Free diffusions and Matrix models with strictly convex interaction

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Abstract

We study solutions to the free stochastic differential equation $dX_t = dS_t - \frac{1}{2}DV(X_t)dt$, where V is a locally convex polynomial potential in m non-commuting variables. We show that for self-adjoint V, the law μ_V of a stationary solution is the limit law of a random matrix model, in which an m-tuple of self-adjoint matrices are chosen according to the law $\exp(-N\text{Tr}(V(A_1,\ldots,A_m)))dA_1\cdots dA_m$. We show that if $V = V_\beta$ depends on complex parameters $\beta_1,\ldots,\beta_k$, then the law μ_V is analytic in β at least for those β for which V_β is locally convex. In particular, this gives information on the region of convergence of the generating function for planar maps.

We prove that the solution dX_t has nice convergence properties with respect to the operator norm as t goes to infinity. This allows us to show that the C^* and W^* algebras generated by an m-tuple with law μ_V share many properties with those generated by a semi-circular system. Among them is lack of projections, exactness, the Haagerup property, and embeddability into the ultrapower of the hyperfinite II$_1$ factor. We show that the microstates free entropy $\chi(\tau_V)$ is finite.

A corollary of these results is the fact that the support of the law of any self-adjoint polynomial in $X_1,\ldots,X_n$ under the law μ_V is connected, vastly generalizing the case of a single random matrix.

We also deduce from this dynamical approach that the convergence of the operator norms of independent matrices from the GUE proved by Haagerup and S. Thorjornsen [14] extends to the context of matrices interacting *via* a convex potential.

1 Introduction

There has been a great deal of interest in studying matrix integrals in physics since the work of 't Hooft who made the connection between the problem of enumerating maps and estimating integrals of the form

$$Z_N(V) = \int e^{-N\text{Tr}(V(X_1,\ldots,X_m))}dX_1\cdots dX_m$$

where dX denotes the Lebesgue measure on $N \times N$ Hermitian matrices and Tr the non-normalized trace $\text{Tr}(A) = \sum_{i=1}^N A_{ii}$. V is a polynomial in m-indeterminates. Let us recall that a map of genus g is a graph which is embedded into a surface of genus g in such a way that the edges

do not intersect and so that disecting the surface along the edges decomposes it into faces, each homeomorphic to a disk. We shall consider maps with colored edges and enumerate them when the degrees of the vertices, as well as the ditribution of color of the edges around each vertex, are prescribed. The number of colors will be m, and the colors will be simply refered by the numbers $\{1, \ldots, m\}$. A vertex with colored half-edges, an orientation and a distinguished half-edge, can be associated bijectivly with a non-commutative monomial $q(X) = X_{i_1} \cdots X_{i_p}$ as follows; the first (or distinguished) half-edge has color i_1, second has color i_2, etc., the last half-edge having color i_p. Such a vertex, equipped with its colored half-edges, distinguished half-edge and orientation, will be called a star of type q. We will denote by $M_g((q_i, k_i)_{1 \leq i \leq n})$ the number of maps with genus g and with k_i stars of type q_i for $1 \leq i \leq n$, the maps being constructed by gluing pairwise the half-edges of the stars (the counting is done up to homomorphism of the surface and stars are labelled).

't Hooft showed that if

$$V(X_1, \ldots, X_m) = W_{(\beta_i, q_i)_{1 \leq i \leq n}}(X_1, \ldots, X_m) = \frac{1}{2} \sum_{i=1}^{m} X_i^2 + \sum_{i=1}^{n} \beta_i q_i(X_1, \ldots, X_m)$$

where q_i, $i = 1, \ldots, n$ are monomials in m-non commutative indeterminates, then

$$\frac{1}{N^2} \log \frac{Z_N(W_{(\beta_i, q_i)_{1 \leq i \leq n}})}{Z_N(W_{(0, q_i)_{1 \leq i \leq n}}} = \sum_{g \geq 0} \frac{1}{N^{2g}} \sum_{k_1, \ldots, k_n \in \mathbb{N}^n} \prod_{1 \leq i \leq n} \frac{(-\beta_i)^{k_i}}{k_i!} M_g((q_i, k_i)_{1 \leq i \leq n})$$

where the equality holds in the sense of formal series. Differentiating formally this equality, one also finds that if we consider the Gibbs measure

$$d\mu_{W_{(\beta_i, q_i)_{1 \leq i \leq n}}}^N(X_1, \ldots, X_m) := \frac{1}{Z_N(W_{(\beta_i, q_i)_{1 \leq i \leq n}})} e^{-N\text{Tr}(W_{(\beta_i, q_i)_{1 \leq i \leq n}}(X_1, \ldots, X_m))} dX_1 \cdots dX_m \quad (1)$$

then we have, for any monomial P, the formal expansion

$$\begin{aligned}
\bar{\mu}_{W_{(\beta_i, q_i)_{1 \leq i \leq n}}}^N(P) &:= \int \frac{1}{N} \text{Tr}\left(P(X_1, \ldots, X_m)\right) d\mu_{W_{(\beta_i, q_i)_{1 \leq i \leq n}}}^N(X_1, \ldots, X_m) \\
&= \sum_{g \geq 0} \frac{1}{N^{2g}} \sum_{k_1, \ldots, k_n} \prod_{1 \leq i \leq n} \frac{(-\beta_i)^{k_i}}{k_i!} M_g((q_i, k_i)_{1 \leq i \leq n}, (P, 1)).
\end{aligned}$$

We refer the reader to the survey papers [13, 19] to see diverse uses of this fact in the physics literature.

In [1, 12] for $m = 1$ and then in [20, 21, 22] for all $m \in \mathbb{N}$, these formal equalities were shown to hold in the sense of large N expansion when V satisfies some convexity hypothesis (or one adds a cutoff to make the integral finite) and the parameters $(\beta_i)_{1 \leq i \leq n}$ are chosen to be small enough. In [20], one of the key steps towards this analysis is to notice that $\bar{\mu}_{W_{(\beta_i, q_i)_{1 \leq i \leq n}}}^N$ converges towards a limit, denoted later $\tau_{W_{(\beta_i, q_i)_{1 \leq i \leq n}}}$, which satisfies the so-called Schwinger-Dyson equation

$$\tau \otimes \tau(\partial_i P) = \tau(D_i V P) \quad (2)$$

for all polynomials P and all $i \in \{1, \ldots, m\}$, and with $V = W_{(\beta_i, q_i)_{1 \leq i \leq n}}$. Here, ∂_i and D_i are respectively the non-commutative derivative and the cyclic derivative with respect to the variable

X_i (see the next section for a definition). One then shows that for sufficiently small parameters β_i, this equation has a unique solution, which is exactly the generating function for planar maps:

$$\tau_{W_{(\beta_i, q_i)_{1 \leq i \leq n}}}(P) = \mathcal{M}_{(\beta_i, q_i)_{1 \leq i \leq n}}(P) := \sum_{k_1, \ldots, k_n} \prod_{1 \leq i \leq n} \frac{(-\beta_i)^{k_i}}{k_i!} M_0((q_i, k_i)_{1 \leq i \leq n}, (P, 1)).$$

It is natural to wonder how to use these expansions to study the numbers $M_g((q_i, k_i)_{1 \leq i \leq n})$ and in particular their asymptotics as $k_1, \ldots, k_n$ go to infinity. The answer to this question is still open in such a general context. It is, however, quite well understood in the case $m = 1$. Let us highlight this point in the case of quadrangulations, corresponding to the potential $V(x) = \beta_4 x^4 + \beta_3 x^3$, $\beta_4, \beta_i \in \mathbb{R}$, even though the enumeration of quadrangulations and triangulations were achieved by direct combinatorial arguments by Tutte [23] long ago. In this case, the Schwinger-Dyson equation (2), taken at $P(x) = (z - x)^{-1}$ shows that the Cauchy transform

$$G(z) = \int \frac{1}{z - x} d\tau_{W_{(\beta_i, x^i)_{i=3,4}}}(x) = \int \frac{1}{z - x} d\mathcal{M}_{(\beta_i, x^i)_{i=3,4}}(x)$$

satisfies an algebraic equation of degree two

$$G(z)^2 = 4\beta z^3 G(z) + zG(z) + P(z) \quad \text{with } P(z) = \sum_{i=3,4} i\beta_i \int \frac{x^{i-1} - z^{i-1}}{x - z} d\tau_{W_{(\beta_i, x^i)_{i=3,4}}}(x) - 1.$$

We can solve this equation in terms of $P(z)$ which is a polynomial of degree two with two unknown coefficients $(\int x^2 d\tau_{W_{(\beta_i, x^i)_{i=3,4}}}(x), \int x d\tau_{W_{(\beta_i, x^i)_{i=3,4}}}(x))$; we then find that G is given by a polynomial plus the square root of a polynomial of degree six, that we denote Q, with two unknown coefficients. Until this point, all the arguments follow the induction relations already found by Tutte. However, the difference now is that we know that $\mathcal{M}_{(\beta_i, x^i)_{i=3,4}} = \tau_{W_{(\beta_i, x^i)_{i=3,4}}}$ is a probability measure on $\mathbb{R}$. Assume we can argue that for sufficiently small β_i's, the support of $\tau_{W_{(\beta_i, x^i)_{i=3,4}}}$ is connected. Then, we see that, because this means that G is analytic outside an interval, the polynomial Q must have two double roots. This actually determines Q, and thus G, uniquely. Since G is the generating function for planar maps, we are done. Hence, we see in this context that the *a priori* information that $\mathcal{M}_{(\beta_i, q_i)_{1 \leq i \leq n}}(P)$ is the Cauchy transform of a probability measure on the real line (which is not clear from its definition as a generating function of maps), and with a connected support, is enough to conclude.

The goal of this article is to push forward the analysis of the limiting tracial state $\tau_{W_{(\beta_i, q_i)_{1 \leq i \leq n}}}$ in the multi-matrix context. We prove in particular that when $W_{(\beta_i, q_i)_{1 \leq i \leq n}}$ satisfies a certain local convexity property (see (3)), the support of the limiting spectral measure of the random matrices with law $\mu^N_{W_{(\beta_i, q_i)_{1 \leq i \leq n}}}$ is connected. In fact, the same is true for an arbitrary non-commutative polynomial in the random matrices. Note that this information is enough to solve the enumeration problem when $m = 1$ as we have seen above for quadrangulations (though connectivity of support can in this case be proved by other techniques, see e.g [10]).

The tracial states $\tau = \tau_V$ under consideration will be solution to the Schwinger-Dyson equation (2) for some general potential V. Non-commutative laws arising as limits of laws of random matrix models given by (1) have also naturally appeared in free probability theory. There, the fact that they satisfy the Schwinger-Dyson type equation is restated as the fact that the free conjugate

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variables of the law are equal to the cyclic gradient of a polynomial potential. It was shown, see [8, 3], that the set of tracial states τ solution of (2) for some polynomial V is dense in the set of tracial states which can be embedded into the ultrapower of the hyperfinite II_1 factor.

In the multi-matrix setting, uniqueness of the solution to the Schwinger-Dyson equation is unclear in general. It was proved in [20] that, when $V = W_{(\beta_i, q_i)_{1 \leq i \leq n}}$, there exists a unique solution such that $|\tau(X_{i_1} \cdots X_{i_k})| \leq R^k$ for all choices of $i_j \in \{1, \cdots, m\}$ and all $k = 1, 2, \ldots$, provided the β_i's are sufficiently small. One of the central result of this paper will be the uniqueness of the solution to Schwinger-Dyson equation for locally strictly convex potentials V, when the domain of strict convexity is large enough.

We also show that if $V = V_\beta = \sum \beta_i q_i$ with some monomials q_i and $\beta = (\beta_i)_{1 \leq i \leq n}$, $\beta \to \tau_{V_\beta}(P)$ is analytic in the whole region of local convexity of V_β, for any monomial P. Because

$$\mathcal{M}_{(\beta_i, q_i)_{1 \leq i \leq n}}(P) = \tau_{W_{(q_i, \beta_i)_{1 \leq i \leq n}}} = \sum_{k_1, \ldots, k_n} \prod_{1 \leq i \leq n} \frac{(-\beta_i)^{k_i}}{k_i!} M_0((q_i, k_i)_{1 \leq i \leq n}, (P, 1)),$$

this result shows that there is no breaking of analyticity of $(\beta_i)_{1 \leq i \leq n} \to \mathcal{M}_{(\beta_i, q_i)_{1 \leq i \leq n}}(P)$ in the domain where $W_{(q_i, \beta_i)_{1 \leq i \leq n}}$ stays locally strictly convex, and thus provides valuable information on the asymptotics of the numbers $M_0((q_i, k_i)_{1 \leq i \leq n}, (P, 1))$.

Of particular interest are the C^*-algebras (A_V, τ_V) and the W^*-algebras (M_V, τ_V) generated by operators having a law satisfying the Schwinger-Dyson equation with a fixed locally convex potential V. We derive several properties of such algebras, showing that they are projectionless, exact [28] and possess the compact approximation property of Haagerup [16], and can be embedded into the ultrapower of the hyperfinite II_1 factor. We also show that the algebras M_V are factors and that the generating operators have a law with finite free entropy. This has as consequences a number of properties of the algebras M_V (including primeness and lack of Cartan subalgebras), see [26] and [15]. All of these properties are similar to (and are often derived from) the corresponding properties of free group factors (which are von Neumann algebras associated to a quadratic potential $V(X_1, \ldots, X_n) = \frac{1}{2} \sum_i X_i^2$). This adds evidence towards a positive answer to Voiculescu's question of whether M_V is isomorphic to a free group factor for a fairly arbitrary potential V.

While somewhat technical, it should be noted that studying properties of A_V and M_V is of substantial interest. Indeed, connectivity of support of limit distributions of random matrix models is directly related to the lack of projections in the C^*-algebra A_V (which in turn is derived from the famous result of Pimsner and Voiculescu [24], essentially dealing with the quadratic V).

Finally, in the remaining sections of the paper we show that the random matrices following the law (1) give a very good approximation to the non-commutative law τ_V. If V is locally convex, we show that the lim sup in the definition of the microstates free entropy of τ_V [25] can be replaced by a limit. In the case that V is (globally) convex, we show that operator norms of arbitrary polynomials in such random matrices almost surely approximate the operator norms of such polynomials computed in the C^*-algebra A_V. This extends the results of [14], which correspond to the case of quadratic V (our proof, though, relies on their result).

The main technical tools used in the present paper involve the study of free Langevin-type diffusion and its convergence to a stationary measure which corresponds to the limit law τ_V of random matrices following the measure (1). We extend some of the results of [5] to the setting of locally convex potentials (see below). This way, we are able to show that operators having a specific limit law can be approximated in the operator norm by continuous functions of free

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Brownian motion. This enables us to carry over a number of properties of the algebra generated by free Brownian motion to the algebras A_V and M_V.

1.1 Definitions, notations and statment of the results.

Let us now state more precisely our setup and results.

We let $\mathbb{C}\langle X_1, \ldots, X_m \rangle$ be the set of polynomials in m non-commutative variables $(X_1, \ldots, X_m)$. We shall not assume in general that $(X_1, \ldots, X_m)$ are self-adjoint but let $(X_1^*, \ldots, X_m^*)$ be their adjoints for some involution $*$. We denote $\mathbb{C}\langle X_1, \ldots, X_m, X_1^*, \ldots, X_m^* \rangle$ the set of polynomials in the non-commutative variables $(X_1, \ldots, X_m, X_1^*, \ldots, X_m^*)$. This set is endowed with the linear involution so that

$$(X_{i_1}^{\varepsilon_1} \cdots X_{i_k}^{\varepsilon_k})^* = X_{i_k}^{-\varepsilon_k} X_{i_{k-1}}^{-\varepsilon_{k-1}} \cdots X_{i_1}^{-\varepsilon_1}$$

where we denoted in short $X_i^1 = X_i$ and $X_i^{-1} = X_i^*$ and the $(\varepsilon_1, \ldots, \varepsilon_k)$ belong to $\{-1, 1\}^k$. We shall denote below for two sets of non-commutative variables $(X_1, \ldots, X_m)$ and $(Y_1, \ldots, Y_m)$ and an involution $*$

$$X.Y = \frac{1}{2} \sum_{i=1}^{m} (X_i Y_i^* + Y_i X_i^*).$$

$\| \cdot \|_\infty$ will denote an operator norm such that the completion of $(\mathbb{C}\langle X_1, \ldots, X_m, X_1^*, \ldots, X_m^* \rangle, *)$ for this norm is a C^*-algebra. For a m-dimensional vector $X = (X_1, \ldots, X_m)$ we denote in short $\|X\|_\infty = \max_{1 \le i \le m} \|X_i\|_\infty$.

We let $V \in \mathbb{C}\langle X_1, \ldots, X_m \rangle$ be a polynomial in m non-commutative variables. We will say that V is (c, M) convex iff for any m-tuples of non-commutative variables $X = (X_1, \ldots, X_m)$ and $Y = (Y_1, \ldots, Y_m)$ in some C^*-algebra $(\mathcal{A}, \| \cdot \|_\infty)$ satisfying $\|X_i\|_\infty, \|Y_i\|_\infty \le M$, $i = 1, \ldots, m$ we have

$$[DV(X) - DV(Y)].(X - Y) \ge c(X - Y).(X - Y) \tag{3}$$

where the inequality is understood in the sense of operators ($X \ge Y$ iff $X - Y$ is self adjoint and has non negative spectrum). $D = (D_1, \ldots, D_m)$ denotes the cyclic gradient which is linear and given, for any monomial P, by

$$D_i P = \sum_{P = P_1 X_i P_2} P_2 P_1.$$

Later, we shall also need the non-commutative gradient ∂ which is given, for any monomial P, any $i \in \{1, \ldots, m\}$ by

$$\partial_i P = \sum_{P = P_1 X_i P_2} P_1 \otimes P_2.$$

We occasionally shall consider polynomials in $\mathbb{C}\langle X_1, \ldots, X_m, X_1^*, \ldots, X_m^* \rangle$; in that case we extend ∂_i and D_i by setting $\partial_i X_j = 1_{i=j} 1 \otimes 1$ and $\partial_i X_j^* = 0 \otimes 0$ whereas we have also the derivative $\partial_{i,*}$ and $D_{i,*}$ with respect to X_i^* which satisfy $\partial_{i,*} X_j = 0 \otimes 0$ but $\partial_{i,*} X_j^* = 1 \otimes 1$, and extending by linearity and Leibnitz rule.

In the case that $X_1, \ldots, X_n$ are self-adjoint, we shall make the convention that $\partial_i X_j = 1_{i=j} 1 \otimes 1$, while $\partial_{i,*} X_j = 0$ for all j (in other words, we shall continue to think of all quantities as functions of $X_1, \ldots, X_n$ alone).

Note that by taking $X = Y + \epsilon Z$ and letting ϵ going to zero we find that for any bounded operator Z and any operator Y with norm strictly smaller than M, the condition that V is (c, M)-convex implies that

$$\sum_{i=1}^{m} \sum_{j=1}^{m} \left(\partial_i D_j V(Y) \sharp Z_i \times Z_j^* + Z_j \times (\partial_i D_j V(Y) \sharp Z_i)^* \right) \geq 2c Z.Z.$$

We shall say that $V \in \mathbb{C}\langle X_1, \ldots, X_m \rangle$ is self-adjoint iff for any self-adjoint variables $X = (X_1, \ldots, X_m)$, $[V(X_1, \ldots, X_m)]^* = V(X_1, \ldots, X_m)$. V is self-adjoint (c, M)-convex if V is self-adjoint and the above holds once restricted to self-adjoint variables, i.e for any m-tuples of self-adjoint variables $X = (X_1, \ldots, X_m)$ and $Y = (Y_1, \ldots, Y_m)$ living in some C^*-algebra $(\mathcal{A}, \| \cdot \|_\infty)$ which are bounded in norm by M, we have

$$[DV(X) - DV(Y)].(X - Y) \geq c(X - Y).(X - Y)$$

In this case $X.Y = \{X, Y\} := \frac{1}{2} \sum_{i=1}^{m} (X_i Y_i + Y_i X_i)$ is simply the anti-commutator of X and Y.

If we specialize this assumption to matrices and consider $\mathcal{A}$ to be the algebra of $N \times N$ matrices with complex entries equipped with the usual involution $(A^*)_{ij} = \bar{A}_{ji}$ and the spectral norm $\| \cdot \|_\infty$, we find that if V is self-adjoint (c, M)-convex, $(X_{ij})_{i \leq j} \to \mathrm{Tr}[V(X)]$ is strictly convex on the set of entries where X is Hermitian and with spectral radius bounded by M since

$$\mathrm{Tr}V(X) - \mathrm{Tr}V(Y) = \int_0^1 \mathrm{Tr}(DV(\alpha X + (1-\alpha)Y).(X - Y)) d\alpha$$

$$= \mathrm{Tr}(DV(Y).(X - Y)) + \int_0^1 \mathrm{Tr}([DV(\alpha X + (1-\alpha)Y) - DV(X)].(X - Y)) d\alpha$$

$$\geq \mathrm{Tr}(DV(Y).(X - Y)) + \frac{c}{2} \mathrm{Tr}((X - Y)^2).$$

Taking $Y = (X + Z)/2$ and X to be X or Z and summing the resulting inequalities gives

$$\mathrm{Tr}V(X) + \mathrm{Tr}V(Z) - 2\mathrm{Tr}V \left(\frac{X + Z}{2} \right) \geq c \mathrm{Tr}((X - Z)^2)$$

and hence the Hessian of $(X_{ij})_{i \leq j} \to \mathrm{Tr}V(X)$ is bounded below by cI, at least on matrices X with norm bounded by M.

This kind of hypothesis was shown to be very useful in [20]. The interest in relaxing the hypothesis of convexity to hold in a bounded domain is related with matrix models where $V = \frac{1}{2} X.X + W$ with $W = \sum_{i=1}^{n} \beta_i q_i$ for some monomials and complex parameters $(\beta_i)_{1 \leq i \leq n} \in \mathbb{C}^n$. It is clear now that for all M finite we can choose the β_i's sufficiently small so that V is $(1/2, M)$-convex (whereas it would not work with no bounds). Indeed, in that case

$$(DV(X) - DV(Y)).(X - Y) = (X - Y).(X - Y) + (DW(X) - DW(Y)).(X - Y)$$

But when the norms of X and Y are bounded by M,

$$|(DW(X) - DW(Y)).(X - Y)| \leq C(M) \max_i |\beta_i|(X - Y).(X - Y)$$

with $C(M)$ a constant which only depends on M and the q_i. Hence, we can now choose t small enough so that $C(M) \max_i |t_i| < 1/2$ and so V is then $(1/2, M)$-convex. This is analogous to what

was done in [20] in case of non-convex interaction; it was shown that then if one adds a cut-off, the large N expansion is still valid provided the parameters in W are small enough.

Hereafter we assume that V is (c, M)-convex. We let $(\mathcal{A}, *, \phi)$ be a non-commutative probability space generated by a free Brownian motion S (we refer to [5, 4] for an introduction to free Brownian motion and its related free Itô calculus). We shall denote by $\|\cdot\|_\infty$ the operator norm in $(\mathcal{A}, \phi)$.

We prove (see Lemma 2.1 and Theorem 2.2):

Theorem 1.1. *Let V be a (c, M)-convex polynomial in $X_1, \ldots, X_m$.*

Then there exist $M_0 = M_0(c, \|DV(0).DV(0)\|_\infty)$, $B_0 = B_0(c, \|DV(0).DV(0)\|_\infty)$, and $b = b(c, \|DV(0).DV(0)\|_\infty, M) \geq B_0$ finite constants, so that whenever $M \geq M_0$ and Z is an m-tuple with $\|Z\| < b$,

i. *There exists a unique solution X_t^Z to*

$$dX_t^Z = dS_t - \frac{1}{2}DV(X_t^Z)dt, \qquad t \in [0, +\infty), \tag{4}$$

with the initial data $X_0^Z = Z$. Moreover, in this case,

$$\|X_t^Z\|_\infty \leq M, \qquad \forall\, t \in [0, +\infty),$$
$$\limsup_{t \to \infty} \|X_t^Z\|_\infty \leq B_0,$$
$$X_t^Z \in C^*(Z, S_q : q \in [0, t]), \qquad \forall\, t \in [0, +\infty).$$

ii. *$\|X_t^Z - X_t^0\|_\infty \to 0$ as $t \to \infty$.*

iii. *The law of $((X_t^Z)^*, X_t^Z)$ converges to a stationary law $\mu_V \in \mathbb{C}\langle X_1, \ldots, X_m, X_1^*, \ldots, X_m^* \rangle'$ as t goes to infinity. μ_V is the non-commutative law of m variables uniformly bounded by B_0. If V is self-adjoint, μ_V is the law of m self-adjointvariables.*

iv. *The restriction $\tau_V = \mu_V|_{\mathbb{C}\langle X_1, \ldots, X_m \rangle}$ of μ_V to $\mathbb{C}\langle X_1, \ldots, X_m \rangle$ satisfies the Schwinger-Dyson equation which states that for all $P \in \mathbb{C}\langle X_1, \ldots, X_m \rangle$ and all $i \in \{1, \ldots, m\}$*

$$\tau_V \otimes \tau_V(\partial_i P) = \tau_V(D_i V P). \tag{5}$$

Moreover, if ν is the law of m variables whose uniform norm is bounded by b and $\nu|_{\mathbb{C}\langle X_1, \ldots, X_m \rangle}$ satisfies (5), then $\nu|_{\mathbb{C}\langle X_1, \ldots, X_m \rangle} = \tau_V$.

The Schwinger-Dyson equation (5) is exactly the same as the one which characterized the enumeration of maps (2) from which we deduce that $\mathcal{M}_{(\beta_i, q_i)_{1 \leq i \leq n}} = \tau_{\frac{1}{2}\sum_{i=1}^m X_i^2 + \sum \beta_i q_i}$ at least for sufficiently small polynomials V. This allows to give some information on the domain of analyticity of $\beta = (\beta_i)_{1 \leq i \leq n} \to \mathcal{M}_{(\beta_i, q_i)_{1 \leq i \leq n}}(P) = \mathcal{M}_\beta(P)$ (see section 3).

Theorem 1.2. *Let $V = V_\beta = \sum_{i=1}^n \beta_i q_i$ be a polynomial, where $\beta = (\beta_i)_{1 \leq i \leq n}$ are (complex) parameters and $(q_i)_{1 \leq i \leq n}$ are monomials. For c, M positive real numbers and $M \geq M_0$ with M_0 as in Theorem 1.1, let $T(c, M) \subset \mathbb{C}^n$ be the interior of the subset of parameters $\beta = (\beta_i)_{1 \leq i \leq n}$ such that V_β is (c, M)-convex.*

Then , for any $P \in \mathbb{C}\langle X_1, \ldots, X_m \rangle$, $\beta \in T(c, M) \to \tau_{V_\beta}(P)$ is analytic. In particular, $\beta \to \mathcal{M}_\beta(P)$ extends analytically to the interior of the set of β_i's where $\frac{1}{2}\sum_{i=1}^m X_i^2 + \sum \beta_i q_i$ is (c, M)-convex for $M \geq M_0(c)$.

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The laws μ_V are interesting in their own. In the free probability language we have proved the following.

Theorem 1.3. *Let V be a (c, M)-convex potential with $M \geq M_0$ the constant of Theorem 1.1. If V is self-adjoint, there exists a non-commutative law in $\mathbb{C}\langle X_1, \ldots, X_m, X_1^*, \ldots, X_m^* \rangle'$ with conjuguate variable $(D_i V)_{1 \leq i \leq m}$. There exists at most one such law satisfying the additional constraint to be the law of variables bounded by b. For non self-adjoint potential, there a unique law $\mu_V \in \mathbb{C}\langle X_1, \ldots, X_m, X_1^*, \ldots, X_m^* \rangle'$ which satisfies for all $P \in \mathbb{C}\langle X_1, \ldots, X_m, X_1^*, \ldots, X_m^* \rangle$ and all $i \in \{1, \ldots, m\}$*

$$\sum_{i=1}^{m} \mu_V \otimes \mu_V \left((\partial_i + \partial_{i,*})(D_i + D_{i,*})P \right) = \sum_{i=1}^{m} \mu_V \left(D_i V D_i P + (D_i V)^* D_{i,*} P \right) \tag{6}$$

with $(D_{i,*}, \partial_{i,*})$ the non-commutative deriavtives with respect to X_i^*. There exists at most one such law satisfying the additional constraint to be the law of variables bounded by b.

This statement is deduced from (2.2)(2). Moreover, we let Z be an m-tuple of operators with law μ_V. We shall prove that the C^*-algebra and the von Neumann algebra generated by Z have many properties in common with the one generated by a semi-circular system.

Theorem 1.4. *Assume that V is (c, M)-convex with $M \geq M_0$ the constant of Theorem 1.1. The C^*-algebra generated by Z is exact, projectionless and its associated von Neumann algebra has the Haagerup approximation property and admits and embedding into the ultrapower of the hyperfinite II_1 factor.*

In particular we have

Corollary 1.5. *Assume that V is (c, M)-convex with $M \geq M_0$ the constant of Theorem 1.1. Let Z be any m-tuple of b-bounded variables with the unique law μ_V satisfying (2). (a) The algebra $C^*(Z)$ has no non-trivial projections. (b) The spectrum of any non-commutative *-polynomial P in the m-tuple Z is connected (in the case that $P(Z)$ is normal, this means that the support of its spectral measure is connected). (c) If P is any polynomial in Z whose value is self-adjoint, then the probability measure given by the law of $P(Z)$ has connected support.*

2 Existence of free diffusions and convergence to their stationary measure

We shall show that if M is chosen large enough (depending on c and $\|DV(0).DV(0)\|_\infty$), we can build a bounded solution to some free stochastic differential equation with drift DV provided that V is (c, M)-convex. This generalizes Langevin dynamics to the context of operators.

Lemma 2.1. *Let V be a (c, M)-convex polynomial in $X_1, \ldots, X_m$. Then there exist finite constants*

$$M_0 = M_0(c, \|DV(0).DV(0)\|_\infty), \qquad B_0 = B_0(c, \|DV(0).DV(0)\|_\infty)$$
$$b = b(c, \|DV(0).DV(0)\|_\infty, M) \geq B_0$$

so that if $M \geq M_0$, and Z is any m-tuple with $\|Z\| < b$, there exists a unique solution X_t to

$$dX_t = dS_t - \frac{1}{2} DV(X_t)dt, \qquad t \in [0, +\infty). \tag{7}$$

with the initial data $X_0 = Z$. Moreover, in this case,

$$\|X_t\| \le M, \qquad \forall\, t \in [0, +\infty),$$

$$\limsup_{t \to \infty} \|X_t\|_\infty \le B_0,$$

$$X_t \in C^*(Z, S_q : q \in [0, +\infty)), \qquad \forall\, t \in [0, +\infty).$$

If V is self-adjoint (c, M)-convex, the above results hold under the additional assumption that $X_0 = Z$ is self-adjoint. In this case, X_t remains self-adjoint for all $t \ge 0$.

Proof. We remind the reader (cf. [5]) that if DV is uniformly Lipschitz for the uniform norm, the existence and uniqueness to (7) is clear by the following Picard argument. For the existence we consider the sequence X_t^n, $n = 0, 1, 2, \dots$ constructed recursively as follows. Set $X_t^0 = Z$ for all t, and having defined X_t^n, let X_t^{n+1} be given by the equation

$$dX_t^{n+1} = dS_t - \frac{1}{2} DV(X_t^n) dt$$

with the initial condition $X_0^{n+1} = Z$.

Subtracting the equations for X_t^{n+1} and X_t^n from each other, we get for all $t \ge 0$,

$$\|X_t^{n+1} - X_t^n\|_\infty \le \frac{1}{2} \int_0^t \|DV(X_s^n) - DV(X_s^{n-1})\|_\infty ds \le \frac{1}{2} \|DV\|_{\mathcal{L}} \int_0^t \|X_s^n - X_s^{n-1}\|_\infty ds$$

where $\|DV\|_{\mathcal{L}}$ denotes the Lipschitz norm of DV

$$\|DV\|_{\mathcal{L}} = \inf\{C : \|DV(Y) - DV(Y')\|_\infty \le C\|Y - Y'\|_\infty\}.$$

Iterating, we deduce that

$$\|X_t^{n+1} - X_t^n\|_\infty \le \left(\frac{\|DV\|_{\mathcal{L}}}{2}\right)^n \frac{t^n}{n!} \sup_{u \le t} \|S_u + uDV(Z)\|_\infty \le \left(\frac{\|DV\|_{\mathcal{L}}}{2}\right)^n \frac{t^n}{n!} (2\sqrt{t} + \frac{t}{2}\|DV(Z)\|_\infty)$$

which proves norm convergence of X_t^n (note that $X_0^n = Z$ for all n). Moreover, this limit satisfies (7). We also see that $X_t^n \in C^*(Z, S_q : q \ge 0)$ and therefore the limit $X_t \in C^*(Z, S_q : q \ge 0)$ as well.

The proof of uniqueness follows the same lines since any two solutions X_t, Y_t satisfy

$$\|X_t - Y_t\|_\infty \le \frac{1}{2} \|DV\|_{\mathcal{L}} \int_0^t \|X_s - Y_s\|_\infty ds$$

which proves that $X = Y$ by Gronwall's argument.

If V is a self-adjoint polynomial, then $D_i V$ is also self-adjoint for all $i \in \{1, \dots, m\}$. Therefore, since S_t is self-adjoint for all $t \ge 0$, we deduce by induction that X_t^n is self-adjoint for all $n \ge 0$ and all $t \ge 0$ and so its limit X_t is also self-adjoint. Hence the solution X_t of the free stochastic equation is self-adjoint for all times.

We now return to the case of a (c, M) convex V. By Lemma 3.2 in [5], we can construct a new function

$$f_R(X) = \frac{1}{2} DV(X) h(\sum_i \|X_i\|_\infty)$$

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so that f_R is uniformly Lipschitz and $f_R(X) = DV(X)$ if $\sum \|X_j\|_\infty \leq R$. The Picard argument above implies existence and uniqueness of a solution X_t^R to

$$dX_t^R = dS_t + f_R(X_t^R)dt$$

for all times t. Clearly, if we show that this solution satisfies $\sum_j \|(X_j^R)_t\|_\infty \leq R$, it will also be a solution to the original equation (7) involving DV. Furthermore, if we start with some initial data X_0 with $\sum \|(X_j)_0\|_\infty < R$, solutions to (7) always exist for small time (at least up until the time that the operator norm of the solution exceeds R). Thus we may consider a solution up to the time that its norm reaches some fixed constant M (with the intent to show that this time is infinite). By free Itô calculus

$$
\begin{aligned}
dX_t.X_t &= 2X_t.dS_t - DV(X_t).X_t dt + 2dt \\
&= 2X_t.dS_t - (DV(X_t) - DV(0)).X_t dt - DV(0).X_t dt + 2dt
\end{aligned}
$$

Therefore, for all $s \geq 0$,

$$
\begin{aligned}
e^{cs} X_s.X_s &= X_0.X_0 - 2\int_0^s e^{ct} DV(0).X_t dt - \int_0^s e^{ct}[(DV(X_t) - DV(0)).X_t - cX_t.X_t]dt \\
&\quad + 2\int_0^s e^{ct} dt + 2\int_0^s e^{ct}.dS_t \\
&\leq X_0.X_0 - \int_0^s e^{ct} DV(0).X_t dt + 2c^{-1}e^{cs} + 2\int_0^s e^{ct}.dS_t
\end{aligned}
$$

where the last inequality holds in the sense of operator and we relied on our hypothesis of (c, M) convexity. Since also $e^{cs}X_s.X_s$ is a non negative operator, we deduce that

$$
\begin{aligned}
A_s &:= \|X_s.X_s\|_\infty \\
&\leq e^{-cs}A_0 + 2c^{-1} + 2\left\|\int_0^s e^{c(t-s)}X_t.dS_t\right\|_\infty + \|DV(0).DV(0)\|_\infty^{\frac{1}{2}} \int_0^s e^{c(t-s)}A_t^{\frac{1}{2}} dt
\end{aligned}
$$

By Theorem 3.2.1 of [4], we know that the free analog of the Burkhölder-Davis inequality for integrals with respect to free Brownian motion holds for the L^p norm even with $p = \infty$. More precisely, the following estimate holds:

$$\left\|\int_0^s e^{c(t-s)}X_t.dS_t\right\|_\infty \leq 2\sqrt{2}\left(\int_0^s e^{2c(t-s)}A_t dt\right)^{\frac{1}{2}} \leq 2\sqrt{2}\left(\int_0^s e^{c(t-s)}A_t dt\right)^{\frac{1}{2}}.$$

Moreover, by the Cauchy-Schwarz inequality, we obtain the bound

$$\left(\int_0^s e^{c(t-s)}A_t^{\frac{1}{2}} dt\right)^2 = \left(\int_0^s e^{\frac{1}{2}c(t-s)} \cdot e^{\frac{1}{2}c(t-s)}A_t^{\frac{1}{2}} dt\right)^2 \leq \frac{1}{c}\int_0^s e^{c(t-s)}A_t dt.$$

Hence, we get the inequality (with $C = \frac{4}{c}(\|DV(0).DV(0)\|_\infty^{\frac{1}{2}} + 4\sqrt{2})^2$ and since $A_0 \leq b^2 m$)

$$
\begin{aligned}
A_s^2 &\leq 4e^{-cs}m^2 b^4 + 2^4 c^{-2} + C\int_0^s e^{c(t-s)}A_t dt \\
&\leq 4e^{-cs}m^2 b^4 + 2^4 c^{-2} + \frac{C}{2}\int_0^s e^{c(t-s)}(B^{-1}A_t^2 + B)dt \\
&\leq 4e^{-cs}m^2 b^4 + 2^4 c^{-2} + CB(2c)^{-1} + CB^{-1}\int_0^s e^{c(t-s)}A_t^2 dt
\end{aligned}
$$

where B is any positive constant (we just used that for all x, $2x \leq B + B^{-1}x^2$).

We now use Gronwall's lemma (or simply iterate in the above inequality) to deduce, with $C' = 2^4 c^{-2} + CB(c)^{-1}$, that

$$A_s^2 e^{cs} \leq [4m^2 b^4 + C' e^{cs}] + \frac{C}{B} \int_0^s e^{\frac{C}{B}(s-u)}[4m^2 b^4 + C' e^{cu}]du$$

We now take B large enough so that $c > \frac{C}{B}$ to conclude that

$$A_s^2 \leq 8m^2 b^4 e^{(\frac{C}{B}-c)s} + C' \frac{c}{c - \frac{C}{B}}$$

If we now choose $B_0^2 = (2^4 c^{-2} + CB(c)^{-1}) \frac{c}{c - \frac{C}{B}}$ for $B = 2C/c$, we have shown that:

- A_s^2 stays bounded by $8m^2 b^4 + B_0^2$. So if $8m^2 b^4 < M^2 - B_0^2$, A_s always stays bounded by M.

- As s goes to infinity, $\limsup_{s\to\infty} A_s$ is bounded by B_0.

- We can choose $b > B_0$ as long as $M > B_0 \sqrt{1 + 8m^2 B_0^2} := M_0$.

- $X_t \in C^*(Z, S_q : q \geq 0)$.

This concludes the proof. □

We now consider the solutions of Lemma 2.1 starting with different initial data.

Theorem 2.2. *Let M_0, B_0 and b be as in Lemma 2.1, and assume that $M \geq M_0$, and that Z is an m-tuple of operators with $\|Z\|_\infty < b$. Consider the unique solutions X_t^Z, X_t^0 to the free SDE*

$$dX_t = dS_t - \frac{1}{2} DV(X_t)dt \tag{8}$$

with initial conditions $X_0^Z = Z$, $X_0^0 = 0$. Then

1. *$\|X_t^Z - X_t^0\|_\infty \to 0 \qquad$ as $t \to \infty$.*

2. *The law of $((X_t^Z)^*, X_t^Z)$ converges to a stationary law $\mu_V \in \mathbb{C}\langle X_1, \ldots, X_m, X_1^*, \ldots, X_m^*\rangle'$ which satisfies for all $P \in \mathbb{C}\langle X_1, \ldots, X_m, X_1^*, \ldots, X_m^*\rangle$ and all $i \in \{1, \ldots, m\}$*

$$\sum_{i=1}^m \mu_V \otimes \mu_V \left((\partial_i + \partial_{i,*})(D_i + D_{i,*})P\right) = \sum_{i=1}^m \mu_V \left(D_i V D_i P + (D_i V)^* D_{i,*} P\right). \tag{9}$$

Moreover, for any $k \in \mathbb{N}$, any $i \in \{1, \ldots, m\}$,

$$\mu_V \left((X_i X_i^*)^k\right) \leq B_0^{2k}. \tag{10}$$

Any law $\nu \in \mathbb{C}\langle X_1, \ldots, X_m, X_1^*, \ldots, X_m^*\rangle'$ of variables bounded in operator norm by b which satisfies (9) equals μ on $\mathbb{C}\langle X_1, \ldots, X_m, X_1^*, \ldots, X_m^*\rangle$.

3. The restriction $\tau_V = \mu_V|_{\mathbb{C}\langle X_1, \ldots, X_m \rangle}$ of μ_V to $\mathbb{C}\langle X_1, \ldots, X_m \rangle$ satisfies for all polynomials $P \in \mathbb{C}\langle X_1, \ldots, X_m \rangle$

$$\sum_{i=1}^{m} \tau_V \otimes \tau_V(\partial_i D_i P) = \sum_{i=1}^{m} \tau_V(D_i V D_i P). \tag{11}$$

Any law $\nu \in \mathbb{C}\langle X_1, \ldots, X_m, X_1^*, \ldots, X_m^* \rangle'$ of variables bounded in operator norm by b whose retriction $\nu|_{\mathbb{C}\langle X_1, \ldots, X_m \rangle}$ satisfies (11) is such that $\nu|_{\mathbb{C}\langle X_1, \ldots, X_m \rangle} = \tau_V$.

Remark 2.3. (a) Note that if Z is an m-tuple of variables, norm bounded by b having the stationary law μ, then X_t^Z is a stationary process (with law given at all times by μ) and $X_t^Z - X_t^0$ converges to zero in operator norm as t goes to infinity by the first point of the theorem.

(b) Recall that a law μ of m non-commutative variables has conjugate variables $(\xi_i)_{1 \leq i \leq m}$ if and only if for any $P \in \mathbb{C}\langle X_1, \ldots, X_m, X_1^*, \ldots, X_m^* \rangle$,

$$\mu \otimes \mu(\partial_i P) = \mu(\xi_i P).$$

Taking the adjoint, we find that we must also have

$$\mu \otimes \mu(\partial_{i,*} P) = \mu(\xi_i^* P).$$

Taking $P = D_i Q$ in the first equality, $P = D_{i,*} Q$ in the second and summing the resulting equalities yields

$$\mu \otimes \mu((\partial_i D_i + \partial_{i,*} D_{i,*}) Q) = \mu(\xi_i D_i Q + \xi_i^* D_{i,*} Q)$$

which differs from (9) by the terms $\mu \otimes \mu(\partial_i D_{i,*} Q + \partial_{i,*} D_i Q)$. Hence, (9) is not equivalent with the fact that μ has conjugate the variable $(D_i V)_{1 \leq i \leq m}$ in the case that V is not self-adjoint. In the self-adjoint case, because of our convention, Q depends only on $X_1, \ldots, X_n$ and so the terms involving $D_{i,*}$ and $\partial_{i,*}$ are equal to zero. In that case, (9) is compatible with the condition that the conjugate variables are equal to the cyclic gradient of V.

Proof. Consider two solutions X_t^Y, X_t^Z with initial data Y and Z, respectively, and assume that $\|Y\|_\infty, \|Z\|_\infty \leq m$. Then

$$d(X_t^Z - X_t^Y) = -\frac{1}{2}[DV(X_t^Z) - DV(X_t^Y)]dt$$

Since by Lemma 2.1, the operator norms of X_t^Z and X_t^Y stay bounded by M for all t and V is (c, M) convex, we find that

$$\begin{aligned} d(X_t^Z - X_t^Y).(X_t^Z - X_t^Y) &\leq -[DV(X_t^Z) - DV(X_t^Y)].(X_t^Z - X_t^Y)dt \\ &\leq -c(X_t^Z - X_t^Y).(X_t^Z - X_t^Y)dt \end{aligned}$$

where the inequality again holds in the sense of operators. This implies that

$$\|(X_t^Z - X_t^Y).(X_t^Z - X_t^Y)\|_\infty \leq e^{-ct}\|Z - Y\|_\infty^2 \tag{12}$$

and so

$$\lim_{t \to \infty} \|(X_t^Z - X_t^Y).(X_t^Z - X_t^Y)\|_\infty = 0.$$

In particular, we can take $Y = 0$ and we have proved that all diffusion solutions starting from different initial data norm-bounded by b will asymptotically be the same as X_t^0.

As t gets large, X_t^0 is bounded by $B_0 < b$ according to Lemma 2.1, and so we can choose $Y = \tilde{X}_t^0$ (with $\tilde{X}$ constructed as X but with some free Brownian motion $\tilde{S}$ on $[0, s]$ and the increments of S on $[s, s+t]$) to deduce

$$\|(X_t^0 - \tilde{X}_{t+s}^0).(X_t^0 - \tilde{X}_{t+s}^0)\|_\infty \le e^{-ct} M. \tag{13}$$

Since $\tilde{X}_{t+s}^0$ has the same law as X_{t+s}^0 we conclude that the joint law of $X_t^0, (X_t^0)^*$ converges as t goes to infinity. We denote this limit by μ. Clearly, μ is a stationary law for the diffusion (since $X_{\infty+s}^0$ and $X_{\infty+t}^0$ have the same law). We now consider X_t^∞ to be the solution of the free SDE starting from X so that (X, X^*) has law μ. Noting that we have

$$dX_t^\infty = dS_t - \frac{1}{2} DV(X_t^\infty) dt \qquad d(X_t^\infty)^* = dS_t - \frac{1}{2}(DV(X_t^\infty))^* dt$$

and applying free Itô's calculus [4], we have that for any $P \in \mathbb{C}\langle X_1, \ldots, X_m, X_1^*, \ldots, X_m^* \rangle$,

$$0 = \partial_t \phi(P(X_t^\infty, (X_t^\infty)^*))$$

$$= \phi \otimes \phi(\frac{1}{2} \sum_{i=1}^m (\partial_{i,*} + \partial_i)(D_{i,*} + D_i) P(X_t^\infty, (X_t^\infty)^*))$$

$$- \frac{1}{2} \phi(DV(X_t^\infty).DP(X_t^\infty) + (DV(X_t^\infty))^*.D_* P(X_t^\infty, (X_t^\infty)^*)))$$

so that for all $*$-polynomials $P \in \mathbb{C}\langle X_1, \ldots, X_m, X_1^*, \ldots, X_m^* \rangle$, μ_V must satisfy (9). τ_V, the restriction of μ_V to $\mathbb{C}\langle X_1, \ldots, X_m \rangle$ must then satisfy (11).

The uniqueness of solutions to this equation is simply due to the fact that if we run the process from Z bounded uniformly by b with law ν satisfying (9), the law of X_t^Z must be stationary, but also converging to μ_V by the previous argument. Hence it must be equal to μ_V. The same argument applies in the case that we only care about the restriction to $\mathbb{C}\langle X_1, \ldots, X_m \rangle$ of some law ν of variables Z bounded by b, since if it satisfies (11), the process X_t^Z will be such that $\partial_t \phi(P(X_t^Z)) = 0$ for all $P \in \mathbb{C}\langle X_1, \ldots, X_m \rangle$ and so the law of X_t^Z restricted to $\mathbb{C}\langle X_1, \ldots, X_m \rangle$ will be stationary (here we use the fact that $DV(X)$ depends only on X and not on its adjoint when speaking of the evolution of the law of X_t^Z restricted to $\mathbb{C}\langle X_1, \ldots, X_m \rangle$). Since it also converges to τ_V, we obtain the desired equality. $\qquad\square$

We shall see in section 3, Corollary 3.2, that in fact μ not only satisfies (11) but actually for all $i \in \{1, \ldots, m\}$, all $P \in \mathbb{C}\langle X_1, \ldots, X_m \rangle$,

$$\mu \otimes \mu(\partial_i P) = \mu(D_i V P).$$

In other words, at least when μ is the law of self-adjoint operators, the conjugate variables of μ are in the cyclic gradient space.

3 Analyticity of the solution to Schwinger-Dyson equation and discussion around phase transition

We show in this section that on the domain where V stays (c, M)-convex for some $c > 0$ and $M \ge M_0$ as in Lemma 2.1, the law τ_V will depend analytically on the parameters of V.

Lemma 3.1. *Let $V = V_\beta = \sum_{i=1}^{n} \beta_i q_i$ be a polynomial, where $\beta = (\beta_i)_{1 \le i \le n}$ are (complex) parameters and $(q_i)_{1 \le i \le n}$ are monomials. For c, M positive real numbers and $M \ge M_0$ with M_0 as in Lemma 2.1 and Theorem 2.2, let $T(c, M) \subset \mathbb{C}^n$ be the interior of the subset of parameters $\beta = (\beta_i)_{1 \le i \le n}$ for which V is (c, M)-convex. Let $\mu_\beta = \mu_{V_\beta}$ be the unique stationary measure of Theorem 2.2 and τ_β the law of $(X_1, \cdots, X_m)$ under μ_β.*

Then for any polynomial $P \in \mathbb{C}\langle X_1, \ldots, X_m \rangle$, the map $\beta \in T(c, M) \to \tau_\beta(P)$ is analytic.

Note that $T(c, M)$ is non empty as soon as the set of monomials $(q_i, 1 \le i \le n)$ contains $(X_i^2, 1 \le i \le m)$. Indeed, if we set $V(X) = \sum_{i=1}^{m} \beta_i X_i^2 + \sum_{i=m+1}^{n} \beta_i q_i(X)$, we always have that for X, Y uniformly bounded by M,

$$\left(\sum_{i=m+1}^{n} \beta_i D q_i(X) - \sum_{i=m+1}^{n} \beta_i D q_i(Y) \right) . (X - Y) \le C(M) \max_{m+1 \le i \le n} |\beta_i| (X - Y) . (X - Y)$$

with a universal constant $C(M)$ which only depends on M and the $(q_i)_{1 \le i \le n}$. Hence, if $\beta_i > 0$ for $i \in \{1, \cdots, m\}$,

$$(DV(X) - DV(Y)) . (X - Y) \ge \left[\min_{1 \le i \le m} \beta_i - C(M) \max_{m+1 \le i \le n} |\beta_i| \right] (X - Y) . (X - Y).$$

Therefore any set of parameters $(\beta_i)_{1 \le i \le n}$ such that

$$\min_{1 \le i \le m} \beta_i - C(M) \max_{m+1 \le i \le n} |\beta_i| \ge c$$

will be such that V is (c, M)-convex.

Proof.

We denote $(X_t^\beta)_{t \ge 0}$ the solution of (8) with potential $V = V_\beta$ and starting from the null operator. We shall show that $\beta \to X_t^\beta$ expands as a sum of uniformly bounded operators. More precisely, we fix β in the interior of $T(c, M)$ and find a family $X_t^{(k_1, \ldots, k_n)}, k_i \in \mathbb{N}, 1 \le i \le n$ of operator-valued processes such that for $\eta \in \mathbb{C}^n$, $|\beta - \eta| := \max_{1 \le i \le n} |\beta_i - \eta_i|$ small enough,

$$X_t^\eta = X_t^\beta + \sum_{\substack{k_1, \ldots, k_n \in \mathbb{N}^n \\ \sum k_i \ge 1}} \prod_{i=1}^{n} (\eta_i - \beta_i)^{k_i} X_t^{(k_1, \ldots, k_n)} \tag{14}$$

Moreover, $X_t^{(k_1, \ldots, k_n)}$ are operator-valued processes such that there exists a constant C which only depends on c, M and the degree of V so that

$$\sup_{t \in \mathbb{R}^+} \| X_t^{(k_1, \ldots, k_n)} \|_\infty \le C^{\sum k_i}. \tag{15}$$

Finally the distribution of $\left(X_t^{(k_1, \ldots, k_n)} \right)_{k_1, \ldots, k_n \in \mathbb{N}^n}$ converges (in the sense of finite marginals, i.e., on polynomials involving only a finite number of the $\left(X_t^{(k_1, \ldots, k_n)} \right)_{k_1, \ldots, k_n \in \mathbb{N}^n}$) towards the law of $\left(X_\infty^{(k_1, \ldots, k_n)} \right)_{k_1, \ldots, k_n \in \mathbb{N}^n}$ as t goes to infinity.

Let us conclude the proof of the lemma assuming (14) and (15). (14) and (15) entail that for all polynomial functions $P \in \mathbb{C}\langle X_1, \ldots, X_m \rangle$, for all $t \geq 0$,

$$\beta \rightarrow \phi(P(X_t^\beta))$$

is analytic in the interior of $T(c, M)$ since it implies that for $\beta \in T(c, M)$,

$$\phi_\beta(P(X_t^\eta)) = \phi \left(P(X_t^\beta + \sum_{k_1, \ldots, k_n} \prod_{i=1}^n (\eta_i - \beta_i)^{k_i} X_t^{(k_1, \ldots, k_n)}) \right)$$

for all η in the domain $B(C, \beta) = \{|\eta - \beta| < 1/C\}$ which does not depend on the time parameter $t \geq 0$. Note also that $\tau(P(X_t^\eta))$ is uniformly bounded independently of $t \in \mathbb{R}^+$ since C does not depend on t. We know that (X_t^η) converges as t goes to infinity towards

$$\tau_\beta(P) := \phi(P(X_\infty^\beta + \sum_{k_1, \ldots, k_n} \prod_{i=1}^n (\eta_i - \beta_i)^{k_i} X_\infty^{(k_1, \ldots, k_n)}))$$

(note here that convergence of the $X^{(k_1, \ldots, k_n)}$ in the sense of finite marginals is sufficient since $\sum_{k_1, \ldots, k_n : \sum k_i \geq K} \prod_{i=1}^n (\eta_i - \beta_i)^{k_i} X_\infty^{k_1, \ldots, k_n}$ goes uniformly to zero as K goes to infinity). But then the limit has to depend analytically on β (as a limit of uniformly bounded functions which are analytic on a fixed domain). This proves the claim.

We now prove (14) and (15). We first check that $\beta \rightarrow X_t^\beta$ is of class $\mathcal{C}^\infty$, then that it is in fact an entire function and bound uniformly its radius of convergence. Finally, we prove that $t \rightarrow (X_t^{(k_1, \ldots, k_n)})_{k_1, \cdots, k_n \in \mathbb{N}^n}$ converges.

Step 1: $\beta \in T(c, M) \rightarrow X_t^\beta$ is of class $\mathcal{C}^\infty$ for all $t \geq 0$.

Let us study the first order differentiability, and first check that $\beta \rightarrow X_t^\beta$ is continuous. In fact, if $1_p(i) = 0$ for $i \neq p$ and $1_p(p) = 1$, we write

$$
\begin{aligned}
X_t^{\beta + \epsilon 1_p} - X_t^\beta &= -\frac{1}{2} \int_0^t [DV_{\beta + \epsilon 1_p}(X_s^{\beta + \epsilon 1_p}) - DV_\beta(X_s^\beta)] ds \\
&= -\frac{1}{2} \int_0^t \int_0^1 \partial DV_{\beta + \epsilon 1_p}((1 - \alpha) X_s^\beta + \alpha X_s^{\beta + \epsilon 1_p}) \sharp [X_s^{\beta + \epsilon 1_p} - X_s^\beta] d\alpha ds \\
&\quad - \frac{1}{2} \int_0^t [DV_{\beta + \epsilon \mathbf{1_p}} - DV_\beta](X_t^\beta).dt
\end{aligned}
$$

We find that if $\beta + \epsilon 1_p$ and β both belong to $T(c, M)$ so that X^β and $X^{\beta + \epsilon 1_p}$ stay uniformly bounded by M, that

$$\|X_t^{\beta + \epsilon 1_p} - X_t^\beta\|_\infty \leq C \int_0^t \|X_s^{\beta + \epsilon 1_p} - X_s^\beta\|_\infty + Ct\epsilon$$

where C only depends on (c, M). So Gronwall's lemma shows that for any time $t \geq 0$, there exists a finite $C(t)$ ($C(t)$ is uniformly bounded on compacts) so that

$$\|X_t^{\beta + \epsilon 1_p} - X_t^\beta\|_\infty \leq C(t)\epsilon. \tag{16}$$

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This suggests that $\beta \to X_t^\beta$ is in fact differentiable. To prove this point, let us introduce the candidate for the corresponding gradient; we define $(\nabla_\beta X_t^{\beta,i})_{1 \le i \le m}$ to be the m-tuple of operators valued processes solution of

$$d\nabla_\beta^p X_t^{\beta,i} = -\sum_{j=1}^m \partial_j D_i V(X_t^\beta) \sharp \nabla_\beta^p X_t^{\beta,j} dt + (d_\beta^p D_i V_\beta)(X_t^\beta) dt \qquad \nabla_\beta^p X_0^{\beta,i} \equiv 0$$

where d_β is the standard gradient with respect to the parameters β (so $d_\beta^j D_i V_\beta = D_i q_j$ for $j \in \{1,\dots,n\}$). Here p is any integer in $\{1,\dots,n\}$. There is a unique solution to this equation (since it is a linear differential equation as X^β is given). By the same type of argument as above, we now prove that for all $t \ge 0$ there exists $C(t)$ finite so that

$$\|X_t^{\beta+\epsilon 1_p} - X_t^\beta - \epsilon\nabla_\beta^p X_t^{\eta,\mathbf{i}}\|_\infty \le C(t)\epsilon^2 \tag{17}$$

Indeed, if we let

$$Y_t^{\epsilon,i} := X_t^{\beta+\epsilon 1_p,i} - X_t^{\beta,i} - \epsilon\nabla_\beta^p X_t^{\beta,i}$$

we find that

$$Y_t^\epsilon = -\int_0^t \int_0^1 \partial DV_{\beta+\epsilon\mathbf{1_p}}((1-\alpha)X_s^\beta + \alpha X_s^{\beta+\epsilon 1_p})\sharp[Y_s^\epsilon]ds$$
$$-\int_0^t \int_0^1 [\partial DV_{\beta+\epsilon\mathbf{1_p}}((1-\alpha)X_s^\beta + \alpha X_s^{\beta+\epsilon 1_p}) - \partial DV_{\beta+\epsilon\mathbf{1_p}}(X_s^\beta)]\sharp[X_s^{\beta+\epsilon 1_p} - X_s^\beta]ds$$
$$-\int_0^t [DV_{\beta+\epsilon\mathbf{1_p}} - DV_\beta - \epsilon d_\beta^p D_i V_\beta](X_t^\beta)dt$$

Using (16) we find that there exists a finite constant $C = C(M)$ such that

$$\max_{1 \le i \le m} \|Y_t^{\epsilon,i}\| \le C\int_0^t \max_{1 \le i \le m} \|Y_s^{\epsilon,i}\|ds + C\epsilon^2 \int_0^t C(s)^2 ds$$

and so Gronwall's lemma gives (17) (note here that $\max_{1 \le i \le m} \|Y_t^{\epsilon,i}\|$ is finite for all $\epsilon > 0$ so that $\beta + \epsilon 1_p \in T(c,M)$).

This shows that $\beta \to X_t^\beta$ is differentiable for all t with first derivative $\nabla_\beta X_t^\beta$. We can continue in the same spirit to show that $\nabla_\beta^p X_t^{\beta,i}$ is differentiable and by induction, we find that $\beta \to X_t^\beta$ is of class $\mathcal{C}^\infty$ in the interior of $T(c,M)$. We next bound uniformly all its derivatives.

Step 2: $\beta \in T(c,M) \to X_t^\beta$ is analytic for all $t \ge 0$.

To this end, we can write X_t^β as a formal series in η in a small ball around $\beta \in T(c,M)$

$$X_t^\eta = \sum_{(k)=(k_1,\dots,k_n)\in\mathbb{N}^n} \prod_{1 \le p \le n} (\eta_p - \beta_p)^{k_p} X_t^{(k)}$$

with $X_t^{(0,\dots,0)} = X_t^\beta$. Indeed, the coefficients of this series are obtained by differentiating the $\mathcal{C}^\infty$ operator X_t^η and $X_t^{(k)} = (k_1!\cdots k_n!)^{-1}\partial_{\eta_1}^{k_1}\cdots\partial_{\eta_n}^{k_n}X_t^\eta|_{\eta=\beta}$. We write $D_i V_\beta(X) = \sum_{j=1}^n \beta_j D_i q_j = \sum_{j=1}^{Dn} \tilde\beta_j q_{ij}$ where $D_i q_j = \sum_{l=1}^D q_{i,l+(j-1)D}$ is the decomposition of $D_i q_j$ as a sum of at most D

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monomials. We denote $q_{ij} = \prod_{1 \leq p \leq d_{ij}}^{\rightarrow} X_{l_{ij}^p}$ with $l_{ij}^p \in \{1, \ldots, m\}$. Moreover, we have $\tilde{\eta}_j = \eta_{[j/D]}$. Plugging this formal series into

$$dX_t^\eta = dS_t - \frac{1}{2} DV_\eta(X_t^\eta)dt$$

we find that $X_{\cdot}^{(k_1, \ldots, k_n)}$ satisfy, for $\sum k_i \geq 1$, the following induction relation

$$
\begin{aligned}
dX_t^{(k),j} &= -\frac{1}{2} \sum_{i=1}^{nD} \sum_{\sum_{p=1}^{d_{ji}} k_r^p = k_r - 1_{r=[i/D]}} \prod_{1 \leq p \leq d_{ji}}^{\rightarrow} X^{(k^p), l_{ji}^p} dt \\
&\quad -\frac{1}{2} \sum_{i=1}^{nD} \tilde{\beta}_i \sum_{\substack{\sum_{p=1}^{d_{ij}} k_r^p = k_r - 1_{r=[i/D]} \\ (k^p) \neq (k) \forall p}} \prod_{1 \leq p \leq d_{ji}}^{\rightarrow} X^{(k^p), l_{ji}^p} dt \\
&\quad -\frac{1}{2} \sum_{l=1}^{m} \partial_l D_j V_\beta(X_t^\beta) \sharp X^{(k),l} dt
\end{aligned}
\tag{18}
$$

Above, the sum over indices k such that $k_r - 1_{r=[i/D]} = -1$ is simply empty. Using the convexity of V_β, we now get a uniform bound by considering $X_t^{(k)}.X_t^{(k)} = \sum_{j=1}^{m} X_t^{(k),j}.X_t^{(k),j}$;

$$
\begin{aligned}
dX_t^{(k)}.X_t^{(k)} &\leq -\sum_{j=1}^{m} \sum_{i=1}^{nD} \sum_{\substack{\sum_{p=1}^{d_{ji}} k_r^p = \\ k_r - 1_{r=[i/D]}}} \left[\left(\prod_{1 \leq p \leq d_{ji}}^{\rightarrow} X^{(k^p), l_{ji}^p} \right)^* X_t^{(k),j} + (X_t^{(k),j})^* \prod_{1 \leq p \leq d_{ji}}^{\rightarrow} X^{(k^p), l_{ji}^p} \right] dt \\
&\quad -\sum_{j=1}^{m} \sum_{i=1}^{nD} \tilde{\eta}_i \sum_{p=1}^{d_{ji}} \sum_{\substack{k_r^p = k_r \\ (k^p) \neq (k) \forall p}} \left[\left(\prod_{1 \leq p \leq d_{ji}}^{\rightarrow} X^{(k^p), l_{ji}^p} \right)^* X_t^{(k),j} + (X_t^{(k),j})^* \prod_{1 \leq p \leq d_{ji}}^{\rightarrow} X^{(k^p), l_{ji}^p} \right] dt \\
&\quad -cX_t^{(k)}.X_t^{(k)} dt
\end{aligned}
\tag{19}
$$

where we have used that by convexity, for any operators X, Z bounded by M,

$$\sum_{i=1}^{m} \sum_{j=1}^{m} (\partial_j D_i V(X) \sharp Z_j).Z_i \geq cZ.Z.$$

Note that in the right hand side of (19), all the $X^{(k^p), l_{ji}^p}$ are such that $\sum k_i^p < \sum k_i$. Hence, we can deduce by induction that $A_t^{(k)} := \max_{1 \leq j \leq m} \|X_t^{(k),j}\|_\infty$ is bounded for all (k) and uniformly on compact sets of the time variable t. Indeed, we proved it in Theorem 2.2 for $(k) = (0, \ldots, 0)$. Let us assume it is true for (k) with $\sum k_i \leq K - 1$ and let us prove it remains true; we simply use

$$\left\| \prod_{1 \leq p \leq d_{ji}}^{\rightarrow} X^{(k^p), l_{ji}^p}.X_t^{(k),j} \right\|_\infty \leq \prod_{1 \leq p \leq d_{ij}} A_t^{(k_p)} A_t^{(k)} \leq B \prod_{1 \leq p \leq d_{ij}} (A_t^{(k_p)})^2 + B^{-1}(A_t^{(k)})^2.$$

Choosing B such that

$$2B^{-1}\sum_{j=1}^{m}\sum_{i=1}^{nD}(1+|\beta_i|)d_{ji}\sum_{\sum k_r^p \le k_r} 1 - c < 0$$

allows to bound $A_t^{(k)}$ uniformly on compact sets by our induction hypothesis. We next show that this bound can be taken uniformly on the time variable. To this end we first consider $Y_t^{(k)} = \sqrt{X_t^{(k)}.X_t^{(k)}}$ and deduce from (19) that

$$
\begin{aligned}
dY_t^{(k)} \le\ & m\sum_{i=1}^{nD}\sum_{\sum_{p=1}^{D}k_r^p=k_r-1_{r=[i/D]}}\prod_{1\le p\le D}A_t^{(k^p)}dt \\
& +mD\sum_{i=1}^{n}|\beta_i|\sum_{\substack{\sum_{p=1}^{D}k_r^p=k_r\\(k^p)\neq(k)\forall p}}\prod_{1\le p\le D}A_t^{(k^p)}dt - cY_t^{(k)}dt
\end{aligned}
\tag{20}
$$

where we eventually added terms (by taking $d_{ij}=D$) which can be done if we assume that $A^{(0)}\ge 1$ which we can always do. Therefore, for $\sum k_i \ge 1$, since $X_0^{(k),j}=0$, we obtain the bound

$$
\begin{aligned}
A_t^{(k)} \le\ & mD\sum_{i=1}^{n}\int_0^t e^{-c(t-s)}\sum_{\sum_{p=1}^{D}k_r^p=k_r-1_{r=i}}\prod_{1\le p\le D}A_s^{(k_p)}ds \\
& +mD\sum_{i=1}^{n}|\beta_i|\int_0^t e^{-c(t-s)}\sum_{\substack{\sum_{p=1}^{D}k_r^p=k_r\\(k_p)\neq(k)\forall p}}\prod_{1\le p\le D}A_s^{(k_p)}ds.
\end{aligned}
\tag{21}
$$

Note that the right hand side of (21) depends only on $A^{(l)}$ for $\sum l_i \le \sum k_i - 1$ (since $k_r^p \le k_r$ for all r but $(k^p)\neq(k)$). Since $A^{(0)}$ is uniformly bounded (as we proved that X_t^β is uniformly bounded for $\beta \in T(c,M)$ and $c>0$), we deduce by induction that $A^{(k)}:=\sup_{t\ge 0}A_t^{(k)}$ is finite and satisfy the induction bound, with $|\beta|=\sum_{i=1}^{n}|\beta_i|$,

$$
\begin{aligned}
A^{(k)} \le\ & mDc^{-1}\sum_{i=1}^{n}\sum_{\sum_{p=1}^{D}k_r^p=k_r-1_{r=i}}\prod_{1\le p\le D}A^{(k_p)} \\
& +mDc^{-1}|\beta|\sum_{\substack{\sum_{p=1}^{D}k_r^p=k_r\\(k_p)\neq(k)\forall p}}\prod_{1\le p\le D}A^{(k_p)}
\end{aligned}
\tag{22}
$$

We rewrite this inequality, since $A^{(k)}$ is obviously finite, as

$$
\begin{aligned}
A^{(k)} \le\ & \frac{mDc^{-1}}{1+mD^2c^{-1}|\beta|(A^{(0)})^{D-1}}\Big\{\sum_{i=1}^{n}\sum_{\sum_{p=1}^{D}k_r^p=k_r-1_{r=i}}\prod_{1\le p\le D}A^{(k_p)} \\
& +mDc^{-1}|\beta|\sum_{\sum_{p=1}^{D}k_r^p=k_r}\prod_{1\le p\le D}A^{(k_p)}\Big\}
\end{aligned}
\tag{23}
$$

where we added the term $mDc^{-1}|\beta|D(A^{(0)})^{D-1}A^{(k)}$ to the last sum. We now want to show that there is a finite $C = C(\beta)$ so that

$$A^{(k)} \le C^{\sum k_i}.$$

To this end we borrow the idea of majorizing sequences as developed by Cartan [9], chapter VII. It goes as follows here. We consider the polynomial in one variable given by

$$\tilde{V}_\kappa(x) = C\kappa x^D \quad \text{with } C := \frac{mDc^{-1}}{1 + mD^2c^{-1}|\beta|(A^{(0)})^{D-1}}.$$

and the the equation

$$x - x_0 = \tilde{V}_\kappa(x) - \tilde{V}_{|\beta|}(x_0)$$

with $x_0 = A^{(0)}$. We claim that for κ in a neighborhood of $|\beta|$, the solution x_κ in the neighborhood of x_0 of this equation is analytic in κ. Indeed, by the implicit function theorem, we only need to check that

$$1 \neq \partial_x \tilde{V}_{|\beta|}(x_0) = \frac{mDc^{-1}|\beta|}{1 + mD^2c^{-1}|\beta|(A^{(0)})^{D-1}} D(A^{(0)})^{D-1}$$

which is always true since $A^{(0)}$ is finite. This implies that there exists a finite constant $C = C(|\beta|)$ such that

$$|\partial_\kappa^k x_\kappa|_{\kappa=|\beta|}| \le k!C^k.$$

But now, $x^{(k)} := (k!)^{-1}\partial_\kappa^k x_\kappa|_{\kappa=|\beta|}$ satisfies the induction relation

$$x^{(k)} = C \sum_{\sum_{i=1}^D k_i = k-1} \prod_{1 \le i \le D} x^{(k_i)} + C|\beta| \sum_{\sum_{i=1}^D k_i = k} \prod_{1 \le i \le D} x^{(k_i)}$$

which implies

$$(1 - C|\beta|Dx_0^{D-1})x^{(k)} = C \sum_{\sum_{i=1}^D k_i = k-1} \prod_{1 \le i \le D} x^{(k_i)} + C|\beta| \sum_{\sum_{i=1}^D k_i = k, k_i < k} \prod_{1 \le i \le D} x^{(k_i)}.$$

Since $1 - C|\beta|Dx_0^{D-1} = 1/(1 + mDc^{-1}|\beta|Dx_0^{D-1}) > 0$, we conclude by induction that $x^{(k)} \ge 0$ for all k (note that $x^{(0)} = A^{(0)} > 0$). But then, comparing (22) and the above inequality, we also prove by induction that

$$A^{(k)} \le x^{(\sum k_i)} \le C^{\sum k_i}$$

which therefore gives the desired bound for the $A^{(k)}$. Hence we have proved (15).

Step 3: *Convergence in law of of $t \to (X_t^{(k)})_{(k)}$ for all (k).*

As we have noticed above, the equations for $X^{(k)}$ are of the form

$$dX_t^{(k)} = -\partial DV_\beta(X_t^\beta)\sharp X_t^{(k)}dt + P_{(k)}(X_t^{(l)}, \sum l_i \le \sum k_i - 1)dt$$

with some polynomial functions $P_{(k)}$. Therefore, if we denote $X_t^{(k),Z}$ the solution of this equation starting from $X_0^{(k)} = Z^{(k)}$, we get from the convexity of V and by induction over $\sum k_i$ that

$$\sup_{\sum k_i \le K} \|X_t^{(k),Z} - X_t^{(k),0}\|_\infty \le C(K)e^{-t}$$

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with some finite constant $C(K)$. Hence, we can start from $Z^{(k)} = X_s^{(k)}$ to see that since $(X_t^{(k),Z})_{k \in \mathbb{N}^n}$ has the same law as $(X_{t+s}^{(k),0})_{k \in \mathbb{N}^n}$, the law of $(X_t^{(k),0}, \sum k_i \leq K)$ converges. $\square$

We now relate the previous result with the absence of phase transition for the generating function of colored planar maps. In [20], the following strong version of Schwinger-Dyson equation was considered; it requires that for all polynomials $P \in \mathbb{C}\langle X_1, \ldots, X_m \rangle$,

$$\tau(D_i V P) = \tau \otimes \tau(\partial_i P), \ 1 \leq i \leq m. \tag{24}$$

It was shown that if $V(X_1, \ldots, X_m) = W_{(q_i, \beta_i)_{1 \leq i \leq n}} = \frac{1}{2} \sum_{i=1}^m X_i^2 + \sum_{i=m+1}^n \beta_i q_i$, there exists a unique solution $\mathcal{M} \in \mathbb{C}\langle X_1, \ldots, X_m \rangle'$ under the condition that

$$|\tau(X_{i_1} \cdots X_{i_k})| \leq R^k$$

for all k and some finite R, provided the β_i's are small enough. We denote $\tau_{W_{(q_i, \beta_i)_{1 \leq i \leq n}}}$ this solution. Note that this solution was not a priori the law of non-commuting variables, except in the case where V is self-adjoint, but just an element of $\mathbb{C}\langle X_1, \ldots, X_m \rangle'$. In particular, E. Maurel Segala and one of the authors always restricted to polynomials in the letters $(X_1, \ldots, X_m)$ and did not consider their adjoints.

Moreover, for all monomial $P \in \mathbb{C}\langle X_1, \ldots, X_m \rangle$,

$$\tau_{W_{(q_i, \beta_i)_{1 \leq i \leq n}}}(P) = \mathcal{M}_{(q_i, \beta_i)_{1 \leq i \leq n}}(P) = \sum_{k_1, \ldots, k_n \in \mathbb{N}^n} \prod_{1 \leq i \leq n} \frac{(-\beta_i)^{k_i}}{k_i!} M_0((q_i, k_i), 1 \leq i \leq n, (P, 1))$$

with $M_0((q_i, k_i), (P, 1))$ the number of planar maps with k_i stars of type q_i for $1 \leq i \leq n$ and one star of type P (we refer the reader to [20], section 2, for a complete description of the numbers $M_0((q_i, k_i), 1 \leq i \leq n, (P, 1))$).

We now claim

Theorem 3.2. *(a) The generating function*

$$(\beta_i)_{1 \leq i \leq n} \in \mathbb{C}^n \rightarrow \mathcal{M}_{(q_i, \beta_i)_{1 \leq i \leq n}}(P) := \sum_{k_1, \ldots, k_n} \prod_{1 \leq i \leq n} \frac{(-\beta_i)^{k_i}}{k_i!} M((q_i, k_i), 1 \leq i \leq n, (P, 1)),$$

which is an absolutely convergent series for $\sum_{i=1}^n |\beta_i|$ small enough, extends analytically in the interior of the domain where $W_{(q_i, \beta_i)_{1 \leq i \leq n}}(X_1, \ldots, X_m) = \frac{1}{2} \sum_{i=1}^m X_i^2 + \sum_{i=1}^n \beta_i q_i$ is (c, M)-convex for some $c > 0$ and $M \geq M_0(c, \|DV(0).DV(0)\|_\infty)$ the constant of Theorem 2.2. The extension of $\mathcal{M}_{(q_i, \beta_i)_{1 \leq i \leq n}}(P)$ is equal to $\tau_{W_{(q_i, \beta_i)_{1 \leq i \leq n}}}$, the restriction to $\mathbb{C}\langle X_1, \ldots, X_m \rangle$ of the invariant measure of Theorem 2.2.
(b) Assume that V is (c, M)-convex with $M \geq M_0$ of Theorem 2.2. The invariant distribution μ_V of Theorem 2.2 not only satisfies (11) but its strong version in the sense that $\tau_V = \mu_V|_{\mathbb{C}\langle X_1, \ldots, X_m \rangle}$ is such that

$$\tau_V(D_i V P) = \tau_V \otimes \tau_V(\partial_i P), \ 1 \leq i \leq m \tag{25}$$

for all polynomials $P \in \mathbb{C}\langle X_1, \ldots, X_m \rangle$.

Hence, the first point of the above theorem shows that the breaking of analyticity (or phase transition) of the map enumeration can not take place when $W_{(q_i, \beta_i)_{1 \leq i \leq n}}$ is (c, M)-convex.

Proof. By [20], if we consider the case where $W_{(q_i,\beta_i)_{1\le i\le n}}$ is self-adjoint, we know that $\mathcal{M}_{(q_i,\beta_i)_{1\le i\le n}}$ is the law of self-adjoint operators which are uniformly bounded by R ($R = R(\beta)$ going to 2 as β goes to zero) when the β's are small enough. As a consequence, $\mathcal{M}_{(q_i,\beta_i)_{1\le i\le n}}$ must coincide with $\tau_{1,(\beta_i)_{1\le i\le n}} = \mu_{1,(\beta_i)_{1\le i\le n}}|_{\mathbb{C}\langle X_1,\dots,X_m\rangle}$ where we put $\mu_{1,(\beta_i)_{1\le i\le n}} := \mu_{W_{(q_i,\beta_i)_{1\le i\le n}}}$ since $\mathcal{M}_{(q_i,\beta_i)_{1\le i\le n}}$ satisfies (11) with potential $W_{(q_i,\beta_i)_{1\le i\le n}}$ (by Theorem 2.2). In particular, $\tau_{1,(\beta_i)_{1\le i\le n}}$ must satisfy (25).

We now show that we can remove the assumption that $W_{(q_i,\beta_i)_{1\le i\le n}}$ is self-adjoint. We denote by $*$ the involution $(z X_{i_1}\cdots X_{i_k})^* = \bar{z} X_{i_k}\cdots X_{i_1}$ so that $W_{(q_i,\beta_i)_{1\le i\le n}}$ is self adjoint iff $W_{(q_i,\beta_i)_{1\le i\le n}} = W^*_{(q_i,\beta_i)_{1\le i\le n}}$. We can always write $W_{(q_i,\beta_i)_{1\le i\le n}}$ in the form $W_{(q_i,\beta_i)_{1\le i\le n},(q_i^*,\beta_i')_{1\le i\le n}} = \frac{1}{2}\sum X_i^2 + \frac{1}{2}\sum \beta_i q_i + \frac{1}{2}\sum \beta_i' q_i^*$. We denote by $\mu_{1,\beta,\beta'}$ the invariant measure of Theorem 2.2 corresponding to such a potential. The situation $V = V^*$ corresponds to $\beta_i' = \bar{\beta}_i$. In that case we have shown that

$$\tau_{1,(\beta_i)_{1\le i\le n},(\beta_i')_{1\le i\le n}}(D_i V P) - \tau_{1,(\beta_i)_{1\le i\le n},(\beta_i')_{1\le i\le n}} \otimes \tau_{1,(\beta_i)_{1\le i\le n},(\beta_i')_{1\le i\le n}}(\partial_i P) = 0. \qquad (26)$$

But recall that if an analytic function of two variables x, y is null on $\Lambda = \{x = \bar{y}, |x| \le \epsilon\}$ for some $\epsilon > 0$, then this function must vanish on its full domain of analyticity since Λ is totally real. Hence, inside the domain of analyticity of $\beta, \beta' \to \tau_{1,(\beta_i)_{1\le i\le n},(\beta_i')_{1\le i\le n}}$ (i.e the domain where $(\beta_i)_{1\le i\le n}, (\beta_i')_{1\le i\le n} \to \tau_{1,(\beta_i)_{1\le i\le n},(\beta_i')_{1\le i\le n}}(P)$ is analytic for all $P \in \mathbb{C}\langle X_1,\dots,X_m\rangle$), (26) is always true. We can now remove the artificial parameters $(\beta_i')_{1\le i\le n}$ to claim that $\tau_{1,(\beta_i)_{1\le i\le n}}$ always satisfy (25) on its domain of analyticity. We finally can remove the condition that $\beta_i = 1$ for $1 \le i \le m$ as follows. We take V (c, M)-convex with $M \ge M_0$. Note first that if we consider $V_\alpha = \frac{\alpha}{2}\sum_{i=1}^m X_i^2 + V$ for some $\alpha > 0$, then the result still holds since by uniqueness of the solution to Schwinger-Dyson equation, we have the scaling property

$$\mu_{V_\alpha} = d_\alpha \sharp \mu_{\frac{1}{2}\sum X_i^2 + d_\alpha \sharp V}$$

with d_α the dilatation $d_\alpha \sharp \mu(P) = \mu(P(\frac{X_1}{\sqrt{\alpha}},\cdots,\frac{X_m}{\sqrt{\alpha}}))$. As $d_\alpha \sharp V$ is always $(0, M)$ convex, $\frac{1}{2}\sum X_i^2 + d_\alpha \sharp V$ satisfies the above hypotheses and so $\tau_{\frac{\alpha}{2}\sum_{i=1}^m X_i^2 + \sum_{i=1}^m \beta_i q_i}$ always satisfies (25). Finally, we can let α going to zero since V_α is (c, M)-convex for all $\alpha \ge 0$ and so $\alpha \to \mu_{V_\alpha}(P)$ is analytic and thus continuous when α goes to zero.

As a conclusion, we have seen that τ_{V_β} satisfies (26) on the domain of analyticity which contain by Lemma 3.1 all the sets $T(c, M)$, $c > 0, M > M_0(c)$.

On the other hand, we also have that $\tau_{1,\beta}$ must agree with $\mathcal{M}_\beta$, the generating function of maps, for all β small enough since $\tau_{1,\beta}$ satisfies (25) and is the law of bounded operators (and it was proved that there is at most one such solution, the generating function of maps, in [20]). We hence conclude that $\beta \to \mathcal{M}_\beta(P)$ extends analytically to the domain of analyticity of $\tau_{1,\beta}$, which contains all β such that $(1, \beta)$ belongs to $T(c, M)$ for $c > 0$ and $M \ge M_0(c)$.

$\square$

4 Connectivity of the support and properties of associated C^* and von Neumann algebras

Throughout the rest of the paper, we shall assume that V is a (c, M) convex potential with $M > M_0$ (so that the hypothesis of Theorem 2.2 holds).

In this section, we denote by μ the unique stationary law for the free stochastic differential equation (8) with drift DV, satisfying (11), where S_q is a free Brownian motion. Lastly, Z will denote some fixed m-tuple of operators, free from $S_q : q \geq 0$, having law μ, and satisfying $\|Z\|_\infty < b$.

The main results of this section concern properties of the C^*-algebra generated by the m-tuple Z with the prescibed law μ_V. We show that this C^*-algebra is exact [28], projectionless and that the associated von Neumann algebra has the Haagerup approximation property [16] and admits and embedding into the ultrapower of the hyperfinite II_1 factor. These properties are shared by (and in fact, in most cases, derived from those of) the C^*-algebra generated by m semicircular systems.

One of the most interesting open problems in operator algebras is a question due to Connes of whether any tracial state has finite approximation in the sense that there exists a norm-bounded sequence of $N \times N$ matrices $(A_1^N, \ldots, A_m^N)$ such that for all *-polynomial function P,

$$\lim_{N \to \infty} \frac{1}{N} \text{Tr}(P(A_1^N, \ldots, A_m^N)) = \tau(P).$$

When $m = 1$, this question is settled by Birkhoff's theorem, but the question is still open when $m \geq 2$. We prove that the laws μ_V have finite-dimensional approximations.

4.1 Approximation of Z by elements from $C^*(S_q : q \geq 0)$.

We first show that, with respect to operator norm, $C^*(S_q : q \geq 0)$ ε-contains a variable with law μ.

Corollary 4.1. *Within the hypothesis and notations of Theorem 2.2, for any $\varepsilon > 0$, there exists a Brownian motion $S_s : s \geq 0$ free from Z, and elements $X' \in C^*(S_s : s \geq 0)$, $X \in C^*(Z, S_s : s \geq 0)$ so that X has the given stationary law μ and $\|X - X'\|_\infty \leq \varepsilon$.*

Proof. Let X_t and X_t^Z be two solutions to (7) with initial data $X_0 = 0$, $X_0^Z = Z$. By Theorem 2.2, $X_t \in C^*(S_s : s \leq t)$ approximates in operator norm the stationary process X_t^Z with marginal distribution μ. Since by Lemma 2.1, $X_t^Z \in C^*(Z, S_q : q \geq 0)$, we may take $X' = X_t$, $X = X_t^Z$ for large enough t. $\qquad\square$

Theorem 4.2. *Let Z be any m-tuple of b-bounded variables with the unique law μ satisfying (11). (a) The algebra $C^*(Z)$ has no non-trivial projections. (b) The spectrum of any non-commutative *-polynomial P in the m-tuple Z is connected (in the case that $P(Z)$ is normal, this means that the support of its spectral measure is connected). (c) If P is any polynomial in Z whose value is self-adjoint, then the probability measure given by the law of $P(Z)$ has connected support.*

Proof. The C^*-algebra $\mathcal{A} = C^*(S_s, s \geq 0)$ can be identified with the C^*-algebra generated by semicircular operators $s(f) : f \in L^2(\mathbb{R}; \mathbb{R})$ where $f \mapsto s(f)$ denotes the free Gaussian functor [4]. It is well known that this C^*-algebra is isomorphic to the infinite reduced free product

$$\mathcal{A} \cong (C[-1, 1], \mu) * \cdots * (C[-1, 1], \mu)$$

where μ denotes the semicircular measure. The algebra $(C[-1, 1], \mu)$ can be unitally embedded in a trace-preserving way into the group C^*-algebra $C^*(\mathbb{Z}) \cong C(\mathbb{T})$ taken with its canonical group trace τ. Indeed, if $u \in C^*(\mathbb{Z})$ denotes the group generator, i.e. $u = \exp(2\pi i\theta) \in C(\mathbb{T})$, then $u + u^*$ generates a copy of $C[-1, 1]$, and the restriction of τ to $C^*(u + u^*)$ is the arcsine law. Hence for

a suitable continuous function f, $f(u + u^*)$ has as its distribution the semicircle law, and we can embed $(C[-1, 1], \mu)$ by sending its generator, multiplication by x, to $f(u + u^*)$. It follows that

$$\mathcal{A} \subset C^*(\mathbb{Z}) *_{\text{red}} \cdots *_{\text{red}} C^*(\mathbb{Z}) \cong C^*_{\text{red}}(\mathbb{F}_\infty).$$

By the results of [24] (see also [17] for a random matrix proof), $C^*(\mathbb{F}_\infty)$ has no non-trivial projections. Thus $\mathcal{A}$ has no non-trivial projections.

Suppose now that $Y \in \mathcal{A}$. Then the spectrum $\sigma(Y)$ must be connected. We sketch the argument, which can be found in standard C^*-algebra literature (see e.g. [6, Proposition 4.6.2 on p. 28]). If $\sigma(Y) = K_1 \cup K_2$ with $K_1 \cap K_2 = \emptyset$ and both K_1, K_2 non-empty, for any contour $\gamma \subset \mathbb{C}$ that contains K_1 but not K_2 and does not intersect $\sigma(Y)$, the integral

$$E = \frac{1}{2\pi i} \int_\gamma (z - Y)^{-1} dz$$

belongs to $\mathcal{A}$, being a norm limit of Riemann sums. Moreover, $E^2 = E$, $E(1 - E) = 0$ but $E \neq 0$, $E \neq 1$. But $E + E^* - 1$ is invertible since $(E + E^* - 1)^2 = 1 + (E - E^*)(E - E^*)^* \geq 1$, and thus

$$P = E(E + E^* - 1)^{-1}$$

is a nonzero self-adjoint projection (since $(1 - E)E = E^*(1 - E^*) = 0$). Clearly, $P \neq 0$ and $(1 - E)P = 0$ so $P \neq 1$. Hence $\mathcal{A}$ would have a non-trivial projection, a contradiction.

Since $X_t^0 \in \mathcal{A}$, it follows that the spectrum of any $*$-polynomial Y in this m-tuple is connected. Since Z and X_t^Z have the same laws at all t, $C^*(Z)$ and $C^*(X_t^Z)$ are isomorphic. So if the support of the spectrum of $P(Z)$ is disconnected, then the support of the spectrum of $P(X_t^Z)$ is disconnected, for all t. Because X_t^0 converges to X_t^Z in operator norm as $t \to \infty$, we find that also spectrum of $P(X_t^0)$ must be disconnected for large enough t. Since the trace-state on $C^*(S_s : s \geq 0)$ is faithful, it follows that for some t large enough, $C^*(S_s : s \geq 0)$ contains an element with disconnected spectrum. But as we saw before, this is impossible. Thus we have proved (b).

Because any non-trivial projection has disconnected spectrum, (a) follows.

Lastly, as the GNS construction for the trace-state on $C^*(Z)$ is faithful (by construction of $C^*(Z)$) and the state is tracial, it follows that the trace vector in the representation is cyclic for both $C^*(Z)$ and its commutant. But this implies that the vector is also separating, so that the trace-state on $C^*(Z)$ is faithful. Thus the support of the law of any self-adjoint operator $X \in C^*(Z)$ is exactly its spectrum. This implies (c). $\square$

4.2 Exactness and the Haagerup property.

We recall that a C^*-algebra A is called exact (cf. [28] and references therein) if there exists a faithful $*$-representation $\pi : A \to B(H)$ with the following property. For any finite subset $F \subset A$ and any $\varepsilon > 0$ there exists a finite-dimensional matrix algebra D and unital completely positive maps $\theta : A \to D$, $\eta : D \to B(H)$ so that

$$\|\eta(\theta(x)) - \pi(x)\|_\infty < \varepsilon, \qquad \forall x \in F.$$

It turns out that this property is equivalent to the statement that the functor $\otimes_{\text{min}} A$ of taking the minimal tensor product with A is exact (i.e., takes exact sequences to exact sequences). Exactness is an important approximation property for a C^*-algebra.

Another important approximation property, this time for a von Neumann algebra, is the Haagerup property. A von Neumann algebra M with a trace τ is said to have the Haagerup property [16] if there exists a sequence of completely-positive maps $\Phi_n : M \to M$, which are unital and trace-preserving, so that the associated maps $\Phi_n : L^2(M, \tau) \to L^2(M, \tau)$ are compact and converge to 1 strongly:

$$\|\Phi_n(x) - x\|_2 \to 0, \qquad n \to \infty$$

for all $x \in M$. For a discrete group von Neumann algebra $L(\Gamma)$, the Haagerup property is equivalent to Gromov's a-T-menability of the group Γ (i.e., to the existence of a cocycle $c : \Gamma \to H$ with values in some unitary representation H of Γ, so that the map $\gamma \mapsto \|c(\gamma)\|_H$ is proper). As was shown by Haagerup, free groups have the Haagerup property.

Theorem 4.3. *Let Z be any b-bounded m-tuple with the unique law μ satisfying* (11). *Then $C^*(Z)$ is exact and $W^*(Z)$ has the Haagerup approximation property.*

Proof. As in Corollary 4.1, let X_t^Z be the solution to the free SDE (8) starting with Z and X_t^0 be the solution starting with zero. Thus $X_t^Z \in C^*(Z, S_q : q \in [0, +\infty))$ and $X_t \in C^*(S_q : q \in [0, +\infty))$, where S_q is a free Brownian motion, free from $C^*(Z)$. Moreover, we have by Theorem 2.2 that $\|X_t^Z - X_t^0\|_\infty \to 0$ as $t \to \infty$.

Let $E : C^*(Z, S_q : q \in [0, +\infty)) \to C^*(S_q : q \in [0, +\infty))$ be the conditional expectation coming from the fact that Z and $S_q, q \in [0, +\infty)$ are freely independent. Then

$$\|E(Z_t - X_t)\|_\infty \leq \|Z_t - X_t\|_\infty \to 0, \tag{27}$$

so that, since $E(X_t) = X_t$,

$$\|E(Z_t) - X_t\|_\infty \to 0.$$

Since μ is stationary, $C^*(X_t^Z) \cong C^*(Z)$; let $\pi_t : C^*(Z) \to C^*(X_t^Z)$ be this isomorphism. Let $H = L^2(W^*(Z, S_q : q \in [0, +\infty)))$. Then as a module over $W^*(Z)$, H is infinite-dimensional, and as a module over $W^*(X_t^Z)$, it is at most infinite-dimensional. Hence there exists an injective unital $*$-homomorphism $\Theta_t : B(H) \to B(H)$ with the property that $\Theta(\pi_t(x)) = x$ for all $t \geq 0$ and $x \in W^*(Z)$ (and thus for $x \in C^*(Z)$).

Let now $F \subset C^*(Z)$ be a finite subset and $\varepsilon > 0$. Then one can find $*$-polynomials $P_x : x \in F$ so that

$$\|P_x(Z) - x\|_\infty < \varepsilon/9.$$

Hence for t sufficiently large, we may assume that

$$\|E(P_x(X_t^Z)) - P_x(X_t^Z)\|_\infty < \varepsilon/9.$$

Since $\pi_t(P_x(Z)) = P_x(X_t^Z)$, we conclude that

$$\|E(\pi_t(x)) - \pi_t(x)\|_\infty < \varepsilon/3, \qquad \forall x \in F.$$

Now for all $x \in F$, $E(\pi_t(x)) \in C^*(S_q : q \in [0, +\infty))$, and hence we can find a finite-dimensional C^*-algebra A and unital completely positive maps $\eta : C^*(S_q : q \in [0, +\infty)) \to A$, $\psi : A \to B(H)$, so that

$$\|\psi \circ \eta(y) - y\|_\infty < \varepsilon/3$$

for all $y \in \pi_t(F)$. Consider now the unital completely positive maps

$$\alpha = \eta \circ E \circ \pi_t : C^*(Z) \to A$$

and

$$\beta = \Theta \circ \psi : A \to B(H).$$

Then

$$\|\beta \circ \alpha(x) - x\|_\infty = \|\eta(\pi_t(x)) - \pi_t(x)\|_\infty < \varepsilon$$

for all $x \in F$. Thus $C^*(Z)$ is exact.

We now turn to the Haagerup property, where we adapt a proof from [7]. Consider the map $\Phi_t : W^*(X_t^Z) \to W^*(X_t^Z)$ which is obtained as the composition

$$\Phi_t = E_{W^*(X_t^Z)} \circ \Psi_t \circ E_{W^*(S_q : q \in [0, +\infty))},$$

where Ψ_t are unital trace-preserving completely positive maps on $W^*(S_q : q \in [0, +\infty))$ so that Ψ_t are compact on L^2 and $\|\Psi_t(x) - x\|_2 \to 0$ for all $x \in W^*(S_q : q \in [0, +\infty))$. Then Φ_t are unital trace-preserving completely-positive maps on $W^*(X_t^Z) \cong W^*(Z)$, and because of (27), one has that $\|\Phi_t(x) - x\|_2 \to 0$ for all $x \in W^*(Z)$. On the other hand, each Φ_t, viewed as a map on $L^2(W^*(Z, S_q : q \in [0, +\infty)))$, is compact (since Ψ_t is compact), and therefore the restriction of Φ_t to $L^2(W^*(X_t^Z))$ is also compact. Thus $W^*(Z)$ has the Haagerup property. $\qquad\square$

4.3 Finite dimensional approximation.

4.3.1 R^ω embeddability for self-adjoint potentials.

In this section, we improve on the results of [20] by showing that if we set

$$\bar{\mu}_V^N(P) = \int \hat{\mu}_N(P) d\mu_V^N(A_1, \ldots, A_m))$$

with

$$\hat{\mu}_N(P) = \frac{1}{N}\mathrm{Tr}(P(A_1, \ldots, A_m))$$

and

$$d\mu_V^N(A_1, \ldots, A_m)) = \frac{1}{Z_N^V} 1_{\|A_i\|_\infty \le M} e^{-N\mathrm{Tr}(V(A_1, \ldots, A_m))} dA_1 \cdots dA_m,$$

then $\bar{\mu}_V^N$ converges towards τ_V for any self-adjoint locally strictly convex potential V. In [20], a similar result was proved when $V = cX.X + W$ with W a 'small' enough polynomial.

Theorem 4.4. *For all $c > 0$, there exists $B_0 < \infty$ and $M_0 < \infty$ so that for any self-adjoint polynomial V which is (c, M)-convex with $M > M_0$ there exists a unique law τ of m self-adjoint variables such that for all $i \in \{1, \ldots, m\}$, all polynomial P,*

$$\tau \otimes \tau(\partial_i P) = \tau(D_i V P) \tag{28}$$

and such that $\tau(X_i^{2k}) \le B_0^{2k}$. Moreover, $\bar{\mu}_V^N$ converges towards τ and therefore τ has finite approximation. In particular, if Z has law τ, then $W^*(Z)$ can be embedded into the ultrapower of the hyperfinite II_1 factor.

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Proof. We can follow the lines of [20] Theorem 3.5 to see that $d\mu_V^N(A_1, \ldots, A_m))$ has a density with respect to the Gaussian law $\prod e^{-\frac{Nc}{2}X_i^2} dX_i / Z_N^c$ which is log-concave. This insures that we can use the Brascamp Lieb inequality which in turn allows us to show that the random matrices under the above Gibbs measures stay bounded in norm by some $B_0 \ll M$ with overwhelming probability. As a consequence, we can perform an infinitesimal change of variables $X_i \to X_i + N^{-1}P_i$ with P_i a self-adjoint polynomial in $\|A\|_\infty \le B_0$ and the null function outside a ball of radius strictly smaller than M. This shows that any almost sure limit points of $\hat{\mu}_N$ under μ_V^N are laws of variables bounded by B_0 and satisfy (28) with the potential V (see [20] for details). Now, if they satisfy (28) they also satisfy (11) (take $P = D_i P$ and sum the equalities) and so by Theorem 2.2, there exists at most one such solution $\mu_V = \tau_V$. Thus, $\hat{\mu}_N$ converges almost surely and therefore in expectation towards τ_V. Hence τ_V has finite approximation. $\qquad\square$

4.3.2 R^ω embeddability for non-self adjoint potentials.

One can give a proof of embeddability of $W^*(Z)$ into the ultrapower of the hyperfinite II_1 factor, based directly on Corollary 4.1. This proof works for arbitrary (c, M)-convex polynomials (without the self-adjoint assumption). Indeed, because of this corollary and with its notations, if $\varepsilon > 0$ is given, and F is a finite collection of $*$-polynomials, then there exists an $X' \in C^*(S_q : q \ge 0)$ with the property that

$$|\tau(P(Z)) - \tau(P(X'))| < \varepsilon, \qquad \forall P \in F$$

(here τ denotes the free product trace-state on $C^*(Z, S_q : q \ge 0)$). Now, $C^*(S_q : q \ge 0)$ is generated by an infinite free semicircular family $\hat{S}_j : j = 1, 2, \ldots$. One can clearly assume that in fact $X' \in C^*(\hat{S}_1, \ldots, \hat{S}_K)$ for some large enough K (since one can replace X' with $E_K(X')$ for K large enough, where $E_K : C^*(\hat{S}_i : i = 1, 2 \ldots) \to C^*(\hat{S}_1, \ldots, \hat{S}_K)$ is the canonical conditional expectation). Thus, by approximating X' with a polynomial in $\hat{S}_1, \ldots, \hat{S}_K$, we may assume that X_k' is a polynomial Q_k in $\hat{S}_1, \ldots, \hat{S}_K$. Since $\hat{S}_1, \ldots, \hat{S}_K$ are free semicircular variables, their law has finite approximations. Thus for N sufficiently large, one can find a K-tuple of $N \times N$ self-adjoint matrices $A_1, \ldots, A_K$ whose law approximates that of $\hat{S}_1, \ldots, \hat{S}_K$ so well that the m-tuple of matrices $B = (B_1, \ldots, B_m) = (Q_1(A_1, \ldots, A_K), \ldots, Q_m(A_1, \ldots, A_K))$ would have the property that

$$|\tau(P(Z)) - \frac{1}{N}\mathrm{Tr}(P(B))| < \varepsilon, \qquad \forall P \in F.$$

5 Free entropy

For the remainder of the paper we shall assume that V is (c, M)-convex and self-adjoint.

In this section we show that for tracial state with conjugate variables given as the cyclic gradient of a self-adjoint (c, M)-convex potential the microstate entropy is the same whether it is defined by a limsup or a liminf.

Theorem 5.1. *Let $c > 0$ and V be a self-adjoint (c, M)-convex potential with $M > M_0$. Let $\tau = \tau_V$ be as in Theorem 4.4. Let $\Gamma(\tau, \epsilon, R, k)$ be the microstates*

$$\Gamma(\tau, \epsilon, R, k) = \{X_1, \cdots, X_m : \|X_i\|_\infty \le R, |\frac{1}{N}Tr(P(X_1, \cdots, X_m)) - \tau(P)| \le \epsilon,$$

$$\textit{for all monomial of degree less than } k\}$$

and let vol denote the volume on the space of m $N \times N$ Hermitian matrices.

Then

$$\chi(\tau) = \limsup_{\epsilon \downarrow 0, k, R \to \infty} \limsup_{N \to \infty} \frac{1}{N^2} \log vol(\Gamma(\tau, \epsilon, R, k)) + \frac{m}{2} \log N$$

$$= \liminf_{\epsilon \downarrow 0, k, R \to \infty} \liminf_{N \to \infty} \frac{1}{N^2} \log vol(\Gamma(\tau, \epsilon, R, k)) + \frac{m}{2} \log N$$

and $\chi(\tau) > -\infty$.

Proof. Note that

$$vol(\Gamma(\tau, \epsilon, R, k)) = \int 1_{A_1, \cdots, A_m \in \Gamma(\tau, \epsilon, R, k)} dA_1 \cdots dA_m \qquad (29)$$

$$\approx e^{N^2 \tau(V)} \int_{\Gamma(\tau, \epsilon, R, k)} e^{-N^2 \hat{\mu}^N(V)} dA_1 \cdots dA_m$$

$$\approx e^{N^2 \tau(V)} \mu_V^N(\Gamma(\tau, \epsilon, R, k)) \int e^{-N^2 \hat{\mu}^N(V)} dA_1 \cdots dA_m \qquad (30)$$

with μ_N^V the Gibbs measure considered in the proof of Lemma 4.4. Here, we used the notation $A(N, \epsilon) \approx B(N, \epsilon)$ when

$$\lim_{\epsilon \downarrow 0} \lim_{N \uparrow 0} \frac{1}{N^2} \log \frac{A(N, \epsilon)}{B(N, \epsilon)} = 1.$$

Since $\mu_V^N(\Gamma(\tau, \epsilon, R, k)) \to 1$ by the proof of Theorem 2.2, we only need to estimate the quantity $\int e^{-N^2 \hat{\mu}^N(V)} dA_1 \cdots dA_m$. To do that we write $V = W + \frac{c}{2} X.X$ with W a $(0, M)$-convex potential so that

$$\partial_t \frac{1}{N^2} \log \int e^{-N^2 \hat{\mu}^N(tW + \frac{c}{2} X.X)} dA_1 \cdots dA_m = -\bar{\mu}_{tW + \frac{c}{2} X.X}^N(W)$$

By the proof of Theorem 4.4, we also see that for all $t \in [0, 1]$

$$\lim_{N \to \infty} \bar{\mu}_{tW + \frac{c}{2} X.X}^N(W) = \tau_{tW + \frac{c}{2} X.X}(W)$$

(note that $tW + \frac{c}{2} X.X$ stays self-adjoint (c, M)-convex). Since everything stays bounded, and since the limit

$$\lim_{N \to \infty} \frac{1}{N^2} \log \int e^{-N^2 \hat{\mu}^N(\frac{c}{2} X.X)} dA_1 \cdots dA_m + \frac{m}{2} \log N$$

is a finite constant $F(c)$, we conclude by bounded convergence theorem that

$$\lim_{N \to \infty} \frac{1}{N^2} \log \int e^{-N^2 \hat{\mu}^N(W + \frac{c}{2} X.X)} dA_1 \cdots dA_m + \frac{m}{2} \log N = F(c) - \int_0^1 \tau_{tW + \frac{c}{2} X.X}(W) dt$$

which shows by (30) that χ can be defined either by limsup or liminf and both are equal to

$$\chi(\tau) = F(c) - \int_0^1 \tau_{tW + \frac{c}{2} X.X}(W) dt + \tau(V)$$

Corollary 5.2. *Let τ be as in Theorem 4.4. Then the von Neumann algebra generated by an m-tuple Z with law τ is a factor, does not have property Γ, is prime and has no Cartan subalgerbas.*

Here we of course use the fact that $\chi(\tau) > -\infty$ and that for an m-tuple Z, $\chi(Z) > -\infty$ implies that $W^*(Z)$ is a factor, non-Γ, has no Cartan subalgebras [26] and is prime [15].

6 Norm convergence

In section 4.3 it was shown that solutions to the Schwinger-Dyson equations (28) are weak limits of finite dimensional approximations. Namely if we let

$$\bar{\mu}_V^N(P) = \int \hat{\mu}_N(P) d\mu_V^N(A_1, \ldots, A_m)$$

with

$$\hat{\mu}_N(P) = \frac{1}{N} \text{Tr}(P(A_1, \ldots, A_m))$$

and

$$d\mu_V^N(A_1, \ldots, A_m)) = \frac{1}{Z_N^V} 1_{\|A_i\| \leq M} e^{-N \text{Tr}(V(A_1, \ldots, A_m))} dA_1 \cdots dA_m,$$

we saw that $\bar{\mu}^N(P)$ converges to the unique solution τ of

$$\tau_V(D_i V P) = \tau_V \otimes \tau_V (\partial_i P)$$

for all $i \in \{1, \ldots, m\}$ and $P \in \mathbb{C}\langle X_1, \ldots, X_m \rangle$. We here show that this convergence holds in norm but for simplicity restrict ourselves to potentials which are uniformly convex (and not only locally convex).

Lemma 6.1. *Let $(X_1, \ldots, X_m)$ be a m-tuple of non-commutative variables with law τ_V and let $(X_1^N, \ldots, X_m^N)$ be random matrices with law μ_V^N. Assume that V is self-adjoint (c, ∞)-convex for some $c > 0$. Then, for any polynomial $P \in \mathbb{C}\langle X_1, \ldots, X_m \rangle$,*

$$\lim_{N \to \infty} \|P(X_1^N, \ldots, X_m^N)\|_\infty = \|P(X_1, \ldots, X_m)\|_\infty \ a.s.$$

This result generalizes the work of Haagerup and S. Thorjornsen [14] where it was proved for $V(X_1, \ldots, X_m) = \frac{1}{2} \sum_{i=1}^m X_i^2$. Our result actually relies on theirs.

Proof. The idea is to use the approximation by processes, and hence the fact that processes are well approximated by polynomials of independent Wigner matrices and then use [14] to conclude that the norm of the latter converge to the limit.

Step 1: Matrix valued diffusions and convergence to the stationary process.

Consider the diffusion with values in the set of Hermitian matrices

$$dX_t^{N,Z} = dH_t^N - \frac{1}{2} DV(X_t^{N,Z}) dt \tag{31}$$

with $X_0^{N,Z} = Z$. Here $(H_t^N, t \geq 0)$ is a m-dimensional Hermitian Brownian motion. In other words, H_t^N are a set of m independent matrix valued process whose matrix entries are given by

$$H_t^N(k, l) = \frac{B_{kl}(t) + i\tilde{B}_{kl}(t)}{\sqrt{2N}} \ k < l, \ H_t^N(l, k) = \bar{H}_t^N(k, l), \ H_t^N(k, k) = \frac{B_{kk}(t)}{\sqrt{N}}$$

where the $B, \tilde{B}$ are independent standard Brownian motions.

A strong solution to (31) exists up to a possible time of explosion (when DV would stop being Lipschitz eventually) since this equation can be seen as a system of classical stochastic differential equation with $N(N+1)/2$ equations driven by independent Brownian motions with a polynomial

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drift. Since this drift derives from a strictly convex potential, it is well known that the time of explosion is almost surely infinite (which can also be show by the arguments of the proof of Lemma 2.1)

Now, let us consider two solutions $X^{N,Z}$ and $X^{N,Z'}$ starting from Z and Z' respectively. Then, we have

$$d(X_t^{N,Z} - X_t^{N,Z'}) = -\frac{1}{2}(DV(X_t^{N,Z}) - DV(X_t^{N,Z'}))dt.$$

Hence, we can apply exactly the same arguments than in the proof of Theorem 2.2 to conclude that

$$\|X_t^{N,Z} - X_t^{N,Z'}\|_\infty \le e^{-ct}\|Z - Z'\|_\infty.$$

If we take Z_N to be random with law μ_V^N we get a stationary process so that

$$\|X_t^{N,Z} - X_t^{N,0}\|_\infty \le e^{-ct}\|Z_N\|_\infty. \tag{32}$$

Now, according to Brascamp-Lieb inequality (see its application on our particular case in [21])

$$\mu_V^N(\|Z_N\|_\infty \ge x) \le e^{-a(c)N(x-x_0(c))}$$

with some $x_0(c) < \infty$ and $a_0(c) > 0$. Therefore, (32) shows that for all $t \ge 0$,

$$\mathbb{P}(\|X_t^{N,0}\|_\infty \ge (e^{-ct} + 1)K) \le 2e^{-a(c)N(K-x_0(c))} \tag{33}$$

since both Z_N and $X_t^{N,Z}$ have law μ_V^N.

Step 2: Uniform bounds on $X^{N,0}$. In this section we want to show that we can also control uniformly the norm of $X_t^N = X_t^{N,0}$ for t in a compact set. To do that let us remind that

$$
\begin{aligned}
dX_t^N.X_t^N &= 2X_t^N.dH_t^N - DV(X_t^N).X_t^N dt + 2dt \\
&\le 2X_t^N.dH_t^N - cX_t^N.X_t^N dt + 2dt - 2DV(0).X_t^N dt \\
&\le 2X_t^N.dH_t^N - \frac{1}{2}cX_t^N.X_t^N dt + Cdt
\end{aligned}
$$

with $C = 2 + \frac{2}{c}DV(0).DV(0)$. Therefore, for any $p \ge 0$, since $X_t^N.X_t^N$ is a non negative matrix for all $s \ge 0$, Itô's calculus yield

$$
\begin{aligned}
\mathrm{Tr}\left((X_s^N.X_s^N)^p\right) &\le 2p\int_0^s \mathrm{Tr}((X_t^N.X_t^N)^{p-1}(X_t^N.dH_t^N - \frac{1}{2}cX_t^N.X_t^N dt + Cdt)) \\
&\quad + 2pN^{-1}\sum_{k=0}^{p-1}\int_0^s \mathrm{Tr}((X_t^N.X_t^N)^k)\mathrm{Tr}((X_t^N.X_t^N)^{p-k-1})dt \\
&\quad + 2pN^{-1}\sum_{k=0}^{p-2}\int_0^s \mathrm{Tr}((X_t^N.X_t^N)^k X_t^N)\mathrm{Tr}((X_t^N.X_t^N)^{p-k-2}X_t^N)dt \tag{34}
\end{aligned}
$$

Now, by Burkhölder-Davis-Gundy inequalities and Chebychev's inequality, there is a finite constant

Λ such that for all $\epsilon > 0$,

$$\mathbb{P}\left(\sup_{s \leq T}\left|\int_0^s \text{Tr}((X_t^N.X_t^N)^{p-1}(X_t^N.dH_t^N)\right| \geq \epsilon N^{\frac{1}{2}}\right) \leq \frac{\Lambda}{\epsilon^4 N^2} E\left[\left(\frac{1}{N}\int_0^T \text{Tr}((X_t^N.X_t^N)^{2p-1}dt)\right)^2\right]$$

$$\leq \frac{\Lambda T}{\epsilon^4 N^2}\int_0^T E\left[\left(\frac{1}{N}\text{Tr}((X_t^N.X_t^N)^{4p-2}dt)\right)\right]dt \leq \frac{\Lambda T^2}{\epsilon^4 N^2}(K^{8p} + 4p\int_K^\infty x^{8p-4}e^{-a(c)N(x-x_0(c))}dx)$$

$$\leq \frac{\Lambda T^2}{\epsilon^4 N^2}(K^{8p} + \frac{(8p-4)!}{(a(c)N)^{8p-3}}) \leq \frac{2\Lambda T^2}{\epsilon^4 N^2}K^{8p}$$

where we used (33) and chose $K = \max\{2x_0(c), 1\}$, $p \leq a(c)NK$. Therefore, if we choose ϵ so that $\epsilon = (K+1)^{2p}T$, we see that if we set

$$A(N,T) = \bigcap_{p \leq a(c)NK}\left\{\sup_{s \leq T}\left|\int_0^s \text{Tr}((X_t^N.X_t^N)^{p-1}(X_t^N.dH_t^N)\right| \leq (K+1)^{2p}TN^{\frac{1}{2}}\right\}$$

then by Borel Cantelli's Lemma,
$$\limsup_{N,T \to \infty}\mathbb{P}(A(N,T)) = 1.$$

Let now restrict ourselves to the set $\cap_{T \geq T_0}\cap_{N \geq N_0} A(N,T)$. We let

$$B(p,N,s) := N^{-1}\text{Tr}\left((X_s^N.X_s^N)^p\right)$$

and observe that these non negative real numbers obey the relation

$$B(p,N,s) \leq B(q,N,s)^{\frac{p}{q}}, \qquad q \geq p.$$

We first control $B(p_0, N, s)$ by (34) which yields for $s \geq 0$,

$$B(p_0,N,s) \leq 2p_0(K+1)^{2p_0}N^{-\frac{1}{2}}s \vee T_0 + \int_0^s (-2cpB(p_0,N,t) + 2p_0(2p_0-1)B(p_0,N,t)^{\frac{p_0-1}{p_0}})dt$$

$$\leq 2p_0(K+1)^{2p_0}N^{-\frac{1}{2}}s \vee T_0 + \int_0^s (-2cpB(p_0,N,t) + 2p_0(2p_0-1)B(p_0,N,t)^{\frac{p_0-1}{p_0}})dt$$

$$\leq 2p_0(K+1)^{2p_0}N^{-\frac{1}{2}}s \vee T_0 + \int_0^s \left\{\left[-2cp_0 + a2p_0(2p_0-1)\right]B(p_0,N,t)\right.$$
$$\left. + \left[\left(\frac{p_0-1}{p_0 a}\right)^{p_0-1} - a\left(\frac{p_0-1}{p_0 a}\right)^{p_0}\right]\right\}dt$$

where we used that $x^{\frac{p_0-1}{p_0}} \leq ax + (\frac{p_0-1}{p_0 a})^{p_0-1} - a(\frac{p_0-1}{p_0 a})^{p_0}$ for all $a > 0$ and $p_0 \geq 1$. Choosing a so that $a(2p_0 + 4p_0(p_0-1)) = cp_0$ we thus have found a finite constant $C(p_0, K)$ such that

$$B(p_0,N,s) \leq C(p_0,K)s \vee T_0 - cp_0\int_0^s B(p_0,N,t)dt$$

which shows that $B(p_0, N, s) \leq \max\{C(p_0,K)/cp_0, C(p_0,K)T_0\}$ is uniformly bounded.

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We now bound $B(p, N, s)$ for $p \geq p_0$ and to this end replace in (34) all $B(q, N, s), q \leq p_0$ by $B(p_0, N, s)^{\frac{p_0}{q}}$. We show by induction over $p \geq p_0$ that $B(p, N, s) \leq C_p C_0^p$ with C_0 a finite constant depending on p_0 and T_0, for all $s \geq 0$ and $p \leq a(c)KN$. Here C_p denotes the Catalan numbers. Indeed, this is satisfied for $p \leq p_0$ and then (34) implies that

$$B(p, N, s) \leq 2p(K+1)^{2p} N^{-\frac{1}{2}} s \vee T_0 + \int_0^s (-2cpB(p, N, t) + 2pB(p, N, t)^{\frac{p-1}{p}}) dt$$

$$+ 2p \sum_{k=0}^{p-1} \int_0^s C_k C_{p-k-1} C_0^{p-1} dt$$

$$+ 2p \sum_{k=0}^{p-2} \int_0^s \left[\left(C_{k+1} C_0^{k+1} \right)^{\frac{k+\frac{1}{2}}{k+1}} \left(C_{p-k-1} C_0^{p-k-1} \right)^{\frac{p-k-\frac{3}{2}}{p-k-1}} \right] dt$$

where we used that $|\frac{1}{N} \mathrm{Tr}((X.X)^k X)| \leq \frac{1}{N} \mathrm{Tr}((X.X)^{k+\frac{1}{2}})$. Because $1 \leq C_k \leq 4C_{k-1}$ and since $\sum_{k=0}^{p-1} C_k C_{p-1-k} = C_p$ for all k and p we conclude that

$$B(p, N, s) \leq 2p(K+1)^{2p} N^{-\frac{1}{2}} s \vee T_0 + \int_0^s (-cpB(p, N, t) + 2pB(p, N, t)^{\frac{p-1}{p}}) dt$$

$$+ 10pC_0^{p-1} C_p s.$$

$$\leq 2p(K+1)^{2p} N^{-\frac{1}{2}} s \vee T_0 + \int_0^s \left(-\frac{1}{2} cpB(p, N, t) + \left(\frac{2}{c} \right)^p \right) dt + 10pC_0^{p-1} C_p s.$$

Thus, if N is large enough so that $N^{-\frac{1}{2}} T_0 \leq 1$, we get that

$$B(p, N, s) \leq \frac{2f(T_0)}{c} \left((K+1)^{2p} N^{-\frac{1}{2}} + \left(\frac{2}{c} \right)^p + 10C_0^{p-1} C_p \right)$$

with $f(T_0) = T_0$ if $s \leq T_0$ and $f(T_0) = \frac{2}{c}$ if $s \geq T_0$. Note here that we used the fact that we have a negative drift growing linearly with p to cancel the multiplication by p. We finally choose C_0 large enough so that

$$\frac{2f(T_0)}{c} \left((K+1)^{2p} N^{-\frac{1}{2}} + \left(\frac{2}{c} \right)^p + 10C_0^{p-1} C_p \right) \leq C_0^p$$

which we can always do.

Hence we have proved that

$$\sup_{t \geq 0} \|X_t^N . X_t^N\|_\infty \leq \min_{p \leq a(c)KN} \sup_{t \geq 0} (NB(p, N, s))^{1/p} \leq 2C'(p_0) < \infty \quad \text{a.s.}$$

In other words, we have proved that $\|X_t^N\|_\infty$ is uniformly bounded almost surely.

Step 3: Convergence of the norm of $P(X_t^{N,0})$ as N goes to infinity. To this end remark that since X_t is always uniformly bounded by M we can always assume V is $\mathcal{C}^\infty$, uniformly bounded and with uniformly Lipschitz cyclic gradient (this amounts to change V outside a place that the diffusion does not see). We let $\tilde{V}$ be equal to V on operators with norm bounded by M and have uniformly Lipschitz cyclic gradient. For instance, we take $\tilde{V}(X_1, \ldots, X_m) = V(f(X_1), \ldots, f(X_m))$

with $f(x) = x$ on $|x| \leq M$, $f(x) = x(1 + (|x| - M)^4)^{-1}$ if $|x| > M$ (since the later is twice continuously differentiable with uniformly bounded derivatives).

Now, by definition if we let

$$\phi_M(X, S)_t = S_t - \frac{1}{2} \int_0^t D\tilde{V}(X_s) ds,$$

then X_t can be expressed as an iterate

$$X_t = \phi_M(X, S)_t = \phi_M(., S)^n(X)_t$$

for all integer numbers n and M greater than the uniform norm on $X_t, t \geq 0$. On the other hand, for two operator valued processes (X, Y)

$$\|\phi_M(X, S)_t - \phi_M(Y, S)_t\|_\infty \leq \frac{1}{2} \|D\tilde{V}\|_{\mathcal{L}} \int_0^t \|X_s - Y_s\|_\infty ds$$

and so we get that

$$
\begin{aligned}
\|X_t - \phi(., S)^n(S)_t\|_\infty &= \|\phi(X, S)_t - \phi(\phi^{n-1}(., S)(S), S)_t\|_\infty \\
&\leq \|D\tilde{V}\|_{\mathcal{L}} \int_0^t \|X_s - \phi^{n-1}(S, .)(S)_s\|_\infty ds \\
&\leq C \frac{\|D\tilde{V}\|_{\mathcal{L}}^n t^{n-1}}{2^n (n-1)!} \sup_{u \leq t} \|X_u - S_u\|_\infty \\
&\leq C \frac{\|D\tilde{V}\|_{\mathcal{L}}^n t^n}{2^n (n-1)!} \|D\tilde{V}\|_\infty
\end{aligned}
$$

for all $n \in \mathbb{N}$.

We next want to show that the norm of $P(X_t^{N,0})$ converges with overwhelming probability for any $P \in \mathbb{C}\langle X_1, \ldots, X_m \rangle$. To do that we approximate $X_t^{N,0}$ by $\phi_M(., H^N)^n(0)_t$ with ϕ as above

We claim that

$$\lim_{N \to \infty} \lim_{M \to \infty} \lim_{n \to \infty} \sup_{t \leq T} \|\phi_M(., H^N)^n(0)_t - X_t^N\|_\infty = 0 \qquad \text{a.s.}$$

Indeed, $\phi_M(., H^N)$ is a contraction for all M finite and $(X_t^N, t \leq T)$ is its unique fixed point as long as $(X_t^N, t \leq T)$ stays uniformly bounded by M. Since we have seen that almost surely $(X_t^N, t \leq T)$ is uniformly bounded, the statement follows.

Step 4: Convergence of the norm of $\phi_M(., H^N)$. For all M, $\phi_M(., H^N)^n(0)_t$ can be approximated uniformly by a polynomial function of H^N on $\sup_{s \leq t} \|H_s^N\|_\infty \leq L\}$ for some L finite, which happens with probability one for some sufficiently large L. We can thus use [14] to conclude that the norm of any polynomial in $\phi_M(., H^N)^n(0)_t$ converges to its analog with H^N replaced by S.

Step 5: Conclusion. We have proved that for all n and M

$$\lim_{N \to \infty} \|P(\phi_M(., H^N)^n(0)_t)\|_\infty = \|P(\phi_M(., S)^n(0)_t)\|_\infty \text{ a.s.}$$

Since X_t^N is uniformly bounded, for N large enough,

$$\lim_{M \to \infty} \lim_{n \to \infty} \sup_{t \leq T} \|\phi_M(., H^N)^n(0)_t - X_t^N\|_\infty = 0 \qquad \text{a.s.}$$

implies
$$\lim_{M\to\infty}\lim_{n\to\infty}\sup_{t\le T}\|P(\phi_M(.,H^N)^n(0)_t) - P(X_t^N)\|_\infty = 0 \qquad \text{a.s.}$$

And finally we have with overwhelming probability (where Z_N with stationary law such that $\|Z_N\| \le K$) for N large enough

$$\|X_t^{N,Z} - X_t^N\|_\infty \le e^{-2ct}\|Z_N\|_\infty \le e^{-ct}K.$$

Let $\epsilon > 0$ be fixed. We fix t so that $e^{-2ct}K = \epsilon$. With overwhelming probability, since $X_t^{N,Z}$ has the same law than Z, $\|X_t^{N,Z}\|_\infty \le K$ and so also $\|X_t^N\|_\infty \le \epsilon + K$. Hence, we have for any polynomial,

$$\lim_{N\to\infty}\|P(X_t^{N,Z}) - P(X_t^N)\|_\infty \le C(K)\epsilon \quad \text{a.s.}$$

Now, we take M, n large enough (greater than some finite random integers) to assure that for N large enough so that

$$\|P(\phi_M(.,H^N)^n(0)_t) - P(X_t^N)\|_\infty \le \epsilon/3$$

and finally

$$\left|\|P(\phi_M(.,H^N)^n(0)_t)\|_\infty - \|P(\phi_M(.,S)^n(0)_t)\|_\infty\right| < \epsilon/3.$$

We have already seen that

$$\|P(\phi_M(.,S)^n(0)_t) - P(X_t)\|_\infty \le \epsilon/3$$

Thus, we have proved

$$\left|\|P(X_t^{N,Z})\|_\infty - \|P(X_t^Z)\|_\infty\right| \le c'(K)\epsilon.$$

This completes the proof since $X_t^{N,Z}$ (resp. X_t^Z) has the same law that Z_N (resp. Z). $\qquad\square$

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