Large deviations upper bounds and central limit theorems for band matrices and non-commutative functionnals of Gaussian large random matrices

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Abstract : We obtain large deviation upper bounds and central limit theorems for non commutative functionnals of large Gaussian Wigner matrices and deterministic diagonal matrices with converging spectral measure. As a consequence, we derive such type of results for Gaussian band matrices and generalized sample covariance matrices.

1. Introduction

During the last decade, the understanding of the asymptotic behaviour of large random matrices has considerably improved since the pioneer works of Wigner [Wig], Arnold [Arn], Wachter [Wac] , Wishart [Wis] and Pastur and Marchenko [P,M]. These papers were mainly motivated by Quantum Physics and proved convergence of the spectral measure of these matrices as their size goes to infinity under diverse assumptions on the distribution of their entries ; Wigner [Wig] studied a random $N \times N$ Hermitian matrix with i.i.d. complex (or real) entries (except for the symmetry constraint), Wishart [Wis] (see also Wachter [Wac])introduced the $N \times N$ Hermitian matrix $X_N X_N^*$ with X_N a $N \times M$ matrix with i.i.d. complex (or real) entries, Pastur and Marchenko considered band matrices where the entries are non zero only on some band surrounding the origin and generalized sample covariance matrices of the form $X_N R X_N^*$ with X_N as above and a $M \times M$ deterministic matrix R with converging spectral distribution (see [Sh],[K,K,P,S] and [BE]). We send the reader to [Bai] and [K,K,P,S] for reviews on the subject.

The fluctuations of the spectral measure around its limit for Wigner's matrix with Gaussian entries were first obtained by K. Johansson [Joh] (see also [C-D]). The fluctuations of the spectral measure around its expectation were studied under much more general assumptions over the entries and for most of the models described above (see [So], [B-M,K],[K,K,P] and references therein). However, such statements are weaker than the result obtained by K. Johansson [Joh] for the Gaussian orthogonal ensemble. In this paper, we shall generalize K. Johansson's type of result to band matrices with Gaussian entries and for polynomial test functions.

Large deviations for the law of the spectral measure of Wigner's matrix with Gaussian entries were obtained in [BA,G] and for related models in [BA,Z] and [H,P]. There is actually no clue how to extend these results to non Gaussian entries. In [G,Z], the authors obtained concentration's inequalities for the spectral measure of the above matrices under various hypotheses on the distribution of the entries. However, even though this paper provides concentration on the right scale, there is no hope to deduce complementary lower bounds. Here, we shall obtain large deviation upper bounds for the deviations of the spectral measure of Gaussian band matrices, which we hope optimal. Note at this point that the joint law of the eigenvalues of Gaussian band matrices is a priori complicated, being given by a $N \times N$ Jacobian which does not lead to simple formulae since the law of Gaussian band matrices are not invariant under the action of a group such as the unitary (or orthogonal)group on the contrary of Wigner's matrices. In particular, the techniques of [BA,G] are useless here. In the direction of interests encountered in free probability, deviations of the non-commutative law of a couple of independent Gaussian Wigner's matrices were studied in [C-D,G] using a functionnal approach based on stochastic calculus. We shall follow a similar approach in this work.

This result will in turn provide a large deviation upper bound for the spectral measure of generalized Gaussian sample covariance matrices.

However, the goal of this paper is not only to consider functions of the spectral measure of large random matrices but more general non-commutative functionnals involving large random matrices and an algebra of deterministic diagonal matrices. Such functionnals were already introduced in [Sh] where the author obtained law of large numbers type of statements for the normalized trace of these functionnals thanks to free probability technics (more precisely the notion of freeness with amalgamation). As a consequence, D. Shlyakhtenko deduced the convergence of the spectral measure for Gaussian band matrices. The strategy followed in this paper is intimately related to the ideas of [Sh] but we shall push forward the analysis to obtain large deviation upper bounds and central limit theorems. In particular, we define a good rate function governing the large deviations of these non-commutative functionnals and a self adjoint positive definite operator defining the covariance of the central limit theorem. Large deviation for non-commutative variables were already obtained in [C-D,G] and a central limit theorem in [C-D] for independent Gaussian Wigner's matrices. The main difference here is that we consider a single random matrix and a deterministic algebra of diagonal matrices. Some of our statements could be interpreted in terms of free probability. However, we shall not discuss this aspect in details here.

The paper is organized as follows ; we begin with the introduction of our notations and results. We then introduce Itô's calculus for band matrices which is the key to all our proofs. In section 4, we state and prove a large deviation upper bound. Studying the minimizer of our rate function, we deduce a law of large numbers theorem in section 5. It is supplemented in section 6 by a Central limit Theorem. We also describe in the next section how these results can be interpreted in terms of inhomogeneous sample covariance matrices.

2. Notations and statement of the results

Hereafter, $\mathcal{M}_N$ will denote the set of $N \times N$ matrices with complex entries. $\mathcal{H}_N$ will be the subset of $\mathcal{M}_N$ of Hermitian matrices. We set $\mathcal{M} = \cup_{N \in I\!N} \mathcal{M}_N$ and $\mathcal{H} = \cup_{N \in I\!N} \mathcal{H}_N$. tr will denote the natural extension of the trace to $\mathcal{M}$ given, for any $A \in \mathcal{M}_N$, $N \in I\!N$, by $\mathrm{tr}(A) = \sum_{i=1}^N A_{ii}$ and tr_N the normalized trace $\mathrm{tr}_N(A) = N^{-1}\mathrm{tr}(A)$ for $A \in \mathcal{M}_N$, $N \in I\!N$. We shall consider, for $N \in I\!N$, the random matrix in $\mathcal{H}_N$

$$(X_N)_{ij} = (H_N)_{ij} \psi_N(i,j)^{\frac{1}{2}}$$

where H_N is a Hermitian matrix with complex Gaussian entries with covariance N^{-1} and ψ_N is a non negative symmetric function on $\{1,..,N\}^2$ which can be decomposed as

$$\psi_N(x,y) = \int \sigma_\tau^N(x)\sigma_\tau^N(y)dp(\tau)$$

with a measure p on a Polish space (Ω, Σ) with finite mass, and bounded functions $(\sigma_\tau^N, \tau \in \Omega)$ on $\{1,..,N\}$ such that $\tau \to \sigma_\tau^N(x)$ is measurable for the sigma-algebra Σ for any $x \in \{1,..,N\}$. We can assume without loss of generality that the total mass of p is one to simplify the notations. We shall assume that, if Δ_τ^N denotes the $N \times N$ matrix with diagonal elements $(\sigma_\tau^N(i), 1 \le i \le N)$,

(H0)*for any* $(\tau_1,..,\tau_n) \in \Omega^n$, $n \in I\!N$, *the joint distribution (in the non-commutative sense)of* $(\Delta_{\tau_1},..,\Delta_{\tau_n})$ *converges, i.e there exists a probability measure* $m_{\tau_1,..,\tau_n}$ *on* $\mathbb{R}$ *so that for every bounded continuous function* f *on* $\mathbb{R}$,

$$\lim_{N\to\infty} \frac{1}{N}\sum_{i=1}^N f\left(\prod_{j=1}^n \sigma_{\tau_j}^N(i)\right) = \int f(x)dm_{\tau_1,..,\tau_n}(x). \tag{2.1}$$

Further, we suppose that

$$T \equiv \sup_{\tau\in\Omega}\sup_{N\in I\!N}\sup_{x\in\{1,..,N\}} |\sigma_\tau^N(x)| < \infty.$$

It is convenient to consider, in order to use the powerful tool of stochastic differential calculus, X_N as the value at time one of the $\mathcal{H}_N$-valued process

$$(X_N(t))_{ij} = (H_N(t))_{ij}\psi_N(i,j)^{\frac{1}{2}}$$

where $H_N(t)$ is the Hermitian Brownian motion which is described on the space $\mathcal{H}_N$ of Hermitian matrices of dimension N as the Markov process $(H_N(t))_{t\in\mathbb{R}^+}$ with values in $\mathcal{H}_N$ and independent complex Brownian motions entries so that

$$E[H_N^{i,j}(t)H_N^{k,l}(s)] = \frac{t\wedge s}{N}\delta_i^l\delta_k^j$$

More precisely, we can construct the entries $\left\{H_N^{i,j}(t), t \ge 0, (i,j) \in \{1,..,N\}\right\}$ via independent real valued Brownian motions $(\beta_{i,j}, \beta'_{k,l})_{1\le i\le j\le N}^{1\le k<l\le N}$ by

$$\begin{aligned} H_N^{k,l} &= \frac{1}{\sqrt{2N}}\left(\beta_{k,l} + i\beta'_{k,l}\right) \text{ if } k<l \\ &= \frac{1}{\sqrt{2N}}\left(\beta_{l,k} - i\beta'_{l,k}\right) \text{ if } k>l \\ &= \frac{1}{\sqrt{N}}\beta_{l,l} \text{ if } k=l. \end{aligned}$$

To take into account the inhomogeneity of the covariance of X_N, we shall, following D. Shlyakhtenko [Sh], consider jointly the matrix-valued process $(X_N(t), t \in$

$[0,1]$) and diagonal matrices. To this end, let us introduce a set $\mathbb{D}$ of sequences Δ of uniformly bounded converging diagonal matrices Δ^N of $\mathcal{H}_N$ (hence with real entries) that is sequences $\Delta = (\Delta^N)_{N\in I\!N}$ so that, if $(\lambda_1^N, .., \lambda_N^N)$ denotes the eigenvalues of Δ^N,

$$\sup_{N\in I\!N} \sup_{i\in\{1,..,N\}} |\lambda_i^N| < \infty \tag{2.2}$$

and $\frac{1}{N}\sum_{i=1}^N \delta_{\lambda_i^N}$ converges as N tends to infinity for the weak topology, i.e there exists a probability measure m_Δ on $\mathbb{R}$ so that for any bounded continuous function f,

$$\lim_{N\to\infty} \frac{1}{N}\sum_{i=1}^N f(\lambda_i^N) = \int f(x)dm_\Delta(x). \tag{2.3}$$

In the sequel, we write in short

$$m(\Delta) = \int x dm_\Delta(x), \qquad \forall \Delta \in \mathcal{D}.$$

We shall consider a subalgebra $\mathcal{D}$, that is stable by product and sum, of $\mathbb{D}$ containing the real vector space generated by the identity and the null matrices as well as the sequences

$$\mathcal{D}_\psi = \{\Delta_\tau = (\Delta_\tau^N = \left(\delta_{i=j}\sigma_\tau^N(i)\right)_{1\le i,j\le N})_{N\in I\!N}, \quad \tau \in \Omega\}.$$

We endow $\mathbb{D}$ with the norm given, for any $\Delta, \tilde{\Delta} \in \mathbb{D}$ by

$$|\Delta - \tilde{\Delta}|_\infty = \sup_{N\in I\!N} \sup_{i\in\{1,..,N\}} |\Delta_i^N - \tilde{\Delta}_i^N|$$

and assume that $\mathcal{D}$ is separable for this norm.

Examples (2.4). *a) The first example one should keep in mind is when*

$$\sigma_\tau^N(i) = \sigma_\tau(\frac{i}{N})$$

with $\sigma_\tau \in \mathcal{C}_b([0,1],\mathbb{R})$ for $\tau \in \Omega$. In the sequel, we shall denote $\Delta(\phi) \in \mathcal{D}_c$ the sequence

$$\Delta(\phi) \equiv \left((\Delta_N(\phi))_{ij} = \delta_{i=j}\phi(\frac{i}{N})\right)_{N\in I\!N}$$

for $\phi \in \mathcal{C}_b([0,1],\mathbb{R})$. One can choose $\mathcal{D}$ to be the set

$$\mathcal{D}_c \equiv \{\Delta : \quad \exists \phi \in \mathcal{C}_b([0,1],\mathbb{R}); \quad \Delta = \Delta(\phi)\}.$$

$\mathcal{D}_c$ is clearly an algebra and is separable for $|\ \ |_\infty$ since $\mathcal{C}_b([0,1],\mathbb{R})$ is separable for the uniform norm. (2.3) is fulfilled since, for any $f \in \mathcal{C}_b(\mathbb{R})$, for any $\phi \in \mathcal{C}_b([0,1],\mathbb{R})$

$$\lim_{N\to\infty} \mathrm{tr}_N\left(f(\Delta_N(\phi))\right) = \lim_{N\to\infty}\frac{1}{N}\sum_{i=1}^N f\left(\phi(\frac{i}{N})\right) = \int_0^1 f(\phi(x))dx = \int f(x)dm_{\Delta(\phi)}(x)$$

with $m_{\Delta(\phi)} = 1_{[0,1]} dx \circ \phi^{-1}$.

b) However, the general scheme proposed above may be useful to include the case where, for instance,

$$\sigma_\tau^N(i) = \sigma_\tau(\frac{i}{N}) + 1_{0 \le \frac{i}{N} \le \frac{M_N}{N}} \tilde{\sigma}_\tau(\frac{i}{N})$$

for some integer number M_N *and bounded continuous functions* σ_τ *and* $\tilde{\sigma}_\tau$. *We assume*

$$\lim_{N\to\infty} \frac{M_N}{N} = \alpha.$$

Choosing for instance $\mathcal{D}$ *to be*

$$\mathcal{D}_d = \left\{ \Delta : \quad \exists \phi, \tilde{\phi} \in \mathcal{C}_b([0,1], \mathbb{R}); (\Delta^N)_{ij} = \delta_{i=j}(\phi(\frac{i}{N}) + 1_{0 \le \frac{i}{N} \le \frac{M_N}{N}} \tilde{\phi}(\frac{i}{N})), N \in I\!N \right\},$$

$\mathcal{D}_d$ *can be seen to be a separable algebra for* $|\ \ |_\infty$. *(2.3) is also easily checked. This second example will appear naturally when we shall consider generalized Wishart's matrices.*

We shall see also an element Δ of $\mathbb{D}$ as a function from $\mathcal{H}$ into $\mathcal{H}$ by setting for any $X \in \mathcal{H}_N$, $N \in I\!N$, $\Delta(X) = \Delta^N$.

In [Sh], the author considered the random variables

$$\left\{ \mathrm{tr}_N \left(P(X_N(1), \Delta_1^N, .., \Delta_n^N) \right), \quad \Delta_1, .., \Delta_n \in \mathcal{D}, n \in I\!N \right\}$$

for non-commutative polynomial functions P of $n+1$ variables, and proved their convergence as N goes to infinity. Because the associated topology inherited for instance on the spectral measure of X_N is not the weak topology, we shall, as in [C-D,G], consider other test functions than polynomials. Such test functions shall belong to the set $\mathcal{E}(\mathbb{C})$ of functions on $\mathcal{H}$ so that for any $N \in I\!N$, $F \in \mathcal{E}(\mathbb{C})$ maps $\mathcal{H}_N$ into $\mathcal{M}_N$. $\mathcal{E}(\mathbb{R})$ will be the subset of Hermitian matrix-valued functions of $\mathcal{E}(\mathbb{C})$. Note that if f is a real function, we can define the function F on $\mathcal{H}$ so that, if $X \in \mathcal{H}$, $X = U^* D U$ for a diagonal matrix D and a unitary matrix U,

$$F(X) = U^* f(D) U, \qquad f(D)_{ij} = \delta_{i=j} f(D_{ii}).$$

It is straightforward that F belongs to $\mathcal{E}(\mathbb{R})$. In particular, for any $z \in \mathbb{C}\backslash\mathbb{R}$, $X \to (z - X)^{-1}$ is an element of $\mathcal{E}(\mathbb{C})$. We shall be particularly interested in the following by the complex vector space $F_{\mathbb{C}}(X, \mathcal{D}) \subset \mathcal{E}(\mathbb{C})$ generated by

$$\{F : \mathcal{H} \to \mathcal{M}; F(X) = \prod_{i=1}^n \frac{1}{z_i - \alpha_i X} \Delta_i(X),$$

$$(z_i)_{1 \le i \le n} \in (\mathbb{C}\backslash\mathbb{R})^n, \alpha_i \in \{0,1\}, \Delta_i \in \mathcal{D}, n \in I\!N\}.$$

Note that $F_{\mathbb{C}}(X, \mathcal{D})$ is an algebra since $\mathcal{D}$ is. Further, it contains 0 and 1 since $\mathcal{D}$ does. $F_{\mathbb{R}}(X, \mathcal{D})$ shall be the real vector space of the Hermitian matrices valued functions of $F_{\mathbb{C}}(X, \mathcal{D})$.

We shall prove the following law of large numbers

Theorem (2.5). *For any $F \in F_{\mathbb{C}}(X,\mathcal{D})$, any $t \in [0,1]$, $\mathrm{tr}_N(F(X_N(t)))$ converges almost surely as N goes to infinity. Its limit, denoted $\mu_t^*(F)$, is described in section 5.*

In particular, if ψ_N is as in examples (2.4) and taking functions of $F_{\mathbb{C}}(X,\mathcal{D})$ which are products of one Stieljes functions and one diagonal matrix, we find that, if ψ is the function on $[0,1]^2$ given by

$$\psi(x,y) = \lim_{N\to\infty} \psi_N([Nx],[Ny]),$$

Corollary (2.6). *For any $t \in [0,1]$, any z in $\mathbb{C}\backslash\mathbb{R}$, any $\phi \in \mathcal{C}_b([0,1],\mathbb{R})$,*

$$N^{-1}\sum_{i=1}^{N} \phi(\frac{i}{N})[(z-X_N(t))^{-1}]_{ii}$$

converges almost surely towards $t^{-\frac{1}{2}}\int_0^1 \phi(x)k(x,t^{-\frac{1}{2}}z)dx$ where, if K is the operator in $L^2([0,1])$ with kernel ψ, k is the unique analytic solution of

$$k(x,z) = (z - K(k(.,z))(x))^{-1}$$

so that $zk(x,z)$ goes to one as $|z|$ goes to infinity for any $x \in [0,1]$.

See Lemma (5.10) for details.

Further, by density of $F_{\mathbb{C}}(X,\mathcal{D})$ in the set of non-commutative polynomial functions and controls of the normalized trace of moments of $X_N(t)$, we shall see that theorem (2.5) implies that

Corollary (2.7) . *For any $t \in [0,1]$, any $\Delta_1,\Delta_2..,\Delta_n \in \mathcal{D}$, any non-commutative polynomial function P of $n+1$ variables, $\mathrm{tr}_N(P(X_N(t),\Delta_1^N,..,\Delta_n))$ converges almost surely towards a well defined limit denoted $\mu_t^*(P)$.*

Hence, we find again the results of [Sh] and [P,M]. This last result is precised in section 6 by a central limit theorem which validity requires the following extra hypotheses that

(H1)*For any $\Delta \in \mathcal{D}$,*

$$N\left(\mathrm{tr}_N(\Delta^N) - m(\Delta)\right)$$

converges as N goes to infinity towards a constant $c(\Delta)$.

Note that as $\mathcal{D}$ must contain $\mathcal{D}_\psi$, this last assumption also applies to $(\Delta_\tau, \tau \in \Omega)$. We shall also impose

(H2)*For any $\Delta_1,..,\Delta_n \in \mathcal{D}$, any $m \in I\!N$, any non-commutative polynomial function P of $n+m$ variables,*

$$\sup_{\tau_1,..,\tau_m\in\Omega}\sup_{N\in I\!N} N\left|\mathrm{tr}_N(P(\Delta_{\tau_1}^N,..,\Delta_{\tau_m}^N,\Delta_1^N,..,\Delta_n^N) - m(P(\Delta_\tau,..,\Delta_{\tau_m},\Delta_1,..,\Delta_n))\right| < \infty.$$

Then, we will show the

Theorem (2.8). *Under (H1) and (H2), for any* $t \in [0,1]$*, any* $n \in I\!N$*; any* $\Delta_1, .., \Delta_n \in \mathcal{D}$*, any non-commutative polynomial function* P *of* $n+1$ *variables* $N\left(\mathrm{tr}_N(P(X_N(t), \Delta_1^N, .., \Delta_n^N)) - \mu_t^*(P)\right)$ *converges in law as* N *goes to infinity towards a (eventually not centered) Gaussian law.*

We send the reader to section 6 for the definition of the mean and the covariance of the above Gaussian law. Let us give the following

Examples (2.9). *We consider again the examples given in (2.4).*

a) In the first example, we consider the case where

$$\sigma_\tau^N(i) = \sigma_\tau(\frac{i}{N}).$$

To obtain a central limit theorem, we shall assume that σ_τ *belongs to* $\mathcal{C}_b^1([0,1], \mathbb{R})$ *for* $\tau \in \Omega$ *and that, if* $|| \quad ||_u$ *is the uniform norm on* $\mathcal{C}_b([0,1], \mathbb{R})$

$$\sup_{\tau \in \Omega} ||\sigma_\tau'||_u < \infty. \tag{2.10}$$

One can then choose $\mathcal{D}$ *to be the set*

$$\mathcal{D}_c' = \left\{\Delta(\phi), \quad \phi \in \mathcal{C}_b^1([0,1], \mathbb{R})\right\}.$$

$\mathcal{D}_c'$ *is clearly an algebra. Further, (H1) is fulfilled since for any* $\phi \in \mathcal{C}_b^1([0,1])$*,*

$$\lim_{N\to\infty} N\left(\frac{1}{N}\sum_{i=1}^N \phi(\frac{i}{N}) - \int_0^1 \phi(x)dx\right) = \frac{1}{2}(\phi(1) - \phi(0)).$$

Also, for (H2), note that for any $\tau_1, .., \tau_m \in \Omega$*, any* $\Delta_1, .., \Delta_n \in \mathcal{D}$*, any non-commutative polynomial function* P *of* $n+m$ *variables,* $P(\Delta_{\tau_1}^N, .., \Delta_{\tau_m}^N, \Delta_1^N, .., \Delta_n^N) = \Delta^N(\phi)$ *for some* $\phi \in \mathcal{C}_b^1([0,1])$ *and that*

$$N|\mathrm{tr}_N(\Delta^N(\phi)) - \int_0^1 \phi(x)dx| \le ||\phi'||_u.$$

The fact that this bound does not depend on the choice of $\tau_1, .., \tau_m \in \Omega$ *is easily derived from (2.10).*

b) In the case where

$$\sigma_\tau^N(i) = \sigma_\tau(\frac{i}{N}) + 1_{0 \le \frac{i}{N} \le \frac{M_N}{N}} \tilde{\sigma}_\tau(\frac{i}{N})$$

for some number M_N *and continuously differentiable functions* σ_τ *and* $\tilde{\sigma}_\tau$*, (H1) and (H2) can also be fulfilled provided*

$$M_N - \alpha N$$

converges towards a constant $c(\alpha)$ and

$$\sup_{\tau\in\Omega} ||\sigma'_\tau||_u < \infty, \qquad \textit{and} \quad \sup_{\tau\in\Omega} ||\tilde{\sigma}'_\tau||_u < \infty.$$

Note that if M_N is an integer number, the first assumption should only be valid along subsequences in general. We then choose $\mathcal{D}$ to be

$$\mathcal{D}'_d = \left\{ D : \quad \exists \phi, \tilde{\phi} \in \mathcal{C}_b^1([0,1]); (\Delta^N)_{ij} = \delta_{i=j}(\phi(\frac{i}{N}) + 1_{0\le \frac{i}{N} \le \frac{M_N}{N}} \tilde{\phi}(\frac{i}{N})), N \in I\!N \right\},$$

To state our large deviation upper bound result, we have to be more precise about the involved topologies and space of measures.

$\mathcal{M}$ is furnished with the operator norm ; if $< \ , \ >_N$ denote the Euclidean scalar product in $\mathbb{C}^N$, $< u, v >_N = \sum_{i=1}^N \bar{u}_i v_i$, and $|| \ \ ||_N$ its associated norm, we define the operator norm $| \ \ |_\infty$ given, for any $A \in \mathcal{M}_N$, $N \in I\!N$, by

$$|A|_\infty = \sup_{||u||_N=1} < u, |A| u >_N = \sup_{||u||_N=1} < u, AA^* u >_N^{\frac{1}{2}} .$$

Recall that $| \ \ |_\infty$ is a norm which satisfies the product property

$$|AB|_\infty \le |A|_\infty |B|_\infty .$$

$\mathcal{M}$ is furnished with the involution *, extension of the usual involution on each $\mathcal{M}_N$, $N \in I\!N$. Also, there is a partial order on $\mathcal{H}$ so that $A \le B$ for $A, B \in \mathcal{H}_N$, $N \in I\!N$, iff $< u, Au >_N \le < u, Bu >_N$ for all $u \in \mathbb{C}^N$.

We can endow $\mathcal{E}(\mathbb{C})$ with the topology inherited from the norm given for any $F \in \mathcal{E}(\mathbb{C})$, by

$$\|F\|_\infty = \sup_{N\ge 1} \sup \{|F(A)|_\infty \ : \ A \in \mathcal{H}_N\} .$$

It is not hard to check (see [C-D,G], Lemma 4.26)that, with (2.2),

Lemma (2.11). *Any $F \in F_{\mathbb{C}}(X, \mathcal{D})$ has finite $|| \ \ ||_\infty$ norm.*

We let $\mathcal{F}_{\mathbb{C}}(X, \mathcal{D})$ (resp. $\mathcal{F}_{\mathbb{R}}(X, \mathcal{D})$) be the closure of $F_{\mathbb{C}}(X, \mathcal{D})$ (resp. $F_{\mathbb{R}}(X, \mathcal{D})$) by the $|| \ \ ||_\infty$ norm. $\mathcal{F}_{\mathbb{C}}(X, \mathcal{D})$ (resp. $\mathcal{F}_{\mathbb{R}}(X, \mathcal{D})$) is a complex (resp. real)Banach space. Further, they are separable. In fact, since $\mathcal{D}$ was assumed separable (note that the norm defined on $\mathbb{D}$ agrees with $|| \ \ ||_\infty$) is a separable set, $\mathcal{F}_{\mathbb{C}}(X, \mathcal{D})$ is separable for $|| \ \ ||_\infty$ with a basis given, for instance, by the set of functions of the form

$$F(X) = \prod_{j=1}^n \frac{1}{i + \alpha_j - \alpha'_j X} \Delta_j(X), X \in \mathcal{H}, \Delta_j \in \mathcal{B}_{\mathcal{D}}, \alpha_j, \alpha'_j \in \mathbb{Q}, n \in I\!N \qquad (2.12)$$

if $\mathcal{B}_{\mathcal{D}}$ is a basis of $\mathcal{D}$.

We can now define the set of non-commutative probability measures ; let $\mathcal{F}_{\mathbb{C}}(X,\mathcal{D})'$ be the algebraic dual of $\mathcal{F}_{\mathbb{C}}(X,\mathcal{D})$, that is the space of linear complex valued forms on $\mathcal{F}_{\mathbb{C}}(X,\mathcal{D})$. Let $\overline{\mathcal{M}}$ be the subset of $\mathcal{F}_{\mathbb{C}}(X,\mathcal{D})'$ with real valued restriction to $\mathcal{F}_{\mathbb{R}}(X,\mathcal{D})$. $\overline{\mathcal{M}}$ is isomorphic to $\mathcal{F}_{\mathbb{R}}(X,\mathcal{D})'$ since for any $\mu \in \overline{\mathcal{M}}$, we can write, with * the natural involution defined by

$$F^*(X) = (F(X))^* \qquad \forall X \in \mathcal{H},$$

$$\mu(F) = \mu\left(\frac{F+F^*}{2}\right) + i\mu\left(\frac{F-F^*}{2i}\right)$$

where $(F+F^*)$ and $(F-F^*)/i \in \mathcal{F}_{\mathbb{R}}(X,\mathcal{D})$. We furnish $\overline{\mathcal{M}}$ with the weak topology induced by $\mathcal{F}_{\mathbb{R}}(X,\mathcal{D})$, denoted $\mathcal{F}_{\mathbb{R}}(X,\mathcal{D})$-topology.

We shall now introduce the analogue of the set of probability measures (that is the notions of boundedness, positivness and mass 1).

For any positive real number a, we denote by $\overline{\mathcal{M}}_a$ the subset of $\overline{\mathcal{M}}$ of linear forms μ such that

$$\forall F \in \mathcal{F}_{\mathbb{C}}(X,\mathcal{D}), |\mu(F)| \leq a\|F\|_\infty \tag{2.13}$$

Further, let us consider the following partial order on $\mathcal{E}(\mathbb{R})$; If $(F,G) \in \mathcal{E}(\mathbb{R})$, $F \leq G$ iff

$$\forall X \in \mathcal{H}, \qquad G(X) \geq F(X).$$

We shall say that a linear form $\mu \in \overline{\mathcal{M}}$ is positive iff

$$\forall F \in \mathcal{F}_{\mathbb{R}}(X,\mathcal{D}) \qquad F \geq 0 \Longrightarrow \mu(F) \geq 0.$$

μ will be said to be tracial if

$$\forall F, G \in \mathcal{F}_{\mathbb{C}}(X,\mathcal{D}) \qquad \mu(GF) = \mu(FG).$$

Let $\overline{\mathcal{M}}_a^+$ be the subset of $\overline{\mathcal{M}}_a$ of positive tracial linear forms. We can define the notion of total mass for any linear form μ of $\overline{\mathcal{M}}_a^+$ by

$$m_\mu = \sup\big\{\mu(F),\ F \in \mathcal{F}_{\mathbb{R}}(X,\mathcal{D}),\ \|F\|_\infty \leq 1\big\} = \mu(1)$$

The analogue of the commutative set of probability measures will be the subset $\overline{\mathcal{M}}_1^=$ of $\overline{\mathcal{M}}_1^+$ of linear form with total mass m_μ exactly equal to one.

By a standard diagonalization procedure, it is not hard to check as in the commutative setting that $\overline{\mathcal{M}}_1^=$ is compact for the $\mathcal{F}_{\mathbb{R}}(X,\mathcal{D})$- topology since $\mathcal{F}_{\mathbb{R}}(X,\mathcal{D})$ is separable. The $\mathcal{F}_{\mathbb{R}}(X,\mathcal{D})$- topology is compatible on $\overline{\mathcal{M}}_1^=$ with the distance

$$\bar{d}(\mu,\nu) = |||\mu-\nu||| \equiv \sum_{p\in\mathbb{N}} \frac{1}{2^p}|\mu(F_p)-\nu(F_p)|.$$

where $(F_p)_{p\in\mathbb{N}}$ is a basis of uniformly bounded functions of $F_{\mathbb{R}}(X,\mathcal{D})$ as described in (2.12).

Hence, $\overline{\mathcal{M}_1^=}$ is a compact metric space, thus Polish.

Let $\hat{\mu}_t^{(N)}$ be given by

$$\hat{\mu}_t^{(N)}(F) = \mathrm{tr}_N(F(X_N(t))) \quad \forall F \in \mathcal{F}_{\mathbb{R}}(X,\mathcal{D}), \quad \forall t \in [0,1].$$

Then, considering $(\hat{\mu}_t^{(N)}, 0 \leq t \leq 1)$ as a continuous $\overline{\mathcal{M}_1^=}$-valued process and endowing the set $\mathcal{C}([0,1],\overline{\mathcal{M}_1^=})$ of such processes with the uniform topology on the time variable and the $\mathcal{F}_{\mathbb{R}}(X,\mathcal{D})$- topology on $\overline{\mathcal{M}_1^=}$, we shall prove that

Theorem (2.14). *The law of $(\hat{\mu}_t^{(N)}, 0 \leq t \leq 1)$ satisfies a large deviation upper bound in the scale N^2 with good rate function S described in theorem (4.1).*

We discuss in section 4 after theorem (4.1) the large deviation upper bound obtained by contraction from theorem (2.14) for the law $\hat{\mu}_1^{(N)}$ and its relation with the non-commutative entropy introduced by D. Shlyakthenko.

Let us make a few remarks about the corollaries of Theorem (2.14) in terms of standard large deviation principle. Since we discussed this point in details in [C-D,G], we shall here be rather sketchy. To this end, we recall the links of $\overline{\mathcal{M}_1^=}$ with standard spaces of probability measures. It is based on the following remark of [C-D,G] (see property 4.32 and Lemma 4.26) that

Property (2.15).

Let $F \in \mathcal{F}_{\mathbb{R}}(X,\mathcal{D})$.

1) For any $f \in \mathcal{C}_b(\mathbb{R})$, $f \circ F$ belongs to $\mathcal{F}_{\mathbb{R}}(X,\mathcal{D})$.

2) The linear functionnal μ_F on $\mathcal{C}_b(\mathbb{R})$ given by

$$\mu_F(f) = \mu(f \circ F)$$

is a compactly supported probability measure on $\mathbb{R}$ for any $\mu \in \overline{\mathcal{M}_1^=}$. Further, the map $\mu \to \mu_F$ from $\overline{\mathcal{M}_1^=}$, furnished with the $\mathcal{F}_{\mathbb{R}}(X,\mathcal{D})$- topology, into $\mathcal{P}(\mathbb{R})$, furnished with the weak topology, is continuous.

As a consequence, the contraction principle and Theorem (2.14) imply

Corollary (2.16). *Let $F \in \mathcal{F}_{\mathbb{R}}(X,\mathcal{D})$. Then, the spectral measure process of $(F(X_N(t)), t \in [0,1])$ satisfies a large deviation upper bound for the weak topology in the scale N^2 with good rate function S_F given for any $\nu \in \mathcal{C}([0,1],\mathcal{P}(\mathbb{R}))$ by*

$$S_F(\nu) = \inf\{S(\mu); (\mu_F)_t = \nu_t \quad \forall t \in [0,1]\}.$$

Note that at this point, we do not obtain a large deviation upper bound for the spectral process of X_N itself since $F(X) = X$ does not belong to $\mathcal{F}_{\mathbb{R}}(X,\mathcal{D})$. To get such a result, we shall prove in addition a tightness criterium which requires the next observations. As in [C-D,G], we can define a probability measure μ_X on $\mathbb{R}$ so that for any $f \in \mathcal{C}_b(\mathbb{R})$, $\mu_X(f) = \mu(f(X))$. In particular, μ_X is countably additive and the monotone convergence theorem holds [1.26, [R]]. Hence, we can set $\mu(X^2) = \mu_X(x^2)$. Let, for $A \in \mathbb{R}^+$, $\mathcal{K}_1^{=}(A)$ be the closed subset of $\overline{\mathcal{M}}_1^{=}$

$$\mathcal{K}_1^{=}(A) \equiv \{\mu \in \overline{\mathcal{M}}_1^{=},\ \mu_X(x^2) \le A\}$$

and

$$\mathcal{K}_1^{=}(\infty) \equiv \bigcup_{A \in I\!N} \mathcal{K}_1^{=}(A) = \{\mu \in \overline{\mathcal{M}}_1^{=},\ \mu_X(x^2) < \infty\}.$$

In Theorem (4.1), $\hat{\mu}^{(N)}$ is considered as an element of $\mathcal{C}([0,1], \mathcal{K}_1^{=}(\infty))$ and we see that all the $\hat{\mu}_t^{(N)}$ belong to $\mathcal{K}_1(A)$ with probability as large as we wish on the exponential scale provided A is large enough (but finite). Also, the processes with entropy S smaller than some M are shown to have covariance uniformly bounded by some constant depending on M. This is enough to see that the $\mathcal{F}_{\mathbb{R}}(X, \mathcal{D})$-topology will be equivalent in our setting with the topology obtained by duality of the set

$$\overline{\mathcal{F}}_{\mathbb{R}}(X, \mathcal{D}) \equiv \{F \in \mathcal{E}(\mathbb{R}); \exists (F_n)_{n \in I\!N} \in \mathcal{F}_{\mathbb{R}}(X, \mathcal{D})^{I\!N}, |F - F_n|(X) \le \frac{1}{n}(X^2 + 1)\}$$

where $|F(X)| = \sqrt{F(X)^2}$. $\overline{\mathcal{F}}_{\mathbb{R}}(X, \mathcal{D})$ contains the canonical process X (approximate X by $X(z + (1/n)X)^{-1} \in \mathcal{F}_{\mathbb{R}}(X, \mathcal{D})$ for some $z \in \mathbb{C} \backslash \mathbb{R}$ as close as needed of 1). Using the following extension of Lemma (2.15)

Lemma (2.17). *Let $F \in \overline{\mathcal{F}}_{\mathbb{R}}(X, \mathcal{D})$ and $\mu \in \mathcal{K}_1^{=}(A)$ for some $A \in \mathbb{R}^+$. Then, we can define*

$$\mu_F(f) = \lim_{n \to \infty} \mu_{F_n}(f) \quad , \quad f \in \mathcal{C}_b^1(\mathbb{R}) \tag{2.18}$$

μ_F is a probability measure on $\mathbb{R}$. Moreover, the map $\mu \to \mu_F$ is continuous from $\mathcal{K}_1^{=}(A)$ into $\mathcal{P}(\mathbb{R})$ for any $A \in \mathbb{R}^+$. Finally, $\mu \to \mu(F)$ is continuous from $\mathcal{K}_1^{=}(A)$ into $\mathbb{R}$ for any $A \in \mathbb{R}^+$.

The proof is the same as that of property 4.33 in [C-D,G].

As a consequence, using Theorem (4.1) and standard exponential approximations described in [D,Z], section 4.2.2 (see the proof of corollary 4.4 of [C-D, G] for details) we obtain

Corollary (2.19). *The conclusions of Corollary (2.16) are valid for any $F \in \overline{\mathcal{F}}_{\mathbb{R}}(X, \mathcal{D})$.*

To complete this introduction, we wish to summarize two applications. First, let us consider the band matrix given by the model studied in this paper with

$$\psi_N(i,j) = \psi(\frac{i}{N}, \frac{j}{N}) = \int \sigma_\tau(\frac{i}{N})\sigma_\tau(\frac{j}{N}) dp(\tau)$$

for bounded continuous functions σ_τ. As quoted in Examples (2.4), we can choose $\mathcal{D} = \mathcal{D}_c$. With such a choice, the law of large number statements (2.5), (2.6) as well as the large deviation upper bounds results (2.14), (2.16) and (2.19) apply. For the central limit theorem, under the hypothesis of Examples (2.9), we can take $\mathcal{D} = \mathcal{D}'_c$ and conclude.

We can also apply our results to the generalized Gaussian Wishart's matrices given by

$$W^N = Y^N T^N (Y^N)^*$$

with Y^N a $N \times M_N$ complex Gaussian matrix with independent entries of covariance $\frac{1}{N}$ and T^N a $M_N \times M_N$ diagonal matrix with non negative eigenvalues. As in [G,Z], we observe that W^N is related to band matrices as follows. If X_N is given by

$$X_N = \begin{pmatrix} 0 & Y^N (T^N)^{\frac{1}{2}} \\ (T^N)^{\frac{1}{2}} (Y^N)^* & 0 \end{pmatrix}, \tag{2.20}$$

the spectrum of $(X_N)^2$ is given by the spectrum of W^N with multiplicity two up to some null eigenvalues since

$$(X_N)^2 = \begin{pmatrix} Y^N T^N (Y^N)^* & 0 \\ 0 & (T^N)^{\frac{1}{2}} (Y^N)^* Y^N (T^N)^{\frac{1}{2}} \end{pmatrix}.$$

Further, X_N has the law of $\left(\psi_N(i,j)^{\frac{1}{2}} H^{N+M_N}(1)_{ij}\right)_{1 \le i,j \le N+M_N}$ with, if $t_1, .., t_{M_N}$ denote the eigenvalues of T^N,

$$\psi_N(i,j) \equiv 1_{N+1 \le i \le N+M_N} 1_{1 \le j \le N} t_i + 1_{N+1 \le j \le N+M_N} 1_{1 \le i \le N} t_j.$$

We assume for simplification that $t_i = t(\frac{i}{N+M_N})$ for some bounded continuous non negative function t. Note that ψ_N can be written

$$\psi_N(i,j) = \int \sigma_\tau^N(i) \sigma_\tau^N(j) dp(\tau) \tag{2.21}$$

with $p(\tau) = \delta_{\tau=1} - \delta_{\tau=2} - \delta_{\tau=3}$ and

$$\sigma_1^N(i) = 1_{\frac{i}{N+M_N} \ge \frac{N+1}{N+M_N}} (t(\frac{i}{N+M_N}) - 1) + 1$$
$$\sigma_2^N(i) = 1_{\frac{i}{N+M_N} \ge \frac{N+1}{N+M_N}} t(\frac{i}{N+M_N})$$
$$\sigma_3^N(i) = 1 - 1_{\frac{i}{N+M_N} \ge \frac{N+1}{N+M_N}}$$

Hence, following the example (b) given in (2.4), if $\frac{N+1}{N+M_N}$ converges as N goes to infinity towards a constant α and if we choose $\mathcal{D}$ to be $\mathcal{D}_d$ described in Examples (2.4).b), the results (2.5), (2.6), (2.14), (2.16) and (2.19) apply to X_N. We note S_T the rate function governing the large deviation of the spectral measure process

$\hat{\mu}^{(N)}_{\cdot}$ in the scale $(N+M_N)^2$ coming from Theorem (2.16) with the above specific choice of function ψ_N. To deduce the same results for W^N, observe that

$$\Delta_3^N (X_N)^2 \Delta_3^N = \begin{pmatrix} W^N & 0 \\ 0 & 0 \end{pmatrix}$$

with the property that for any $F \in \mathcal{F}_{\mathbb{C}}(X, \mathcal{D})$,

$$G(X) = F(\Delta_3 X^2 \Delta_3) \quad \in \mathcal{F}_{\mathbb{C}}(X, \mathcal{D}). \tag{2.22}$$

This last property can be deduced from the observation that for any $z \in \mathbb{C} \backslash \mathbb{R}$,

$$(z - \Delta_3 X^2 \Delta_3)^{-1} = \Delta_3 (z^{\frac{1}{2}} + iX)^{-1} (z^{\frac{1}{2}} - iX)^{-1} \Delta_3 + (I - \Delta_3) z^{-1}$$

with any choice of the square root $z^{\frac{1}{2}}$ of z. Hence, if μ_T (resp. μ)is defined by

$$\mu_T(F) = \mu(F(\Delta_3 X^2 \Delta_3)), \qquad \mu(F) = \mu(F(X))$$

for $F \in \mathcal{F}_{\mathbb{C}}(X, \mathcal{D})$,the map $\mu \to \mu_T$ in $\mathcal{M}_1^{=}$ furnished with the $\mathcal{F}_{\mathbb{R}}(X, \mathcal{D})$-topology is continuous. We can hence deduce from the contraction principle and theorem (2.14) the following result. Consider $W^N = Y^N T^N (Y^N)$ with Y^N a $N \times M_N$ matrix with i.i.d complex Gaussian entries and T^N a diagonal matrix with entries $(t(\frac{1}{N+M_N}), ..., t(1))$ for a continuous function t. We set

$$\hat{\mu}_T^{(N)} = \mathrm{tr}_N(F(W^N)).$$

Corollary (2.23). *Assume that M_N/N converges towards a positive constant α. Then, the law of $\hat{\mu}_T^{(N)}$ satisfies a large deviation upper bound in the scale $(N+M_N)^2$ for the $\mathcal{F}_{\mathbb{R}}(X, \mathcal{D})$-topology. If S_T is the rate function of Theorem (4.1) with the specific choice $p(\tau) = \delta_{\tau=1} - \delta_{\tau=2} - \delta_{\tau=3}$ and, for $x \in [0, 1]$,*

$$\begin{aligned} \sigma_1(x) &= 1_{x \geq \frac{1}{1+\alpha}} (t(x) - 1) + 1 \\ \sigma_2(x) &= 1_{x \geq \frac{1}{1+\alpha}} t(x) \\ \sigma_3(x) &= 1 - 1_{x \geq \frac{1}{1+\alpha}}, \end{aligned}$$

the good rate function governing the large deviations upper bounds of $\hat{\mu}_T^{(N)}$ is given by

$$\begin{aligned} I_T(\mu) = \inf \{ S_T(\nu) \quad ; \\ \nu_1(F(\Delta_3 X^2 \Delta_3)) = \mu(F) + \nu_0((I - \Delta_3) F(0) (I - \Delta_3)) \quad \forall F \in \mathcal{F}_{\mathbb{R}}(X, \mathcal{D}) \}. \end{aligned}$$

The central limit Theorem for X^N and W^N can also be deduced from Theorem (2.8) under the hypothesis that $N + 1 - \alpha(N + M_N)$ converges as underlined in Examples (2.9), which requires considering subsequences in general. This hypothesis is needed to insure the convergence of the expectation of $N(\mathrm{tr}_N - \mu_1^*)P$ but would not be required if we would consider the fluctuations of the spectral measure around its mean. We will not detail these results here.

The central ingredient to prove the previous theorems is an Itô's formula for $(\hat{\mu}_t^{(N)}, t \in [0, 1])$. We shall prove it in the next section.

3. Itô's calculus

To present the stochastic differential calculus for the process X_N, we need first to define a few differential operators. Most of them can be already encountered in [C-D,G] where the reader can find a more detailed introduction.

3.1 Differential Operators

Let us first recall the definition of the non-commutative derivation. It is the linear map D_X from $F_{\mathbb{C}}(X,\mathcal{D})$ into $\mathcal{E}(\mathbb{C})\otimes\mathcal{E}(\mathbb{C})$ so that for any $F,G\in F_{\mathbb{C}}(X,\mathcal{D})$,

$$\lim_{\epsilon\to 0}\epsilon^{-1}\left(F(X+\epsilon G(X))-F(X)\right)=D_X F\sharp G(X)$$

with the notation $(A\otimes B)\sharp C=ACB$ and where $\otimes$ denotes the standard tensor product. D_X can be equivalently described by the the non commutative Leibnitz rule and its action on basic functionnals. The non commutative Leibnitz rule says that for every $F,G\in F_{\mathbb{C}}(X,\mathcal{D})$, any $A\in\mathcal{H}$,

$$D_X(FG)(A)=D_X(F)(A)\times 1\otimes G(A)+F(A)\otimes 1\times D_X(G)(A).$$

Here $\times$ denotes the multiplication in the tensor product space so that for any $N\in I\!N$, any $A,B,C,D\in\mathcal{M}_N$, $A\otimes B\times C\otimes D=AC\otimes BD$. Then, D_X is uniquely defined if we set for any $A\in\mathcal{H}$, any $z\in\mathbb{C}\backslash\mathbb{R}$, any $\alpha\in\mathbb{R}$,

$$D_X(\frac{1}{z-\alpha X})(A)=\alpha\frac{1}{z-\alpha A}\otimes\frac{1}{z-\alpha A}$$

and for any $\Delta\in\mathcal{D}$,

$$D_X(\Delta)(A)=0.$$

Note that

$$D_X\left(F_{\mathbb{C}}(X,\mathcal{D})\right)\subset F_{\mathbb{C}}(X,\mathcal{D})\otimes F_{\mathbb{C}}(X,\mathcal{D}). \tag{3.1}$$

We can thus define a second order operator D_X^2 from $F_{\mathbb{C}}(X,\mathcal{D})\otimes F_{\mathbb{C}}(X,\mathcal{D})$ into $F_{\mathbb{C}}(X,\mathcal{D})\otimes F_{\mathbb{C}}(X,\mathcal{D})\otimes F_{\mathbb{C}}(X,\mathcal{D})$ by

$$D_X^2\equiv\frac{1}{2}\left(D_X\otimes 1+1\otimes D_X\right)\circ D_X.$$

Let, for $\tau\in\Omega$, M_τ be the map from $\mathcal{M}_N\otimes\mathcal{M}_N\otimes\mathcal{M}_N$ into $\mathcal{M}_N\otimes\mathcal{M}_N$ for any $N\in I\!N$ so that for any $A,B,C\in\mathcal{M}_N$,

$$M_\tau(A\otimes B\otimes C)\equiv\Delta_\tau^N B\otimes A\Delta_\tau^N C$$

for any $N\in I\!N$. We set for $X\in\mathcal{H}$,

$$(\mathbb{L}_\tau F)(X)\equiv M_\tau\left(D_X^2F(X)\right) \tag{3.2}$$

and

$$(\mathbb{L}F)(X)=\int(\mathbb{L}_\tau F)(X)dp(\tau) \qquad (3.3).$$

It is also natural to define the derivation $\mathcal{D}_X$ from $F_{\mathbb{C}}(X,\mathcal{D})$ into $\mathcal{E}(\mathbb{C})$ so that for any trace $\mu \in \mathcal{M}_1^{=}$, any $F, G \in F_{\mathbb{C}}(X,\mathcal{D})$,

$$\lim_{\epsilon \to 0} \epsilon^{-1} \mu \left(F(X + \epsilon G(X)) - F(X) \right) = \mu(\mathcal{D}_X F \times G(X)).$$

It was already noticed in [C-D,G] that if m is the map from $\mathcal{M}_N \otimes \mathcal{M}_N$ into $\mathcal{M}_N$ for all $N \in I\!N$, so that $m(A \otimes B) = BA$,

$$\mathcal{D}_X = m \circ D_X.$$

Also, in view of (3.1) ans since $F_{\mathbb{C}}(X,\mathcal{D})$ is an algebra,

$$\mathcal{D}_X \left(F_{\mathbb{C}}(X,\mathcal{D}) \right) \subset F_{\mathbb{C}}(X,\mathcal{D}). \tag{3.4}$$

We also set $\mathcal{D}^*$ to be the linear operator on $F_{\mathbb{C}}(X,\mathcal{D})$ so that

$$\mathcal{D}^*(F)(X) = \left(\mathcal{D} F(X) \right)^* \qquad \forall X \in \mathcal{H}.$$

Finally, if we let $m_\tau : \mathcal{M}_N \to \mathcal{M}_N$ for all $N \in I\!N$ be the left hand side multiplication by Δ_τ, that is for any $A \in \mathcal{H}_N$, $N \in I\!N$,

$$m_\tau(A) \equiv \Delta_\tau^N A,$$

we set

$$\mathcal{L}_\tau \equiv \frac{1}{2} m_\tau \otimes m_\tau \circ D_X \circ \mathcal{D}_X.$$

Then, we define the operator from $F_{\mathbb{C}}(X,\mathcal{D})$ into $F_{\mathbb{C}}(X,\mathcal{D}) \otimes F_{\mathbb{C}}(X,\mathcal{D})$

$$\mathcal{L} = \int \mathcal{L}_\tau dp(\tau)$$

that is that for every test function $F \in F_C(X,\mathcal{D})$, any $A \in \mathcal{H}$,

$$\mathcal{L} F(A) = \int (\mathcal{L}_\tau F)(A) dp(\tau).$$

3.2 Itô's formula

Let $\mathcal{C}^1([0,1], F_{\mathbb{R}}(X,\mathcal{D}))$ be the set of time-continuously differentiable functions with values in $F_{\mathbb{R}}(X,\mathcal{D})$ and time derivative in $F_{\mathbb{R}}(X,\mathcal{D})$. We next show the

Lemma (3.5).

(1) Itô's formula for the matrix-valued process X_N : for every $F \in \mathcal{C}^1([0,1], F_{\mathbb{C}}(X,\mathcal{D}))$, any $t \in [0,1]$,

$$F_t(X_N(t)) = F_0(X_N(0)) + \int_0^t \mathrm{tr}_N \otimes Id \left(\mathbb{L}(F_s)(X_N(s)) \right) ds$$

$$+\int_0^t \partial_s F_s(X_N(s))ds + \int_0^t D_X F(X_N(s))\sharp dX_N(s). \qquad (3.6)$$

(2) Itô's formula for the measure-valued process : for every $F \in \mathcal{C}^1([0,1], F_{\mathbb{R}}(X,\mathcal{D}))$, any $t \in [0,1]$,

$$Q_F^{(N)}(t) = \mathrm{tr}_N F_t(X_N(t)) - \mathrm{tr}_N F_0(X_N(0)) - \int_0^t \mathrm{tr}_N[\partial_s F_s(X_N(s))]ds$$
$$- \int_0^t (\mathrm{tr}_N \otimes \mathrm{tr}_N)[\mathcal{L}F_s(X_N(s))]ds$$

is a real-valued martingale with bracket

$$< Q_F^{(N)} >_t = \frac{1}{N^2}\int_0^t \int \mathrm{tr}_N[m_\tau(\mathcal{D}_X F_s(X_N(s)))m_\tau(\mathcal{D}_X^* F_s(X_N(s)))]ds.$$

Proof. The proof follows multi-dimensional Itô's formula. Indeed, considering $F_t(X_N)$ as a function of the entries of H_N, note that for any $i,j \in \{1,..,N\}$, and with $(X_N)_{ij} = h_{ij}\psi_N(i,j)^{\frac{1}{2}}$,

$$\partial_{h_{ij}} F_t(X_N) = D_X F_t(X_N)\sharp(\partial_{h_{ij}} X_N),$$

and for any $k,l \in \{1,..,N\}$,

$$\partial_{h_{kl}}\partial_{h_{ij}} F_t(X_N) = D_X \otimes 1 \circ D_X F_t(X_N)\sharp\left(\partial_{h_{kl}} X_N, \partial_{h_{ij}} X_N\right)$$
$$+ 1 \otimes D_X \circ D_X F_t(X_N)\sharp\left(\partial_{h_{ij}} X_N, \partial_{h_{kl}} X_N\right)$$

where we have noted $A \otimes B \otimes C\sharp(D,E) = ADBEC$. Also, remark that

$$(\partial_{h_{ij}} X_N)_{kl} = \delta_{ij=kl}\psi_N(i,j)^{\frac{1}{2}}. \qquad (3.7)$$

Now, recall that multi-dimensional Itô's calculus yields, since $< (H_N)_{kl}, (H_N)_{ij} >_t = N^{-1}\delta_{kl=ji}t$,

$$dF_t(X_N(t)) = \partial_t F_t(X_N(t))dt + \sum_{i,j=1}^N \partial_{h_{ij}} F_t(X_N(t))(dX_N(t))_{ij}$$
$$+\frac{1}{2N}\sum_{i,j=1}^N \psi_N(i,j)\partial_{h_{ji}}\partial_{h_{ij}} F_t(X_N(t))dt \qquad (3.8)$$

where above the action of the operator is to be understood entries by entries, e.g for any $k,l \in \{1,..,N\}$,

$$\left(\sum_{i,j=1}^N \psi_N(i,j)\partial_{h_{ji}}\partial_{h_{ij}} F_t(X)\right)_{kl} = \sum_{i,j=1}^N \psi_N(i,j)\partial_{h_{ji}}\partial_{h_{ij}}\left(F_t(X)\right)_{kl}.$$

But, according to (3.7), for any $k,l,m,n,o,p\in\{1,..,N\}$, if we denote $(A\otimes B\otimes C)_{klmnop}=A_{kl}B_{mn}C_{op}$,

$$\begin{array}{rl} & \left(\frac{1}{2N}\sum_{i,j=1}^N \psi_N(i,j)\partial_{h_{ji}}\partial_{h_{ij}}F_t(X)\right)_{kl} \\ = & \frac{1}{N}\sum_{i,j=1}^N \psi_N(i,j)\left(D_X^2F\right)_{kijjil} \\ = & \int\left(\frac{1}{N}\sum_{i,j=1}^N \sigma_\tau^N(i)\sigma_\tau^N(j)\left(D_X^2F\right)_{kijjil}\right)dp(\tau) \\ = & \int\left(\mathrm{tr}_N\otimes Id(\mathbb{L}_\tau F(X))\right)_{kl}dp(\tau) \end{array} \tag{3.9}$$

giving the first part of the lemma.

For the second part, we need only to take the trace on both sides of (3.6) to obtain

$$d\mathrm{tr}_N F_t(X_N(t)) = \mathrm{tr}_N(\partial_t F_t(X_N(t)))dt + \sum_{i,j=1}^N \mathrm{tr}_N\left(D_XF(X_N(t))\sharp dX_N(t)\right)$$

$$+\int \mathrm{tr}_N\otimes\mathrm{tr}_N\left(\mathbb{L}_\tau F(X_N(t))\right)dp(\tau)dt. \tag{3.10}$$

The first term in (3.8) gives the martingale term

$$\mathrm{tr}_N(D_XF_t(X_N(t))\sharp dX_N(t)) = \mathrm{tr}_N(\mathcal{D}_XF_t(X_N(t))dX_N(t)).$$

For the second, note that

$$\mathrm{tr}_N\otimes\mathrm{tr}_N\left(\mathbb{L}_\tau F(X)\right) = \mathrm{tr}_N\otimes\mathrm{tr}_N\left(\mathcal{L}_\tau F(X)\right). \tag{3.11}$$

Indeed, denoting (F_i^1,F_i^2) the family of functions in $F_{\mathbb{C}}(X,\mathcal{D})$ so that

$$D_XF=\sum_i F_i^1\otimes F_i^2$$

we find that

$$D_X^2F=\frac{1}{2}\sum_i\sum_j\left((F_i^1)_j^1\otimes(F_i^1)_j^2\otimes F_i^2+F_i^1\otimes(F_i^2)_j^1\otimes(F_i^2)_j^2\right)$$

so that

$$M_\tau(D_X^2F)=\frac{1}{2}\sum_i\sum_j\left(\Delta_\tau(F_i^1)_j^2\otimes(F_i^1)_j^1\Delta_\tau F_i^2+\Delta_\tau(F_i^2)_j^1\otimes F_i^1\Delta_\tau(F_i^2)_j^2\right).$$

On the other hand,

$$\begin{aligned}\mathcal{L}_\tau F &= \frac{1}{2}m_\tau\otimes m_\tau\circ D_X(\sum_i F_i^2F_i^1) \\ &= \frac{1}{2}\sum_i\sum_j\left(\Delta_\tau(F_i^1)_j^2\otimes\Delta_\tau F_i^2(F_i^1)_j^1+\Delta_\tau(F_i^2)_j^1\otimes\Delta_\tau(F_i^2)_j^2F_i^1\right)\end{aligned}$$

so that taking the trace having the property

$$\mathrm{tr}_N(\Delta_\tau(F_i^2)_j^2 F_i^1) = \mathrm{tr}_N(F_i^1 \Delta_\tau(F_i^2)_j^2)$$

gives (3.11).

Hence,

$$Q_F^{(N)}(t) = \mathrm{tr}_N(\mathcal{D}_X F_t(X_N(t)) dX_N(t))$$

is a martingale. Its bracket is easily computed by

$$< (dX_N)_{ij}, (dX_N)_{kl} >_t = \delta_{ij=lk} N^{-1} \psi_N(i,j) dt.$$

The fact that the martingale is real valued is clear since, as $F_t \in F_{\mathbb{R}}(X, \mathcal{D})$,

$$\mathrm{tr}_N(F_t) = \mathrm{tr}_N(F_t^*) = \overline{\mathrm{tr}_N(F_t)}$$

since tr_N is invariant by transposition.

□

4. Large deviation upper bound

We shall prove a large deviation upper bound for non-commutative functionnals of the process of $(X_N(t))_{t\in[0,1]}$ in this section. The rate function for these deviations is defined as follows. First, we define the empty state δ_0 to be the element of $\mathcal{K}_1^=(\infty)$ so that for any $F \in F_{\mathbb{R}}(X, \mathcal{D})$, $F(X) = \prod_{i=1}^n \frac{1}{z_i - \alpha_i X} \Delta_i$,

$$\delta_0(F) = m(F(0))$$

where $\Delta_i \in \mathcal{D}, 1 \le i \le n$ and $m(F(0))$ is defined by (2.3) since $F(0) = \prod_{i=1}^n z_i^{-1} \Delta_i \in \mathcal{D}$ by construction. We let $\mathcal{C}_s([0,1], \mathcal{K}_1^=(\infty))$ be the subset of $\mathcal{C}([0,1], \mathcal{K}_1^=(\infty))$ of continuous $\mathcal{K}_1^=(\infty)$-valued processes μ so that $\mu_0 = \delta_0$ and for any $\Delta \in \mathcal{D}$, any $t \in [0,1]$,

$$\mu_t(\Delta) = \delta_0(\Delta).$$

Then, S is defined by

$$S(\mu) = \begin{cases} +\infty \text{ if } \mu \notin \mathcal{C}_s([0,1], \mathcal{K}_1^=(\infty)) \\ \sup_{0 \le s \le t \le 1} S^{s,t}(\mu) \text{ otherwise,} \end{cases}$$

with, if for $F, G \in \mathcal{C}^1([0,1], \mathcal{F}_{\mathbb{R}}(X, \mathcal{D}))$, we define for any times $0 \le s \le t \le 1$, any $\mu \in \mathcal{C}([0,1], \mathcal{K}_1^=(\infty))$,

$$S^{s,t}(F, \mu) = \mu_t(F_t) - \mu_s(F_s) - \int_s^t \mu_u(\partial_u F_u) du - \int_s^t \mu_u(\mathcal{L} F_u) du,$$

$$\ll F, G \gg_\mu^{s,t} = \int_s^t \int \mu_u \left(m_\tau(\mathcal{D}_X F_u) m_\tau(\mathcal{D}_X^* G_u) \right) dp(\tau),$$

$$S^{s,t}(\mu) = \sup_{F \in \mathcal{C}^1([0,1], \mathcal{F}_{\mathbb{R}}(X,\mathcal{D}))} \left(S^{s,t}(F,\mu) - \frac{1}{2} \ll F, F \gg_{\mu}^{s,t} \right).$$

Let us denote $\hat{\mu}_t^{(N)}$ the linear map on $\mathcal{F}_{\mathbb{C}}(X,\mathcal{D})$ so that for any $F \in \mathcal{F}_{\mathbb{C}}(X,\mathcal{D})$, any $t \in [0,1]$,

$$\hat{\mu}_t^{(N)}(F) = \mathrm{tr}_N(F(X_N(t))).$$

We infer that $\hat{\mu}^{(N)}$ belongs to $\mathcal{C}([0,1], \mathcal{K}_{=}^{1}(\infty))$.

We shall prove in this section that

Theorem (4.1). $\hat{\mu}^{(N)} \in \mathcal{C}([0,1], \mathcal{K}_{=}^{1}(\infty))$ *satisfies a large deviation upper bound in the scale* N^2 *with good rate function* S *, that is*

1) S *is a non negative function which has compact level sets for the* $\mathcal{F}_{\mathbb{R}}(X,\mathcal{D})$*-topology. Further, for any* $M > 0$*, there exists a* $A > 0$ *so that*

$$E_M = \{S \leq M\} \subset \mathcal{C}([0,1], \mathcal{K}_1^{=}(A)).$$

2)

$$\limsup_{A\to\infty} \limsup_{N\to\infty} \frac{1}{N^2} \log \mathbb{P}(\hat{\mu}^{(N)} \in \mathcal{C}([0,1], \mathcal{K}_1^{=}(A))^c) = -\infty.$$

3) For any closed subset F *of* $\mathcal{C}([0,1], \overline{\mathcal{M}}_{=}^{1})$,

$$\limsup_{N\to\infty} \frac{1}{N^2} \log \mathbb{P}\left(\hat{\mu}^{(N)} \in F\right) \leq -\inf_F S.$$

In particular, since the application $\mu \in \mathcal{C}([0,1], \overline{\mathcal{M}}_{=}^{1}) \to \mu_1 \in \overline{\mathcal{M}}_{=}^{1}$ is continuous, we deduce from the contraction principle that

Corollary (4.2). $\hat{\mu}_1^{(N)}$ *satisfies a large deviation upper bound in the scale* N^2 *with good rate function given for* $\mu \in \overline{\mathcal{M}}_{=}^{1}$ *by*

$$S_1(\mu) = \inf\{S(\nu); \quad \nu \in \mathcal{C}([0,1], \overline{\mathcal{M}}_{=}^{1}) : \quad \nu_1 = \mu\}$$

It is natural that the above infimum should be achieved at the limit process μ^b obtained by conditionning the entries at time 1. It satisfies the differential equation

$$\partial_t \mu_t^b(F) = -\mu_t^b \otimes \mu_t^b(\mathcal{L}F) + \mu_t^b(\frac{X}{t}\mathcal{D}_X F).$$

μ_t^b can also be constructed as the law of $tA + X_{t(1-t)}$ where $(X_s, s \in [0,1])$ is the limit of $(X_N(s), s \in [0,1])$ and A has law μ_1 and is free from $(X_N(s), s \in [0,1])$. We then deduce an upper bound for S_1 given by

$$S_1(\mu) \leq S(\mu^b) \leq \int_0^1 u^{-1} J(\tilde{\mu}_u^b) du$$

with J the Fisher's information

$$J(\mu) = \sup_{F \in F_{\mathbb{R}}(X,\mathcal{D})} \{2\mu \otimes \mu(\mathcal{L}F) - \mu(X\mathcal{D}_X F) - \frac{1}{2}\mu(||\mathcal{D}_X F||^2)\}$$

and $\tilde{\mu}_u^b$ the image of μ_u^b by the homothety of ratio $u^{-\frac{1}{2}}$. By a translation on the function F, we find

$$J(\mu) = \sup_{F \in F_{\mathbb{R}}(X,\mathcal{D})} \{\mu \otimes \mu(\int dp(\tau) m_\tau \otimes m_\tau D_X \circ \mathcal{D}_X F) - \frac{1}{2}\mu(||\mathcal{D}_X F||^2)\} + \frac{1}{2}\mu(X^2) - 1.$$

Thus, $J(\mu)$ is finite iff $\mu(X^2) < \infty$ and, by Riesz's theorem, if there exists $H \in \overline{\mathcal{D}_X(F_{\mathbb{R}}(X,\mathcal{D}))}^{L^2(\mu)}$ so that for all $F \in \mathcal{D}_X(F_{\mathbb{R}}(X,\mathcal{D}))$

$$\mu \otimes \mu(\int dp(\tau) \Delta_\tau \otimes \Delta_\tau \times D_X F) = \mu(FH),$$

and then

$$J(\mu) = \frac{1}{2}\mu(H^2) + \frac{1}{2}\mu(X^2) - 1.$$

Thus, the natural Fisher entropy is here given in terms of the image by the adjoint of D_X of $\int dp(\tau)\Delta_\tau \otimes \Delta_\tau$ (compare with Wigner's matrices where one takes the image of $1 \otimes 1$ by the same adjoint (see [Vo3])). This Fisher's entropy is related to that defined by D. Shlyakhtenko [Sh2] ; they are equal when $D_X^*(\int dp(\tau)\Delta_\tau \otimes \Delta_\tau)$ belongs to the gradient space $\overline{\mathcal{D}_X(F_{\mathbb{R}}(X,\mathcal{D}))}^{L^2(\mu)}$.

The proof of this theorem follows the usual scheme ; we first study the rate function S and prove that it is a good rate function. We then show that $\hat{\mu}^{(N)}$ is exponentially tight and provide then a weak large deviation upper bound.

4.1 Study of the rate function

Lemma (4.3). *S is a non negative function which has compact level sets for the $\mathcal{F}_{\mathbb{R}}(X,\mathcal{D})$-topology. Further, for any $M > 0$, there exists a $A > 0$ so that*

$$E_M = \{S \leq M\} \subset \mathcal{C}([0,1], \mathcal{K}_1^{=}(A)). \tag{4.4}$$

Proof. First note that S is non negative since, for μ with $S(\mu) < \infty$, we have

$$\begin{aligned} S(\mu) &= \sup_{0 \leq s \leq t \leq 1} \sup_{F \in \mathcal{C}^1([0,1], F_{\mathbb{R}}(X,\mathcal{D}))} \left(S^{s,t}(F,\mu) - \frac{1}{2} \ll F, F \gg_\mu^{s,t} \right) \\ &= \sup_{0 \leq s \leq t \leq 1} \sup_{F \in \mathcal{C}^1([0,1], F_{\mathbb{R}}(X,\mathcal{D}))} \sup_{\lambda \in \mathbb{R}} \left(\lambda S^{s,t}(F,\mu) - \frac{\lambda^2}{2} \ll F, F \gg_\mu^{s,t} \right) \\ &= \frac{1}{2} \sup_{0 \leq s \leq t \leq 1} \sup_{F \in \mathcal{C}^1([0,1], F_{\mathbb{R}}(X,\mathcal{D}))} \frac{\left(S^{s,t}(F,\mu)\right)^2}{\ll F, F \gg_\mu^{s,t}} \end{aligned}$$

is non negative as $S^{s,t}(F,\mu), \ll F, F \gg_{\mu}^{s,t} \in \mathbb{R} \times \mathbb{R}^+$. Further, for any $F \in \mathcal{C}^1([0,1], F_{\mathbb{R}}(X,\mathcal{D}))$, $\mu \to S^{s,t}(F,\mu)$ is continuous by the stability properties of (3.1)-(3.4). For the same reason, $\mu \to \ll F, F \gg_{\mu}^{s,t}$ is continuous and hence $S^{0,1}$, as a supremum of continuous functions, is lower semi-continuous that is has closed level sets. Since $\mathcal{M}_1^{=}$ is compact, the precompact subsets of $\mathcal{C}^1([0,1], \overline{\mathcal{M}}_{=}^{1})$ can be included, following [D,G], lemma 5.4, in compact sets of the form $\mathcal{K} = \bigcap_{n\in I\!N} \mathcal{K}'_n$ with $\mathcal{K}'_n = \{\nu \in \mathcal{C}([0,1], \overline{\mathcal{M}_1^{=}})/$ the function $(u \to \nu_u(F_n))$ belongs to $K'_n\}$ if $(K'_n)_{n\in I\!N}$ a sequence of compact subsets of $\mathcal{C}_b([0,1],\mathbb{R})$ and $(F_n)_{n\in I\!N}$ a basis of $F_{\mathbb{R}}(X,\mathcal{D})$. In view of Arzéla-Ascoli theorem, the compact subsets K' of $\mathcal{C}([0,1],\mathbb{R})$ are such that there exists a finite constant $C > 0$, a family ϵ_n of positive real numbers ϵ_n, $\epsilon_n \to 0$ as n goes to infinity, a family of positive real numbers δ_n such that

$$K' = \{f \in \mathcal{C}_b([0,1],\mathbb{R}), ||f||_u \leq C, \sup_{|t-s|\leq\delta_n} |f(t) - f(s)| \leq \epsilon_n, \quad \forall n \in I\!N\}.$$

Hence, to prove that the level sets E_M can be included into some $\mathcal{K}$, we need to show that for every $F \in F_{\mathbb{R}}(X,\mathcal{D})$, and every $m > 0$, there exists $\delta_m^M(F)$ so that

$$\forall \nu \in E_M \quad \sup_{|t-s|\leq \delta_m^M(F)} |\nu_t(F) - \nu_s(F)| \leq \frac{1}{m}$$

Since by definition we have

$$\forall F \in F_{\mathbb{R}}(X,\mathcal{D}), \ \forall \nu \in E_M$$

$$S^{s,t}(\nu,F)^2 \leq 2S^{0,1}(\nu) \ll F, F \gg_{\nu}^{s,t} \leq 2M \ll F, F \gg_{\nu}^{s,t}$$

we deduce

$$|\nu_t(F) - \nu_s(F)| \leq \left|\int_s^t \nu_u \otimes \nu_u(\mathcal{L}F)du\right| + \sqrt{2M \ll F, F \gg_{\nu}^{s,t}}$$

By definition of $F_{\mathbb{R}}(X,\mathcal{D})$, (3.1)-(3.4) and Lemma (2.11), all the functions appearing in the above right hand side are uniformly bounded for $|| \quad ||_\infty$ so that we conclude that there exists a finite constant $\Delta^M(F)$ so that

$$|\nu_t(F) - \nu_s(F)| \leq \Delta^M(F)(\sqrt{|t-s|} + |t-s|)$$

Finally, to prove (4.4), we take

$$F(X) = \frac{X^2}{1+\epsilon X^2} = \frac{X}{i+\sqrt{\epsilon}X}\frac{X}{-i+\sqrt{\epsilon}X} \in F_{\mathbb{R}}(X,\mathcal{D})$$

and compute $\mathcal{D}_X F(X) = 2X(1+\epsilon X^2)^{-2}$ and

$$\begin{aligned}\mathcal{L}F(X,X) \qquad &= \int \Delta_\tau \otimes (\Delta_\tau(1+\epsilon X^2)^{-2})dp(\tau) \\ &+ \int \Delta_\tau X \otimes \Delta_\tau \times D_X(1+\epsilon X^2)^{-2}dp(\tau)\end{aligned} \tag{4.5}$$

with $X \otimes 1 D_X(1+\epsilon X^2)^{-2}$ given by

$$\epsilon \sum_{n=0,1;p=1,2} ((1+\epsilon X^2)^{-p}X^n) \otimes ((1+\epsilon X^2)^{-3+p}X^{1-n}) \tag{4.6}$$

easily checked to be the sum of tensor product of bounded operators with norm bounded above independently of ϵ.

Hence, there exists a finite constant C so that if $\mu \in E_M$, for all $t \in [0,1]$,

$$\mu_t(F) \le M + C + \int_0^t \mu_s((\Delta_\tau \frac{2X}{(1+\epsilon X^2)^2})^2)ds$$

It is not hard to verify that by the trace and positivness properties of μ_s, Cauchy-Schwartz's inequality type statements are valid and that $\forall F, G \in \mathcal{F}_{\mathbb{R}}(X, \mathcal{D}), \quad F \ge 0$,

$$\mu_s(GF) \le \|G\|_\infty \mu_s(F).$$

Hence, we compute

$$\mu_s((\Delta_\tau \frac{2X}{(1+\epsilon X^2)^2})^2) \le 4||\Delta_\tau||_\infty^2 \mu_s((\frac{X}{(1+\epsilon X^2)^2})^2) \le 4||\Delta_\tau||_\infty^2 \mu_s(F)$$

so that we conclude, since the operator norm of Δ_τ is uniformly bounded by T by assumption (H0), that

$$\mu_t(F) \le (C+M) + 4T^2 \int_0^t \mu_s(F)ds$$

and hence by Gronwall's lemma

$$\sup_{t\in[0,1]} \mu_t(F) \le (C+M)e^{4T^2}.$$

We can now let $\epsilon \downarrow 0$ and conclude that $\sup_{t\in[0,1]} \mu(X_t^2) \le (T^2+M)e^{4T^2}$ which proves the second point of the lemma. $\square$

4.2 Exponential Tightness

Note first that $\hat{\mu}^{(N)}$ belongs to $\mathcal{C}([0,1], \mathcal{K}_1^=(\infty))$ almost surely. Further

Lemma (4.7). *There exists compact subsets $\mathcal{K}_L$, $L \in I\!N$, of $\mathcal{C}([0,1], \mathcal{M}_1^=)$ so that*

$$\limsup_{N\to\infty} \frac{1}{N^2} \log \mathbb{P}\left(\hat{\mu}^{(N)} \in \mathcal{K}_L^c\right) \le -L.$$

The proof follows the description of the precompact sets $\mathcal{C}([0,1], \mathcal{K}_1^=(\infty))$ given in the last part and is given in details in [C-D,G] in a slightly different context. We shall not detail it here.

Also

Lemma (4.8).

$$\limsup_{A\to\infty}\limsup_{N\to\infty}\frac{1}{N^2}\log\mathbb{P}\left(\hat{\mu}^{(N)}\in\mathcal{C}([0,1],\mathcal{K}_A)^c\right)=-\infty.$$

Proof. This amounts to prove that

$$\limsup_{A\to\infty}\limsup_{N\to\infty}\frac{1}{N^2}\log\mathbb{P}\left(\sup_{t\in[0,1]}\hat{\mu}_t^{(N)}(X^2)\geq A\right)=-\infty$$

But

$$\sup_{t\in[0,1]}\hat{\mu}_t^{(N)}(X^2)=\sup_{t\in[0,1]}\frac{1}{N}\sum_{i,j=1}^{N}\psi_N(i,j)|H_N(t)_{i,j}|^2$$

$$\leq\frac{1}{N^2}\sum_{i,j=1}^{N}\psi_N(i,j)\sup_{t\in[0,1]}((\beta_t^{i,j})^2+(\tilde{\beta}_t^{i,j})^2)$$

Since ψ_N is uniformly bounded and $\sup_{t\in[0,1]}(\beta_t^{i,j})^2$ has the same law that $(\beta_1^{i,j})^2$, we find an $\alpha>0$ and a finite constant C_α so that

$$\mathbb{P}[e^{\alpha N^2\sup_{t\in[0,1]}\hat{\mu}_t^{(N)}(X^2)}]\leq C_\alpha^{N^2}$$

which, thanks to Chebyshev's inequality, allows us to conclude.

□

4.3 Weak large deviation upper bound

In view of Lemma (4.7), we can get a large deviation upper bound by means of a weak large deviation upper bound which is an easy consequence of

Lemma (4.9).

$$\limsup_{\delta\downarrow 0}\limsup_{N\to\infty}\frac{1}{N^2}\log\mathbb{P}\left(\overline{\mathcal{D}}(\nu,\hat{\mu}^{(N)})<\delta\right)\leq -S(\nu)\tag{4.10}$$

for any $\nu\in\mathcal{C}([0,1],\overline{\mathcal{M}_1^=})$.

Proof. Note that, at time 0,

$$\hat{\mu}_0^{(N)}(F)=\mathrm{tr}_N(F(0))$$

converges, as $F(0)\in\mathcal{D}$, towards $m(F(0))$ by (H0). Thus, for any $\eta>0$, for N large enough $d(\hat{\mu}_0^{(N)},\delta_0)\leq\delta$. Hence, with $\hat{\mu}_t^{(N)}(F)=\hat{\mu}_0^{(N)}(F)$ for any $F\in\mathcal{D}$, we deduce that

$$\limsup_{\delta\downarrow 0}\limsup_{N\to\infty}\frac{1}{N^2}\ln\mathbb{P}(\hat{\mu}^{(N)}\in B_\delta(\nu))=-\infty$$

if $\nu \notin \mathcal{C}_s([0,1],\mathcal{K}_1^{=})$. Therefore, we shall assume hereafter that $\nu \in \mathcal{C}_s([0,1],\mathcal{K}_1^{=})$. We shall follow the ideas developed in [K,O,V]. To this end, we define a family of positives super-martingales $\{\zeta_F^{(N)}, F \in \mathcal{C}^1([0,1],F_{\mathbb{R}}(X,\mathcal{D}))\}$, equal to 1 at $t=0$, thanks to lemma (3.5) :

$$\begin{aligned}\zeta_F^{(N)}(t) &= \exp\Big(N^2 Q_F^{(N)}(t) - \frac{N^4}{2}\langle Q_F^{(N)}\rangle_t\Big)\\ &= \exp\Big(N^2 (S^{0,t}(\hat{\mu}^{(N)},F) - \frac{1}{2} \ll F,F \gg_{\hat{\mu}^{(N)}}^{0,t})\Big)\end{aligned}$$

Let $\nu \in \mathcal{C}([0,1],\mathcal{P}(\mathbb{R}))$ and $F \in \mathcal{C}([0,1],F_{\mathbb{R}}(X,\mathcal{D}))$; then for any $0 \le s \le t \le 1$, if $\zeta_F^{(N)}(t,s) = \zeta_F^{(N)}(t)\zeta_F^{(N)}(s)^{-1}$,

$$\begin{aligned}\mathbb{P}\big(\hat{\mu}^{(N)} \in B(\nu,\delta)\big) &= \mathbb{E}\Big[1_{\hat{\mu}^{(N)}\in B(\nu,\delta)}\frac{\zeta_F^{(N)}(t,s)}{\zeta_F^{(N)}(t,s)}\Big]\\ &\le \sup_{\nu' \in B(\nu,\delta)} \exp\Big(-N^2\big(S^{s,t}(\nu',F) - \frac{1}{2} \ll F,F \gg_{\nu'}^{s,t}\big)\Big)\\ &= \exp\Big(-N^2 \inf_{\nu'\in B(\nu,\delta)}\big(S^{s,t}(\nu',F) - \frac{1}{2} \ll F,F \gg_{\nu'}^{s,t}\big)\Big)\end{aligned}$$

where we have used $\mathbb{E}[\zeta_F^{(N)}(t,s)] \equiv 1$. Notice that if F belongs to $\mathcal{C}^1([0,1],F_{\mathbb{R}}(X,\mathcal{D}))$, the function $\nu' \to S^{0,1}(\nu',F) - \frac{1}{2} \ll F,F \gg_{\nu'}^{0,1}$ is continuous. Thus, for any function $F \in \mathcal{C}^1([0,1],F_{\mathbb{R}}(X,\mathcal{D}))$

$$\limsup_{\delta\downarrow 0}\limsup_{N\to\infty}\frac{1}{N^2}\ln \mathbb{P}(\hat{\mu}^{(N)} \in B(\nu,\delta)) \le -(S^{0,1}(\nu,F) - \frac{1}{2} \ll F,F \gg_{\nu}^{0,1})$$

We conclude by taking the supremum over F that

$$\limsup_{\delta\downarrow 0}\limsup_{N\to\infty}\frac{1}{N^2}\ln \mathbb{P}(\hat{\mu}^{(N)} \in B(\nu,\delta))$$

$$\le - \sup_{F\in\mathcal{C}^1([0,1],F_{\mathbb{R}}(X,\mathcal{D}))}(S^{0,1}(\nu,F) - \frac{1}{2} \ll F,F \gg_{\nu}^{0,1})$$

□

5 Law of large numbers

According to the large deviation upper bound of the previous section, we know that $\hat{\mu}^{(N)}$, as an element of $\mathcal{C}([0,1],\overline{\mathcal{M}_1^{=}})$, concentrates almost surely towards the minimizers of S. In this section, we prove that S admits a unique minimizer and study it. We then deduce a law of large numbers theorem for bounded test functions which we strenghten in a second time to include polynomial functions.

5.1 Study of the minimizers of S

Since S is a good rate function, it achieves its minimum value, which is zero. Its minimizers are hence characterized as the $\mu \in \mathcal{C}_s([0,1], \mathcal{K}_1^{=}(\infty))$ satisfying

$$S^{0,1}(\mu, F) = 0 \tag{5.1}$$

for all test functions F. We shall prove that

Lemma (5.2). *(5.1) admits a unique solution* $\mu^* \in \mathcal{C}_s([0,1], \mathcal{K}_{=}^1(\infty))$.

To prove lemma (5.2), we first show that the minimizers have finite moments and provide bounds for them ;

Lemma (5.3). *There exists a finite constant C so that if μ minimizes S,*

$$\sup_{t\in[0,1]} \mu_t(X^{2n}) \leq n! C^n, \qquad \forall n \in I\!N.$$

In particular, as a standard probability measure, $\mu_t(X^{2n})$ is defined by its moments.

Proof. Set, for $\epsilon > 0$, $F(X) = \frac{X^2}{1+\epsilon X^2}$. Following (4.5), we have

$$\mathcal{D}_X F^n(X) = 2nF^{n-1}\frac{X}{(1+\epsilon X^2)^2}$$

and

$$D_X \circ \mathcal{D}_X F^n(X) = 4n \sum_{k=0}^{n-2} F^k(X) \frac{X}{(1+\epsilon X^2)} \otimes \frac{1}{(1+\epsilon X^2)} F^{n-2-k}(X) \frac{X}{(1+\epsilon X^2)^2}$$

$$+ \sum_{k=0}^{n-1} F^k(X) \otimes 1 \times D_X \circ \mathcal{D}_X F(X) \times 1 \otimes F^{n-1-k}(X). \tag{5.4}$$

Noticing that

$$||\Delta_\tau||_\infty \leq T, \qquad \frac{2\sqrt{\epsilon}X}{1+\epsilon X^2} \leq 1 \quad \forall \quad \epsilon \in \mathbb{R}^+, \quad \forall X \in \mathcal{H},$$

and recalling from (4.6) that $D_X \circ \mathcal{D}_X F$ is uniformly bounded in the tensor product space, we find a finite constant C so that for any $\mu \in \overline{\mathcal{M}_1^{=}}$,

$$\mu \otimes \mu (\mathcal{L} F^n) \leq Cn \sum_{k=0}^{n-2} \mu\left(F^{k+1} + F^k\right) \mu\left(F^{n-k-1} + F^{n-k-2}\right).$$

Hence, if μ satisfies (5.1), and $m_n(t) \equiv \sup_{\epsilon\in[0,1]} \sup_{k\leq n} \mu_t(F^k)$, we have

$$m_n(t) \leq (4n^2 C) \int_0^t m_{n-1}(s)^2 ds,$$

so that with $m_n = \sup_{t\in[0,1]} m_n(t)$,

$$m_n \le (4n^2C)m_{n-1}^2 \le \prod_{p=1}^{n}(4p^2C)^{2^{n-p}}.$$

Hence, $\mu(X^{2n}) = \lim_{\epsilon\downarrow 0}\mu[(\frac{X^2}{1+\epsilon X^2})^n]$ exists and is finite for all n. We can therefore extend (5.1) by talking F to be polynomial. We then get the easier formula

$$\begin{aligned}&\partial_t\mu_t(\prod_{i=1}^{n}\Delta_i X)\\&=\sum_{k=1}^{n}\sum_{l=0}^{n-k-1}\int dp(\tau)\mu_t(\Delta_\tau\prod_{i=k+1}^{k+l}\Delta_i X\Delta_{k+l+1})\mu_t(\Delta_\tau\prod_{i=k+l+2}^{n}\Delta_i X\prod_{i=1}^{k-1}\Delta_i X\Delta_k)\\&+\sum_{k=1}^{n}\sum_{l=1}^{k-1}\int dp(\tau)\mu_t(\Delta_\tau\prod_{i=k+1}^{n}\Delta_i X\prod_{i=1}^{l}\Delta_i X\Delta_l)\mu_t(\Delta_\tau\prod_{i=l+1}^{k-1}\Delta_i X\Delta_k)\end{aligned}$$

for any $(\Delta_1,..,\Delta_n)\in\mathcal{D}$. By induction over n, we deduce that $\mu_t(\prod_{i=1}^{n}\Delta_i X) = 0$ if n is odd for every $\Delta_i\in\mathcal{D}$. Taking $\Delta_i = 1$, we get if n is even,

$$\partial_t\mu_t(X^n) \le T^2 n\sum_{k=0}^{n-2/2}\mu_t(X^{2k})\mu_t(X^{n-2k-2}). \tag{5.5}$$

Let

$$u_n(t) = \frac{1}{n!}\mu_t(X^{2n}).$$

Then, (5.5) implies

$$\partial_t u_n(t) \le T^2\sum_{k=0}^{n-1}(C_{n-1}^k)^{-1}u_k(t)u_{n-1-k}(t). \tag{5.6}$$

with $C_n^k = (n!/k!(n-k)!)$. Observe that $c = \sup_{n\in I\!N}\sum_{k=0}^{n-1}(C_{n-1}^k)^{-1} < \infty$ so that by induction we see that there exists $C<\infty$ ($C\le cT^2$) such that

$$\sup_{t\in[0,1]} u_n(t) \le C^n$$

which finishes the proof of the Lemma. □

We are now in position to prove Lemma (5.2) :

Proof of Lemma (5.2). Finally, the moments of μ_t are uniquely determined since, if μ, ν are two solutions,

$$\Delta_k(t) = \sup_{n\le k}\sup_{\Delta\in\mathcal{D},||\Delta||_\infty\le 1}|\nu_t(\prod_{i=1}^{n}\Delta_i X) - \mu_t(\prod_{i=1}^{n}\Delta_i X)|,$$

we have by the above equation

$$\Delta_k(t) \le 2k^2\sqrt{k!C^k}T^2 \int_0^t \Delta_{k-1}(s)ds \le 2k^2\sqrt{k!C^k}T^2 \int_0^t \Delta_k(s)ds$$

which, by Gronwall's lemma implies $\Delta_k(t) = 0$. In view of lemma (5.3), this is enough to guarantee directly that for any $\xi_1, .., \xi_n \in \mathbb{C}$, $z_1, .., z_i \in \mathbb{C}$

$$\mu_t\left(\prod_{i=1}^n \Delta_i e^{\xi_i(X-z_i)}\right)$$

is uniquely defined (check that the expansion indeed converges) and then by integration over the ξ_k's when $\Im(z_k) \neq 0$ (with $\mathrm{sgn}(\Im(z_k))\xi_k \in (-\infty, 0]$),

$$\mu_t\left(\prod_{i=1}^n \Delta_i \frac{1}{z_i - X}\right)$$

are uniquely determined, insuring the uniqueness of μ_t as an element of $\mathcal{M}_1^{=}$ for any $t \in [0,1]$.

□

Let us notice that $(\mu_t^*, t \in [0,1])$ satisfies a scaling property

Lemma (5.7). *For any $t \in [0,1]$, if for $F \in F_{\mathbb{C}}(X, \mathcal{D})$, $F^t(X) = F(\sqrt{t}X)$,*

$$\mu_t^*(F) = \mu_1^*(F^t). \tag{5.8}$$

Proof. Indeed, $\mathcal{L}$, as a second order differential operator on X, satisfies for any $F \in F_{\mathbb{C}}(X, \mathcal{D})$,

$$\mathcal{L}F^t(X) = t(\mathcal{L}F)(\sqrt{t}X). \tag{5.9}$$

Hence, since μ_t^* is uniquely characterized by (5.1), we have for any $t \in [0,1]$, $\lambda \in (0, t^{-1}]$, for any function $F \in F_{\mathbb{C}}(X, \mathcal{D})$

$$\begin{aligned}\mu_{\lambda t}^*(F) &= \delta_0(F) + \int_0^{\lambda t} \mu_s^* \otimes \mu_s^*(\mathcal{L}F)ds \\ &= \delta_0(F) + \int_0^t \mu_{\lambda s}^* \otimes \mu_{\lambda s}^*(\lambda\mathcal{L}F)ds \\ &= \delta_0(F) + \int_0^t \mu_{\lambda s}^* \otimes \mu_{\lambda s}^*((\mathcal{L}F^\lambda)(\sqrt{\lambda}^{-1}X))ds\end{aligned}$$

Thus, $(\mu_t^\lambda, t \in [0,1])$ given by

$$\mu_t^\lambda(F) = \mu_{\lambda t}^*(F(\sqrt{\lambda}^{-1}X)), \quad F \in F_{\mathbb{C}}(X, \mathcal{D})$$

satisfies

$$\mu_t^\lambda(F) = \delta_0(F) + \int_0^t \mu_s^\lambda \otimes \mu_s^\lambda(\mathcal{L}F)ds.$$

Since we have seen in the previous section that this equation characterized μ^*, we deduce that $\mu_t^\lambda = \mu_t^*$ for $t \in [0,1]$. Taking $\lambda = t_0^{-1}$ for $t_0 \in [0,1]$, we deduce $\mu_1(F(\sqrt{t_0}^{-1}X)) = \mu_{t_0}(F)$ or equivalently (5.8).

□

In the setting of examples (2.4).a) or b) we can more precisely identify the limit law of the spectral measure of $(X_N(t), t \in [0,1])$. In fact, let

$$\psi(x,y) = \lim_{N\to\infty} \psi_N([Nx],[Ny])$$

and denote K the operator in $L^2([0,1])$ with kernel ψ. Then

Lemma (5.10). *Let $k : [0,1] \times \mathbb{C}\backslash\mathbb{R} \to \mathbb{C}$ be the unique analytic solution of the non linear equation*

$$k(x,z) = (z - K(k(.,z))(x))^{-1}$$

so that $zk(x,z)$ goes to one as $|z|$ goes to infinity for any $x \in [0,1]$. Then, for any $\phi \in \mathcal{C}_b([0,1],\mathbb{R})$, any $t \in [0,1]$,

$$\mu_t^*(\Delta(\phi)(z-X)^{-1}) = \frac{1}{\sqrt{t}} \int_0^1 \phi(x) k(x, \frac{z}{\sqrt{t}}) dx.$$

This result is analoguous to that found in [Sh] and [C,G].

Proof. Note first that by (5.1),

$$\partial_t \mu_t^*(\Delta(\phi)(z-X)^{-1}) = -\frac{1}{2}\partial_z \int dp(\tau) \mu_t^*(\Delta(\phi\sigma_\tau)(z-X)^{-1}) \mu_t^*(\Delta(\sigma_\tau)(z-X)^{-1}). \tag{5.11}$$

Further, according to lemma (5.7),

$$\mu_t^*(\Delta(\phi)(z-X)^{-1}) = \mu_1^*(\Delta(\phi)(z-\sqrt{t}X)^{-1}) = \sqrt{t}^{-1} \mu_1^*(\Delta(\phi)(\sqrt{t}^{-1}z - X)^{-1}) \tag{5.12}$$

so that we get by derivation over $t \in [0,1]$,

$$\partial_t \mu_t^*(\Delta(\phi)(z-X)^{-1}) = -\frac{1}{2t}\mu_t^*(\Delta(\phi)(z-X)^{-1}) - \frac{z}{2t^{\frac{3}{2}}} \partial_z \mu_t^*(\Delta(\phi)(z-X)^{-1}). \tag{5.13}$$

(5.11) and (5.13) result with

$$\partial_z \left(z\mu_t^*(\Delta(\phi)(z-X)^{-1}) \right) = t\partial_z \left(\int dp(\tau) \mu_t^*(\Delta(\phi\sigma_\tau)(z-X)^{-1}) \mu_t^*(\Delta(\sigma_\tau)(z-X)^{-1}) \right).$$

Noting that $\lim_{|z|\to\infty} z\mu_t^*(\Delta(\phi)(z-X)^{-1}) = \delta_0(\Delta(\phi))$, we get by integration over z,

$$z\mu_t^*(\Delta(\phi)(z-X)^{-1}) = \delta_0(\Delta(\phi)) + t\int dp(\tau) \mu_t^*(\Delta(\phi\sigma_\tau)(z-X)^{-1}) \mu_t^*(\Delta(\sigma_\tau)(z-X)^{-1}). \tag{5.14}$$

Now, observe that for any $t \in [0,1]$ and $z \in \mathbb{C}\backslash\mathbb{R}$, $\phi \to \mu_t^*(\Delta(\phi)(z-X)^{-1})$ is a linear bounded map on $L^2([0,1])$ since, by Cauchy-Schwartz's inequality

$$|\mu_t^*(\Delta(\phi)(z-X)^{-1})| \leq |\Im(z)|^{-1}\delta_0(\Delta(\phi)^2)^{\frac{1}{2}} = |\Im(z)|^{-1}|\phi|_{L^2([0,1])}.$$

Hence, Riesz's theorem shows that there exists $k_t(.,z) \in L^2([0,1])$ such that for any $\phi \in L^\infty([0,1])$

$$\mu_t^*(\Delta(\phi)(z-X)^{-1}) = \int_0^1 \phi(x)k_t(x,z)dx. \tag{5.15}$$

We deduce from (5.12) that for almost all $x \in [0,1]$,

$$k_t(x,z) = \sqrt{t}^{-1}k_1(x,\sqrt{t}^{-1}z) \tag{5.16}$$

and from (5.14) that for any $\phi \in L^2([0,1])$,

$$\int_0^1 dx\phi(x)\left(zk_1(x,z) - k_1(x,z)K(k_1(.,z))(x)\right)dx = \int_0^1 \phi(x)dx$$

so that for almost all $x \in [0,1]$,

$$k_1(x,z) = (z - K(k(.,z))(x)k(x,z))^{-1}. \tag{5.17}$$

(5.15) and (5.17) give lemma (5.10).

□

5.2 Law of large numbers

As a direct consequence of lemma (5.2),

Lemma (5.18). *For any $F \in \overline{\mathcal{F}}_{\mathbb{C}}(X,\mathcal{D})$, $(\mathrm{tr}_N(F(X_N(t))))_{t\in[0,1]}$ converges almost surely towards $(\mu_t^*(F))_{t\in[0,1]}$.*

We can also improve the law of large numbers stated in lemma (5.18) by enlarging the set of test functions. Indeed, denoting $P_{\mathbb{C}}(X,\mathcal{D})$ the set of non-commutative polynomial functions of X and elements of $\mathcal{D}$, we have

Lemma (5.19). *For any polynomial function P of $P_{\mathbb{C}}(X,\mathcal{D})$, $(\hat{\mu}_t^{(N)}(P))_{t\in[0,1]}$ converges in $L^{\infty-}(\mathbb{P}) = \cap_{q\in\mathbb{N}}L^q(\mathbb{P})$ towards $(\mu_t^*(P))_{t\in[0,1]}$. In other words, for any $q \in \mathbb{N}$,*

$$\lim_{N\to\infty}\sup_{t\in[0,1]}\mathbb{E}[|\hat{\mu}_t^{(N)}(P) - \mu_t^*(P)|^q] = 0.$$

Proof. We can of course restrict ourselves to

$$P(X) = \prod_{i=1}^n X\Delta_i(X)$$

for $(\Delta_i)_{1\le i\le n} \in \mathcal{D}$ since the Δ_i's can be identically equal to identity. Set, for $\epsilon > 0$,

$$P_\epsilon(X) = (i)^n \prod_{i=1}^n \frac{X}{i+\epsilon X}\Delta_i(X) \in F_{\mathbb{C}}(X, \mathcal{D}).$$

Then, for any $t \in [0,1]$,

$$|\mathrm{tr}_N(P_\epsilon(X_N(t))) - \mathrm{tr}_N(P(X_N(t)))| \le \epsilon n \prod_{i=1}^n ||\Delta_i||_\infty \mathrm{tr}_N[X_N(t)^{2n}]^{\frac{1}{2}}. \tag{5.20}$$

Note that

$$\mathbb{E}[\mathrm{tr}_N[X_N(t)^{2n}]] = \frac{1}{N}\sum_{\gamma;|\gamma|=2n} \mathbb{E}[\prod_{b_i\in\gamma} X^N_{b_i}(t)] \tag{5.21}$$

with γ a set of connected bonds $b=(i,j)$ in $\{1,..,N\}^2$ of length $2n$ so that $b_{i+1}\simeq b_i$ if

$$b\simeq b' \leftrightarrow b=(i,j), b'=(j,k), i,j,k\in\{1,..,N\}^3$$

and $b_{2n}\simeq b_1$. In the right hand side of (5.21), only the contours γ so that if $b=(i,j)\in\gamma$, $b^*=(j,i)\in\gamma$ with equal degree contribute, so that

$$\mathbb{E}[\mathrm{tr}_N[X_N(t)^{2n}]] = \frac{1}{N}\sum_{\gamma;|\gamma|=2n} \mathbb{E}[\prod_{b,b^*\in\gamma} X^N_b(t)X^N_{b^*}(t)] \tag{5.22}$$

But, with $b=(i,j)$,

$$X^N_b(t)X^N_{b^*}(t) = \psi_N(i,j)H^N_b(t)H^N_{b^*}(t)$$

so that, since $H^N_b(t)H^N_{b^*}(t)\ge 0$ for any $b\in\{1,..,N\}$, we deduce from (5.21) that

$$\mathbb{E}[\mathrm{tr}_N[X_N(t)^{2n}]] \le T^n\mathbb{E}[\mathrm{tr}_N[H_N(t)^{2n}]] \tag{5.23}$$

It is well known (see for instance [S,S], Theorem 2) that for any $n\in I\!N$,

$$\sup_{N\in I\!N}\sup_{t\in[0,1]} \mathbb{E}[\mathrm{tr}_N[H_N(t)^{2n}]] < \infty$$

so that (5.22) results with, for any $n\in I\!N$,

$$\sup_{N\in I\!N}\sup_{t\in[0,1]} \mathbb{E}[\mathrm{tr}_N[X_N(t)^{2n}]] < \infty. \tag{5.24}$$

With (5.20), we find, for any $q\in I\!N$, a finite constant $C(P,q)$ so that

$$\sup_{N\in I\!N}\sup_{t\in[0,1]} \mathbb{E}[|\mathrm{tr}_N(P_\epsilon(X_N(t))) - \mathrm{tr}_N(P(X_N(t)))|^q] \le C(P,q)\epsilon^q. \tag{5.25}$$

Recalling by the previous proof that

$$\lim_{\epsilon\downarrow 0}\sup_{t\in[0,1]} |\mu^*_t(P_\epsilon) - \mu^*_t(P)| = 0$$

and by Lemma (5.18) for any $\epsilon>0$ (since P_ϵ is uniformly bounded so that dominated convergence theorem applies)

$$\lim_{N\to\infty}\mathbb{E}[\sup_{t\in[0,1]} |\mathrm{tr}_N(P_\epsilon(X_N(t))) - \mu^*_t(P_\epsilon(X_N(t)))|^q] = 0$$

we deduce from (5.25) that for any $q\in I\!N$,

$$\lim_{N\to\infty}\sup_{t\in[0,1]} \mathbb{E}[|\mathrm{tr}_N(P(X_N(t))) - \mu^*_t(P(X_N(t)))|^q] = 0.$$

□

6. Central limit theorem

In this section, we shall assume that $\mathcal{D}$ satisfies additionnally the hypotheses (H1) and (H2) of section 2. The goal of this section is to study the fluctuations of $\hat{\mu}_t^{(N)}(P)$ for $t \in [0,1]$ and $P \in P_{\mathbb{C}}(X,\mathcal{D})$. This is equivalent, by the scaling property to study the fluctuations of $\{\hat{\mu}_1^{(N)}(P), P \in P_{\mathbb{C}}(X,\mathcal{D})\}$. This result is slightly less powerful than what T. Cabanal Duvillard proved in [C-D] for the fluctuations of non-commutative functionnals of independent Gaussian Wigner's matrices who obtained fluctuations on path space. However, to our point of view, the exhibited covariance functions are simpler here and the generalization to path space result somehow not so much motivated.

To describe the mean and the covariance of the limiting Gaussian variables, we shall introduce the following operators on $P_{\mathbb{C}}(X,\mathcal{D})$.

We first let $X.\partial_X$ be the differential operator in $P_{\mathbb{C}}(X,\mathcal{D})$ given by

$$X.\partial_X P = D_X P \sharp X = \partial_t P(tX)|_{t=1}.$$

As a counter part, we let $\mathcal{I}$ be given by

$$\mathcal{I}(P)(X) = \int_0^1 P(\sqrt{u}X)du.$$

We define second order operators by

$$L^M = (\mu_1^* \circ \mathcal{I} \otimes I + I \otimes \mu_1^* \circ \mathcal{I}) \circ \mathcal{L}$$

and

$$\Xi = X.\partial_X - 2\mu_1^* \otimes I \circ \mathbb{L}.$$

Let $P_{\mathbb{R}}(X,\mathcal{D})$ be the subset of $P_{\mathbb{C}}(X,\mathcal{D})$ of Hermitian valued polynomial functions. We recall that according to lemma (5.3), any $P \in P_{\mathbb{R}}(X,\mathcal{D})$ belongs to $L^2(\mu_1^*)$.

We shall prove that

Theorem (6.1).

(1) $I + \Xi$ as an operator from $P_{\mathbb{R}}(X,\mathcal{D})$ into $P_{\mathbb{R}}(X,\mathcal{D})$ is symmetric and invertible. Further, $(I+\Xi)^{-1}$ is a non negative operator from $P_{\mathbb{R}}(X,\mathcal{D})$ into $P_{\mathbb{R}}(X,\mathcal{D})$, e.g. for any $P \in P_{\mathbb{R}}(X,\mathcal{D})$,

$$< P, (I+\Xi)^{-1}P >_{L^2(\mu_1^*)} \geq 0.$$

Further, if we set $(A \otimes B)^t = B \otimes A$ and for $Q \in P_{\mathbb{R}}(X,\mathcal{D})$, $DQ^t(X) = (DQ(X))^t$ for all $X \in \mathcal{H}$, we have the more explicit formula for all $P, Q \in P_{\mathbb{R}}(X,\mathcal{D})$,

$$\mu_1^*(P\Xi Q) = \int \mu_1^* \circ m_\tau \otimes \mu_1^* \circ m_\tau \left(DQ \times (DP)^t\right) dp(\tau).$$

(2) If (H1) and (H2) are verified, for any $P \in P_{\mathbb{R}}(X, \mathcal{D})$, $N\left(\hat{\mu}_1^{(N)}(P) - \mu_1^*(P)\right)$ converges in law towards a Gaussian variable with covariance

$$C(P) = \int \mu_1^* \left(m_\tau(\mathcal{D}_X P)(I + \Xi)^{-1} m_\tau(\mathcal{D}_X P) \right) dp(\tau)$$

and mean

$$M(P) = c_0(e^{L^M} P(0)).$$

Before going any further, let us detail the above result in the classical Wigner's case.

Remark (6.2). *In Wigner's case where $\Delta_\tau \equiv 1$ and P is a polynomial function of X only, note that we find the result originally due to K. Johansson* [Joh] *and in this form in* [C-D]. *Note first that in this case $c_0 \equiv 0$ and the asymptotic Gaussian law is centered. In fact, in this case μ_1^* is the semicircle law $\pi^{-1}\sqrt{4 - x^2}dx$ and $\mathbb{L}$ can be seen as the operator from $P[X]$ into $P[X, Y]$ given by*

$$\mathbb{L}f(x, y) = (y - x)^{-1} \left(f'(y) - \frac{f(y) - f(x)}{y - x} \right).$$

But, if PV denotes the principal value, the Hilbert transform $H(\mu_1^*)(y) = PV \int (y - x)^{-1} d\mu_1^*(x)$ is well known to be equal to $H(\mu_1^*)(y) = 2^{-1}y$ on the support Λ_1^* of μ_1^*. Thus, we obtain on Λ_1^* that

$$\begin{aligned}
\Xi(f)(x) &= xf'(x) - 2 \int d\mu_1^*(y)(x - y)^{-1} \left(f'(x) - \frac{f(y) - f(x)}{y - x} \right) \\
&= 2PV \int d\mu_1^*(y) \frac{f(x) - f(y)}{(y - x)^2} \\
&= -f(x) - 2PV \int \frac{f(y)}{(x - y)^2} d\mu_1^*(y).
\end{aligned}$$

In the last line, we used $PV \int (y - x)^{-2} d\mu_1^*(x) = -2^{-1}$ which can be obtained by formal derivation from the definition of the Hilbert transform of the semi-circular law. It can look at first false because it states that the integral of a non negative quantity is negative, but one should be careful that we have to take the principal value and actually justify these equalities by going back to the definition of principal values.

From the second formula it is clear that Ξ is a symmetric non negative operator in $L^2(\mu_1^*)$ with

$$\mu_1^*(f \Xi g) = \int d\mu_1^*(x) d\mu_1^*(y) \left(\frac{f(x) - f(y)}{x - y} \right) \left(\frac{g(x) - g(y)}{x - y} \right)$$

giving the identification of Ξ of the theorem since Df can be seen as the symmetric function of two variables

$$Df(x, y) = \frac{f(x) - f(y)}{x - y}.$$

Further, from the last formula, we obtain that

$$(I+\Xi)(f)(x) = -2PV\int \frac{f(y)}{(x-y)^2}d\mu_1^*(y)$$

so that if we note K the symmetric operator in $L^2(\mu_1^*)$ given by

$$K(f) = \int \log|x-y|^{-1} f(y)d\mu_1^*(y),$$

we find for $x \in \Lambda_1^*$,

$$(I+\Xi)(f)(x) = -2\partial_x K\left((p_1^* f)'/p_1^*\right)(x)$$

with $\mu_1^*(dx) = p_1^*(x)dx$. Hence, $(I+\Xi)$ is a symmetric operator in $L^2(\mu_1^*)$ and for any continuously differentiable functions f, g,

$$\mu_1^*(g(I+\Xi)(f)) = 2\mu_1^*\left(\frac{(p_1^* g)'}{p_1^*}K\left(\frac{(p_1^* f)'}{p_1^*}\right)\right).$$

In particular,

$$C(P) = \mu_1^*(P'(I+\Xi)^{-1}P') = \frac{1}{2}\mu_1^*(PK^{-1}P).$$

The proof of theorem (6.1) follows two steps ; we first show that $N\left(\hat{\mu}_1^{(N)}(P) - \mu_1^*(P)\right)$ converges in law towards a centered Gaussian variable and then identifies the covariance of this Gaussian law.

6.1 A central limit theorem

Since Itô's calculus is again the basis of our approach, let us first quote that we can extend $\mathcal{L}$ and $\mathbb{L}$ to $P_{\mathbb{C}}(X,\mathcal{D})$ by saying that D_X satisfies the non-commutative Leibnitz rule on $P_{\mathbb{C}}(X,\mathcal{D})$ and that for any $A \in \mathcal{H}$

$$D_X X(A) = 1 \otimes 1, \qquad D_X \Delta = 0 \otimes 0 \quad \forall \Delta \in \mathcal{D},$$

If $\mathcal{C}^1([0,1], P_{\mathbb{R}}(X,\mathcal{D}))$ denotes the space of time continuously differentiable polynomial functions, we can extend naturally lemma (3.5) by

Lemma (6.3). *For any $F \in \mathcal{C}^1([0,1], P_{\mathbb{R}}(X,\mathcal{D}))$, the statements of lemma (3.5) are true.*

Let us define, for $s \in [0,1]$, the differential operator L_s on $P_{\mathbb{R}}(X,\mathcal{D})$ given by

$$L_s = (\mu_s^* \otimes 1 + 1 \otimes \mu_s^*)\mathcal{L}. \tag{6.4}$$

Note that L_s reduces by one the degree of any polynomial function $P \in P_{\mathbb{R}}(X,\mathcal{D})$ as a function of $(X,\mathcal{D})$, and of two as a function of X. Hence, for any polynomial function $P \in P_{\mathbb{R}}(X,\mathcal{D})$, any $t \in [0,1]$, we can define

$$P_t(X) = e^{\int_t^1 L_s ds}P(X) \in \mathcal{C}^1([0,1], P_{\mathbb{R}}(X,\mathcal{D})) \tag{6.5}$$

as the unique solution of the differential equation

$$\partial_t P_t(X) = -L_t P_t(X), \qquad P_1 = P.$$

We shall prove that

Lemma (6.6). *Under hypotheses (H1) and (H2), for any $P \in P_{\mathbb{R}}(X, \mathcal{D})$, $N\left(\hat{\mu}_1^{(N)}(P) - \mu_1^*(P)\right)$ converges in law towards a Gaussian variable with covariance*

$$\tilde{C}(P) = \int_0^1 \int \mu_t^*[m_\tau(\mathcal{D}_X P_t) m_\tau(\mathcal{D}_X^* P_t)] dp(\tau) dt$$

and mean $c(P_0(0))$.

In the next section we shall show that $\tilde{C}(P)$ coincides with $C(P)$ defined in theorem (6.1). Note that, by definition of L^M, we already have $c(e^{L^M}(P)(0)) = c(P_0(0))$.

Proof of lemma (6.6). Let us first notice that (5.1) implies that

$$\begin{aligned} \partial_t \mu_t^*(P_t) &= \mu_t^*(\partial_t P_t) + \mu_t^* \otimes \mu_t^*(\mathcal{L} P_t) \\ &= -\mu_t^* \otimes \mu_t^*(\mathcal{L} P_t) \end{aligned}$$

so that lemma (6.3) gives

$$dN(\hat{\mu}_t^{(N)} - \mu_t^*)(P_t) = N(\hat{\mu}_t^{(N)} - \mu_t^*) \otimes (\hat{\mu}_t^{(N)} - \mu_t^*)(\mathcal{L} P_t) dt + N dQ_P^{(N)}(t) \qquad (6.7)$$

with $(NQ_P^{(N)}(t))_{t \in [0,1]}$ a real-valued martingale with bracket

$$< NQ_P^{(N)} >_t = \int_0^t \int \mathrm{tr}_N [m_\tau(\mathcal{D}_X P_s(X_N(s))) m_\tau(\mathcal{D}_X^* P_s(X_N(s)))] dp(\tau) ds. \qquad (6.8)$$

To show that the first term in the r.h.s. of (6.7) goes to zero in $L^{\infty-}$ as N goes to infinity, we shall prove by induction that

Lemmata (6.9). *For any $n \in I\!N$, any $P_1, .., P_n \in P_{\mathbb{C}}(X, \mathcal{D})$,*

$$\sup_{t \in [0,1]} \sup_{\tau_1, .., \tau_n \in \Sigma} \sup_{N \in I\!N} \mathbb{E}[(N(\hat{\mu}_t^{(N)} - \mu_t^*)(\prod_{i=1}^n \Delta_{\tau_i} P_i))^q] < \infty. \qquad (6.10)$$

Proof. Let $|P|$ be the degree of a polynomial function P that is, if

$$P(X) = \sum_{k=1}^M \beta_k \left(\prod_{i=1}^{n_k} \Delta_i^k X \right) \Delta_i^{n_k+1},$$

for some $n_k \in I\!N$, $\Delta_i^k \in \mathcal{D} \backslash \{0\}, \beta_k \in \mathbb{R}$,

$$|P| \equiv \max_{k \in \{1, .., M\}} n_k.$$

We let $P_{\mathbb{C}}^M(X, \mathcal{D})$ be the polynomial functions with degree less or equal to M. For $P \in P_{\mathbb{C}}^0(X, \mathcal{D})$, $P \in \mathcal{D}$ and our induction hypothesis is fulfilled under (H2). Assume now that (6.10) has been proved for any any choice of $P_1,..,P_n \in P_{\mathbb{C}}(X,\mathcal{D})$ so that $\sum_{i=1}^n |P_i| = M$ for some $M \in I\!N$. Take $P_1,..,P_n \in P_{\mathbb{C}}(X,\mathcal{D})$ so that

$$P = P_{\underline{\tau}} = \prod_{i=1}^n \Delta_{\tau_i} P_i$$

has degree $M+1$. By lemma (6.3), we find that

$$N(\hat{\mu}_t^{(N)} - \mu_t^*)(P) = N(\hat{\mu}_0^{(N)} - \mu_0^*)(P) + \int_0^t \left(N(\hat{\mu}_s^{(N)} - \mu_s^*)\right) \otimes \hat{\mu}_s^{(N)}(\mathcal{L}P)ds$$
$$+ \int_0^t \mu_s^* \otimes \left(N(\hat{\mu}_s^{(N)} - \mu_s^*)\right)(\mathcal{L}P)ds + NQ_P^{(N)}(t)$$

so that, by Jensen's inequality, for any $n \in 2I\!N$,

$$\mathbb{E}[(N(\hat{\mu}_t^{(N)} - \mu_t^*)(P))^n] \leq 4^n \mathbb{E}[(N(\hat{\mu}_0^{(N)} - \mu_0^*)(P))^n]$$
$$+4^n \int_0^t \mathbb{E}[((N(\hat{\mu}_s^{(N)} - \mu_s^*) \otimes \hat{\mu}_s^{(N)}(\mathcal{L}P))^n]ds \tag{6.11}$$
$$+4^n \int_0^t \mathbb{E}[(\mu_s^* \otimes (N(\hat{\mu}_s^{(N)} - \mu_s^*)(\mathcal{L}P))^n]ds + 4^n \mathbb{E}[(NQ_P^{(N)}(t))^n]$$

with a martingale $(NQ_P^{(N)}(u), 0 \leq u \leq t)$ with bracket

$$< NQ_P^{(N)} >_u = \int_0^u \int \mathrm{tr}_N[m_\tau(\mathcal{D}_X P(X_N(s)))m_\tau(\mathcal{D}_X^* P(X_N(s)))]ds. \tag{6.12}$$

Notice that since $P_i(0) \in \mathcal{D}$ for $i \in \{1,..,n\}$, (H2) implies that

$$\sup_{\tau_1,..\tau_n \in \Omega} \sup_{N \in I\!N} \mathbb{E}[(N(\hat{\mu}_0^{(N)} - \mu_0^*)(\prod_{i=1}^n \Delta_{\tau_i} P_i(0)))^n] < \infty \tag{6.13}$$

for any $P_1,..,P_n \in P_{\mathbb{R}}(X,\mathcal{D})$.

Moreover, observe that

(1) For any P in $P_{\mathbb{C}}^M(X,\mathcal{D})$, $M \in I\!N$, $\mathcal{L}P \in P_{\mathbb{C}}^{M-1}(X,\mathcal{D}) \otimes P_{\mathbb{C}}^{M-1}(X,\mathcal{D})$.

(2) From (2.2) and the uniform bound hypothesis on the operator norm of $(\Delta_\tau)_{\tau\in\Omega}$, we find that for any $P_1,..,P_n \in P_{\mathbb{C}}(X,\mathcal{D})$,

$$\sup_{\tau_1,..,\tau_n \in \Omega} \sup_{t\in[0,1]} \sup_{N \in I\!N} \mathbb{E}[\hat{\mu}_t^{(N)}\left(P_{\underline{\tau}}\right)] < \infty. \tag{6.14}$$

From these two points and our induction hypothesis (with the uniform property with respect to the τ's in Ω), we infer that

$$\sup_{\tau_1,..\tau_n\in\Omega}\sup_{N\in I\!N}\int_0^1 \mathbb{E}[((N(\hat\mu_s^{(N)}-\mu_s^*)\otimes\hat\mu_s^{(N)}(\mathcal{L}P_{\underline{\tau}}))^n]<\infty$$

as well as

$$\sup_{\tau_1,..\tau_n\in\Omega}\sup_{N\in I\!N}\int_0^1 \mathbb{E}[((N(\hat\mu_s^{(N)}-\mu_s^*)\otimes\mu_s^*(\mathcal{L}P_{\underline{\tau}}))^n]<\infty. \quad (6.15)$$

(3) The third term in (6.11) can be bounded by Burkholder-Davis-Gundy inequality which asserts that there exists for any $n\in I\!N$ a finite constant c_n so that

$$\mathbb{E}[\sup_{0\le s\le t}(NQ_P^{(N)}(s))^n]\le c_n\mathbb{E}[< NQ^{(N)}_{P_{\underline{\tau}}}>_t^{\frac{n}{2}}]$$
$$\le c_n\int_0^t\int \mathbb{E}\left[\mathrm{tr}_N[m_\tau(\mathcal{D}_X P_{\underline{\tau}}(X_N(s)))m_\tau(\mathcal{D}^*_X P_{\underline{\tau}}(X_N(s)))]^{\frac{n}{2}}\right]dp(\tau)ds$$

where we have used in the last line (6.12). In view of remark (2) above, we deduce that

$$\sup_{s\in[0,1]}\sup_{\tau_1,..\tau_n\in\Omega}\sup_{N\in I\!N}\mathbb{E}\left[\hat\mu_s^{(N)}[m_\tau(\mathcal{D}_X P_{\underline{\tau}})m_\tau(\mathcal{D}^*_X P_{\underline{\tau}})]^{\frac{n}{2}}\right]<\infty$$

and hence

$$\sup_{\tau_1,..\tau_n\in\Omega}\sup_{N\in I\!N}\mathbb{E}[\sup_{0\le s\le t}(NQ^{(N)}_{P_{\underline{\tau}}}(s))^n]<\infty. \quad (6.16).$$

Plugging (6.13) (6.15), (6.16) into (6.11) bound $\mathbb{E}[(N(\hat\mu_t^{(N)}-\mu_t^*)(\Delta_\tau P_{\underline{\tau}}))^n]$ uniformly in $t\in[0,1]$, $\tau_1,..\tau_n\in\Omega$ and $N\in I\!N$ and thus completes the proof of the lemmata.

□

We can now finish the proof of lemma (6.6) Following (6.7), for any $P\in P(X,\mathcal{D})$,

$$N(\hat\mu_1^{(N)}-\mu_1^*)(P)=N(\hat\mu_0^{(N)}-\mu_0^*)(P_0)+R_N(P)+NQ_P^{(N)}(1) \quad (6.17)$$

where $R_N(P)$ is some reminder term. Indeed, observe that P_s is for any $s\in[0,1]$ a polynomial function with degree less or equal to $M+1$ and with coefficients uniformly bounded in time according to lemma (5.3). The same observation holds for $\mathcal{L}P_s$ which coefficients on the monomial basis of $P^M_{\mathbb{C}}(X,\mathcal{D})\otimes P^M_{\mathbb{C}}(X,\mathcal{D})$ can be uniformly bounded in time. As a consequence, lemmata (6.9) implies that for any $n\in 2I\!N$,

$$\sup_{N\in I\!N}N^n\mathbb{E}[|R_N(P)|^n]<\infty. \quad (6.18)$$

In particular, $R_N(P)$ converges almost surely towards zero by Borel-Cantelli's lemma. Recall now that $P_0(0)$ belongs to $\mathcal{D}$ so that,

$$\lim_{N\to\infty}N(\hat\mu_0^{(N)}-\mu_0^*)(P_0)=c(P_0(0)). \quad (6.19)$$

Turning to the study of the last term in the r.h.s. of (6.17), recall that we have defined $(NQ_P^{(N)}(t), t \in [0,1])$, as a martingal with bracket defined in (6.8). Again, by the above remarks on the structure of P_s and lemma (5.19), we see that $< NQ_P^{(N)} >_t$, for $t \in [0,1]$, converges in $L^{\infty-}$ (and in particular in probability) towards

$$\tilde{C}_t(P) = \int_0^t \int \mu_s^*[m_\tau(\mathcal{D}_X P_s) m_\tau(\mathcal{D}_X^* P_s)] dp_\tau ds.$$

Note that $\tilde{C}_t(P)$ is bounded as a consequence of lemma (5.3). This classicaly implies that $NQ_P^{(N)}(1)$ converges in law towards a centered Gaussian process with covariance $\tilde{C}(P)$. Indeed, taking $\lambda \in \mathbb{R}$, we know that, $(NQ_P^{(N)}(t), t \in [0,1])$ being a local martingale, $(\exp\{i\lambda NQ_P^{(N)}(t)\}, t \in [0,1])$ is a semi-martingale and for $t \in [0,1]$,

$$E[\exp\{i\lambda NQ_P^{(N)}(t)\}] e^{\frac{\lambda^2}{2}\tilde{C}_t(P)} = 1$$

$$-\frac{\lambda^2}{2} \int_0^t \int_\Sigma E\left[\exp\{i\lambda NQ_P^{(N)}(s)\}((\hat{\mu}_s^{(N)} - \mu_s^*)[m_\tau(\mathcal{D}_X P_s) m_\tau(\mathcal{D}_X^* P_s))]\right] dp_\tau ds.$$

By lemma (5.19), the last term in the above right hand side goes to zero as N goes to infinity. Thus, for any $\lambda \in \mathbb{R}$,

$$\lim_{N\to\infty} E[\exp\{i\lambda NQ_P^{(N)}(1)\}] = e^{-\frac{\lambda^2}{2}\tilde{C}_1(P)},$$

that is $NQ_P^{(N)}(1)$ converges in law towards a centered Gaussian variable with covariance $\tilde{C}(P) = \tilde{C}_1(P)$. This result with (6.19) and (6.18) gives lemma (6.6). □

6.2 Study of the covariance

In this last section, we give a more explicit formula for the covariances driving the previous central limit theorems. The first step of which is to study the operator Ξ introduced in theorem (6.1).

6.2.1 Study of some operators in $L^2(\mu_1^*)$

Define the map on $\cup \mathcal{M}_N \otimes \mathcal{M}_N$ given by $(A \otimes B)^t = B \otimes A$ and set $(DQ)^t(X) = (DQ(X))^t$ for all $X \in \mathcal{H}$. Then

Lemma (6.20).

(1) For any $P, Q \in P_\mathbb{C}(X, \mathcal{D})$,

$$\mu_1^*\left(Q\mathcal{D} \circ (\mu_1^* \otimes I + I \otimes \mu_1^*) \circ \mathcal{L}(P)\right) = \mu_1^*\left(Q\mu_1^* \otimes I \circ \mathbb{L} \circ \mathcal{D}(P)\right). \qquad (6.21)$$

(2) For any $P, Q \in P_\mathbb{C}(X, \mathcal{D})$, by

$$\mu_1^*\left(P\left(\mu_1^* \otimes I \circ \mathbb{L} - \frac{1}{2}X.\partial_X\right)Q\right) = -\frac{1}{2}\int \mu_1^* \circ m_\tau \otimes \mu_1^* \circ m_\tau\left(DQ \times (DP)^t\right) dp(\tau).$$

(3) $\Xi = X.\partial_X - 2\mu_1^* \otimes I \circ \mathbb{L}$ *is a symmetric operator from* $P_{\mathbb{R}}(X,\mathcal{D})$ *into* $P_{\mathbb{R}}(X,\mathcal{D})$. $I + \Xi : P_{\mathbb{R}}(X,\mathcal{D}) \to P_{\mathbb{R}}(X,\mathcal{D})$ *is invertible. Its inverse* $(I+\Xi)^{-1} : P_{\mathbb{R}}(X,\mathcal{D}) \to P_{\mathbb{R}}(X,\mathcal{D})$ *is symmetric non negative for the scalar product* $< , >_{L^2(\mu_1^*)}$, *e.g. for any polynomial functions* $P, Q \in P_{\mathbb{R}}(X,\mathcal{D})$,

$$< P, (I+\Xi)^{-1} Q >_{L^2(\mu_1^*)} = < Q, (I+\Xi)^{-1} P >_{L^2(\mu_1^*)}, \qquad \textit{and} \ < P, (I+\Xi)^{-1} P >_{L^2(\mu_1^*)} \geq 0.$$

Proof. Unfortunately, we could not prove this lemma directly from the equation (5.1) defining the minimum μ_1^*. Instead, we shall go back to properties of the Hermitian Brownian motion and deduce it by taking the large N limit.

To prove the first point, let us take $P \in P_{\mathbb{C}}(X,\mathcal{D})$, and consider the derivatives of $\mathrm{tr}_N \otimes \mathrm{tr}_N \circ \mathcal{L}(P(X_N))$ with respect to the entries of the self adjoint matrix $X_N = (x_{ij})_{1 \leq i,j \leq N}$ with

$$x_{ij} = (1/\sqrt{2N}) \psi_N(i,j)^{\frac{1}{2}} (h_{ij} + \sqrt{-1} \tilde{h}_{ij})$$

when $i < j$.

We first note that for any $i, j \in \{1, .., N\}$, with $(\Delta_{ij})_{kl} = \delta_{kl=ij}$,

$$\partial_{x_{ij}} \mathrm{tr}(P(X_N)) = \mathrm{tr}(DP(X_N) \sharp \Delta_{ij}) = (\mathcal{D}P(X_N))_{ji}. \tag{6.22}$$

Now, recall that from (3.9) and (3.11),

$$\mathrm{tr}_N \otimes \mathrm{tr}_N \mathcal{L}(P)(X_N) = \frac{1}{2N} \sum_{i,j=1}^{N} \psi_N(i,j) \partial_{h_{ji}} \partial_{h_{ij}} \mathrm{tr}_N(P)(X_N) \tag{6.23}$$

implying with (6.22)¸ that since $\partial_{x_{ij}}$ commutes with $\partial_{h_{kl}}$, for any $i, j \in \{1, .., N\}$, any $P \in P_{\mathbb{C}}(X,\mathcal{D})$,

$$\partial_{x_{ij}} \mathrm{tr}_N \otimes \mathrm{tr}_N \mathcal{L}(P) = \frac{1}{2N} \sum_{k,l=1}^{N} \psi_N(k,l) \partial_{h_{kl}} \partial_{h_{lk}} (\mathcal{D}P)(X_N)_{ji}. \tag{6.24}$$

Since $\mathcal{L}P \in P_{\mathbb{C}}(X,\mathcal{D}) \otimes P_{\mathbb{C}}(X,\mathcal{D})$, (6.22) gives

$$\partial_{x_{ij}} \mathrm{tr}_N \otimes \mathrm{tr}_N \mathcal{L}(P) = \left((\mathcal{D} \otimes \mathrm{tr}_N + \mathrm{tr}_N \otimes \mathcal{D})(\mathcal{L}(P)) \right)_{ji}. \tag{6.25}$$

Further, by (3.11),

$$\frac{1}{2N} \sum_{k,l=1}^{N} \psi_N(k,l) \partial_{h_{kl}} \partial_{h_{lk}} (\mathcal{D}P)(X_N)_{ji} = \left(\mathrm{tr}_N \otimes I \circ \mathbb{L}(\mathcal{D}P)(X_N) \right)_{ji}$$

proving with (6.24) and (6.25) that

$$(\mathcal{D} \otimes \mathrm{tr}_N + \mathrm{tr}_N \otimes \mathcal{D})(\mathcal{L}(P))(X_N) = \mathrm{tr}_N \otimes I \circ \mathbb{L}(\mathcal{D}P)(X_N).$$

As a consequence, for any $Q \in P_{\mathbb{C}}(X,\mathcal{D})$, we obtain

$$\begin{aligned}&\mathrm{tr}_N\left[Q(X_N(1))\mathcal{D}\circ(I\otimes\mathrm{tr}_N+\mathrm{tr}_N\otimes I)(\mathcal{L}(P))(X_N(1))\right]\\&\qquad=\mathrm{tr}_N\left[Q(X_N(1))\mathrm{tr}_N\otimes I\circ\mathbb{L}(\mathcal{D}P)(X_N(1))\right].\end{aligned}$$

Hence, using the law of large numbers theorem (5.19), we obtain at the large N limit Lemma (6.20).(1).

To prove the second part of the lemma, we recall first that the Ornstein-Uhlenbeck process

$$dy_t=\frac{1}{\sqrt{2N}}d\beta(t)-\frac{1}{2}y_tdt \tag{6.26}$$

with initial distribution γ_N, the centered Gaussian law with covariance $(2N)^{-1}$, is stationnary. We let X_N^{OU} be the matrix-valued process constructed as X_N but with, instead of independent Brownian motions $(\frac{1}{\sqrt{2N}}\beta_{i,j},\frac{1}{\sqrt{2N}}\beta'_{k,l})^{1\le k<l\le N}_{1\le i<j\le N}$ and $(\frac{1}{\sqrt{N}}\beta_{i,i})_{1\le i\le N}$, independent copies $(y_{i,j},y'_{k,l})^{1\le k<l\le N}_{1\le i<j\le N}$ and $(\sqrt{2}y_{i,i})_{1\le i\le N}$ of the Ornstein-Uhlenbeck process (6.26). Note that for any time $t\in[0,1]$, $X_N^{OU}(t)$ has the same law that $X_N(1)$. Let L_N be the infinitesimal generator of $(y_{i,j},y'_{k,l})^{1\le k<l\le N}_{1\le i\le j\le N}$,

$$L_N=\frac{1}{4N}\sum_{i\le j}\left((1+1_{i=j})\partial_{y_{ij}^2}+1_{i\neq j}\partial_{(y'_{ij})^2}\right)-\frac{1}{2}\sum_{i\le j}\left(y_{ij}\partial_{y_{ij}}+1_{i\neq j}y'_{ij}\partial_{y'_{ij}}\right).$$

It is well known that L_N is a symmetric operator in $L^2(\gamma_N^{\otimes N^2})$ and that, for any $f,g:\mathbb{R}^{N^2}\to\mathbb{R}$,

$$\gamma_N^{\otimes N^2}\left(f(-L_N)(g)\right)=\frac{1}{4N}\sum_{i\le j}\gamma_N^{\otimes N^2}\left((1+1_{i=j})\partial_{y_{ij}}f\partial_{y_{ij}}g+1_{i\neq j}\partial_{y'_{ij}}f\partial_{y'_{ij}}g\right). \tag{6.27}$$

Now, one can check as in (3.11) that for any $P\in P_{\mathbb{C}}(X,\mathcal{D})$,

$$L_NP=\left(\mathrm{tr}_N\otimes I\mathbb{L}-\frac{1}{2}X.\partial_X\right)P.$$

Hence, (6.27) implies that for any $P,Q\in P_{\mathbb{C}}(X,\mathcal{D})$,

$$\begin{aligned}&\gamma_N^{\otimes N^2}\left(\mathrm{tr}_N(Q(X_N)\left(\mathrm{tr}_N\otimes I\mathbb{L}-\frac{1}{2}X.\partial_X\right)P(X_N))\right)\\&=\gamma_N^{\otimes N^2}\left(\mathrm{tr}_N(Q(X_N)L_NP(X_N))\right)\\&=\gamma_N^{\otimes N^2}\left(\mathrm{tr}_N(P(X_N)L_NQ(X_N))\right)\\&=\gamma_N^{\otimes N^2}\left(\mathrm{tr}_N(P(X_N)\left(\mathrm{tr}_N\otimes I\mathbb{L}-\frac{1}{2}X.\partial_X\right)Q(X_N))\right).\end{aligned}$$

Thus, applying again lemma (5.19) since X_N has, under $\gamma_N^{\otimes N^2}$ the same law that $X_N(1)$, we find

$$\mu_1^*\left(Q\left(\mu_1^*\otimes I\mathbb{L}-\frac{1}{2}X.\partial_X\right)P\right)=\mu_1^*\left(P\left(\mu_1^*\otimes I\mathbb{L}-\frac{1}{2}X.\partial_X\right)Q\right)$$

that is the symmetry of the operator $\left(\mu_1^* \otimes I\mathbb{L} - \frac{1}{2}X.\partial_X\right)$ in $L^2(\mu_1^*)$. We can also find another definition of this symmetric operator thanks again to (6.27) which gives

$$\gamma_N^{\otimes N^2}\left(\mathrm{tr}_N(Q(X_N)L_NP(X_N))\right) = \frac{-1}{4N^2}\sum_{l,k=1}^{N}\sum_{i\leq j}\gamma_N^{\otimes N^2}\Big(\partial_{y_{ij}}(Q(X_N))_{lk}\partial_{y_{ij}}(P(X_N))_{kl} + 1_{i\neq j}\partial_{y'_{ij}}(Q(X_N))_{lk}\partial_{y'_{ij}}(P(X_N))_{kl}\Big)$$

Now observe that, if $(\Delta_{ij})_{kl} = \delta_{(ij)=kl}(\psi_N(i,j))^{\frac{1}{2}}$,

$$\begin{aligned}
\partial_{y_{ij}}(Q(X_N))_{lk} &= (DQ\sharp(\Delta_{ij}+\Delta_{ij}^*))_{lk} \text{ if } i<j\\
\partial_{y_{ii}}(Q(X_N))_{lk} &= (DQ\sharp\Delta_{ii})_{lk}\\
\partial_{y'_{ij}}(Q(X_N))_{lk} &= (DQ\sharp(\sqrt{-1}\Delta_{ij}-\sqrt{-1}\Delta_{ij}^*))_{lk} \text{ if } i<j
\end{aligned}$$

yielding

$$\begin{aligned}
&\gamma_N^{\otimes N^2}\left(\mathrm{tr}_N(Q(X_N)L_NP(X_N))\right)\\
&= -\frac{1}{2N^2}\sum_{l,k=1}^{N}\sum_{i,j=1}^{N}\psi_N(i,j)\gamma_N^{\otimes N^2}\left((DQ\sharp\Delta_{ij})_{lk}(DP\sharp\Delta_{ji})_{kl}\right)\\
&= -\frac{1}{2N^2}\int\gamma_N^{\otimes N^2}\left(\mathrm{tr}_N\circ m_\tau\otimes\mathrm{tr}_N\circ m_\tau\left(DQ(X_N)\times DP^t(X_N)\right)\right)dp(\tau).
\end{aligned}$$

Now, we can again use lemma (5.19) to take the limit $N\to\infty$ and conclude that

$$\mu_1^*\left(Q\left(\mu_1^*\otimes I\mathbb{L}-\frac{1}{2}X.\partial_X\right)P\right) = -\frac{1}{2}\int\mu_1^*\circ m_\tau\otimes\mu_1^*\circ m_\tau\left(DG\times(DP)^*\right)dp(\tau) \tag{6.28}$$

which achieves the proof of the lemma.

For the last point of the lemma, let us first recall that $\Xi(P_{\mathbb{C}}(X,\mathcal{D}))\subset P_{\mathbb{C}}(X,\mathcal{D})$. Further, if $P\in P_{\mathbb{R}}(X,\mathcal{D})$, $(\Xi(P)(X))^* = \Xi(P)(X)$ because

- $X.\partial_X P = \lim_{\epsilon\downarrow 0}\epsilon^{-1}(P^{(1+\epsilon)^2}-P) = \lim_{\epsilon\downarrow 0}\epsilon^{-1}(P^{(1+\epsilon)^2}-P)^* = (X.\partial_X P)^*$.
- Similarly, $\mathbb{L}(P) = (\mathbb{L}(P))^*$ if $(A\otimes B)^* = B^*\otimes A^*$ from which one sees that

$$(\mu_1^*\otimes I\circ\mathbb{L}(P)(X))^* = \mu_1^*\otimes I\circ\mathbb{L}(P)(X).$$

Moreover, if we define formally

$$(I+\Xi)^{-1}\equiv\sum_{n\geq 0}(-\Xi)^n,$$

then $(I+\Xi)^{-1}$ is well defined on $P_{\mathbb{C}}(X,\mathcal{D})$ since for any $P\in P_{\mathbb{C}}(X,\mathcal{D})$, for n large enough, $\Xi^n P\equiv 0$. Further, it is not hard to check that for any $P\in P_{\mathbb{C}}(X,\mathcal{D})$,

$$(I+\Xi)(I+\Xi)^{-1}P = (I+\Xi)^{-1}(I+\Xi)P = P,$$

implying that $I+\Xi$ is invertible with inverse $(I+\Xi)^{-1}: P_{\mathbb{R}}(X,\mathcal{D}) \to P_{\mathbb{R}}(X,\mathcal{D})$. Clearly, the symmetry of $\Xi : P_{\mathbb{R}}(X,\mathcal{D}) \to P_{\mathbb{R}}(X,\mathcal{D})$ proved at point (2) implies that $(I+\Xi)^{-1}$. Finally, for any polynomial function $P \in P_{\mathbb{R}}(X,\mathcal{D})$, if we let $Q = (I+\Xi)^{-1}P \in P_{\mathbb{R}}(X,\mathcal{D})$,

$$< P, (I+\Xi)^{-1}P >_{L^2(\mu_1^*)} = < (I+\Xi)Q, Q >_{L^2(\mu_1^*)} \geq 0$$

since by (2), $< \Xi Q, Q >_{L^2(\mu_1^*)} \geq 0$ for any $Q \in\in P_{\mathbb{R}}(X,\mathcal{D})$. The proof of the Lemma is complete.

□

6.2.2 Identification of the covariance

Hereafter, a polynomial function $Q \in P_{\mathbb{R}}(X,\mathcal{D})$ will be fixed and we shall denote $Q_s(X) = e^{\int_s^1 L_u du} Q$. Set, for any $s \in [0,1]$, any $\tau \in \Sigma$,

$$\Lambda(s,\tau) = \mu_s^*[m_\tau(\mathcal{D}_X Q_s) m_\tau(\mathcal{D}_X^* Q_s)].$$

Note first that by lemma (5.7), for any $s \in [0,1]$, any $\tau \in \Sigma$,

$$\Lambda(s,\tau) = \mu_1^*[m_\tau((\mathcal{D}_X Q_s)(\sqrt{s}X)) m_\tau((\mathcal{D}_X^* Q_s)(\sqrt{s}X))] \tag{6.29}$$

Further, since $\mathcal{D}_X$ is a derivation,

$$(\mathcal{D}_X P^s)(X) = \sqrt{s}(\mathcal{D}_X P)^s(X).$$

Thus, (6.29) reads

$$\Lambda(s,\tau) = \frac{1}{s}\mu_1^*[m_\tau((\mathcal{D}_X Q_s^s)) m_\tau((\mathcal{D}_X^* Q_s^s))]. \tag{6.30}$$

Now, by definition of $X.\partial_X$,

$$\partial_s Q_s^s(X) = (-L_s + \frac{1}{2s} X.\partial_X) Q_s(\sqrt{s}X). \tag{6.31}$$

But, since $\mathcal{L}$ is a second order operator, for any $s \in [0,1]$, $\mathcal{L}(P^s) = s(\mathcal{L}(P))^s$, we find

$$(L_s P)^s = s^{-1} L_1(P^s).$$

Thus, we deduce from (6.31) that

$$\partial_s Q_s^s(X) = s^{-1}(-L_1 + \frac{1}{2} X.\partial_X)(Q_s^s)(X)$$

so that, for any $s > 0$,

$$Q_s^s(X) = e^{\log(s)(-L_1 + \frac{1}{2}X.\partial_X)}(Q)(X). \tag{6.32}$$

Remark that we can compute the commutator of $X.\partial_X$ and $\mathcal{D}_X$ since

$$\begin{aligned}\mathcal{D}_X \circ X.\partial_X P &= \lim_{\epsilon\to 0} \epsilon^{-1}\mathcal{D}_X(P^{(1+\epsilon)^2} - P)\\ &= \lim_{\epsilon\to 0} \epsilon^{-1}\left((1+\epsilon)(\mathcal{D}_X P)^{(1+\epsilon)^2} - \mathcal{D}_X P\right)\\ &= (X.\partial_X \circ \mathcal{D}_X + \mathcal{D}_X)P\end{aligned}$$

Thus,

$$\mathcal{D}_X \circ (-L_1 + \frac{1}{2}X.\partial_X) = (-\tilde{L}_1 + \frac{1}{2}X.\partial_X + \frac{1}{2}I) \circ \mathcal{D}_X \tag{6.33}$$

with

$$\tilde{L}_1 = \frac{1}{2}\mathcal{D}_X \circ (\mu_1 \otimes 1 + 1 \otimes \mu_1)\int dp(\tau) m_\tau \otimes m_\tau D_X.$$

Now, as an operator on $P_{\mathbb{C}}(X, \mathcal{D})$, we observed in lemma (6.20).(1) that

$$\Xi = -2\mu_1^* \otimes I\mathbb{L} + X.\partial_X = -2\tilde{L}_1 + X.\partial_X. \tag{6.34}$$

Plugging (6.32), (6.33) and (6.34) in (6.30) yields, with the observation that m_τ commutes with Ξ,

$$\Lambda(s,\tau) = \frac{1}{s}\mu_1^*[e^{\frac{1}{2}\log(s)(I+\Xi)} m_\tau(\mathcal{D}_X Q) e^{\frac{1}{2}\log(s)(I+\Xi)} m_\tau(\mathcal{D}_X Q)]. \tag{6.35}$$

Hence, we find that, since $I + \Xi$ is symmetric definite positive,

$$\begin{aligned}\tilde{C}(Q) &= \int_0^1 \int \Lambda(s,\tau) dp(\tau) ds\\ &= \int_0^1 \frac{1}{s}\int \mu_1^*[e^{\frac{1}{2}\log(s)(I+\Xi)} m_\tau(\mathcal{D}_X Q) e^{\frac{1}{2}\log(s)(I+\Xi)} m_\tau(\mathcal{D}_X Q)] dp(\tau) ds\\ &= \int_0^\infty \int \mu_1^*[e^{-u(I+\Xi)} m_\tau(\mathcal{D}_X Q) m_\tau(\mathcal{D}_X Q)] dp(\tau) du\\ &= \int \mu_1^*\left(m_\tau(\mathcal{D}_X Q)(I+\Xi)^{-1}(m_\tau(\mathcal{D}_X Q))\right) dp(\tau)\end{aligned}$$

which is by definition $C(Q)$. Here, one can check that the last line agrees with our definition of $(I+\Xi)^{-1} = \sum_{n\geq 0}(-\Xi)^n$ by expending the exponential in Ξ (yielding only a finite sum since) and integrating the polynomial function in u.

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