqslant 1$, and a sequence of positive numbers $b_0, b_1, b_2, \cdots, b_{\mathsf{T}-1}$. The transition probability at time $0 \leqslant \mathsf{t} \leqslant \mathsf{T}-1$ is given in the notations of Theorem 6.3 as

$$\rho: \ \alpha_i = 1, \ 1 \leqslant i \leqslant N; \qquad \rho': \beta_1 = b_{\mathsf{t}}, \tag{6.16}$$ {{eq_principal_

with all other parameters setting to 0. The transition probability at time $0 \leqslant \mathsf{t} \leqslant \mathsf{T}-1$ is given by

$$\mathbb{P}(\boldsymbol{\lambda}(\mathsf{t}+1) = \boldsymbol{\lambda} | \boldsymbol{\lambda}(\mathsf{t}) = \boldsymbol{\mu}) = \frac{1}{H(1^N, \beta_1 = b_{\mathsf{t}})} \frac{J_{\boldsymbol{\lambda}}(1^N;\theta)}{J_{\boldsymbol{\mu}}(1^N;\theta)} \widetilde{J}_{\boldsymbol{\lambda}/\boldsymbol{\mu}}(\beta_1 = b_{\mathsf{t}};\theta) \tag{6.17}$$ {{e:Jprocess}}

Then the transition probability gives the skew Jack polynomials

$$\mathbb{P}(\boldsymbol{\lambda}(\mathsf{T}) = \boldsymbol{\lambda} | \boldsymbol{\lambda}(0) = \boldsymbol{\mu}) = \frac{1}{H(1^N, \boldsymbol{\beta} = (b_0, \cdots, b_{\mathsf{T}-1}))} \frac{J_{\boldsymbol{\lambda}}(1^N;\theta)}{J_{\boldsymbol{\mu}}(1^N;\theta)} \widetilde{J}_{\boldsymbol{\lambda}/\boldsymbol{\mu}}(\boldsymbol{\beta} = (b_0, \cdots, b_{\mathsf{T}-1});\theta) \tag{6.18}$$ {{e:jackdynamic

The following claim states that we can encode the Jack ascending process 6.17 of Young diagrams as an N-particle non-intersecting θ-Bernoulli rank walk

{c:Jackdensity}

Claim 6.5. *The transition probability of the Jack process 6.17 is non-degenerate only for partitions $\boldsymbol{\lambda}, \boldsymbol{\mu}$ with at most N parts, i.e. $\boldsymbol{\lambda} = (\lambda_1 \geqslant \lambda_2 \geqslant \ldots \geqslant \lambda_N)$ and $\boldsymbol{\mu} = (\mu_1 \geqslant \mu_2 \geqslant \ldots \geqslant \mu_N)$. Further, if we identify*

$$\mathsf{x} = (\mathsf{x}_1, \mathsf{x}_2, \ldots, \mathsf{x}_N) \in \mathbb{W}_\theta^N, \quad \mathsf{x}_i = \mu_i - \theta(i-1), \quad \mathsf{x}_i + e_i = \lambda_i - \theta(i-1), \quad 1 \leqslant i \leqslant N, \tag{6.19}$$ {{e:x1}}

then $e_i \in \{0,1\}$ and the transition probability is

$$\mathbb{P}(\mathsf{x} + \boldsymbol{e} | \mathsf{x}) = \frac{1}{(1+b_{\mathsf{t}})^N} \prod_{1 \leqslant i < j \leqslant N} \frac{(\mathsf{x}_i + \theta e_i) - (\mathsf{x}_j + \theta e_j)}{\mathsf{x}_i - \mathsf{x}_j} b_{\mathsf{t}}^{\sum_{i=1}^N e_i}. \tag{6.20}$$ {{e:drift_dynam

6.4 Proof of Theorem 1.6

{s:proofmain2}

Thanks to the relation (6.18), the large deviation asymptotics of the skew Jack polynomials can be obtained from the large deviation principle of the dynamics (6.20), {e:drift_dynamics}

$$\frac{1}{N^2} \ln \widetilde{J}_{\boldsymbol{\lambda}/\boldsymbol{\mu}}(\boldsymbol{\beta} = (b_0, \cdots, b_{\mathsf{T}-1});\theta) = \frac{1}{N^2} \ln \left(\mathbb{P}(\boldsymbol{\lambda}(\mathsf{T}) = \boldsymbol{\lambda} | \boldsymbol{\lambda}(0) = \boldsymbol{\mu}) H(1^N, \boldsymbol{\beta} = (b_0, \cdots, b_{\mathsf{T}-1}))\right)$$
$$+ \frac{1}{2\theta}\left(\int_{\mathbb{R}^2} \ln|x-y| \mathrm{d}h(x,0)\mathrm{d}h(y,0) - \int_{\mathbb{R}^2} \ln|x-y| \mathrm{d}h(x,T)\mathrm{d}h(y,T)\right) + \mathrm{o}(1), \tag{6.21}$$ {{e:jackdynamic

where we used Equation (C.9). The dynamics (6.20) and the non-intersecting θ-Bernoulli random walks (1.10) differ by the following drift {e:jackdynamics2}

$$D(\boldsymbol{x}) := \prod_{0 \leq \mathsf{t} \leq \mathsf{T}-1} 2^N b_{\mathsf{t}}^{\sum_{i=1}^N e_i(\mathsf{t})}. \tag{6.22}$$

Theorem 1.6 follows from plugging the following lemma to Equation (6.21).

{l:drift}

Lemma 6.6. *Adopt the notations and assumptions of Theorem 1.6, the following holds*

$$\frac{1}{N^2}\ln D(\boldsymbol{x}) = \frac{1}{\theta}\mathcal{F}^f(H(\boldsymbol{x}))T\ln 2 + \mathrm{O}(1/N)$$

with

$$\mathcal{F}^f(H) := -\int_0^T\int_{\mathbb{R}} f(s)\partial_s H(y,s)\mathrm{d}y\mathrm{d}s \tag{6.23}$$ {{defd}}

Moreover, $\mathcal{F}^f$ is continuous on the space of Lipschitz functions with uniform norm.

Proof. We notice that since $e_i(\mathsf{t}) = \mathsf{x}_i(\mathsf{t}+1) - \mathsf{x}_i(\mathsf{t})$ and $b_{\mathsf{t}} = e^{f(\mathsf{t}/N)}$, we have

$$\begin{aligned}
\ln D(\boldsymbol{x}) &= N\mathsf{T}\ln 2 + \sum_{0\le \mathsf{t}\le \mathsf{T}-1} f\left(\frac{\mathsf{t}}{N}\right)\sum_{i=1}^N(\mathsf{x}_i(\mathsf{t}+1) - \mathsf{x}_i(\mathsf{t}))\\
&= N\mathsf{T}\ln 2 + \sum_{1\le \mathsf{t}\le \mathsf{T}} f\left(\frac{\mathsf{t}-1}{N}\right)\sum_{i=1}^N \mathsf{x}_i(\mathsf{t}) - \sum_{0\le \mathsf{t}\le \mathsf{T}-1} f\left(\frac{\mathsf{t}}{N}\right)\sum_{i=1}^N \mathsf{x}_i(\mathsf{t})\\
&= N\mathsf{T}\ln 2 + \sum_{1\le \mathsf{t}\le \mathsf{T}-1}\left(f\left(\frac{\mathsf{t}-1}{N}\right) - f\left(\frac{\mathsf{t}}{N}\right)\right)\sum_{i=1}^N \mathsf{x}_i(\mathsf{t}) + f\left(\frac{\mathsf{T}}{N}\right)\sum_{i=1}^N \mathsf{x}_i(\mathsf{T}) - f(0)\sum_{i=1}^N \mathsf{x}_i(0)\\
&= N\mathsf{T}\ln 2 + \int_0^T \partial_s f(s)\sum_{i=1}^N \mathsf{x}_i(\lfloor \mathsf{N}s\rfloor)\mathrm{d}s + f(T)\sum_{i=1}^N \mathsf{x}_i(\mathsf{T}) - f(0)\sum_{i=1}^N \mathsf{x}_i(0)
\end{aligned} \tag{6.24}$$

On the other hand, for $t\in[\mathsf{t}/N, (\mathsf{t}+1)/N]$, we have set $x(t) = \mathsf{x}(\mathsf{t})/N + (Nt-\mathsf{t})e(\mathsf{t})/N$, so that for all integer number t,

$$\sup_{t\in[\mathsf{t}/N,(\mathsf{t}+1)/N]} |\sum_{i=1}^N \mathsf{x}_i(\lfloor \mathsf{N}t\rfloor) - N\sum_{i=1}^N x_i(t)| \le N$$

Therefore, we deduce that

$$\ln D(\boldsymbol{x}) \quad = \quad N\mathsf{T}\ln 2 - N\int_0^T \partial_s f(s)\sum_{i=1}^N x_i(s)\mathrm{d}s + f(T)\sum_{i=1}^N \mathsf{x}_i(\mathsf{T}) - f(0)\sum_{i=1}^N \mathsf{x}_i(0) + \mathrm{O}(N),$$

where the $\mathrm{O}(N)$ error is given as $N\theta^{-1}\int_0^T|\partial_s f(s)|\mathrm{d}s$. Now it is easy to check that

$$\begin{aligned}
\int y\rho(y;\boldsymbol{x}(t))\mathrm{d}y &= \int y\sum_{i=1}^N 1_{y\in[x_i(t),x_i(t)+\theta/N]}\mathrm{d}y\\
&= \frac{1}{2}\sum_{i=1}^N\left(\left(x_i(t)+\frac{\theta}{N}\right)^2 - x_i(t)^2\right)\\
&= \frac{\theta}{N}\sum_{i=1}^N x_i(t) + \mathrm{O}(1/N).
\end{aligned}$$

By integration by parts we also have since $H(x;\boldsymbol{x}(t)) = \int_{-\infty}^x \rho(y;\boldsymbol{x}(t))\mathrm{d}y$,

$$\frac{\theta}{N}\sum_{i=1}^N x_i(t) + \mathrm{O}(1/N) = \int y\rho(y,\boldsymbol{x}(t))\mathrm{d}y = [yH(y;\boldsymbol{x}(t))]_{-K}^K - \int_{-K}^K H(y;\boldsymbol{x}(t))\mathrm{d}y$$

where we notice that we can restrict the integral on the left to a compact set where the particles lives, say $[-K, K]$ that can be chosen independent of the time (provided the particles at time 0 are bounded and T is also bounded). Then $[yH(y;\boldsymbol{x}(t))]_{-K}^{K} = \theta K$.

As a conclusion, we get:

$$\frac{1}{N^2}\ln D(\boldsymbol{x}) = \frac{1}{\theta}\left(\int_0^T \partial_s f(s)\int_{-K}^{K} H(y,s)\mathrm{d}y\mathrm{d}s - f(T)\int_{-K}^{K} H(y,T)\mathrm{d}y + f(0)\int_{-K}^{K} H(y,0)\mathrm{d}y\right)$$
$$+ T\ln 2 + \mathrm{O}(1/N).$$

Noticing that $H(y,s)$ is Lipschitz, thus differentiable. The claim (6.23) follows from an integration by part. Finally, it is clear that if $\partial_s f$ is bounded, $\mathcal{F}^f$ is continuous for the uniform norm. □

A Dynamical Loop Equation

{s:DynamicalLoc

We define the particle configuration:

$$\mathbb{W}^n := \{(x_1,\ldots,x_n)\in\mathbb{R}^n \mid x_i > x_{i+1},\quad i=1,2,\ldots,n-1\}. \tag{A.1}$$ {{e:defWtheta}}

Given a particle configuration $\boldsymbol{x} = (x_1, x_2, \ldots, x_n) \in \mathbb{W}^n$, an interaction function $b(z)$, and weight functions $\phi^+(z), \phi^-(z)$. Our central object is the following transition probability

$$\mathbb{P}(\boldsymbol{x}+\boldsymbol{e}|\boldsymbol{x}) = \frac{1}{Z(\boldsymbol{x})}\prod_{1\leqslant i<j\leqslant n}\frac{b(x_i+\theta e_i)-b(x_j+\theta e_j)}{b(x_i)-b(x_j)}\prod_{i=1}^{n}\phi^+(x_i)^{e_i}\phi^-(x_i)^{1-e_i}, \tag{A.2}$$ {{e:m1ccopy}}

where $\boldsymbol{e} = (e_1, e_2, \ldots, e_n) \in \{0,1\}^n$ and $Z(\boldsymbol{x})$ is a normalization constant.

{a:asymp}

Assumption A.1. *Fix constants $\mathsf{N} > 0$ and $\boldsymbol{l}, \boldsymbol{r}$. We introduce a small parameter $\varepsilon = \mathsf{N}/n \ll 1$, and assume the particle configuration $\boldsymbol{x} = (x_1, x_2, \ldots, x_n) \in \widetilde{\mathbb{W}}_\theta^n$ satisfies $\boldsymbol{l} \leqslant \varepsilon x_n \leqslant \varepsilon x_1 \leqslant \boldsymbol{r} - \varepsilon$. We further assume*

1. *There exists an open complex neighborhood Λ of $[\boldsymbol{l}, \boldsymbol{r}]$, such that for $z = \varepsilon\xi$,*

$$b(\xi) = z \quad \text{and} \quad \phi^\pm(\xi) = \varphi^\pm(z), \quad z \in \Lambda. \tag{A.3}$$ {{e:asymp}}

   where z and $\varphi^\pm(z)$ are holomorphic functions on Λ. In addition, we require z to be conformal (injective and biholomorphic) and $\mathsf{b}(\bar z) = \bar z$.

2. *The functions z, $[\partial_z z]^{-1}$, and $\varphi^\pm(z)$ are uniformly bounded, namely there exists a universal[1] constant $C > 0$ such that*

$$|z| \leqslant C, \quad |\partial_z z| \geqslant 1/C, \quad |\varphi^\pm(z)| \leqslant C, \quad z \in \Lambda. \tag{A.4}$$ {{e:derbba}}

Remark A.2. *In [19], the particle configurations are assumed to be inside the lattice:*

$$\mathbb{W}_\theta^n := \{(x_1,\ldots,x_n)\in\mathbb{R}^n \mid x_1\in\mathbb{Z},\quad x_i - x_{i+1} \in \theta + \mathbb{Z}_{\geqslant 0},\quad i=1,2,\ldots,n-1\}. \tag{A.5}$$

Here we relaxed the particle configuration to (A.1).

The dynamical loop equation states that the following observable is a holomorphic function.

{t:loopeq}

Theorem A.3 (Dynamical Loop Equation)**.** *Choose an open set $U \subset \mathbb{C}$, a particle configuration $\boldsymbol{x} = (x_1 > x_2 > \cdots > x_n) \in \mathbb{R}^n$ such that the interval $[x_n, x_1] \subset U$, a parameter $\theta > 0$, two holomorphic functions $\phi^+(z)$, $\phi^-(z)$ on U and a conformal (i.e., holomorphic and injective) function $b(z)$ on U. Assume that the random n–tuple $\boldsymbol{e} = (e_1, e_2, \ldots, e_n) \in \{0,1\}^n$ is distributed according to the transition probability (A.2). Then the following observable is a holomorphic function of $z \in U$:*

$$\mathbb{E}\left[\phi^+(z)\prod_{j=1}^{n}\frac{b(z+\theta)-b(x_j+\theta e_j)}{b(z)-b(x_j)} + \phi^-(z)\prod_{j=1}^{n}\frac{b(z)-b(x_j+\theta e_j)}{b(z)-b(x_j)}\right]. \tag{A.6}$$ {{e:sum1}}

[1]By "universal" we mean not depending on n, ε, or $(x_1, \ldots, x_n)$.

Proof. See [19, Proof of Theorem 1.1] □

Our last assumption for the asymptotic analysis involves the following functions defined in terms of the empirical density (1.14) and functions $\phi^{\pm}(z)$ of Assumption A.1: {e:sum1}

$$\mathcal{B}(z) = \mathcal{G}(z)\varphi^{+}(z) + \varphi^{-}(z), \qquad \mathcal{G}(z) = \exp\left[\theta \int_{\boldsymbol{l}}^{\boldsymbol{r}} \frac{\rho(s;\boldsymbol{x})}{z-s}\mathrm{d}s\right]. \tag{A.7}$$ {{e:B_function}

Note that $\mathcal{B}(z)$ is a holomorphic function for $z \in \Lambda \setminus [\boldsymbol{l},\boldsymbol{r}]$.

{a:stable}

Assumption A.4. *There exists annular set $S \subset \big(\Lambda \setminus [\boldsymbol{l},\boldsymbol{r}]\big)$ (containing a contour surrounding $[\boldsymbol{l},\boldsymbol{r}]$), a small universal constant $c > 0$ such that for $z \in S$ we have $c < |\mathcal{B}(z)| < c^{-1}$. Moreover, for any closed contour $\omega \subset \big(\Lambda \setminus [\boldsymbol{l},\boldsymbol{r}]\big)$, we have*

$$\frac{1}{2\pi \mathrm{i}} \oint_{\omega} \frac{\partial_z \mathcal{B}(z)}{\mathcal{B}(z)}\mathrm{d}z = 0, \tag{A.8}$$ {{e:aB0}}

which implies that there exists a well-defined single-valued branch of the function $\ln \mathcal{B}(z)$ in S.

{t:loopstudy}

Theorem A.5. *Consider transition probability (A.2) with parameters satisfying Assumptions A.1 and A.4 for all small enough ε. Then for any $z \in \Lambda \setminus [\boldsymbol{l},\boldsymbol{r}]$ we have as $\varepsilon \to 0$:* {e:aB0}

$$\frac{1}{\varepsilon}\int_{\boldsymbol{l}}^{\boldsymbol{r}} \frac{(\rho(s;\boldsymbol{y}) - \rho(s;\boldsymbol{x}))}{z-s}\mathrm{d}s = \Delta\mathcal{M}(z) + \frac{1}{2\pi \mathrm{i}\theta}\oint_{\omega_-} \frac{\ln \mathcal{B}(w)\mathrm{d}w}{(w-z)^2} + \mathrm{O}(\varepsilon), \tag{A.9}$$ {{e:dmg2}}

Moreover, $\Delta\mathcal{M}(z)$ are mean 0 random variables such that $\{\varepsilon^{-1/2}\Delta\mathcal{M}(z)\}_{z\in\Lambda\setminus[\boldsymbol{l},\boldsymbol{r}]}$ are asymptotically Gaussian with covariance given by

$$\begin{aligned}
&\lim_{\varepsilon\to 0}\mathbb{E}\left[\varepsilon^{-1/2}\Delta\mathcal{M}(z_1), \varepsilon^{-1/2}\Delta\mathcal{M}(z_2)\right] \\
&= \frac{1}{2\pi \mathrm{i}\theta}\oint_{\omega_-} \frac{\mathcal{G}(w)\varphi^{+}(w)}{\mathcal{B}(w)} \frac{\mathsf{b}'(z_1)}{(w-\mathsf{b}(z_1))^2}\frac{\mathsf{b}'(z_2)}{(w-\mathsf{b}(z_2))^2}\mathrm{d}w,
\end{aligned} \tag{A.10}$$ {{e:covT}}

where the contour $\omega_- \subset \Lambda$ encloses $[\boldsymbol{l},\boldsymbol{r}]$, but not z_1, z_2. The higher order joint moments of $\{\varepsilon^{-1/2}\Delta\mathcal{M}(z)\}_{z\in\Lambda\setminus[\boldsymbol{l},\boldsymbol{r}]}$ converge as $\varepsilon \to 0$ to the Gaussian joint moments.

The implicit constants in the $\mathrm{O}(\varepsilon)$ are uniform in all the involved parameters, as long as the constants C and c of Assumptions A.1 and A.4 are fixed and z belongs to a compact subset of $\Lambda \setminus [\boldsymbol{l},\boldsymbol{r}]$.

B Proofs for Results from Section 2

{s:setupproof}

Proof of Lemma 2.1. In this proof, for simplicity of notations, we omit the dependence on N and simply write $\mathbf{y} = \mathbf{y}^{(N)}$ and $\mathbf{z} = \mathbf{z}^{(N)}$. By our assumption $\mathcal{P}(\mathbf{y}, \mathbf{z}; \mathsf{T}) \neq \emptyset$, there exists a non-intersecting Bernoulli paths $\{\boldsymbol{y}(t)\}_{0\leqslant t\leqslant T}$ from $\boldsymbol{y}(0) = \mathbf{y}/N$ to $\boldsymbol{y}(T) = \mathbf{z}/N$.

Next we construct a non-intersecting Bernoulli paths $\{\boldsymbol{z}(t)\}_{0\leqslant t\leqslant T}$, such that its height function approximates H^*. First we can construct the level line of H^* as

$$\gamma(y,t) = \inf\{x : H^*(x,t) > y\}, \quad 0 \leqslant y < \theta. \tag{B.1}$$

Since H^* is 2-Lipschitz we have $H^*(\gamma(y,t),t) = y$ for any $0 \leqslant y < \theta$. Since $\nabla H^* \in \overline{\mathcal{T}}$, for any $s \geqslant t$,

$$H^*(\gamma(y,t),s) \leqslant H^*(\gamma(y,t),t) \leqslant H^*(\gamma(y,t) + (s-t), s). \tag{B.2}$$

We conclude that $\gamma(y,t) \leqslant \gamma(y,s) \leqslant \gamma(y,t) + (s-t)$. It follows that $\gamma(y,t)$ is 1-Lipschitz in t, and $\partial_t\gamma(y,t) \in [0,1]$.

We take $m = \lceil \varepsilon N/(2\theta)\rceil$, and construct the nonintersecting Bernoulli paths $\{\boldsymbol{x}(t)\}_{0\leqslant t\leqslant T}$ in the following way: the path $\{x_i(t)\}_{0\leqslant t\leqslant T}$ is obtained from $\{y_i(t)\}_{0\leqslant t\leqslant T}$ by truncating from left using

$\gamma(\theta(i-m)/N, t)$, and from right using $\gamma(\theta(i+m)/N, t)$. Here we make the convention $\gamma(y, s) = -\infty$ if $y < 0$, and $\gamma(y, s) = +\infty$ if $y > 1$. Formally, $\{\boldsymbol{x}(t)\}_{0 \leqslant t \leqslant T}$ is given as for any $1 \leqslant i \leqslant N$ and $t \in N^{-1}[\![0, \mathsf{T}]\!]$

$$x_i(t) = \operatorname{argmin}\{|x - y_i(t)| : x - y_i(t) \in N^{-1}\mathbb{Z}, \gamma(\theta(i-m)/N, t) \leqslant x \leqslant \gamma(\theta(i+m)/N, t)\}. \tag{B.3}$$ {{e:constructx}}

We denote the height function of $\{\boldsymbol{x}(t)\}_{0 \leqslant t \leqslant T}$ as $\mathcal{H}$.

Next we check that $\{\boldsymbol{x}(t)\}_{0 \leqslant t \leqslant T}$ forms a non-intersecting θ-Bernoulli walk from $\mathsf{y}^{(N)}/N$ to $\mathsf{z}^{(N)}/N$. Under assumption (1.22), for N large enough

$$\sup_{x \in \mathbb{R}} |\mathcal{H}(x, 0) - h(x, 0)|, \sup_{x \in \mathbb{R}} |\mathcal{H}(x, T) - h(x, T)| \leqslant \varepsilon/(4\theta). \tag{B.4}$$

It follows that for $t \in \{0, T\}$,

$$\begin{aligned} h(y_i(t), t) &\geqslant \mathcal{H}(y_i(t), t) - \varepsilon/4 = \frac{\theta(i-1)}{N} - \varepsilon/4 \geqslant h(\gamma(\theta(i-m)/N, t), \\ h(y_i(t), t) &\leqslant \mathcal{H}(y_i(t), t) + \varepsilon/4 = \frac{\theta(i-1)}{N} + \varepsilon/4 \leqslant h(\gamma(\theta(i+m)/N, t), \end{aligned}$$

And thus $\gamma(\theta(i-m)/N, t) \leqslant y_i(t) = x_i(t) \leqslant \gamma(\theta(i+m)/N, t)$ for $t \in \{0, T\}$.

We need to show $x_i(t+1/N) - x_i(\mathsf{t}/N) \in \{0, 1\}$ and $x_{i+1}(t) - x_i(t) \geqslant \theta/N$ for any $t = \mathsf{t}/N \in N^{-1}[\![0, \mathsf{T}]\!]$. By symmetry, we assume $y_i(t) \geqslant \gamma_i(t)$, so it is far from $\gamma_{i-m}(t)$. In fact we have

$$y_i(t), y_{i+1}(t), y_i(t+1/N), y_{i+1}(t+1/N) \geqslant \gamma_{i-m}(t) + \varepsilon/2. \tag{B.5}$$ {{e:asump}}

Notice that under (B.5) we only need to deal with the possible truncation caused by $\gamma_{i+m}(t)$. There are still several cases

1. If $\gamma(\theta(i-m)/N, t) \leqslant y_i(t) \leqslant \gamma(\theta(i+m)/N, t)$, then $x_i(t) = y_i(t)$. If also $x_i(t+1/N) = y_i(t+1/N)$, then $x_i(t+1/N) - x_i(t) = y_i(t+1/N) - y_i(t) \in \{0, 1/N\}$; otherwise $y_i(t+1/N) > \gamma(\theta(i+m)/N, t+1/N)$. Recall that $\partial_s \gamma(y, s) \in [0, 1]$, we must have

$$y_i(t+1/N) > \gamma(\theta(i+m)/N, t+1/N) \geqslant \gamma(\theta(i+m)/N, t) \geqslant y_i(t) \Rightarrow y_i(t+1/N) = y_i(t) + 1/N. \tag{B.6}$$

In this case $x_i(t+1/N) = y_i(t)$, and $x_i(t+1/N) - x_i(t) = 0$.

If $x_{i+1}(t) = y_{i+1}(t)$, then $x_{i+1}(t) - x_i(t) = y_{i+1}(t) - y_i(t) \geqslant \theta/N$; otherwise $y_{i+1}(t) > \gamma(\theta(i+m+1)/N, t)$. Recall that $\gamma(y+\theta/N, s) - \gamma(y, s) \geqslant \theta/N$, we must have

$$y_{i+1}(t) > \gamma(\theta(1+i+m)/N, t+1/N) \geqslant \gamma(\theta(i+m)/N, t) + \theta/N \geqslant y_i(t) + \theta/N. \tag{B.7}$$

In this case $x_{i+1}(t) \geqslant y_i(t) + \theta/N = x_i(t) + \theta/N$.

2. If $y_i(t) > \gamma(\theta(i+m)/N, t)$, then $x_i(t) = y_i(t) - \mathbb{Z}_{\geqslant 0}$. For $y_i(t+1/N)$ there are two cases: If $y_i(t+1/N) \leqslant \gamma(\theta(i+m)/N, t+1/N)$, recalling that $\partial_s \gamma(y, s) \in [0, 1]$, then we must have

$$y_i(t+1/N) \leqslant \gamma(\theta(i+m)/N, t+1/N) \leqslant \gamma(\theta(i+m)/N, t) + 1/N < y_i(t) + 1/N. \tag{B.8}$$

In this case $y_i(t) = y_i(t+1/N) = x_i(t+1/N)$, and it follows

$$y_i(t) - 1/N = y_i(t+1/N) - 1/N \leqslant \gamma(\theta(i+m)/N, t+1/N) - 1/N \leqslant \gamma(\theta(i+m)/N, t). \tag{B.9}$$

So $x_i(t) = y_i(t) - 1/N$ and $x_i(t+1/N) - x_i(t) = 1/N$.

Otherwise if $y_i(t+1/N) \geqslant \gamma(\theta(i+m)/N, t+1/N)$, recall $\partial_t \gamma(y, t) \in [0, 1]$, we have

$$x_i(t) \leqslant x_i(t+1/N) \leqslant \gamma(\theta(i+m)/N, t+1/N) \leqslant \gamma(\theta(i+m)/N, t) + 1/N < x_i(t) + 1/N + 1/N. \tag{B.10}$$

In this case we still have $x_i(t+1/N) - x_i(t) \in \{0, 1/N\}$.

For $y_{i+1}(t)$ there are two cases: If $y_{i+1}(t) \leqslant \gamma(\theta(1+i+m)/N, t+1/N)$, then $x_{i+1}(t) = y_{i+1}(t)$ and $x_{i+1}(t) - x_i(t) \geqslant y_{i+1}(t) - y_i(t) \geqslant \theta/N$. Otherwise if $y_{i+1}(t) > \gamma(\theta(1+i+m)/N, t+1/N)$, recall that $\gamma(y+\theta/N, s) - \gamma(y, s) \geqslant \theta/N$, so

$$x_i(t) + \theta/N \leqslant \gamma(\theta(i+m)/N, t) + \theta/N \leqslant \gamma(\theta(1+i+m)/N, t), \tag{B.11}$$

and $x_{i+1}(t) \geqslant x_i(t) + \theta/N$.

Finally we check that the height function $\mathcal{H}$ of $\{\boldsymbol{x}(t)\}_{0\leqslant t\leqslant T}$ as constructed in (B.3) satisfies (2.1). For any $x_i(t) \leqslant x < x_{i+1}(t)$, then $\gamma(\theta(i-m)/N) \leqslant x < \gamma(\theta(1+i+m)/N)$

$$\left|\mathcal{H}(x,t) - \frac{\theta i}{N}\right| \leqslant \frac{\theta}{N}. \tag{B.12}$$ {{e:Hbound1}}

and

$$\max\left\{0, \frac{\theta(i-m)}{N}\right\} \leqslant H^*(x,t) \leqslant \min\left\{\theta, \frac{\theta(1+i+m)}{N}\right\}, \tag{B.13}$$ {{e:Hbound2}}

The claim (2.1) fo{{e:Hbound2}}(B.12) and (B.13). □

Proof of Lemma 2.4. The statement (5.51) follows from (4.11) and $\nabla\mathcal{A}(x,t) = (\varrho, -\varrho v)\mathbf{1}(tv \leqslant x \leqslant 1+tv)$. On the interval $[tv, \ell+tv]$, we have

$$\kappa_t(x) = \int_{tv}^{\ell+tv} \frac{\delta \mathrm{d}y}{(y-x)^2+\delta^2} \asymp 1 \tag{B.14}$$

and

$$1-\kappa_t(x) = \int_{[tv,\ell+tv]^\complement} \frac{\delta \mathrm{d}y}{(y-x)^2+\delta^2} \asymp \frac{\delta}{\delta + \operatorname{dist}(x, \{tv, \ell+tv\})}. \tag{B.15}$$

Outside the interval $[tv, \ell+tv]$, we have

$$\kappa_t(x) = \int_{tv}^{\ell+tv} \frac{\delta \mathrm{d}y}{(y-x)^2+\delta^2} \lesssim \frac{\delta}{\delta + \operatorname{dist}(x, [tv, \ell+tv]) + \operatorname{dist}(x, [tv, \ell+tv])^2}, \tag{B.16}$$

and

$$1-\kappa_t(x) = \int_{[tv,\ell+tv]^\complement} \frac{\delta \mathrm{d}y}{(y-x)^2+\delta^2} \asymp 1. \tag{B.17}$$

□

Proof of Lemma 2.5. From the expression (2.6),

$$\begin{aligned}\kappa_t(x) = -\operatorname{Im}\int_{tv}^{\ell+tv} \frac{\mathrm{d}y}{(x+\mathrm{i}\delta)-y} &= \frac{\mathrm{i}}{2}\left(\int_{tv}^{\ell+tv} \frac{\mathrm{d}y}{(x+\mathrm{i}\delta)-y} - \int_{tv}^{\ell+tv} \frac{\mathrm{d}y}{(x-\mathrm{i}\delta)-y}\right) \\ &= \frac{\mathrm{i}}{2}\left(\ln\frac{(x+\mathrm{i}\delta)-(\ell+tv)}{(x+\mathrm{i}\delta)-tv} - \ln\frac{(x-\mathrm{i}\delta)-(\ell+tv)}{(x-\mathrm{i}\delta)-tv}\right),\end{aligned} \tag{B.18}$$ {{e:kappat}}

where $\ln(\cdot)$ is the branch on $\mathbb{C}\setminus[-\infty, 0]$. And

$$\begin{aligned}\operatorname{Hib}(\kappa_t)(x) = \operatorname{Re}\int_{tv}^{\ell+tv} \frac{\mathrm{d}y}{(x+\mathrm{i}\delta)-y} &= \frac{1}{2}\left(\int_{tv}^{\ell+tv} \frac{\mathrm{d}y}{(x+\mathrm{i}\delta)-y} + \int_{tv}^{\ell+tv} \frac{\mathrm{d}y}{(x-\mathrm{i}\delta)-y}\right) \\ &= \frac{1}{2}\left(\ln\frac{(x+\mathrm{i}\delta)-(\ell+tv)}{(x+\mathrm{i}\delta)-tv} + \ln\frac{(x-\mathrm{i}\delta)-(\ell+tv)}{(x-\mathrm{i}\delta)-tv}\right)\end{aligned} \tag{B.19}$$ {{e:Hibkappa}}

The expression (B.18) is well-defined {e:Hibkappa} for $|\operatorname{Im}[z]| < \delta$. Thus $\kappa_t(x)$ extends analytically to $\kappa_t(z)$ for $|\operatorname{Im}[z]| < \delta$. Moreover,

$$\begin{aligned}
\partial_z \kappa_t(z) &= \frac{\mathrm{i}}{2}\partial_z\left(\ln\frac{(z+\mathrm{i}\delta)-(\ell+tv)}{(z+\mathrm{i}\delta)-tv} - \ln\frac{(z-\mathrm{i}\delta)-(\ell+tv)}{(z-\mathrm{i}\delta)-tv}\right)\\
&= \frac{\mathrm{i}}{2}\left(\frac{1}{(z+\mathrm{i}\delta)-(\ell+tv)} - \frac{1}{(z+\mathrm{i}\delta)-tv} - \frac{1}{(z-\mathrm{i}\delta)-(\ell+tv)} + \frac{1}{(z-\mathrm{i}\delta)-tv}\right)\\
&= \frac{\delta}{(z-\ell-tv)^2+\delta^2} - \frac{\delta}{(z-tv)^2+\delta^2}.
\end{aligned}\tag{B.20}$$

{{e:dzkappa}}

If $\operatorname{dist}(z, \{tv, \ell+tv\})$, the first statement in the claim (2.12) follows from {e:dzkappa} (B.20).

For $z = x + \mathrm{i}\eta$ with $|\operatorname{Im}[z]| \leqslant \delta/2$

$$\left|\frac{\delta}{(z-tv)^2+\delta^2}\right| \leqslant \frac{\delta}{(x-tv)^2+\delta^2-\operatorname{Im}[z]^2} \lesssim \frac{\delta}{\delta^2+(x-tv)^2},\tag{B.21}$$

{{e:term1}}

and similarly

$$\left|\frac{\delta}{(z-\ell-tv)^2+\delta^2}\right| \lesssim \frac{\delta}{\delta^2+(x-\ell-tv)^2},\tag{B.22}$$

{{e:term2}}

We conclude that by plugging (B.21) and (B.22) {e:term2} into (B.20),

$$|\partial_z\kappa_t(z)| \lesssim \frac{\delta}{\delta^2+\operatorname{dist}(x,\{tv,\ell+tv\})^2} \lesssim \frac{\min\{\kappa_t(x), 1-\kappa_t(x)\}}{\delta},\tag{B.23}$$

{{e:dzkz}}

where the last inequality is from Lemma 2.4. The claim (2.10) follows from {e:dzkz} integrating (B.23) from x to $x + \mathrm{i}\eta$.

The expression of the Hilbert transform (B.24) also extends analytically to $\operatorname{Hib}(\kappa_t)(z)$ for $|\operatorname{Im}[z]| < \delta$,

$$\operatorname{Hib}(\kappa_t)(z) = \frac{1}{2}\left(\ln\frac{(z+\mathrm{i}\delta)-(\ell+tv)}{(z+\mathrm{i}\delta)-tv} + \ln\frac{(z-\mathrm{i}\delta)-(\ell+tv)}{(z-\mathrm{i}\delta)-tv}\right).\tag{B.24}$$

{{e:Hibkappa}}

For $z = x + \mathrm{i}\eta$ with $|\eta| \leqslant \delta/2$, we have

$$\left|\ln\frac{(z+\mathrm{i}\delta)-(\ell+tv)}{(z+\mathrm{i}\delta)-tv}\right| \leqslant \ln\left|1+\frac{\ell}{(z+\mathrm{i}\delta)-tv}\right| + \pi \leqslant \ln(\ell/\delta) + \mathrm{O}(1),\tag{B.25}$$

and we have the same estimate for the second term on the righthand side of (B.24). This gives the first statement of (2.11). To prove the second statement of (2.11), we can rewrite (B.24) as,

$$\operatorname{Hib}(\kappa_t)(z) = \frac{1}{2}\left(\ln((z-\ell-tv)^2+\delta^2) - \ln((z-tv)^2+\delta^2)\right).\tag{B.26}$$

{{e:Hibkappa2}}

If $\operatorname{dist}(z, \{tv, \ell+tv\}) \gtrsim \ell$, from (B.24), we have {e:Hibkappa2}

$$\operatorname{Hib}(\kappa_t(z) \lesssim \frac{\delta}{\operatorname{dist}(z,\{tv,\ell+tv\})},\tag{B.27}$$

this gives the second statement in (2.12).

For $z = x + \mathrm{i}\eta$ with $|\eta| \leqslant \delta/2$, we have

$$|\operatorname{Im}\ln((z-\ell-tv)^2+\delta^2)| = |\operatorname{Im}\ln(\mathrm{i}2\eta(x-\ell-tv)+(x-\ell-tv)^2+(\delta^2-\eta^2))|\tag{B.28}$$

$$= \left|\operatorname{Im}[\ln(x-\ell-tv)^2+(\delta^2-\eta^2)] + \operatorname{Im}\ln\left(1+\frac{\mathrm{i}2\eta(x-\ell-tv)}{(x-\ell-tv)^2+(\delta^2-\eta^2)}\right)\right| \lesssim \frac{|\eta|}{\sqrt{\delta^2-\eta^2}} \lesssim \frac{\eta}{\delta}\tag{B.29}$$

where in the last two inequalities we used that $(x-\ell-tv)^2+(\delta^2-\eta^2) \geqslant 2|x-\ell-tv|\sqrt{\delta^2-\eta^2}$. We have the same estimate for the second term on the righthand side of (B.26). This gives the second statement of (2.11). □

C Asymptotics for Macdonald Polynomials

{s:Macdonald}

In this section, we explain the correspondence between non-intersecting θ-Bernoulli walk ensembles and certain Macdonald ascending process (C.4). This correspondence will be used to derive large deviation asymptotics for (skew) Macdonald polynomials. We collect some basic properties of Macdonald symmetric functions in Appendix C. We recall the Macdonald ascending process in Appendix C.2. Our main references are [41] and [53]. In Appendix C.3, we prove Theorem 1.7.

C.1 Macdonald polynomials and specializations

We use the following notations:

$$(a;q)_\infty=\prod_{i=1}^{\infty}(1-aq^{i-1}),\quad f(u)=\frac{(tu;q)_\infty}{(qu;q)_\infty},\quad \Gamma_q(x)=(1-q)^{1-x}\frac{(q;q)_\infty}{(q^x;q)_\infty}. \tag{C.1}$$

{{e:bracketf}}

The next class of Markov chains is built out of the specializations of Macdonald polynomials. We refer to [40, Section VI] and [8] for definitions and properties of Macdonald symmetric functions and use them as a black box in this section. Macdonald symmetric functions P and Q are indexed by partitions and implicitly depend on two parameters $q,t\in(0,1)$. The coefficients for the symmetric functions are in $\mathbb{Q}[q,t]$. Macdonald symmetric functions $P_{\boldsymbol\lambda}(\boldsymbol{x};q,t)$ and $Q_{\boldsymbol\lambda}(\boldsymbol{x};q,t)$ are elements of the algebra Λ of the symmetric functions in infinitely many variables $(x_i)_{i=1}^\infty$ uniquely determined by the following two properties:

1. $P_{\boldsymbol\lambda}$, $|\boldsymbol\lambda|=m$, can be expressed in terms of the monomial symmetric functions via a strictly upper unitriangular transition matrix:

$$P_{\boldsymbol\lambda}=m_{\boldsymbol\lambda}+\sum_{\boldsymbol\mu<\boldsymbol\lambda\in\mathbb{Y}_m}R_{\boldsymbol\lambda\boldsymbol\mu}P_{\boldsymbol\mu},$$

   where $R_{\boldsymbol\lambda\boldsymbol\mu}$ are functions of q,t and $\boldsymbol\mu<\boldsymbol\lambda$ is comparison in the dominance order on the set $\mathbb{Y}_m$ of all partitions of m (equivalently, Young diagrams with m boxes).

2. They are pairwise orthogonal with respect to the scalar product defined on the power sums via

$$\langle p_{\boldsymbol\lambda},p_{\boldsymbol\mu}\rangle_{q,t}=\delta_{\boldsymbol\lambda\boldsymbol\mu}z_{\boldsymbol\lambda}(q,t),\qquad p_\lambda=\prod_{i=1}^{\infty}p_{\lambda_i},\qquad z_{\boldsymbol\lambda}(q,t)=\prod_{i\geqslant1}i^{m_i}(m_i)!\prod_{i=1}^{\ell(\boldsymbol\lambda)}\frac{1-q^{\lambda_i}}{1-t^{\lambda_i}},$$

   where $\boldsymbol\lambda=1^{m_1}2^{m_2}\ldots$, i.e. m_i is the multiplicity of i in $\boldsymbol\lambda$, $\ell(\boldsymbol\lambda)$ is the number of rows, and $p_k=(x_1)^k+(x_2)^k+\ldots$, $k\geqslant1$.

We further define $Q_{\boldsymbol\lambda}=\frac{P_{\boldsymbol\lambda}}{\langle P_{\boldsymbol\lambda},P_{\boldsymbol\lambda}\rangle_{q,t}}$. Finally, the skew Macdonald polynomials $P_{\boldsymbol\lambda/\boldsymbol\mu}$ and $Q_{\boldsymbol\lambda/\boldsymbol\mu}$ are defined through the expansions:

$$P_{\boldsymbol\lambda}(x_1,x_2,\ldots,y_1,y_2,\ldots;q,t)=\sum_{\boldsymbol\mu}P_{\boldsymbol\lambda/\boldsymbol\mu}(x_1,x_1,\ldots;q,t)P_{\boldsymbol\mu}(y_1,y_2,\ldots;q,t),$$
$$Q_{\boldsymbol\lambda}(x_1,x_2,\ldots,y_1,y_2,\ldots;q,t)=\sum_{\boldsymbol\mu}Q_{\boldsymbol\lambda/\boldsymbol\mu}(x_1,x_1,\ldots;q,t)Q_{\boldsymbol\mu}(y_1,y_2,\ldots;q,t).$$

C.2 Macdonald ascending processes

{s:Macdonaldpro

Computations in the algebra of symmetric functions Λ can be converted into numeric identities by means of *specializations*, which are algebra homomorphism from Λ to the set of complex numbers. Specialization ρ is uniquely determined by its values on any set of algebraic generators of Λ and we use $(p_k)_{k=1}^\infty$ as such generators. The value of ρ on a symmetric function f is denoted $f(\rho)$. Given two specializations ρ, ρ', we define their union (ρ, ρ') through the formula:

$$p_k(\rho, \rho') = p_k(\rho) + p_k(\rho'), \quad k \geqslant 1.$$

A specialization ρ is called *Macdonald nonnegative* if its values on all (skew) Macdonald symmetric functions are non-negative, i.e., if for all partitions $\boldsymbol{\lambda}$ and $\boldsymbol{\mu}$,

$$P_{\boldsymbol{\lambda}/\boldsymbol{\mu}}(\rho; q, t) \geqslant 0,$$

The description of Macdonald nonnegative specializations was conjectured in [38] and proven in [42]:

{Theorem_Macdon

Theorem C.1 ([42])**.** *For any fixed $q, t \in (0,1)$, Macdonald nonnegative specializations can be parameterized by triplets $(\boldsymbol{\alpha} = \{\alpha_i\}_{i\geqslant 1}, \boldsymbol{\beta} = \{\beta_i\}_{i\geqslant 1}, \gamma)$ of nonnegative numbers satisfying $\sum_{i=1}^{\infty}(\alpha_i + \beta_i) < \infty$. The specialization ρ corresponding to a triplet $(\boldsymbol{\alpha}, \boldsymbol{\beta}, \gamma)$ is defined by for $k \geqslant 2$*

$$p_1(\rho) = \sum_{i=1}^{\infty} \alpha_i + \frac{1-q}{1-t}\left(\gamma + \sum_{i=1}^{\infty} \beta_i\right), \qquad p_k(\rho) = \sum_{i=1}^{\infty}(\alpha_i)^k + (-1)^{k-1}\frac{1-q^k}{1-t^k}\sum_{i=1}^{\infty}(\beta_i)^k.$$

Following [8, Section 2.3], we create Markov chains out of the Macdonald-positive specializations:

{Definition_Mac

Definition C.2. *Given two specializations ρ and ρ' we define the ascending transition through*

$$\mathbb{P}(\boldsymbol{\lambda} \mid \boldsymbol{\mu}) = \frac{1}{\Pi(\rho; \rho')}\frac{P_{\boldsymbol{\lambda}}(\rho; q, t)}{P_{\boldsymbol{\mu}}(\rho; q, t)} Q_{\boldsymbol{\lambda}/\boldsymbol{\mu}}(\rho'; q, t), \tag{C.2}$$ {{eq_ascending_

where $\boldsymbol{\lambda}$ and $\boldsymbol{\mu}$ are partitions with $\boldsymbol{\mu} \subset \boldsymbol{\lambda}$ and $\Pi(\rho; \rho')$ is the result of applying ρ to the x_i variables and ρ' to the y_j variables in the infinite product

$$\Pi = \prod_{i,j\geqslant 1} \frac{(t x_i y_j; q)_\infty}{(x_i y_j; q)_\infty}.$$

To obtain the asymptotics of Skew Macdonald polynomials, we study the following special Macdonald ascending process. We fix $N, \mathsf{T} \geqslant 1$, and a sequence of positive numbers $b_0, b_1, b_2, \cdots, b_{\mathsf{T}-1}$. The transition probability at time $0 \leqslant t \leqslant \mathsf{T}-1$ is given in the notations of Theorem C.1 as

$$\rho: \ \alpha_i = t^{i-1}, \, 1 \leqslant i \leqslant N; \qquad \rho_{\mathsf{t}} : \beta_1 = b_{\mathsf{t}}, \tag{C.3}$$ {{eq_principal_

with all other parameters set to 0. This gives a Markov process on Young diagrams $\boldsymbol{\lambda}(0), \boldsymbol{\lambda}(1), \cdots, \boldsymbol{\lambda}(\mathsf{T})$

$$\mathbb{P}(\boldsymbol{\lambda}(\mathsf{t}+1) = \boldsymbol{\lambda} | \boldsymbol{\lambda}(\mathsf{t}) = \boldsymbol{\mu}) = \frac{1}{\Pi(\rho, \beta_1 = b_t)}\frac{P_{\boldsymbol{\lambda}}(\rho; q, t)}{P_{\boldsymbol{\mu}}(\rho; q, t)} Q_{\boldsymbol{\lambda}/\boldsymbol{\mu}}(\beta_1 = b_t; q, t) \tag{C.4}$$ {{e:Mprocess}}

Then the transition probability gives the skew Macdonald polynomials

$$\mathbb{P}(\boldsymbol{\lambda}(\mathsf{T}) = \boldsymbol{\lambda} | \boldsymbol{\lambda}(0) = \boldsymbol{\mu}) = \frac{1}{\Pi((\rho, \boldsymbol{\beta} = (b_0, \cdots, b_{\mathsf{T}-1}))}\frac{P_{\boldsymbol{\lambda}}(\rho; q, t)}{P_{\boldsymbol{\mu}}(\rho; q, t)} Q_{\boldsymbol{\lambda}/\boldsymbol{\mu}}(\boldsymbol{\beta} = (b_0, \cdots, b_{\mathsf{T}-1}); q, t) \tag{C.5}$$ {{e:MDdynamics1

The evaluation of the Macdonald polynomial under the principle specialization $\rho = (1, t, \ldots, t^{N-1})$ is explicit, see [40, Chapter VI, (6.11)]:

$$P_{\boldsymbol{\lambda}}(1, t, \ldots, t^{N-1}; q, t) = t^{\sum_{i=1}^{N}(i-1)\lambda_i} \prod_{i<j\leqslant N} \frac{(q^{\lambda_i - \lambda_j} t^{j-i}; q)_\infty}{(q^{\lambda_i - \lambda_j} t^{j-i+1}; q)_\infty} \frac{(t^{j-i+1}; q)_\infty}{(t^{j-i}; q)_\infty}, \tag{C.6}$$ {{e:MDP}}

for $\boldsymbol{\lambda} = \lambda_1 \geqslant \lambda_2 \ldots \geqslant \lambda_N \geqslant 0$; $P_{\boldsymbol{\lambda}}(\rho; q, t) = 0$ if $\lambda_{N+1} > 0$. We further use the $w_{u,v}$ automorphism of the algebra of symmetric functions Λ defined on the power sums by:

$$w_{u,v}(p_k) = (-1)^{k-1}\frac{1-u^k}{1-v^k} p_k.$$

As shown in [40, Chapter VI, Section 7],

$$w_{t,q}\big(P_{\boldsymbol{\lambda}'/\boldsymbol{\mu}'}(\cdot;t,q)\big) = Q_{\boldsymbol{\lambda}/\boldsymbol{\mu}}(\cdot;q,t),$$

where $\boldsymbol{\lambda}'$ and $\boldsymbol{\mu}'$ are transposed Young diagrams $\boldsymbol{\lambda}$ and $\boldsymbol{\mu}$, respectively. Hence, with the specialization $\boldsymbol{\beta} = (b_0, b_1, \cdots, b_{\mathsf{T}-1})$, as in (C.3), we have {e:MDR}

$$Q_{\boldsymbol{\lambda}/\boldsymbol{\mu}}(\boldsymbol{\beta} = (b_0, b_1, \cdots, b_{\mathsf{T}-1}); q, t) = P_{\boldsymbol{\lambda}'/\boldsymbol{\mu}'}(\boldsymbol{\alpha} = (b_0, b_1, \cdots, b_{\mathsf{T}-1}); t, q).$$

{c:mdensity2}

Claim C.3. *The transition probability of Definition C.2 under the specializations (C.3) is non-degenerate only for partitions $\boldsymbol{\lambda}, \boldsymbol{\mu}$ with at most n parts, i.e. $\boldsymbol{\lambda} = (\lambda_1 \geqslant \lambda_2 \geqslant \ldots \geqslant \lambda_n)$ and $\boldsymbol{\mu} = (\mu_1 \geqslant \mu_2 \geqslant \ldots \geqslant \mu_n)$. Further, if we set $t = q^\theta$ and identify*

$$\boldsymbol{x} = (\mathsf{x}_1, \mathsf{x}_2, \ldots, \mathsf{x}_N) \in \mathbb{W}_\theta^N, \quad \mathsf{x}_i = \mu_i - \theta(i-1), \quad \mathsf{x}_i + e_i = \lambda_i - \theta(i-1), \quad 1 \leqslant i \leqslant n, \tag{C.7}$$

{{e:x1}}

then $e_i \in \{0, 1\}$ and the transition probability is given by

$$\mathbb{P}(\mathbf{x} + \boldsymbol{e}|\mathbf{x}) = \frac{1}{\Pi(\rho, \boldsymbol{\beta} = b_{\mathsf{t}})} \prod_{1 \leqslant i < j \leqslant n} \frac{q^{\mathsf{x}_i + \theta e_i} - q^{\mathsf{x}_j + \theta e_j}}{q^{\mathsf{x}_i} - q^{\mathsf{x}_j}} b_{\mathsf{t}}^{\sum_{i=1}^n e_i}. \tag{C.8}$$

{{e:drift_MDdyn

The following lemma gives the asymptotics of the Macdonald polynomial $P_{\boldsymbol{\lambda}}(1, t, \ldots, t^{N-1}; q, t)$.

{l:Prhoformula}

Lemma C.4. *Given a Young diagram $\boldsymbol{\lambda} \in \mathbb{Y}_N$, we identify it as an particle configuration*

$$\mathbf{x} = (\mathsf{x}_1, \mathsf{x}_2, \ldots, \mathsf{x}_N) \in \mathbb{W}_\theta^N, \quad \mathsf{x}_i = \lambda_i - \theta(i-1) \quad 1 \leqslant i \leqslant N, \tag{C.9}$$

{{e:defxfromla}

then

$$P_{\boldsymbol{\lambda}}(1, t, \ldots, t^{N-1}; q, t) = q^{\sum_{i=1}^N (i-1)\theta(\mathsf{x}_i + (i-1)\theta)} \prod_{i < j \leqslant N} \frac{\Gamma_q(\mathsf{x}_i - \mathsf{y}_j)}{\Gamma_q(\mathsf{x}_i - \mathsf{x}_j + \theta)} \prod_{i=1}^N \frac{\Gamma_q(i\theta)}{\Gamma_q(\theta)}, \tag{C.10}$$

{{e:MDP2}}

Given sequences a sequence of Young diagrams

$$\boldsymbol{\lambda}^{(N)} = (\lambda_1^{(N)} \geqslant \lambda_2^{(N)} \geqslant \cdots \geqslant \lambda_N^{(N)}) \in \mathbb{Y}_N, \quad N \geqslant 1 \tag{C.11}$$

such that

1. *There exists a constant $C > 0$, $\lambda_1^{(N)} \leqslant CN$*

2. *There exists a 1-Lipschitz nondecreasing function $h: \mathbb{R} \mapsto [0, \theta]$, and when $N \to \infty$*

$$\frac{\theta}{N} \sum_{i=1}^N \delta\left(\frac{\lambda_i^{(N)} - (i-1)\theta}{N}\right) \to \partial_x h(x), \tag{C.12}$$

{{e:hxdef}}

in distribution. Then

$$\begin{aligned}\lim_{N\to\infty} \frac{1}{N^2} \ln P_{\boldsymbol{\lambda}}(1, t, \ldots, t^{N-1}; q, t) = \frac{1}{2\theta} \int \ln(1 - e^{\kappa|x-y|}) \mathrm{d}h(x)\mathrm{d}h(y) - \theta \int_0^1 x \ln(1 - e^{\kappa\theta x}) \mathrm{d}x \\ -\kappa\theta^2 \int \frac{(1-x)x}{e^{-\kappa\theta x} - 1} \mathrm{d}x + \frac{\kappa}{\theta} \int x h(x) \mathrm{d}h(x) + \frac{\kappa\theta^2}{3} + \mathrm{o}(1).\end{aligned} \tag{C.13}$$

{{e:MDP3}}

Proof. We can reorganize (C.6), and {e:MDP3} rewrite it in terms of the q-Gamma functions

$$\begin{aligned}
&\prod_{i<j\leqslant N}\frac{(q^{\lambda_i-\lambda_j}t^{j-i};q)_\infty}{(q^{\lambda_i-\lambda_j}t^{j-i+1};q)_\infty}\frac{(t^{j-i+1};q)_\infty}{(t^{j-i};q)_\infty} &\text{(C.14)}\\
&=\prod_{i<j\leqslant N}\frac{(q^{\mathsf{x}_i-\mathsf{y}_j};q)_\infty}{(q^{\mathsf{x}_i-\mathsf{x}_j+\theta};q)_\infty}\frac{(q^{\theta(j-i+1)};q)_\infty}{(q^{\theta(j-i)};q)_\infty} &\text{(C.15)}\\
&=\prod_{i<j\leqslant N}\frac{\Gamma_q(\mathsf{x}_i-\mathsf{y}_j+\theta)}{\Gamma_q(\mathsf{x}_i-\mathsf{x}_j)}\frac{\Gamma_q(\theta(j-i))}{\Gamma_q(\theta(j-i+1))} &\text{(C.16)}\\
&=\prod_{i<j\leqslant N}\frac{\Gamma_q(\mathsf{x}_i-\mathsf{y}_j+\theta)}{\Gamma_q(\mathsf{x}_i-\mathsf{x}_j)}\prod_{i=1}^N\frac{\Gamma_q(\theta)}{\Gamma_q(i\theta)}. &\text{(C.17)}
\end{aligned}$$

The claim (C.10) follows. For the asymptotics (6.10), simply write $\lambda_i^{(N)}-(i-1)\theta=\mathsf{x}_i$. By our assumption (C.12), the exponent in the first term on the righthand side of (C.10) is

$$\frac{\kappa}{N^3}\sum_{i=1}^N(i-1)\theta(\mathsf{x}_i+(i-1)\theta)=\frac{\kappa}{N}\sum_{i=1}^N\frac{\theta i}{N}\frac{\mathsf{x}_i}{N}+\frac{\kappa\theta^2}{3}+\mathrm{O}(1/N)=\frac{\kappa}{\theta}\int xh(x)\mathrm{d}h(x)+\frac{\kappa\theta^2}{3}+\mathrm{o}(1) \quad \text{(C.18)}$$ {{e:Gammat0}}

We recall the following estimates for the q-Gamma function for $q\leqslant 1$, the first statement follows from [43, Corollary], and the second follows from [43, Theorem 1].

$$\begin{aligned}
&\left|\ln\Gamma_q(z)-(z-1/2)\ln\left(\frac{q^z-1}{q-1}\right)-\frac{1}{\ln q}\int_{-\ln q}^{-z\ln q}\frac{u\mathrm{d}u}{e^u-1}\right|\leqslant C,\quad z\geqslant\theta &\text{(C.19)}\\
&\left|\partial_z\ln\Gamma_q(z)-\ln\left(\frac{q^z-1}{q-1}\right)\right|\leqslant\frac{C}{z},\quad z\geqslant\theta. &\text{(C.20)}
\end{aligned}$$

So we have

$$\begin{aligned}
&\frac{1}{N^2}\sum_{i=1}^N\ln\Gamma_q(\theta)-\ln\Gamma_q(i\theta)=-\frac{1}{N^2}\sum_{i=1}^N\left(i\theta\ln\left(\frac{q^{i\theta}-1}{q-1}\right)+\frac{1}{\ln q}\int_0^{-i\theta\ln q}\frac{u\mathrm{d}u}{e^u-1}\right)+\mathrm{O}(1/N)\\
&=\frac{\theta}{2}\ln\left(\frac{\kappa}{N}\right)-\theta\int_0^1x\ln(1-e^{\kappa\theta x})\mathrm{d}x-\kappa\theta^2\int\frac{(1-x)x}{e^{-\kappa\theta x}-1}\mathrm{d}x+\mathrm{O}(\ln N/N).
\end{aligned} \quad \text{(C.21)}$$ {{e:Gammat1}}

and

$$\begin{aligned}
&\frac{1}{N^2}\sum_{i<j}\ln\frac{\Gamma(\mathsf{x}_i-\mathsf{x}_j+\theta)}{\Gamma(\mathsf{x}_i-\mathsf{x}_j)}=\frac{1}{N^2}\sum_{i<j}\theta\ln\left(\frac{q^{\mathsf{x}_i-\mathsf{x}_j}-1}{q-1}\right)+\mathrm{O}\left(\sum_{i<j}\frac{1}{\mathsf{x}_i-\mathsf{x}_j}\right)\\
&=\frac{1}{N^2}\sum_{i<j}\theta\ln\left(\frac{q^{\mathsf{x}_i-\mathsf{x}_j}-1}{q-1}\right)+\mathrm{O}\left(\ln N/N\right)\\
&=-\frac{\theta}{2}\ln\left(\frac{\kappa}{N}\right)+\frac{1}{2\theta}\int\ln(1-e^{\kappa|x-y|})\mathrm{d}h(x)\mathrm{d}h(y)+\mathrm{O}(\ln N/N).
\end{aligned} \quad \text{(C.22)}$$ {{e:Gammat2}}

where we used that $\mathsf{x}_i-\mathsf{x}_j\geqslant\theta(i-j)$ for the second inequality; in the last equality we used Lemma 2.8. The claim (C.13) follows {e:Gammat2} from plugging (C.18), (C.21) and (C.22) into (C.10). □

C.3 Proof of Theorem 1.7 {s:proofmain3}

Thanks to the relation (C.5), the large deviation asymptotics of the skew Macdonald polynomials can be obtained from the large deviation principle of the dynamics (C.8),

$$\begin{aligned}\frac{1}{N^2}\ln Q_{\boldsymbol{\lambda}/\boldsymbol{\mu}}(\boldsymbol{\beta}=(b_0,\cdots,b_{\mathsf{T}-1});q,t)&=\frac{1}{N^2}\ln\left(\mathbb{P}(\boldsymbol{\lambda}(\mathsf{T})=\boldsymbol{\lambda}|\boldsymbol{\lambda}(0)=\boldsymbol{\mu})\Pi(\rho,\boldsymbol{\beta}=(b_0,\cdots,b_{\mathsf{T}-1}))\right)\\&+\frac{1}{2\theta}\left(\int_{\mathbb{R}^2}\ln(1-e^{\kappa|x-y|})\mathrm{d}h(x,t)\mathrm{d}h(y,t)+2\kappa\int xh(x,t)\mathrm{d}h(x,t)\right)\Bigg|_{t=T}^{t=0}+\mathrm{o}(1),\end{aligned}\tag{C.23}$$ {{e:MDdynamics2

where we used Lemma C.4. The dynamics (6.20) and non-intersecting θ-Bernoulli random walks {e:MDdynamics2} (1.10) differ by the drift $E(\boldsymbol{x})D(\boldsymbol{x})$ with

$$D(\boldsymbol{x}):=\prod_{0\le\mathsf{t}\le\mathsf{T}-1}2^N b_{\mathsf{t}}^{\sum_{i=1}^N e_i(\mathsf{t})}\tag{C.24}$$

$$E(\boldsymbol{x}):=\prod_{0\le\mathsf{t}\le\mathsf{T}}\prod_{i<j}\frac{\mathsf{x}_i(\mathsf{t})-\mathsf{x}_j(\mathsf{t})}{\mathsf{x}_i(\mathsf{t})+\theta e_i(\mathsf{t})-(\mathsf{x}_j(\mathsf{t})+\theta e_j(\mathsf{t}))}\frac{q^{\mathsf{x}_i(\mathsf{t})+\theta e_i(\mathsf{t})}-q^{\mathsf{x}_j(\mathsf{t})+\theta e_j(\mathsf{t})}}{q^{\mathsf{x}_i(\mathsf{t})}-q^{\mathsf{x}_j(\mathsf{t})}}\tag{C.25}$$

Theorem 1.7 follows from plugging Lemma 6.6 and the following lemma to (C.23). We remark that all the terms involving κ cancels out perfectly.

Lemma C.5. *Adopt the notations and assumptions of Theorem 1.7, the following holds*

$$\frac{1}{N^2}\ln E(\boldsymbol{x})=\frac{1}{2\theta}\left(\iint\ln\frac{1-e^{\kappa|x-y|}}{-\kappa|x-y|}\mathrm{d}h(y,t)\mathrm{d}h(x,t)+2\kappa\int xh(x,t)\mathrm{d}h(x,t)\right)\Bigg|_{t=0}^{t=T}+\mathrm{o}(1)$$

Proof. We observe that

$$\frac{q^x-q^y}{x-y}=\frac{1}{\ln q}\int_0^1 q^{\alpha x+(1-\alpha)y}\mathrm{d}\alpha\tag{C.26}$$

so that

$$\begin{aligned}&\frac{\mathsf{x}_i(\mathsf{t})-\mathsf{x}_j(\mathsf{t})}{\mathsf{x}_i(\mathsf{t})+\theta e_i(\mathsf{t})-(\mathsf{x}_j(\mathsf{t})+\theta e_j(\mathsf{t}))}\frac{q^{\mathsf{x}_i(\mathsf{t})+\theta e_i(\mathsf{t})}-q^{\mathsf{x}_j(\mathsf{t})+\theta e_j(\mathsf{t})}}{q^{\mathsf{x}_i(\mathsf{t})}-q^{\mathsf{x}_j(\mathsf{t})}}\\&=\frac{q^{\theta e_j(\mathsf{t})}\int_0^1 q^{\alpha(\mathsf{x}_i(\mathsf{t})+\theta e_i(\mathsf{t})-\mathsf{x}_j(\mathsf{t})-\theta e_j(\mathsf{t}))}q\mathrm{d}\alpha}{\int_0^1 q^{\alpha(\mathsf{x}_i(\mathsf{t})-\mathsf{x}_j(\mathsf{t}))}\mathrm{d}\alpha}=q^{\theta e_j(\mathsf{t})}\mathbb{E}_{q^{\mathsf{x}_i(\mathsf{t})-\mathsf{x}_j(\mathsf{t})}}[q^{\alpha\theta(e_i(\mathsf{t})-e_j((t)))}]\end{aligned}\tag{C.27}$$

where for a real number β, $\mathbb{E}_\beta$ denotes the expectation over the variable $\alpha\in[0,1]$ given, for any bounded measurable function f on $[0,1]$, by

$$\mathbb{E}_\beta[f(\alpha)]=\frac{\int_0^1 f(\alpha)\beta^\alpha\mathrm{d}\alpha}{\int_0^1\beta^\alpha\mathrm{d}\alpha}.$$

We recall that $q=e^{\kappa/N}$, and denote $t=\mathsf{t}/N$, so that

$$\mathbb{E}_{q^{\mathsf{x}_i(\mathsf{t})-\mathsf{x}_j(\mathsf{t})}}[q^{\alpha\theta(e_i(\mathsf{t})-e_j((t)))}]=e^{\frac{\theta\kappa(e_i(\mathsf{t})-e_j(\mathsf{t}))}{N}\mathbb{E}_{e^{\kappa(x_i(t)-x_j(t))}}[\alpha]+\mathrm{O}\left(\frac{1}{N^2}\right)}$$

where $\mathrm{O}(1/N^2)$ is uniform over the $x_i(\mathsf{t}),x_i(\mathsf{t})$ and over θ in a bounded set. Recall that $e_i(\mathsf{t})=\mathsf{x}_i(\mathsf{t}+1)-\mathsf{x}_i(\mathsf{t})$. Hence, we deduce that if $f(y):=\mathbb{E}_{e^y}[\alpha]$,

$$\begin{aligned}E(\boldsymbol{x})=&\prod_{0\leqslant\mathsf{t}\leqslant\mathsf{T}-1}e^{\frac{\kappa}{N}\sum_{1\leqslant i\leqslant N}\theta(i-1)e_i(\mathsf{t})}\times\\&\times e^{\theta\kappa\sum_{0\le\mathsf{t}\le\mathsf{T}-1:t=\mathsf{t}/N}\sum_{i<j}f(\kappa(x_i(t)-x_j(t)))((x_i(t+1/N)-x_j(t+1/N))-(x_i(t)-x_j(t)))+\mathrm{O}(N)}.\end{aligned}\tag{C.28}$$ {{dens}}

For the first term on the righthand side of (C.28), we have {dens}

$$\begin{aligned} &\frac{\kappa}{N^3}\sum_{\mathsf{t}=0}^{\mathsf{T}-1}\sum_{i=1}^{N}(i-1)\theta(\mathsf{x}_i(\mathsf{t}+1)-\mathsf{x}_i(\mathsf{t})) = \frac{\kappa}{N^2}\sum_{i=1}^{N}(i-1)\theta(x_i(T)-x_i(0)) \\ &= \frac{\kappa}{N}\sum_{i=1}^{N}\frac{\theta i}{N}(x_i(T)-x_i(0)) + \mathrm{O}(1/N) = \frac{\kappa}{\theta}\int xh(x,t)\mathrm{d}h(x,t)\Big|_{t=0}^{t=T} + \mathrm{o}(1) \end{aligned} \tag{C.29}$$ {{e:tt1}}

For the second term on the righthand side of (C.28), let {e:tt1}

$$F(x) = \int_0^{\kappa x} f(y)\mathrm{d}y = \int_0^{\kappa x}\partial_y \ln\int_0^1 e^{\alpha y}\mathrm{d}\alpha\mathrm{d}y = \ln\int_0^1 e^{\alpha\kappa x}\mathrm{d}\alpha = \ln\frac{1-e^{\kappa x}}{-\kappa x} \tag{C.30}$$

so that $F'(x) = \kappa f(\kappa x)$, and

$$\begin{aligned} &\kappa f(\kappa(x_i(t)-x_j(t)))(x_i(t+1/N)-x_j(t+1/N)-(x_i(t)-x_j(t))) \\ &= F(x_i(t+1/N)-x_j(t+1/N)) - F(x_i(t)-x_j(t)) + \mathrm{O}\left(\frac{1}{N^2}\right) \end{aligned} \tag{C.31}$$

where the error is again uniform since f is uniformly Lipschitz as $|\partial_y f(y)| \leq 2$. Therefore, for the second term on the righthand side of (C.28), we have

$$\begin{aligned} &\frac{\theta}{N^2}\sum_{i<j}F(x_i(T)-x_j(T)) - F(x_i(0)-x_j(0)) + \mathrm{o}(1) \\ &= \frac{1}{2\theta}\iint F(|x-y|)\rho(y;\boldsymbol{x}(t))\rho(x;\boldsymbol{x}(t))\mathrm{d}y\mathrm{d}x\Big|_{t=0}^{t=T} + \mathrm{O}(1/N) \end{aligned} \tag{C.32}$$ {{e:tt2}}

where we finally used that F is uniformly Lipschitz on compacts. Finally, we can see that the above right hand side is a continuous function of $H(y;\boldsymbol{x}(T))$ and $H(y;\boldsymbol{x}(0))$ equipped with the uniform topology, provided its derivative is a probability measure. The proof follows the ideas from Lemma 2.8, so we omit the details. It follows from plugging (C.29) and (C.32) into (C.28), {e:tt2}

$$\frac{1}{N^2}\ln E(\boldsymbol{x}) = \frac{1}{2\theta}\left(\iint \ln\frac{1-e^{\kappa|x-y|}}{-\kappa|x-y|}\mathrm{d}h(y,t)\mathrm{d}h(x,t) + 2\kappa\int xh(x,t)\mathrm{d}h(x,t)\right)\Big|_{t=0}^{t=T} + \mathrm{o}(1) \tag{C.33}$$

□

References

[1] A. Aggarwal, A. Borodin, L. Petrov, and M. Wheeler. Free fermion six vertex model: symmetric functions and random domino tilings. *Selecta Mathematica*, 29(3):36, 2023.

[2] A. Aggarwal, A. Borodin, and M. Wheeler. Colored fermionic vertex models and symmetric functions. *Communications of the American Mathematical Society*, 3(08):400–630, 2023.

[3] G. W. Anderson, A. Guionnet, and O. Zeitouni. *An introduction to random matrices*, volume 118 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 2010.

[4] K. Astala, E. Duse, I. Prause, and X. Zhong. Dimer models and conformal structures. *arXiv preprint arXiv:2004.02599*, 2020.

[5] S. Belinschi, A. Guionnet, and J. Huang. Large deviation principles via spherical integrals. *Probability and Mathematical Physics*, 3(3):543–625, 2022.

[6] A. Borodin. On a family of symmetric rational functions. *Advances in Mathematics*, 306:973–1018, 2017.

[7] A. Borodin, A. Bufetov, and M. Wheeler. Between the stochastic six vertex model and hall-littlewood processes. *arXiv preprint arXiv:1611.09486*, 2016.

[8] A. Borodin and I. Corwin. Macdonald processes. *Probability Theory and Related Fields*, 158(1):225–400, 2014.

[9] A. Borodin and V. Gorin. General β-jacobi corners process and the Gaussian free field. *Communications on Pure and Applied Mathematics*, 68(10):1774–1844, 2015.

[10] A. Borodin and L. Petrov. Higher spin six vertex model and symmetric rational functions. *Selecta Mathematica*, 24(2):751–874, 2018.

[11] A. Bufetov and V. Gorin. Representations of classical lie groups and quantized free convolution. *Geometric and Functional Analysis*, 25:763–814, 2015.

[12] A. Bufetov and V. Gorin. Fluctuations of particle systems determined by schur generating functions. *Advances in Mathematics*, 338:702–781, 2018.

[13] A. Bufetov and V. Gorin. Fourier transform on high-dimensional unitary groups with applications to random tilings. *Duke Mathematical Journal*, 168(13):2559 – 2649, 2019.

[14] A. Bufetov and A. Knizel. Asymptotics of random domino tilings of rectangular aztec diamonds. In *Annales de l'Institut Henri Poincaré, Probabilités et Statistiques*, volume 54, pages 1250–1290. Institut Henri Poincaré, 2018.

[15] H. Cohn, R. Kenyon, and J. Propp. A variational principle for domino tilings. *Journal of the American Mathematical Society*, 14(2):297–346, 2001.

[16] C. Cuenca. Pieri integral formula and asymptotics of jack unitary characters. *Selecta Mathematica*, 24(3):2737–2789, 2018.

[17] C. Cuenca et al. Asymptotic formulas for macdonald polynomials and the boundary of the (q, t)-gelfand-tsetlin graph. *SIGMA. Symmetry, Integrability and Geometry: Methods and Applications*, 14:001, 2018.

[18] V. Gorin. *Lectures on random lozenge tilings*, volume 193. Cambridge University Press, 2021.

[19] V. Gorin and J. Huang. Dynamical loop equation. *arXiv preprint arXiv:2205.15785*, 2022.

[20] V. Gorin and K. Liechty. Boundary statistics for the six-vertex model with dwbc. *arXiv preprint arXiv:2310.12735*, 2023.

[21] V. Gorin and G. Panova. Asymptotics of symmetric polynomials with applications to statistical mechanics and representation theory. *The Annals of Probability*, 43(6):3052 – 3132, 2015.

[22] V. Gorin and L. Petrov. Universality of local statistics for noncolliding random walks. *The Annals of Probability*, 47(5):2686–2753, 2019.

[23] V. Gorin and M. Shkolnikov. Multilevel dyson brownian motions via jack polynomials. *Probability Theory and Related Fields*, 163(3-4):413–463, 2015.

[24] A. Guionnet. Large deviations upper bounds and central limit theorems for non-commutative functionals of gaussian large random matrices. In *Annales de l'Institut Henri Poincare (B) Probability and Statistics*, volume 38, pages 341–384. Elsevier, 2002.

[25] A. Guionnet. First order asymptotics of matrix integrals; a rigorous approach towards the understanding of matrix models. *Comm. Math. Phys.*, 244(3):527–569, 2004.

[26] A. Guionnet and J. Huang. Asymptotics of rectangular spherical integrals. *Journal of Functional Analysis*, 285(11):110144, 2023.

[27] A. Guionnet and J. Husson. Asymptotics of k dimensional spherical integrals and applications. *ALEA Lat. Am. J. Probab. Math. Stat.*, 19(1):769–797, 2022.

[28] A. Guionnet and M. Maïda. A Fourier view on the R-transform and related asymptotics of spherical integrals. *J. Funct. Anal.*, 222(2):435–490, 2005.

[29] A. Guionnet and O. Zeitouni. Large deviations asymptotics for spherical integrals. *J. Funct. Anal.*, 188(2):461–515, 2002.

[30] A. Guionnet and O. Zeitouni. Addendum to: "Large deviations asymptotics for spherical integrals" [J. Funct. Anal. **188** (2002), no. 2, 461–515; mr1883414]. *J. Funct. Anal.*, 216(1):230–241, 2004.

[31] J. Huang. β-nonintersecting poisson random walks: Law of large numbers and central limit theorems. *International Mathematics Research Notices*, 2017.

[32] J. Huang. Law of large numbers and central limit theorems through jack generating functions. *Advances in Mathematics*, 380:107545, 2021.

[33] J. Huang and C. McSwiggen. Asymptotics of generalized bessel functions and weight multiplicities via large deviations of radial dunkl processes. *arXiv preprint arXiv:2305.04131*, 2023.

[34] H. Jack. I.?a class of symmetric polynomials with a parameter. *Proceedings of the Royal Society of Edinburgh Section A: Mathematics*, 69(1):1–18, 1970.

[35] H. Jack. A surface integral and symmetric functions. *Proceedings of the Royal Society of Edinburgh Section A: Mathematics*, 69(4):347–364, 1972.

[36] R. Kenyon and A. Okounkov. Limit shapes and the complex Burgers equation. *Acta Mathematica*, 199(2):263 – 302, 2007.

[37] S. Kerov, A. Okounkov, and G. Olshanski. The boundary of the Young graph with Jack edge multiplicities. *Internat. Math. Res. Notices*, (4):173–199, 1998.

[38] S. V. Kerov and N. Tsilevich. *Asymptotic representation theory of the symmetric group and its applications in analysis*, volume 219. American Mathematical Society Providence, RI, 2003.

[39] W. König, N. O'Connell, and S. Roch. Non-colliding random walks, tandem queues, and discrete orthogonal polynomial ensembles. *Electronic Journal of Probability*, 7, 2002.

[40] I. G. Macdonald. *Symmetric functions and Hall polynomials.* Oxford university press, 1998.

[41] I. G. Macdonald. *Symmetric functions and Hall polynomials.* Oxford Classic Texts in the Physical Sciences. The Clarendon Press, Oxford University Press, New York, second edition, 2015. With contribution by A. V. Zelevinsky and a foreword by Richard Stanley.

[42] K. Matveev. Macdonald-positive specializations of the algebra of symmetric functions: Proof of the kerov conjecture. *Annals of Mathematics*, 189(1):277–316, 2019.

[43] D. S. Moak. The q-analogue of stirling's formula. *The Rocky Mountain Journal of Mathematics*, pages 403–413, 1984.

[44] A. Okounkov. Infinite wedge and random partitions. *Selecta Mathematica*, 7(1):57, 2001.

[45] A. Okounkov and G. Olshanski. Asymptotics of Jack polynomials as the number of variables goes to infinity. *International Mathematics Research Notices*, 1998(13):641–682, 01 1998.

[46] A. Okounkov and N. Reshetikhin. Correlation function of schur process with application to local geometry of a random 3-dimensional young diagram. *Journal of the American Mathematical Society*, 16(3):581–603, 2003.

[47] I. Pak and G. Panova. Bounds on kronecker coefficients via contingency tables. *Linear Algebra and its Applications*, 602:157–178, 2020.

[48] I. Pak and G. Panova. Durfee squares, symmetric partitions and bounds on kronecker coefficients. *Journal of Algebra*, 629:358–380, 2023.

[49] I. Pak, G. Panova, and D. Yeliussizov. On the largest kronecker and littlewood–richardson coefficients. *Journal of Combinatorial Theory, Series A*, 165:44–77, 2019.

[50] L. Petrov. Asymptotics of random lozenge tilings via gelfand–tsetlin schemes. *Probability theory and related fields*, 160(3-4):429–487, 2014.

[51] L. Petrov. Asymptotics of uniformly random lozenge tilings of polygons. Gaussian free field. *The Annals of Probability*, 43(1):1–43, 2015.

[52] D. D. Silva and O. Savin. Minimizers of convex functionals arising in random surfaces. *Duke Mathematical Journal*, 151(3):487 – 532, 2010.

[53] R. P. Stanley. Some combinatorial properties of Jack symmetric functions. *Adv. Math.*, 77(1):76–115, 1989.

[54] R. P. Stanley. Enumerative combinatorics volume 1 second edition. *Cambridge studies in advanced mathematics*, 2011.

[55] A. M. Vershik and S. V. Kerov. Characters and factor representations of the infinite unitary group. In *Doklady Akademii Nauk*, volume 267, pages 272–276. Russian Academy of Sciences, 1982.

[56] D. Zagier. The dilogarithm function. In *Frontiers in Number Theory, Physics, and Geometry II: On Conformal Field Theories, Discrete Groups and Renormalization*, pages 3–65. Springer, 2007.