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Large Random Matrices: Lectures on Macroscopic Asymptotics

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Contents

Preface

These notes include the material from a series of nine lectures given at the Saint-Flour probability summer school in 2006. The two other lecturers that year were Maury Bramson and Steffen Lauritzen.

The topic of these lectures was large random matrices, and more precisely the asymptotics of their macroscopic observables such as the empirical measure of their eigenvalues. The interest in such questions goes back to Wishart and Wigner, in the twenties and fifties respectively. Large random matrices have been since then intensively studied in theoretical physics, in connection with various fields such as QCD, quantum chaos, string theory or quantum gravity.

Since the nineties, several key mathematical results have been obtained and the theory of large random matrices expanded in various directions, in connection with combinatorics, operator algebra theory, number theory, algebraic geometry, integrable systems etc. I felt that the time was right to summarize some of them, namely those which connect with the asymptotics of macroscopic observables, with a particular emphasis on their relation with combinatorics and operator algebra theory.

I wish to thank Jean Picard for organizing the Saint-Flour school and helping me through the preparation of these notes, an the other participants of the school, in particular for their useful comments to improve these notes. I am very grateful to several collaborators with whom I consulted on various points, in particular Greg Anderson, Edouard Maurel Segala, Dima Shlyakhtenko and Ofer Zeitouni.

Lyon, France *Alice Guionnet*
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Notation

• $\mathcal{C}_b(\mathbb{R})$ (resp. $\mathcal{C}_b^1(\mathbb{R}^N, \mathbb{R})$) denotes the space of bounded continuous functions on $\mathbb{R}$ (resp. k times continuously differentiable functions from $\mathbb{R}^N$ into $\mathbb{R}$). If f is a real-valued function on a metric space (X, d),

$$\|f\|_\infty = \sup_{x \in X} |f(x)|$$

denotes its supremum norm, whereas we set the Lipschitz norms to be

$$\|f\|_{\mathcal{L}} = \sup_{x \neq y} \frac{|f(x) - f(y)|}{d(x, y)} + \sup_x |f(x)|, \quad |f|_{\mathcal{L}} = \sup_{x \neq y} \frac{|f(x) - f(y)|}{d(x, y)}.$$

For $x \in \mathbb{R}^N$, and $f \in \mathcal{C}_b^1(\mathbb{R}^N, \mathbb{R})$, we let

$$\|x\|_2 = \left(\sum_{i=1}^N (x_i)^2 \right)^{\frac{1}{2}}, \quad \|\nabla f\|_2 = \left(\sum_{i=1}^N (\partial_{x_i} f(x))^2 \right)^{\frac{1}{2}}.$$

• $\mathcal{P}(X)$ denote the set of probability measures on the metric space (X, d). $\mu(f)$ is a shorthand for $\int f(x) d\mu(x)$. We shall call the weak topology on $\mathcal{P}(X)$ the topology so that $\mu \to \mu(f)$ is continuous if f is bounded continuous on (X, d). The moments topology refers to the continuity of $\mu \to \mu(x^k)$ for all $k \in \mathbb{N}$. Even though both topologies coincide if X is compact subset of $\mathbb{R}$, they can be different in general.

• If (X, d) is a metric space, Dudley's distance d_D on $\mathcal{P}(X)$ (which is compatible with the weak topology on $\mathcal{P}(X)$) is given by

$$d_D(\mu, \nu) := \sup_{\|f\|_{\mathcal{L}} \leq 1} \left| \int f(x) d\mu(x) - \int f(x) d\nu(x) \right| \tag{0.1}$$

• $\mathcal{M}_N(\mathbb{C})$ (resp.$\mathcal{H}_N^{(1)}$, resp. $\mathcal{H}_N^{(2)}$) denotes the set of $N \times N$ (resp. symmetric, resp. Hermitian) matrices with complex (resp. real, resp. complex) coefficients. $\mathcal{M}_N(\mathbb{C})$ is equipped with the trace Tr:

$$\mathrm{Tr}(A) = \sum_{i=1}^N A_{ii}.$$

• If A is an $N \times N$ Hermitian matrix, we denote by $(\lambda_k(A))_{1 \leq k \leq N}$ its eigenvalues.

• For A an $N \times N$ matrix, we define

$$\|A\|_2 = \left(\sum_{i,j=1}^N |A_{ij}|^2 \right)^{\frac{1}{2}} \quad \text{and} \quad \|A\|_\infty = \lim_{n \to \infty} \left(\mathrm{Tr}((AA^*)^n) \right)^{\frac{1}{2n}}.$$

The latter norm also coincides with the spectral radius of A which we denote by $\lambda_{\max}(A)$. 1 or I will denote the identity in $\mathcal{M}_N(\mathbb{C})$ and when no confusion is possible, for any constant c, c denotes $c1$.

• $\mathbb{C}\langle X_1, \ldots, X_m \rangle$ denotes the set of polynomials in m non-commutative indeterminates $(X_1, \ldots, X_m)$, $\mathbb{C}\langle X_1, \ldots, X_m \rangle_{sa}$ the subset of polynomials such that $P = P^*$ for some involution $*$ defined on $\mathbb{C}\langle X_1, \ldots, X_m \rangle$.

• Often, bold symbols will indicate vectors, e.g., $\mathbf{X} = (X_1, \ldots, X_m)$ or matrices e.g., $\mathbf{A} = (A_{ij})_{1 \leq i,j \leq N}$. The letters $(\mathbf{A}, \mathbf{B})$ in general refer to random matrices, whereas (X, Y, Z), to generic (eventually non-commutative) indeterminates.

Introduction

Random matrix theory was introduced in statistics by Wishart [206] in the thirties, and then in theoretical physics by Wigner [205]. Since then, it has developed separately in a wide variety of mathematical fields, such as number theory, probability, operator algebras, convex analysis etc.

Therefore, lecture notes on random matrices can only focus on special aspects of the theory; for instance, the well-known book by Mehta [153] displays a detailed analysis of the classical matrix ensembles, and in particular of their eigenvalues and eigenvectors, the recent book by Bai and Silverstein [10] emphasizes the results related to sample covariance matrices, whereas the book by Hiai and Petz [117] concentrates on the applications of random matrices to free probability and operator algebras. The book in progress [6] in collaboration with Anderson and Zeitouni will try to take a broader and more elementary point of view, but still without relations to number theory or Riemann Hilbert approach for instance. The first of these topics is reviewed briefly in [126] and the second is described in [73].

The goal of these notes is to present several aspects of the asymptotics of random matrix "macroscopic" quantities (RMMQ) such as

$$L_N(X_{i_1}\cdots X_{i_p}) := \frac{1}{N}\mathrm{Tr}(\mathbf{A}_{i_1}^N\cdots\mathbf{A}_{i_p}^N)$$

when $(i_k \in \{1,\dots,m\}, 1 \le k \le p)$ and $(\mathbf{A}_p^N)_{1\le p\le m}$ are some $N\times N$ random matrices whose size N goes to infinity. We will study their convergence, their fluctuations, their concentration towards their mean and, as much as possible in view of the states of the art, their large deviations and the asymptotics of their Laplace transforms. We will in particular stress the relation of the latest to enumeration questions. We shall focus on the case where $(\mathbf{A}_p^N)_{1\le p\le m}$ are Wigner matrices, that is Hermitian matrices with independent entries, although several results of these notes can be extended to other classical random matrices such as Wishart matrices.

When $m = 1$, $L_N(X_1^p)$ is the normalized sum of the pth power of the eigenvalues of X_1, that is the pth moment of the spectral measure of X_1. In his seminal article [205], Wigner proved that $E[L_N(X_1^p)]$ converges for any integer number p provided the entries of $\sqrt{N}\mathbf{A}_1$ have all their moments finite, are centered and have constant variance equal to one. We shall investigate this convergence in Chapter 1. We will show that it holds almost surely and that the hypothesis on the entries can be weakened. This result extends to several matrices, as shown by Voiculescu [197], see Section 3.2.

One of the interesting aspects of this convergence is the relation between the limits of the RMMQ and the enumeration of interesting graphs. Indeed, a key observation is that the empirical moment $L_N(X_1^{2p})$ converges towards the Catalan number C_p, the number of rooted trees with p edges, or equivalently the number of non-crossing pair partitions of $2p$ ordered points. As shown by Voiculescu [197], words in several matrices lead to the enumeration of *colored* trees. Considering the central limit theorem for such macroscopic quantities, we shall see also that their limiting covariances can be expressed in terms of numbers of certain planar graphs. It turns out that if the matrices have complex Gaussian entries, this relation extends far beyond the first two moments. Harer and Zagier [113] showed that the expansion of $E[L_N(X_1^p)]$ in terms of the dimension N can be seen as a topological expansion (i.e., as a generating function with parameter N^{-2} and coefficients which count graphs sorted by their genus). We shall see in these notes that also Laplace transforms of RMMQ's can be interpreted as generating functions (with parameters the dimension and the parameters of the Laplace transform) of interesting numbers.

This idea goes back to Brézin, Itzykson, Parisi and Zuber [50] (see also 't Hooft [187]) who considered matrix integrals given by

$$Z_N(P) = E[e^{N\mathrm{Tr}(P(\mathbf{A}_1^N,\ldots,\mathbf{A}_m^N))}]$$

with a polynomial function P and independent copies $\mathbf{A}_i^N$ of an $N\times N$ matrix $\mathbf{A}^N$ with complex Gaussian entries. Then, they showed that if $P=\sum t_i q_i$ with some (complex) parameters t_i and some monomials q_i, $\log Z_N(P)$ expands formally (as a function of the parameters t_i and the dimension N of the matrices). The weight $N^{-2g}\prod_i (t_i)^{k_i}/k_i!$ will have a coefficient which counts the number of graphs with k_i vertices depending on the monomial q_i, for $i\geq 0$, that can be embedded properly in a surface of genus g. This relation is based on Feynman diagrams (see a review by A. Zvonkin [211]) and we shall describe it more precisely in Section 7.4. Matrix integrals were used widely in physics to solve problems in connection with the enumeration of maps [50, 42, 125, 209, 77, 76]. Part of these notes (mostly Part III) will show that, under appropriate assumptions, such formal equalities can be proved to hold as well asymptotically. In particular, we will see that $N^{-2}\log Z_N(\lambda P)$ converges for sufficiently small λ and the limit is a generating function for graphs embedded into the sphere.

In the second part of these notes, we show how to estimate Laplace transforms of traces of matrices in non-perturbative situation (that is estimate $N^{-2}\log Z_N(\lambda P)$ for large λ's). In this case, it is no longer clear whether matrix integrals are related to the enumeration of graphs (except when P satisfies some convexity property, in which case it was shown in [106] that the free energy is the analytic extension of the enumeration of planar maps found for $Z_N(\lambda P)$ and λ small). Thus, different tools have to be introduced to estimate $Z_N(P)$ in general. First we consider one-matrix integrals and derive the large deviation principle for the spectral measure of Gaussian Wigner matrices. We then introduce a dynamical point of view to extend the previous result to *shifted* Gaussian Wigner matrices. The latter is applied to estimate some two-matrix integrals (such as the Ising model on random graphs) and Schur functions. We show in the last part of these notes how dynamics and large deviations techniques can be used to study the more general problem of estimating free entropy (see Chapter 18). The question of computing the free entropy remains open.

The outline of this book is as follows.

In the first part of these notes, we study the convergence of the RMMQ's and more precisely the convergence of the spectral measure of a Wigner matrix. We follow Wigner's original approach to study this question and estimate moments. This moments method can be refined to prove a central limit theorem (Section 2.1) or study the largest eigenvalue of random matrices, as proposed initially by Sinaï and Soshnikov (Section 2.2). Finally, we show that Wigner's theorem can be generalized to several matrices.

In the second part of these notes, we study concentration inequalities. These inequalities have provided a very powerful tool to control the probability of deviations of diverse random variables from their mean or their median (see some applications in [188]). After introducing some basic notions and results of concentration of measure theory, we specialize them to random matrices. In particular, we deduce concentration of the spectral measure or of the largest eigenvalue of Wigner matrices with nice entries. We also apply Brascamp–Lieb inequalities to random matrices.

In the third part, we study Gaussian matrix integrals in a perturbative regime. We give sufficient conditions so that they converge as the size of the matrices goes to infinity, study the first order correction to this convergence and relate the limits to the enumeration of graphs. The inequalities developed in Part II are important tools for this analysis.

In the fourth part of these notes, we concentrate on the eigenvalues of Gaussian random matrices (mainly the so-called Gaussian unitary or orthogonal ensembles). We remind the reader that their joint law is given as the law of Gaussian random variables interacting via a Coulomb gas potential. This joint law is key to many detailed analysis of the spectrum of the Gaussian ensembles, such has the study of the spacing fluctuations in the bulk or at the edge [153, 191], the interpretation of the limit has a determinantal process [119, 182, 39] etc. In these notes, we will only focus again on the RMMQ and deduce large deviation principles for the spectral measure and the largest eigenvalue.

In the fifth part, we start addressing the question of proving large deviation principles for the laws of RMMQ's in a multi-matrix setting. We obtain a large deviations principle for the law of the Hermitian Brownian motion, from which we deduce estimates on Schur functions and Harish–Chandra–Itzykson–Zuber integrals. We apply these results to the related enumeration questions of the Ising model on random graphs for instance.

In the last part, we discuss the natural generalization of these questions to a general multi-matrix setting, namely analyzing free entropy. We introduce a free probability set-up and the notion of freeness. We then obtain bounds on free entropy.

As a conclusion, the goal of these notes is to present an overview of the study of macroscopic quantities of random matrices (law of large numbers, central limit theorems etc.) with a special emphasis on large deviations questions. I tried to give proofs as elementary and complete as possible, based on "standard tools" of probability (concentration, large deviations, etc.) which we shall, however, recall in some detail to help non-probabilists to understand proofs. Some proofs are new, some are improved versions of the proofs taken from articles and others are inspired from a book in progress with G. Anderson and O. Zeitouni [6]. In comparison with that book, these notes focus on matrix models and large deviations questions, whereas [6] attempts to give a more complete picture of random matrix theory, including local properties of the spectrum.

Part I

Wigner matrices and moments estimates

In this part, we follow the strategy introduced by Wigner [205] to study the spectrum of random matrices: we estimate moments of traces of polynomials in these random matrices. We prove in this way several key results. First, we obtain the convergence (in expectation and almost surely) of the spectral measure (for the moments or the weak topology) of Wigner matrices. We also study its fluctuations around the limit. We generalize the convergence to a multi-matrix setting by showing that the trace of words in several matrices converges in the limit where the dimension goes to infinity. Finally, we generalize the estimation of moments to the case where the exponent blows up with the dimension N of the matrices, but more slowly than $\sqrt{N}$. This is enough to bound the distance between the largest eigenvalue and its limit.

1

Wigner's theorem

We consider in this section an $N \times N$ matrix $\mathbf{A}^N = \left(A_{ij}^N\right)_{1\le i,j\le N}$ with real or complex entries such that $\left(A_{ij}^N\right)_{1\le i\le j\le N}$ are independent and $\mathbf{A}^N$ is self-adjoint; $A_{ij}^N = \bar{A}_{ji}^N$. We assume further that

$$\mathbb{E}[A_{ij}^N] = 0, \lim_{N\to\infty} \frac{1}{N^2} \sum_{1\le i,j\le N} |N\mathbb{E}[|A_{ij}^N|^2] - 1| = 0.$$

We shall show in this chapter that the eigenvalues $(\lambda_1, \ldots, \lambda_N)$ of $\mathbf{A}^N$ satisfy the almost sure convergence

$$\lim_{N\to\infty} \frac{1}{N} \sum_{i=1}^{N} f(\lambda_i) = \int f(x) d\sigma(x)$$

where f is a bounded continuous function or a polynomial function, when the entries have finite moments. σ is the semicircle law

$$\sigma(dx) = \frac{1}{2\pi}\sqrt{4-x^2}1_{|x|\le 2}dx.$$

We shall first prove this convergence for polynomial functions and rely on the fact that for all $k \in \mathbb{N}$, $\int x^k d\sigma(x)$ is null when k is odd and given by the Catalan number $C_{k/2}$ when k is even. We thus start this chapter by discussing the properties and characterizations of Catalan numbers.

1.1 Catalan numbers, non-crossing partitions and Dick paths

We will encounter first the Catalan numbers as the number of (oriented) rooted trees. We shall define more precisely this object in the next paragraph. Actually, Catalan numbers count many other combinatorial objects. In a first part, we shall see that they also enumerate non-crossing partitions as well as Dick paths, a fact which we shall use later. As a warm-up to matrix models, we will also state the bijection with planar maps with one star. Then, we will study the Catalan numbers, their generating function, and relate them to the moments of the semicircle law.

1.1.1 Catalan numbers enumerate oriented rooted trees

Let us recall that a graph is given by a set of vertices (or nodes) $V = \{i_1, \ldots, i_k\}$ and a set E of edges $(e_i)_{i \in I}$. An edge is a couple $e = (i_{j_1}, i_{j_2})$ for some $j_1, j_2 \in \{1, \ldots, k\}^2$. An edge $e = (i_p, i_\ell)$ is directed if (i_p, i_ℓ) and (i_ℓ, i_p) are distinct when $i_p \neq i_\ell$, which amounts to writing edges as directed arrows. It is undirected otherwise. A cycle (or loop) is a collection of distinct undirected edges $e_i = (v_i, v_{i+1})$, $1 \leq i \leq p$ such that $v_1 = v_{p+1}$ for some $p \geq 1$. A graph is connected if any two vertices (v_1, v_2) of the graph are connected by a path (that is that there exists a collection of edges $e_i = (a_i, b_i), 1 \leq i \leq n$ such that $v_1 = a_1$, $b_i = a_{i+1}$, $b_n = v_2$).

A *tree* is a connected graph with no loops (or cycles).

We will say that a tree is oriented if it is drawn (or embedded) into the plane; it then inherits the orientation of the plane. A tree is rooted if we specify one oriented edge, called the root. Note that if each edge of an oriented tree is seen as a double (or fat) edge, the connected path drawn from these double edges surrounding the tree inherits the orientation of the plane (see Figure 1.1). A root on this oriented tree then specifies a starting point in this path. This path will be intimately connected with the Dick path that we consider next.

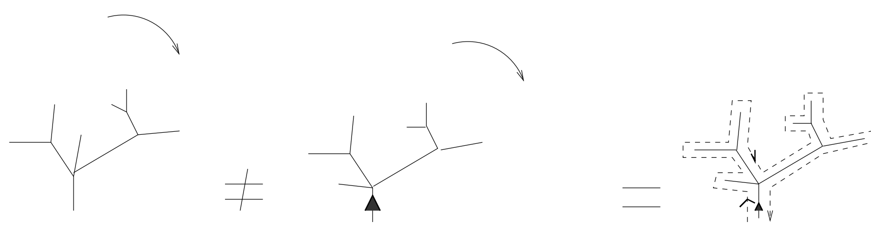


Fig. 1.1. Embedding rooted trees into the plane

Let us give the following well-known characterization of trees among connected graphs.

Lemma 1.1. *Let $G = (V, E)$ be a connected graph with E a set of undirected edges, and denote by $|A|$ the number of distinct elements of a finite discrete set A. Then,*

$$|V| \leq |E| + 1. \tag{1.1}$$

Moreover, $|V| = |E| + 1$ iff G is a tree.

Proof. (1.1) is straightforward when $|V| = 1$ and can be proven by induction as follows. Assume $|V| = n$ and consider one vertex v of V. This vertex is contained in $l \geq 1$ edges of E that we denote $(e_1, \ldots, e_l)$. The graph G then decomposes into $(v, e_1, \ldots, e_l)$ and $r \leq l$ connected graphs $(G_1, \ldots, G_r)$. We denote $G_j = (V_j, E_j)$ for $j \in \{1, \ldots, r\}$. We have

$$|V| - 1 = \sum_{j=1}^{r} |V_j|, \quad |E| - l = \sum_{j=1}^{r} |E_j|.$$

Applying the induction hypothesis to the connected graphs $(G_j)_{1 \leq j \leq r}$ gives

$$|V| - 1 \leq \sum_{i=1}^{r} (|E_j| + 1) = |E| + r - l \leq |E|, \tag{1.2}$$

which proves (1.1). In the case where $|V| = |E| + 1$, we claim that G is a tree, namely does not have any loop. In fact, for the equality to hold, we need to have equalities when performing the previous decomposition of the graph, a decomposition which can be reproduced until all vertices have been considered. If the graph contains a loop, the first time that the decomposition considers a vertex v of this loop, v must be the end point of at least two different edges, with end points belonging to the same connected graph (because they belong to the loop). Hence, we must have $r < l$ so that a strict inequality occurs in the right-hand side of (1.2). ☐

Definition 1.2. *We denote by C_k the number of rooted oriented trees with k edges.*

Equivalently, we shall see in the following two paragraphs that C_k is the number of Dick paths of length $2k$, or the number of non-crossing pair partitions of $2k$ elements, or the number of planar maps with one star of type x^{2k}.

Exercise 1.3. *Show that $C_2 = 2$ and $C_3 = 5$ by drawing the corresponding graphs.*

1.1.2 Bijection with Dick paths

Definition 1.4. *A Dick path of length $2n$ is a path starting and ending at the origin, with increments $+1$ or -1, and that stays above the non-negative real axis.*

We shall prove:

Property 1.5. *There exists a bijection between the set of rooted oriented trees and the set of Dick paths.*

Proof. To construct a Dick path from a rooted oriented tree, let us define a walk on the tree (or a closed path around the tree) as follows. We regard the oriented tree as a *fat tree*, which amounts to replacing each edge by a double edge (the double edge is made of two parallel edges surrounding the original edge, see Figure 1.1) while keeping the same set of vertices. The union of these double edges defines a path that surrounds the tree. The walk on the tree is defined by putting the orientation of the plane on this curve and starting from the root as the first step of the Dick path (see Figure 1.2). To define the Dick path, one follows the walk and counts a unit of time each time one meets a vertex; then adds $+1$ to the Dick path when one meets an (non-oriented) edge that has not yet been visited and -1 otherwise. Since to add a -1, one must have added a $+1$ corresponding to the first visit of the edge, the Dick path is non-negative and since at the end all edges are visited exactly twice, the path constructed will come back at 0 at time $2n$. This defines a bijection (see Figure 1.2) since, given a Dick path, we can recover the rooted tree by first gluing the couples of steps where one step up is followed by one step down and representing each couple of glued steps by one edge;

we then obtain a path decorated with edges. Continuing this procedure until all steps have been glued two by two provides a rooted tree.

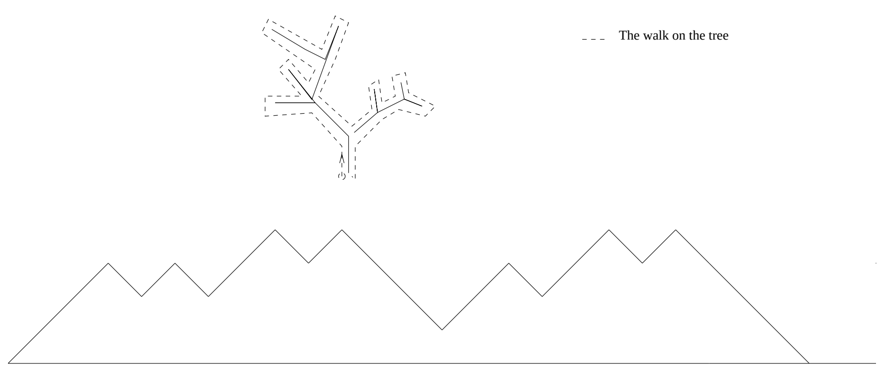


Fig. 1.2. Bijection between trees and Dick paths

1.1.3 Bijection with non-crossing pair partitions

Let us recall the following definition:

Definition 1.6. • *A partition of the set* $S := \{1, \ldots, n\}$ *is a decomposition*

$$\pi = \{V_1, \ldots, V_r\}$$

of S *into disjoint and non-empty sets* V_i.

- *The set of all partitions of S is denoted by $\mathcal{P}(S)$, and for short by $\mathcal{P}(n)$ if $S := \{1, \ldots, n\}$.*
- *The $V_i, 1 \leq i \leq r$ are called the blocks of the partition and we say that $p \sim_\pi q$ if p, q belong to the same block of the partition π.*
- *A partition π of $\{1, \ldots, n\}$ is said to be crossing if there exist $1 \leq p_1 < q_1 < p_2 < q_2 \leq n$ with*

$$p_1 \sim_\pi p_2 \not\sim_\pi q_1 \sim_\pi q_2.$$

  It is non-crossing otherwise.
- *A partition is a pair partition if all blocks have cardinality two.*

The bijection between oriented rooted trees with n edges and non-crossing pair-partitions of $2n$ elements goes as follows. On each edge of the tree we draw an arc going from one side of the edge to the other side and that does not cross the tree. We start from the root and draw one arc in such a way that the part of the tree visited by the walk before arriving for the second time at the first edge is contained in the ball with boundary given by the arc. We then continue this procedure, drawing the arcs in such a way that they do not cross, till no edge is left. Finally, we think of the tree as being drawn by the folding of a cord with both ends at the root; in other words, we replace the tree by the fat tree designed from the trajectory of the walk as shown in Figure 1.2. Unfolding the cord while keeping the arcs gives a pair-partition. A less colorful way to say the same thing is to label each side of the edges starting from the root and following the orientation and to write down the pair-partition with pairings given by the labels of the two sides of the edges. For instance, the drawing below represents the pair-partition of $\{1, \ldots, 24\}$ given by $(1, 24)$, $(2, 13)$, $(3, 4)$, $(5, 6)$, $(7, 12)$, $(8, 9)$, $(10, 11)$, $(14, 23)$, $(15, 16)$, $(17, 22)$, $(18, 19)$, $(20, 21)$.

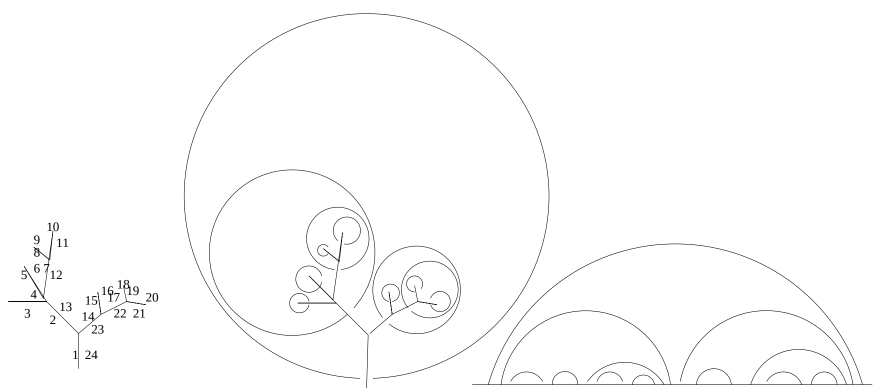


Fig. 1.3. Drawing the partitions on the tree and unfolding the tree

We claim that the resulting partition is non-crossing. Indeed, if we take two edges of a tree, say $e_1 = (a_1, b_1)$ and $e_2 = (a_2, b_2)$, let T_1 be the subtree visited, when one follows the orientation on the tree, between the time it visits the two sides of the edge e_1. Then, either $e_2 \in T_1$, and then $a_1 < a_2 < b_2 < b_1$, or $e_2 \notin T_1$, corresponding either to $a_2 < b_2 < a_1 < b_1$ or $a_1 < b_1 < a_2 < b_2$. We have thus shown:

Property 1.7. *To each oriented rooted tree with n edges we can associate bijectively a non-crossing pair partition of $2n$ elements.*

Remark 1. Observe that in the bijection, the elements of the partition are the edges of the tree seen as double (or fat) edges, as for the definition of the walk on the tree (see Figure 1.2).

Let us finally remark that there is an alternative way to draw non-crossing partitions that we shall use later. Instead of drawing the points of the partition on the real line, we can draw them on the circle, provided we mark, say, the place where we put the first element and provide the circle with an orientation corresponding to the orientation on the real line. With this mark and orientation, we have again a bijection. The drawing of the partition then becomes a series of arcs which can be drawn either outside of the annulus or inside (see Figure 1.4). As a matter of fact, the circle is irrelevant here, the only thing that matters are the points, the marked point and the orientation. So, we can also see one such point as the end point of a half-edge, all the half-edges intersecting in one vertex in the center of the previous circle. Thus, we can draw our set of the k points on the real line as a vertex with k half-edges, one marked half-edge and an orientation. We shall later call the set of these edges, marked half-edge and orientation a star. In this picture, the pair partition corresponds to the gluing of these half-edges two by two and the fact that the partition is non-crossing exactly means that the edges (obtained by the gluing of two half-edges) do not cross.

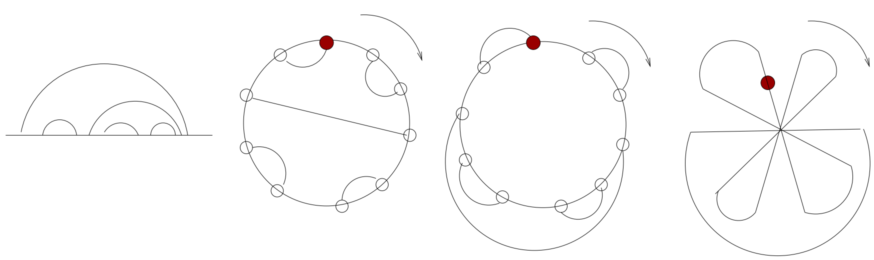


Fig. 1.4. Non-crossing partitions and stars

The last drawing in Figure 1.4 is a planar map; that is, a connected graph that is embedded into the sphere.

Definition 1.8. *A star of type x^k is a vertex with k half-edges, one marked half-edge and an orientation. A map is a connected graph that is embedded into a surface in such a way that the edges do not intersect and if we cut the surface along the edges, we get a disjoint union of sets that are homeomorphic to an open disk (these sets are called the faces of the map). A map with stars $x^{q_1}, \ldots, x^{q_p}$ is a graph where the half-edges of the stars $x^{q_1}, \ldots, x^{q_p}$ have been glued pair-wise, the orientation of each pair of edge agreeing, hence providing to the full graph one orientation.*

The genus g of the map is the genus of such a surface; it satisfies

$$2 - 2g = \sharp vertices + \sharp faces - \sharp edges.$$

A planar map is a map with genus zero.

For more details on maps, we refer to the review [211]. Note that once a graph is embedded into a surface, the natural orientation of the surface induces an orientation around each vertex of the graph (more precisely a cyclic order on the end points of the half-edges of its vertices). This fact has its counterpart since (cf. [211, Proposition 4.7]) an orientation around each vertex of a graph uniquely determines its embedding into a surface. This shows that, modulo the notion of marked points, the notion of a star is intimately related to the idea of embedding the corresponding graph into a surface. Prescribing a marked half-edge will be useful later to describe how we will count these graphs (the orientation and the marked point of the stars being equivalent to a labeling of its half-edges).

To find out the genus of a map with only one vertex of degree k, one can also recall that the end points of the half-edges of the star represent the middle of the edges of the fat tree. Drawing these edges and gluing them pairwise according to the map allows one to visualize the surface on which one can embed the map (in the figure below, the lines on the surface are now the boundary of the polygon).

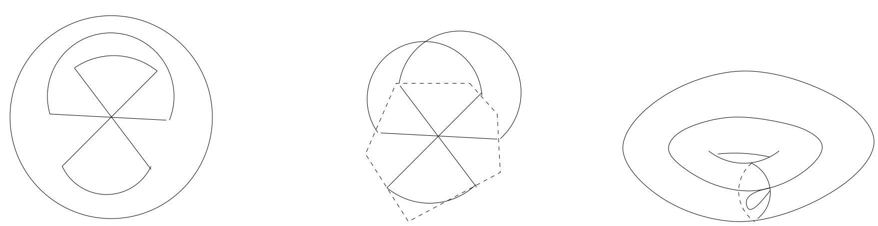


Fig. 1.5. Partitions and maps

1.1.4 Induction relation

We next show that the Catalan numbers satisfy the following induction relation.

Property 1.9. $C_0 = 1$ *and for all* $k \geq 1$

$$C_k = \sum_{l=0}^{k-1} C_{k-l-1} C_l. \tag{1.3}$$

Proof. By convention, C_0 will be taken to be equal to one and we consider an oriented tree T rooted at $r = (i_1, i_2)$ with $k \geq 1$ edges. Starting from the root r and following the orientation, we let t_1 be the first time that we return to i_1 following the walk on T. The subgraph T_1 of the tree we have investigated is a tree, with only the edge $r = (i_1, i_2)$ attached to i_1. We let r_1 be the first edge (according to the orientation of the plane) attached to i_2. Removing the edge r from T_1, we obtain an oriented tree T_1' rooted at r_1. We denote by $l_1 \leq k-1$ the number of its edges. $T_2 = T \backslash T_1$ is an oriented rooted tree (at the first edge attached to i_1) with $k-1-l_1$ edges. Therefore, any oriented rooted tree with k edges can be decomposed into an edge and two oriented rooted trees with respectively l_1 and $k - l_1 - 1$ vertices for some $l_1 \in \{0, \ldots, k-1\}$. This proves (using $C_0 = 1$) that (1.3) holds with $l = l_1$. □

Property (1.9) defines uniquely the Catalan numbers by induction. We can also give the more explicit formula:

Property 1.10. *For all* $k \geq 0$, $C_k \leq 2^{2k}$ *and*

$$C_k = \frac{\binom{2k}{k}}{k+1}.$$

Proof. Note that since C_k is also the number of Dick paths with length $2k$, it is smaller than the number of walks (that is, the number of connected paths with steps equal to $+1$ or -1) starting at the origin with length $2k$, that is, 2^{2k}. In particular, if we define

$$S(z) := \sum_{k=0}^{\infty} C_k z^k,$$

$S(z)$ is absolutely convergent in $|z| < 4^{-1}$. We can therefore multiply both sides of equality (1.3) by z^k and sum the resulting equalities for $k \in \mathbb{N} \backslash \{0\}$. We arrive at

$$S(z) - 1 = z S(z)^2.$$

As a consequence,

$$S(z) = \frac{1 \pm \sqrt{1-4z}}{2z}.$$

Since $S(0) = 1$, we conclude that

$$S(z) = \frac{1 - \sqrt{1-4z}}{2z}. \tag{1.4}$$

We can now expand $\sqrt{1-4z}$ in a Taylor series around the origin to obtain

$$\sqrt{1-4z} = 1 - 2z - \sum_{k=1}^{n} \frac{(2^{-1})^{k+1}(2k-1)(2k-3)\cdots(1)}{(k+1)!}(4z)^{k+1} + o(z^{n+1})$$

yielding

$$S(z) = 1 + 2\sum_{k=1}^{n} \frac{(2^{-1})^{k+1}(2k-1)(2k-3)\cdots(1)}{(k+1)!}(4z)^{k} + o(z^{n}).$$

Therefore, by identifying each term of the series we find

$$C_k = 2\frac{4^k(2^{-1})^{k+1}(2k-1)(2k-3)\cdots(1)}{(k+1)!} = \frac{2k!}{(k+1)!k!} = \frac{\binom{2k}{k}}{k+1}.$$

□

1.1.5 The semicircle law and Catalan numbers

The standard semicircle law is given by

$$\sigma(dx) = \frac{1}{2\pi}\sqrt{4-x^2}\mathbf{1}_{|x|\leq 2}dx.$$

Property 1.11. *Let $m_k = \int x^k d\sigma(x)$. Then for all $k \geq 0$,*

$$m_{2k} = C_k.$$

Proof. By the change of variables $x = 2\sin(\theta)$

$$\begin{aligned} m_{2k} &= \int_{-2}^{2} x^{2k}\sigma(x)dx = \frac{2\cdot 2^{2k}}{\pi}\int_{-\pi/2}^{\pi/2} \sin^{2k}(\theta)\cos^2(\theta)d\theta \\ &= \frac{2\cdot 2^{2k}}{\pi}\int_{-\pi/2}^{\pi/2} \sin^{2k}(\theta)d\theta - (2k+1)m_{2k}\,. \end{aligned}$$

Hence,

$$(2k+2)m_{2k} = \frac{2\cdot 2^{2k}}{\pi}\int_{-\pi/2}^{\pi/2} \sin^{2k}(\theta)d\theta = 4(2k-1)m_{2k-2}\,,$$

from which, together with $m_0 = 1$, one concludes that

$$m_{2k} = \frac{4(2k-1)}{(2k+2)}m_{2k-2}\,, \tag{1.5}$$

leading to the claimed assertion that $m_{2k} = C_k$ by Property 1.10. □

Corollary 1.12. *For $z \in \mathbb{C} \backslash \mathbb{R}$, let*

$$G_\sigma(z) := \int \frac{1}{z-x} d\sigma(x)$$

be the Stieltjes transform of the semicircle law. Then, for $z \in \mathbb{C} \backslash [-2, 2]$

$$G_\sigma(z) = \frac{1}{2}\left(z - \sqrt{z^2 - 4}\right).$$

Proof. When $|z| > 2$, we can write

$$\begin{aligned} G_\sigma(z) &= \frac{1}{z} \int \frac{1}{1 - z^{-1}x} d\sigma(x) = \frac{1}{z} \sum_{k \geq 0} z^{-k} \int x^k d\sigma(x) \\ &= \frac{1}{z} \sum_{k \geq 0} z^{-2k} C_k = \frac{1}{z} S(z^{-2}) \\ &= \frac{1}{z} \left(\frac{1 - \sqrt{1 - 4z^{-2}}}{2z^{-2}} \right) = \frac{1}{2}\left(z - \sqrt{z^2 - 4}\right) \end{aligned}$$

where we finally used (1.4). This equality extends to the whole domain of analyticity of G_σ, i.e., $\mathbb{C} \backslash [-2, 2]$. □

1.2 Wigner's theorem

We consider an $N \times N$ matrix $\mathbf{A}^N$ with real or complex entries such that $\left(A^N_{ij}\right)_{1 \leq i \leq j \leq N}$ are independent and $\mathbf{A}^N$ is self-adjoint; $A^N_{ij} = \bar{A}^N_{ji}$. We assume that

$$\mathbb{E}[A^N_{ij}] = 0,\ 1 \leq i, j \leq N,\ \lim_{N \to \infty} \frac{1}{N^2} \sum_{1 \leq i,j \leq N} |N\mathbb{E}[|A^N_{ij}|^2] - 1| = 0. \tag{1.6}$$

In this section, we use the same notation for complex and for real entries since both cases will be treated at once and yield the same result. The aim of this section is to prove the convergence of the quantities $N^{-1}\mathrm{Tr}\left((\mathbf{A}^N)^k\right)$ as N goes to infinity and k is any positive integer number. Since $\mathrm{Tr}\left((\mathbf{A}^N)^k\right) = \sum_{i=1}^N \lambda_i^k$ if $(\lambda_1, \ldots, l_N)$ are the eigenvalues of $\mathbf{A}_N$, this prove the convergence in moments of the spectral measure of $\mathbf{A}_N$.

Theorem 1.13 (Wigner's theorem). *[205] Assume that* (1.6) *holds and for all $k \in \mathbb{N}$,*

$$B_k := \sup_{N \in \mathbb{N}} \sup_{ij \in \{1,\ldots,N\}^2} \mathbb{E}[|\sqrt{N} A^N_{ij}|^k] < \infty. \tag{1.7}$$

Then,

$$\lim_{N \to \infty} \frac{1}{N} \mathrm{Tr}\left((\mathbf{A}^N)^k\right) = \begin{cases} 0 & \text{if } k \text{ is odd,} \\ C_{\frac{k}{2}} & \text{otherwise,} \end{cases}$$

where the convergence holds in expectation and almost surely. $(C_k)_{k \geq 0}$ are the Catalan numbers.

Proof. We start the proof by showing the convergence in expectation. The strategy is simply to expand the expectation of the trace of the matrix in terms of the expectation of its entries. We then use some (easy) combinatorics on trees to find out the main contributing term in this expansion. The almost sure convergence is obtained by estimating the variance of the considered random variables.

1. *Expanding the expectation.*
   Setting $\mathbf{B}^N = \sqrt{N}\mathbf{A}^N = (B_{ij})_{1\le i,j\le N}$, we have
   $$\mathbb{E}\left[\frac{1}{N}\mathrm{Tr}\left((\mathbf{A}^N)^k\right)\right] = \sum_{i_1,\ldots,i_k=1}^{N} N^{-\frac{k}{2}-1}\mathbb{E}[B_{i_1i_2}B_{i_2i_3}\cdots B_{i_ki_1}] \quad (1.8)$$
   where $(B_{ij})_{1\le i,j\le N}$ denote the entries of $\mathbf{B}^N$ (which may eventually depend on N). We denote $\mathbf{i} = (i_1,\ldots,i_k)$ and set
   $$P(\mathbf{i}) := \mathbb{E}[B_{i_1i_2}B_{i_2i_3}\cdots B_{i_ki_1}].$$
   By (1.7) and Hölder's inequality, $P(\mathbf{i})$ is bounded uniformly by B_k. Since the random variables $(B_{ij})_{1\le i\le j\le N}$ are independent and centered, $P(\mathbf{i})$ vanishes unless for any edge (i_p, i_{p+1}), $p\in\{1,\ldots,k\}$, there exists $l\neq p$ such that $(i_p,i_{p+1}) = (i_l,i_{l+1})$ or (i_{l+1},i_l). Here, we used the convention $i_{k+1} := i_1$. We next show that the set of indices that contributes to the first order in the right-hand side of (1.8) is described by trees.
2. *Connected graphs and trees.*
   $V(\mathbf{i}) = \{i_1,\ldots,i_k\}$ will be called the vertices. An edge is a pair (i,j) with $i,j\in\{1,\ldots,N\}^2$. At this point, edges are directed in the sense that we distinguish (i,j) from (j,i) when $j\neq i$. We denote by $E(\mathbf{i})$ the collection of the k edges $(e_p)_{p=1}^k = (i_p,i_{p+1})_{p=1}^k$ with $i_{k+1} = i_1$.
   We consider the graph $G(\mathbf{i}) = (V(\mathbf{i}),E(\mathbf{i}))$. $G(\mathbf{i})$ is connected since there exists an edge between any two vertices i_ℓ and $i_{\ell+1}$, $\ell\in\{1,\ldots,k-1\}$. Note that $G(\mathbf{i})$ may contain loops (e.g., cycles, for instance edges of type (i,i)) and multiple undirected edges.
   The skeleton $\tilde{G}(\mathbf{i})$ of $G(\mathbf{i})$ is the graph $\tilde{G}(\mathbf{i}) = \left(\tilde{V}(\mathbf{i}),\tilde{E}(\mathbf{i})\right)$ where $\tilde{V}(\mathbf{i})$ is the set of different vertices of $V(\mathbf{i})$ (without multiplicities) and $\tilde{E}(\mathbf{i})$ is the set of undirected edges of $E(\mathbf{i})$, also taken without multiplicities.
3. *Convergence in expectation.*
   Since we noticed that $P(\mathbf{i})$ equals zero unless each edge in $E(\mathbf{i})$ is repeated at least twice, we have that
   $$|\tilde{E}(\mathbf{i})| \le \frac{k}{2} \Rightarrow |\tilde{E}(\mathbf{i})| \le \left[\frac{k}{2}\right],$$
   and so by (1.1) applied to the skeleton $\tilde{G}(\mathbf{i})$ we find

$$|\tilde{V}(\mathbf{i})| \leq \left[\frac{k}{2}\right] + 1$$

where $[x]$ is the integer part of x. Thus, since the indices are chosen in $\{1, \ldots, N\}$, there are at most $N^{[\frac{k}{2}]+1}$ indices that contribute to the sum (1.8) and so we have

$$\left|\mathbb{E}\left[\frac{1}{N}\mathrm{Tr}\left((\mathbf{A}^N)^k\right)\right]\right| \leq B_k N^{[\frac{k}{2}]-\frac{k}{2}}.$$

where we used (1.7). In particular, if k is odd,

$$\lim_{N\to\infty} \mathbb{E}\left[\frac{1}{N}\mathrm{Tr}\left((\mathbf{A}^N)^k\right)\right] = 0.$$

If k is even, the only indices that will contribute to the first order asymptotics in the sum are those such that

$$|\tilde{V}(\mathbf{i})| = \frac{k}{2} + 1,$$

which, by Lemma 1.1, implies that:
a) $\tilde{G}(\mathbf{i})$ is a tree.
b) $|\tilde{E}(\mathbf{i})| = 2^{-1}|E(\mathbf{i})| = \frac{k}{2}$ and so each edge in $E(\mathbf{i})$ appears exactly twice.
Thus, $G(\mathbf{i})$ appears as a fat tree where each edge of $\tilde{G}(\mathbf{i})$ is repeated exactly twice.
$G(\mathbf{i})$ is rooted (a root is given by the directed edge (i_1, i_2)). These edges are directed by the natural order on the indices. Because $G(\mathbf{i})$ is a tree, we see that each pair of directed edges corresponding to the same undirected edge in $\tilde{E}(\mathbf{i})$ is of the form $\{(i_p, i_{p+1}), (i_{p+1}, i_p)\}$. Moreover, the order on the indices induces a cyclic order on the fat tree that uniquely prescribes the way this fat tree can be embedded into the plane, the orientation of the plane agreeing with the orientation on the fat tree (see Figure 1.1). Therefore, for these indices, $P(\mathbf{i}) = \prod_{e\in\tilde{E}(\mathbf{i})} E[|\sqrt{N}A_e^N|^2]$. We write $G(\mathbf{i}) \simeq G(\mathbf{j})$ if $G(\mathbf{i})$ and $G(\mathbf{j})$ corresponds to the same rooted tree (but with eventually different values of the indices). By (1.6), for any fixed rooted tree G,

$$\frac{1}{N^{\frac{k}{2}+1}} \sum_{\mathbf{i}:G(\mathbf{i})\simeq G} |\prod_{e\in\tilde{E}(\mathbf{i})} E[|\sqrt{N}A_e^N|^2] - 1| \leq \frac{kB_2^{\frac{k}{2}-1}}{N^2} \sum_{i,j=1}^{N} |E[|B_{ij}|^2] - 1|$$

goes to zero as N goes to infinity. Hence, we deduce that

$$\lim_{N\to\infty} \mathbb{E}\left[\frac{1}{N}\mathrm{Tr}\left((\mathbf{A}^N)^k\right)\right] = \sharp\{\text{rooted oriented trees with } k/2 \text{ edges}\}.$$

4. *Almost sure convergence.* To prove the almost sure convergence, we estimate the variance and then use the Borel–Cantelli lemma. The variance is given by

$$\mathrm{Var}((\mathbf{A}^N)^k) := \mathbb{E}\left[\frac{1}{N^2}\left(\mathrm{Tr}\left((\mathbf{A}^N)^k\right)\right)^2\right] - \mathbb{E}\left[\frac{1}{N}\mathrm{Tr}\left((\mathbf{A}^N)^k\right)\right]^2$$
$$= \frac{1}{N^{2+k}} \sum_{\substack{i_1,\ldots,i_k=1\\ i'_1,\ldots,i'_k=1}}^{N} [P(\mathbf{i},\mathbf{i}') - P(\mathbf{i})P(\mathbf{i}')]$$

with
$$P(\mathbf{i},\mathbf{i}') := \mathbb{E}[B_{i_1 i_2} B_{i_2 i_3} \cdots B_{i_k i_1} B_{i'_1 i'_2} \cdots B_{i'_k i'_1}].$$

We denote by $G(\mathbf{i},\mathbf{i}')$ the graph with vertices $V(\mathbf{i},\mathbf{i}') = \{i_1,\ldots,i_k,i'_1,\ldots,i'_k\}$ and edges $E(\mathbf{i},\mathbf{i}') = \{(i_p,i_{p+1})_{1\le p\le k},(i'_p,i'_{p+1})_{1\le p\le k}\}$. For the indices $(\mathbf{i},\mathbf{i}')$ to contribute to the leading order of the sum, $G(\mathbf{i},\mathbf{i}')$ must be connected. Indeed, if $E(\mathbf{i}) \cap E(\mathbf{i}') = \emptyset$, $P(\mathbf{i},\mathbf{i}') = P(\mathbf{i})P(\mathbf{i}')$. Moreover, as before, each edge must appear at least twice to give a non-zero contribution so that $|\tilde{E}(\mathbf{i},\mathbf{i}')| \le k$. Therefore, we are in the same situation as before, and if $\tilde{G}(\mathbf{i},\mathbf{i}') = (\tilde{V}(\mathbf{i},\mathbf{i}'),\tilde{E}(\mathbf{i},\mathbf{i}'))$ denotes the skeleton of $G(\mathbf{i},\mathbf{i}')$, we have the relation

$$|\tilde{V}(\mathbf{i},\mathbf{i}')| \le |\tilde{E}(\mathbf{i},\mathbf{i}')| + 1 \le k+1. \tag{1.9}$$

This already shows that the variance is at most of order N^{-1} (since $P(\mathbf{i},\mathbf{i}') - P(\mathbf{i})P(\mathbf{i}')$ is bounded by $2B_{2k}$ uniformly), but we need a slightly better bound to prove the almost sure convergence. To improve our bound let us show that the case where $|\tilde{V}(\mathbf{i},\mathbf{i}')| = |\tilde{E}(\mathbf{i},\mathbf{i}')| + 1 = k+1$ cannot occur. In this case, we have seen that $\tilde{G}(\mathbf{i},\mathbf{i}')$ must be a tree since then equality holds in (1.9). Also, $|\tilde{E}(\mathbf{i},\mathbf{i}')| = k$ implies that each edge appears with multiplicity exactly equals to 2. For any contributing set of indices $\mathbf{i},\mathbf{i}'$, $\tilde{G}(\mathbf{i},\mathbf{i}') \cap G(\mathbf{i})$ and $\tilde{G}(\mathbf{i},\mathbf{i}') \cap G(\mathbf{i}')$ must share at least one edge (i.e., one edge must appear with multiplicity one in each of this subgraph) since otherwise $P(\mathbf{i},\mathbf{i}') = P(\mathbf{i})P(\mathbf{i}')$. This is a contradiction. Indeed, if we equip $\tilde{G}(\mathbf{i},\mathbf{i}')$ with the orientation of the indices from $\mathbf{i}$ and the root (i_1,i_2), we may define the walk on $\tilde{G}(\mathbf{i},\mathbf{i}') \cap G(\mathbf{i})$ as in Figure 1.2 (it is simply the path $i_1 \to i_2 \cdots \to i_k \to i_1$). Since this walk comes back to i_1, either it visits each edge twice, which is impossible if $\tilde{G}(\mathbf{i},\mathbf{i}') \cap G(\mathbf{i})$ and $\tilde{G}(\mathbf{i},\mathbf{i}') \cap G(\mathbf{i}')$ share one edge (and all edges have multiplicity two), or it has a loop, which is also impossible since $\tilde{G}(\mathbf{i},\mathbf{i}')$ is a tree. Therefore, we conclude that for all contributing indices,

$$|\tilde{V}(\mathbf{i},\mathbf{i}')| \le k$$

which implies

$$\mathrm{Var}((\mathbf{A}^N)^k) \le 2B_k N^{-2}.$$

Applying Chebychev's inequality gives for any $\delta > 0$

$$\mathbb{P}\left(\left|\frac{1}{N}\mathrm{Tr}\left((\mathbf{A}^N)^k\right) - \mathbb{E}\left[\frac{1}{N}\mathrm{Tr}\left((\mathbf{A}^N)^k\right)\right]\right| > \delta\right) \le \frac{2B_k}{\delta^2 N^2},$$

and so the Borel–Cantelli lemma implies

$$\lim_{N\to\infty}\left|\frac{1}{N}\mathrm{Tr}\left((\mathbf{A}^N)^k\right) - \mathbb{E}\left[\frac{1}{N}\mathrm{Tr}\left((\mathbf{A}^N)^k\right)\right]\right| = 0 \quad a.s.$$

The proof of the theorem is complete.

□

Exercise 1.14. *Take for $L \in \mathbb{N}$, $\mathbf{A}^{N,L}$ the $N \times N$ self-adjoint matrix such that $\mathbf{A}^{N,L}_{ij} = (2L)^{-\frac{1}{2}} 1_{|i-j|\le L} A_{ij}$ with $(A_{ij}, 1 \le i \le j \le N)$ independent centered random variables having all moments finite and $E[A_{ij}^2] = 1$. The purpose of this exercise is to show that for all $k \in \mathbb{N}$,*

$$\lim_{L\to\infty}\lim_{N\to\infty}\mathbb{E}\left[\frac{1}{N}\mathrm{Tr}((\mathbf{A}^{N,L})^k)\right] = C_{k/2}$$

with C_x null if x is not integer. Hint: Show that for $k \ge 2$

$$\mathbb{E}\left[\frac{1}{N}\mathrm{Tr}((\mathbf{A}^{N,L})^k)\right] = (2L)^{-k/2} \sum_{\substack{|i_2-L|\le L,\\ |i_{p+1}-i_p|\le L, p\ge 2}} \mathbb{E}[A_{Li_2}\cdots A_{i_kL}] + o(1).$$

Then prove that the contributing indices to the above sum correspond to the case where $G(L, i_2, \cdot, i_k)$ is a tree with $k/2$ vertices and show that being given a tree there are approximately $(2L)^{\frac{k}{2}}$ possible choices of indices $i_2, \ldots, i_k$.

1.3 Weak convergence of the spectral measure

Let $(\lambda_i)_{1\le i\le N}$ be the N (real) eigenvalues of $\mathbf{A}^N$ and define

$$L_{\mathbf{A}^N} := \frac{1}{N}\sum_{i=1}^N \delta_{\lambda_i}$$

to be the *spectral measure* of $\mathbf{A}^N$. $L_{\mathbf{A}^N}$ belongs to the set $\mathcal{P}(\mathbb{R})$ of probability measures on $\mathbb{R}$. We claim the following:

Theorem 1.15. *Assume that (1.7) holds for all $k \in \mathbb{N}$. Then, for any bounded continuous function f,*

$$\lim_{N\to\infty}\int f(x)dL_{\mathbf{A}^N}(x) = \int f(x)d\sigma(x) \quad a.s.$$

Proof. . Let $B > 2$ and $\delta > 0$ be fixed. By Weierstrass' theorem, we can find a polynomial P_δ such that

$$\sup_{|x|\le B} |f(x) - P_\delta(x)| \le \delta.$$

Then

$$\left|\int f(x)d(L_{\mathbf{A}^N}(x) - \sigma(x))\right| \le \left|\int P_\delta(x)d(L_{\mathbf{A}^N}(x) - \sigma(x))\right| \\ +\delta + \left|\int_{|x|\ge B} (f - P_\delta)(x)dL_{\mathbf{A}^N}(x)\right| \quad (1.10)$$

where we used that $1_{|x|\ge B}d\sigma(x) = 0$ since $B > 2$. By Theorem 1.13,

$$\lim_{N\to\infty} |\int P_\delta(x)d(L_{\mathbf{A}^N}(x) - \sigma(x))| = 0 \quad \text{a.s.} \quad (1.11)$$

Moreover, using that f is bounded, if p denotes the degree of P_δ, we can find a finite constant $C = C(\delta, B)$ so that

$$|\int_{|x|\ge B} (f - P_\delta)(x)dL_{\mathbf{A}^N}(x)| \le C\int_{|x|\ge B} (1 + |x|^p)dL_{\mathbf{A}^N}(x) \\ \le 2CB^{-p-2q}\int x^{2(p+q)}dL_{\mathbf{A}^N}(x)$$

where we finally used Chebychev's inequality with some $q \ge 0$. Using again Theorem 1.13, we find that

$$\limsup_{N\to\infty} |\int_{|x|\ge B} (f - P_\delta)(x)dL_{\mathbf{A}^N}(x)| \le 2CB^{-p-2q}\int x^{2(p+q)}d\sigma(x) \\ \le CB^{-p-2q}2^{2(p+q+1)} \quad \text{a.s.}$$

We let q go to infinity to conclude, since $B > 2$, that

$$\limsup_{N\to\infty} |\int_{|x|\ge B} (f - P_\delta)(x)dL_{\mathbf{A}^N}(x)| = 0 \quad \text{a.s.}$$

Finally, let δ go to zero to conclude from (1.10) and (1.11) that

$$\limsup_{N\to\infty} |\int f(x)d(L_{\mathbf{A}^N}(x) - \sigma(x))| = 0 \quad \text{a.s.}$$

$\square$

1.4 Relaxation of the hypotheses over the entries –universality

In this section, we relax the assumptions on the moments of the entries while keeping the hypothesis that $(A_{ij}^N)_{1\le i\le j\le N}$ are independent. Generalizations of Wigner's theorem to possibly mildly dependent entries can be found for instance in [45].

1.4.1 Relaxation over the number of finite moments

A nice, simple, but finally optimal way to relax the assumption that the entries of $\sqrt{N}\mathbf{A}^N$ possess all their moments, relies on the following observation.

Lemma 1.16. *Let A, B be $N\times N$ Hermitian matrices, with eigenvalues $\lambda_1(A)\ge\lambda_2(A)\ge\cdots\ge\lambda_N(A)$ and $\lambda_1(B)\ge\lambda_2(B)\ge\cdots\ge\lambda_N(B)$. Then,*

$$\sum_{i=1}^N |\lambda_i(A)-\lambda_i(B)|^2 \le \mathrm{Tr}(A-B)^2\,.$$

Proof. Since $\mathrm{Tr}A^2=\sum_i(\lambda_i(A))^2$ and $\mathrm{Tr}B^2=\sum_i(\lambda_i(B))^2$, the lemma amounts to showing that

$$\mathrm{Tr}(AB)\le\sum_{i=1}^N\lambda_i(A)\lambda_i(B)$$

for all A,B as above, or equivalently, since if $A=U\mathrm{diag}(\lambda_1(A),\ldots,\lambda_N(A))U^*$ with a unitary matrix U,

$$\mathrm{Tr}(AB)=\sum_{i,j=1}^N\lambda_k(A)\lambda_j(B)|U_{ij}|^2,$$

that

$$\sum_{i=1}^N\lambda_i(A)\lambda_i(B)=\sup_{v_{ij}\ge0:\sum_j v_{ij}=1,\sum_i v_{ij}=1}\sum_{i,j}\lambda_i(A)\lambda_j(B)v_{ij}\,. \tag{1.12}$$

An elementary proof can be given (see [6]) by showing by induction over N that the optimizing matrix v above has to be the identity matrix. Indeed, this is true for $N=1$, and one proceeds by induction: if $v_{11}=1$ then the problem is reduced to $N-1$, while if $v_{11}<1$, there exists a j and a k with $v_{1j}>0$ and $v_{k1}>0$. Set $v=\min(v_{1j},v_{k1})>0$ and define $\bar v_{11}=v_{11}+v$, $\bar v_{kj}=v_{kj}+v$ and $\bar v_{1j}=v_{1j}-v$, $\bar v_{k1}=v_{k1}-v$, and $\bar v_{ab}=v_{ab}$ for all other pairs ab. Then,

$$\begin{aligned}
&\sum_{i,j}\lambda_i(A)\lambda_j(B)(\bar{v}_{ij}-v_{ij})\\
&\quad= v(\lambda_1(A)\lambda_1(B)+\lambda_k(A)\lambda_j(B)-\lambda_k(A)\lambda_1(B)-\lambda_1(A)\lambda_j(B))\\
&\quad= v(\lambda_1(A)-\lambda_k(A))(\lambda_1(B)-\lambda_j(B))\geq 0\,.
\end{aligned}$$

Thus, $\bar{V}=\{\bar{v}_{ij}\}$ satisfies the constraints, is also a maximum, and the number of zero elements in the first row and column of $\bar{V}$ is larger by 1 at least from the corresponding one for V. If $\bar{v}_{11}=1$, the conclusion follows by the induction hypothesis, while if $\bar{v}_{11}<1$, one repeats this (at most $2N-2$ times since the operation sends one entry to zero in the first column or the first line) to conclude. □

Corollary 1.17. *Assume that the entries $\{\sqrt{N}A^N_{ij}, i\leq j\}$ are independent and are either equidistributed with finite variance or such that*

$$\sup_{N\in\mathbb{N}}\sup_{1\leq i,j\leq N}\mathbb{E}[|\sqrt{N}A^N_{ij}|^4]<\infty. \tag{1.13}$$

Assume also that $\{\sqrt{N}A^N_{ij}, i\leq j\}$ are centered and for all

$$\lim_{N\to\infty}\max_{1\leq i\leq j\leq N}|\mathbb{E}[(\sqrt{N}A^N_{ij})^2]-1|=0.$$

Then, for any bounded continuous function f

$$\lim_{N\to\infty}\int f(x)dL_{\mathbf{A}^N}(x)=\int f(x)d\sigma(x)\quad a.s.$$

Remark. When the entries are not equidistributed, the convergence in probability can be proved when $(\sqrt{N}A^N_{ij})_{1\leq i\leq j\leq N}$ are uniformly integrable. We strengthen here the hypotheses to have the almost sure convergence of the law of large numbers theorem.

Proof. Fix a constant C and consider the matrix $\hat{A}_N$ whose elements satisfy, for $i\leq j$ and $i=1,\ldots,N$,

$$\hat{A}^N_{ij}=\frac{1}{\sigma^N_{ij}(C)}\left(A^N_{ij}\mathbf{1}_{\sqrt{N}|A^N_{ij}|\leq C}-E(A^N_{ij}\mathbf{1}_{\sqrt{N}|A^N_{ij}|\leq C})\right)$$

with

$$\sigma^N_{ij}(C)^2:=\mathbb{E}\left[\left(A^N_{ij}\mathbf{1}_{\sqrt{N}|A^N_{ij}|\leq C}-E(A^N_{ij}\mathbf{1}_{\sqrt{N}|A^N_{ij}|\leq C})\right)^2\right].$$

$\hat{\mathbf{A}}^N$ satisfies the hypothesis of Theorem 1.15 for any $C\in\mathbb{R}^+$, so that

$$\lim_{N\to\infty}\int f(x)dL_{\hat{\mathbf{A}}^N}(x)=\int f(x)d\sigma(x)\text{ a.s.} \tag{1.14}$$

Assume now that f is bounded Lipschitz, with Lipschitz constant

$$\|f\|_{\mathcal{L}} = \sup_{x \neq y} \frac{|f(x) - f(y)|}{|x - y|} + \sup_x |f(x)|.$$

Then,

$$\begin{aligned}
&\left| \int f(x) dL_{\hat{\mathbf{A}}^N}(x) - \int f(x) dL_{\mathbf{A}^N}(x) \right| \\
&\qquad \leq \frac{\|f\|_{\mathcal{L}}}{N} \sum_{i=1}^{N} |\lambda_i(\mathbf{A}^N) - \lambda_i(\hat{\mathbf{A}}^N)| \\
&\qquad \leq \|f\|_{\mathcal{L}} \left(\frac{1}{N} \sum_{i=1}^{N} |\lambda_i(\mathbf{A}^N) - \lambda_i(\hat{\mathbf{A}}^N)|^2 \right)^{\frac{1}{2}}
\end{aligned}$$

regardless of the order on the eigenvalues. We conclude that

$$\left| \int f(x) dL_{\hat{\mathbf{A}}^N}(x) - \int f(x) dL_{\mathbf{A}^N}(x) \right| \leq \|f\|_{\mathcal{L}} \left(\frac{1}{N} \mathrm{Tr}(\mathbf{A}^N - \hat{\mathbf{A}}^N)^2 \right)^{\frac{1}{2}}$$

with

$$(\mathbf{A}^N - \hat{\mathbf{A}}^N)_{ij} = \frac{1}{\sigma_{ij}^N(C)} \left(A_{ij}^N 1_{\sqrt{N} A_{ij}^N \geq C} - \mathbb{E}[A_{ij}^N 1_{\sqrt{N} A_{ij}^N \geq C}] \right) + (1 - \sigma_{ij}^N(C)) \hat{A}_{ij}^N \tag{1.15}$$

where we used that $\mathbb{E}[A_{ij}^N] = 0$ for all i, j. Under the assumption (1.13) or when $\{\sqrt{N} A_{ij}^N, i \leq j\}$ are independent and equidistributed and with finite variance, we can use the strong law of large numbers to get that

$$\limsup_{N \to \infty} \frac{1}{N} \sum_{i,j=1}^{N} |(A^N - \hat{A}^N)_{ij}|^2 \leq \limsup_{N \to \infty} \max_{1 \leq i \leq j \leq N} \mathbb{E}[|\sqrt{N}(A^N - \hat{A}^N)_{ij}|^2] \quad \text{a.s.} \tag{1.16}$$

Thus, by Lemma 1.16,

$$\begin{aligned}
&\limsup_{N \to \infty} \left| \int f(x) dL_{\hat{\mathbf{A}}^N}(x) - \int f(x) dL_{\mathbf{A}^N}(x) \right| \\
&\qquad \leq \|f\|_{\mathcal{L}} \limsup_{N \to \infty} \max_{1 \leq i \leq j \leq N} \mathbb{E}[((A^N - \hat{A}^N)_{ij})^2] \quad \text{a.s.}
\end{aligned}$$

Letting C go to infinity shows that the above right-hand side goes to zero (by (1.15) and since $(\sqrt{N} A_{ij}^N)_{i \leq j}$ is uniformly integrable under our assumptions) and therefore

$$\limsup_{C \to \infty} \limsup_{N \to \infty} \left| \int f(x) dL_{\hat{\mathbf{A}}^N}(x) - \int f(x) dL_{\mathbf{A}^N}(x) \right| = 0 \quad \text{a.s.}$$

We conclude with (1.14) that for all Lipschitz functions f,

$$\lim_{N\to\infty} \int f(x) dL_{\mathbf{A}^N}(x) = \int f(x) d\sigma(x) \quad \text{a.s.}$$

Now, taking any non negative Lipschitz function that vanishes on $[-2, 2]$ and equals one on $[-3, 3]^c$, we deduce that

$$\lim_{N\to\infty} L_{\mathbf{A}^N}([-3, 3]^c) = 0 \quad \text{a.s.}$$

Since by the Weierstrass theorem, Lipschitz functions are dense in the set of continuous functions on the compact set $[-3, 3]$, we can approximate any bounded continuous function f on $[-3, 3]$ by a sequence of Lipschitz functions f_δ up to an error δ (for the supremum norm on $[-3, 3]$). We choose f_δ with uniform norm bounded by that of f on the whole real line. We now conclude that for any bounded continuous function f,

$$\begin{aligned}
\limsup_{N\to\infty} & \left| \int f(x) dL_{\mathbf{A}^N}(x) - \int f(x) d\sigma(x) \right| \\
& \le 2\|f\|_\infty \limsup_{N\to\infty} \left(L_{\mathbf{A}^N}([-3, 3]^c) + \sigma([-3, 3]^c) \right) \\
& \qquad + \limsup_{N\to\infty} \left| \int f_\delta(x) dL_{\mathbf{A}^N} - \int f_\delta(x) d\sigma(x) \right| + \delta \\
& = \delta.
\end{aligned}$$

Letting δ go to zero finishes the proof. $\square$

Remark. Let us remark that if $\sqrt{N} A^N(ij)$ has no moments of order 2, the theorem is no longer valid (see the heuristics of Cizeau and Bouchaud [64] and rigorous studies in [208] and [26]). Even though under appropriate assumptions the spectral measure of the matrix A^N, once properly normalized, converges, its limit is not the semicircle law but a heavy-tailed law with unbounded support.

1.4.2 Relaxation of the hypothesis on the centering of the entries

The last generalization concerns the hypothesis on the mean of the variables $\sqrt{N} A^N_{ij}$ which, as we shall see, is irrelevant in the statement of Corollary 1.17. More precisely, we shall prove the following lemma (taken from [109]).

Lemma 1.18. *Let $\mathbf{A}^N, \mathbf{B}^N$ be $N \times N$ Hermitian matrices for $N \in \mathbb{N}$ such that $\mathbf{B}^N$ has rank $r(N)$. Assume that $N^{-1} r(N)$ converges to zero as N goes to infinity. Then, for any bounded continuous function f with compact support,*

$$\limsup_{N\to\infty} \left| \int f(x) dL_{\mathbf{A}^N + \mathbf{B}^N}(x) - \int f(x) dL_{\mathbf{A}^N}(x) \right| = 0.$$

If moreover $(L_{\mathbf{A}^N}, N \in \mathbb{N})$ is tight in $\mathcal{P}(\mathbb{R})$, equipped with its weak topology, the above holds for any bounded continuous function.

Proof. We first prove the statement for bounded increasing functions. To this end, we shall first prove that for any Hermitian matrix $\mathbf{Z}^N$, any $e \in \mathbb{C}^N$, $\lambda \in \mathbb{R}$, and for any bounded measurable increasing function f,

$$\left| \int f(x) dL_{\mathbf{Z}^N}(x) - \int f(x) dL_{\mathbf{Z}^N + \lambda e e^*}(x) \right| \le \frac{2}{N} \|f\|_\infty. \tag{1.17}$$

We denote by $\lambda_1^N \le \lambda_2^N \cdots \le \lambda_N^N$ (resp. $\eta_1^N \le \eta_2^N \cdots \le \eta_N^N$) the eigenvalues of $\mathbf{Z}^N$ (resp. $\mathbf{Z}^N + \lambda e e^*$). By Lidskii's theorem 19.3, the eigenvalues λ_i and η_i are interlaced;

$$\lambda_1^N \le \eta_2^N \le \lambda_3^N \cdots \le \lambda_{2[\frac{N-1}{2}]+1}^N \le \eta_{2[\frac{N}{2}]}^N,$$

$$\eta_1^N \le \lambda_2^N \le \eta_3^N \cdots \le \eta_{2[\frac{N-1}{2}]+1}^N \le \lambda_{2[\frac{N}{2}]}^N.$$

Therefore, if f is an increasing function,

$$\sum_{i=1}^N f(\lambda_i^N) \le \sum_{i=2}^N f(\eta_i^N) + \frac{1}{N}\|f\|_\infty \le \sum_{i=1}^N f(\eta_i^N) + \frac{2}{N}\|f\|_\infty$$

but also

$$\begin{aligned} \sum_{i=1}^N f(\lambda_i^N) = f(\lambda_1^N) + \sum_{i=2}^N f(\lambda_i^N) &\ge f(\lambda_1^N) + \sum_{i=2}^N f(\eta_{i-1}^N) \\ &= f(\lambda_1^N) - f(\eta_i^N) + \sum_{i=1}^N f(\eta_i^N). \end{aligned}$$

These two bounds prove (1.17).

Now, let us denote by $(e_1^N, \cdots, e_{r(N)}^N)$ an orthonormal basis of the vector space of eigenvectors of $\mathbf{B}^N$ with non-zero eigenvalues so that

$$\mathbf{B}^N = \sum_{i=1}^{r(N)} \eta_i^N e_i^N (e_i^N)^*$$

with some real numbers $(\eta_i^N)_{1 \le i \le r(N)}$. Iterating (1.17) shows that for any bounded increasing function f,

$$\left| \int f(x) dL_{\mathbf{A}^N}(x) - \int f(x) dL_{\mathbf{A}^N + \mathbf{B}^N}(x) \right| \le \frac{2r(N)}{N} \|f\|_\infty. \tag{1.18}$$

Therefore, for any increasing bounded continuous function, when $N^{-1}r(N)$ goes to zero,

$$\limsup_{N \to \infty} \left| \int f(x) dL_{\mathbf{A}^N + \mathbf{B}^N}(x) - \int f(x) dL_{\mathbf{A}^N}(x) \right| = 0. \tag{1.19}$$

Of course, the result immediately extends to decreasing functions by $f \to -f$. Now, note that any Lipschitz function f that vanishes outside of a compact set $K = [-k, k]$ can be written as the difference of two bounded continuous functions (this is in fact true as soon as f has bounded variations) since it is almost surely (with respect to Lebesgue measure) differentiable with derivative bounded by $|f|_{\mathcal{L}}$ and

$$f(x) - f(0) = \int_0^x f'(x) 1_{f'(x)\geq 0} dx - \int_0^x (-f'(x)) 1_{f'(x)<0} dx.$$

Hence, (1.19) extends to the case of compactly supported Lipschitz functions, and then to any bounded compactly supported continuous functions (by density for the supremum norm).

To remove the assumption that f is compactly supported we assume $(L_{\mathbf{A}^N})_{N\in\mathbb{N}}$ tight so that $\sup_N L_{\mathbf{A}^N}([-k,k]^c)$ goes to zero as k goes to infinity. Now, taking $f(x) = (x-k) \wedge 1 \vee 0$ for some finite k, we deduce that

$$\begin{aligned}\limsup_{N\to\infty} L_{\mathbf{A}^N+\mathbf{B}^N}([k+1,\infty[) &\leq \limsup_{N\to\infty} \int (x-k)\wedge 1 \vee 0 \, dL_{\mathbf{A}^N+\mathbf{B}^N}(x) \\ &= \limsup_{N\to\infty} \int (x-k)\wedge 1 \vee 0 \, dL_{\mathbf{A}^N}(x) \\ &\leq \limsup_{N\to\infty} L_{\mathbf{A}^N}([k,\infty[) \leq \varepsilon_k\end{aligned}$$

where ε_k is a sequence going to zero with k, which exists by the assumption that $(L_{\mathbf{A}^N}, N \in \mathbb{N})$ is tight. We apply the same argument for $L_{\mathbf{A}^N+\mathbf{B}^N}(]-\infty,-k-1])$ with the decreasing function $f(x) = (-k-x) \wedge 1 \vee 0$ and deduce that

$$\limsup_{k\to\infty} \limsup_{N\to\infty} L_{\mathbf{A}^N+\mathbf{B}^N}([-k,k]^c) = 0.$$

This allows us to finish the proof of the lemma for any bounded continuous function f since we also have $\limsup_{k\to\infty} \limsup_{N\to\infty} L_{\mathbf{A}^N}([-k,k]^c) = 0$. □

By Corollary 1.17 and Lemma 1.18, we find the following:

Corollary 1.19. *Assume that the matrix* $\left(\mathbb{E}[A^N_{ij}]\right)_{1\leq i,j\leq N}$ *has rank* $r(N)$ *so that* $N^{-1}r(N)$ *goes to zero as* N *goes to infinity, and that the variables* $\sqrt{N}(A^N_{ij} - \mathbb{E}[A^N_{ij}])$ *satisfy* (1.13) *and have variance* 1*. Then, for any bounded continuous function* f*,*

$$\lim_{N\to\infty} \int f(x) dL_{\mathbf{A}^N}(x) = \int f(x) d\sigma(x) \quad a.s.$$

This result holds in particular if $\mathbb{E}[A^N_{ij}] = x^N$ *is independent of* $i,j \in \{1,\ldots,N\}^2$*, in that case* $r(N) = 1$*.*

Bibliographical notes. Since the convergence of the spectral measure was proved by Wigner [205] when the entries possess moments of all orders, many papers have improved this result. The optimal hypothesis for the convergence of the spectral measure of Wigner matrices to the smicircle law is that the entries have a finite second moment, since if they do not, the asymptotics of the spectral measure described in [26] show that the renormalization of the eigenvalues must depend on the tail of the entries and the limit is a heavy tailed law rather than the semicircle law. More precise results have been derived; for instance, when the entries have only a finite fourth moment, Bai [11] proved the convergence of the spectral measure and showed that some distance to the limit is at most of order $N^{-\frac{1}{4}}$ (this result was improved to N^{-1} under stronger hypotheses in [96]). Bai used a method directly based on estimations of the Cauchy–Stieljes transform of the spectral measure, rather than on moments. The convergence of the spectral measure of diverse classical ensembles of matrices were shown; for instance for Wishart matrices [145], for Wigner matrices with correlated entries [45], Toeplitz matrices [53, 112], or for non symmetric matrices (with a complex spectrum) such as Ginibre ensemble [95, 12]. We refer the reader to [13] for more examples.

2

Wigner's matrices; more moments estimates

In this chapter, we elaborate upon the previous computation of moments in two directions. First we give a better estimate of the error to the previous limit and prove a central limit theorem. Second, we consider the case were moments are taken at powers that blow up with the dimension of the matrices; we basically show that if this power is small compared to the square root of the dimension, the first-order contribution is still given, in the moment expansion, by graphs that are trees.

2.1 Central limit theorem

In the previous section, we proved Wigner's theorem by evaluating $\int x^p dL_{\mathbf{A}^N}(x)$ for $p \in \mathbb{N}$. We shall push this computation one step further here and prove a central limit theorem. Namely, setting

$$\int x^k d\bar{L}_{\mathbf{A}^N}(x) := \mathbb{E}\left[\int x^k dL_{\mathbf{A}^N}(x)\right],$$

we shall prove that

$$M_k^N := N\left(\int x^k dL_{\mathbf{A}^N}(x) - \int x^k d\bar{L}_{\mathbf{A}^N}(x)\right) = \sum_{i=1}^N \left(\lambda_i^k - \mathbb{E}[\lambda_i^k]\right)$$

converges in law to a centered Gaussian variable. Since in Chapter III we shall give a complete and detailed proof of the central limit theorem in the case of Gaussian entries with a weak interaction, we will be rather sketchy here. We refer to [7] for a complete and clear treatment and [6] for a simplified exposition of the full proof of the theorem we state below. To simplify, we assume here that $\mathbf{A}^N$ is a Wigner matrix with

$$A_{ij}^N = \frac{B_{ij}}{\sqrt{N}},$$

where $(B_{ij}, 1 \le i \le j \le N)$ are independent real equidistributed random variables. Their marginal distribution μ has all moments finite (in particular (1.7) is satisfied) and satisfies

$$\int x d\mu(x) = 0 \text{ and } \int x^2 d\mu(x) = 1.$$

We shall show why the following statement holds.

Theorem 2.1. *Let*

$$\sigma_k^2 = k^2 [C_{\frac{k-1}{2}}]^2 + \frac{k^2}{2} [C_{\frac{k}{2}}]^2 \left[\int x^4 d\mu(x) - 1 \right] + \sum_{r=3}^{\infty} \frac{2k^2}{r} \left(\sum_{\substack{k_i \ge 0 \\ 2\sum_{i=1}^r k_i = k-r}} \prod_{i=1}^r C_{k_i} \right)^2,$$

In this formula, C_x equals zero if x is not an integer and otherwise is equal to the Catalan number.

Then, M_k^N converges in moments to the centered Gaussian variable with variance σ_k^2, i.e., for all $l \in \mathbb{N}$,

$$\lim_{N\to\infty} \mathbb{E}\left[(M_k^N)^l\right] = \frac{1}{\sqrt{2\pi}\sigma_k} \int x^l e^{-\frac{x^2}{2\sigma_k^2}} dx.$$

Remark. Unlike the standard central limit theorem for independent variables, the variance here depends on $\mu(x^4)$.

Outline of the proof.

- *We first prove that the statement is true when $l = 2$.* (It is clearly true for $k = 1$ since A_k^N is centered.) We thus want to show

$$\sigma_k^2 = \lim_{N\to\infty} \mathbb{E}\left[(M_k^N)^2\right]. \tag{2.1}$$

Below (1.9), we proved that $\mathbb{E}\left[(A_k^N)^2\right]$ is bounded, uniformly in N. Furthermore, we can write

$$\mathbb{E}\left[(M_k^N)^2\right] = \frac{1}{N^k} \sum_{\mathbf{i},\mathbf{i}'} [P(\mathbf{i},\mathbf{i}') - P(\mathbf{i})P(\mathbf{i}')]$$

where the sum over $\mathbf{i}, \mathbf{i}'$ will hold on graphs $\tilde{G}(\mathbf{i},\mathbf{i}') = (\tilde{V}(\mathbf{i},\mathbf{i}'), \tilde{E}(\mathbf{i},\mathbf{i}'))$ so that

$$|\tilde{V}(\mathbf{i},\mathbf{i}')| \le k, \quad |\tilde{E}(\mathbf{i},\mathbf{i}')| \le k.$$

Since $[P(\mathbf{i},\mathbf{i}') - P(\mathbf{i})P(\mathbf{i}')]$ is uniformly bounded, the only contributing graphs to the leading order will be those such that $|\tilde{V}(\mathbf{i},\mathbf{i}')| = k$. Then, since we always have $|\tilde{V}(\mathbf{i},\mathbf{i}')| \le |\tilde{E}(\mathbf{i},\mathbf{i}')| + 1$, we have two cases:

- $|\tilde{E}(\mathbf{i},\mathbf{i}')| = k - 1$ in that case the skeleton $\tilde{G}(\mathbf{i},\mathbf{i}')$ will again be a tree but with one edge less than the total number possible; this means that one edge appears with multiplicity four and belongs to $\tilde{E}(\mathbf{i}) \cap \tilde{E}(\mathbf{i}')$, the other edges appearing with multiplicity 2. Hence, the graphs of $\tilde{E}(\mathbf{i})$ and $\tilde{E}(\mathbf{i}')$ are both trees (so that k must be even); there are $C_{\frac{k}{2}}^2$ such trees, and they are glued by a common edge, to choose among $\frac{k}{2}$ edges in each of the tree. Finally, there are two possible choices to glue the two trees according to the orientation. Thus, there are

$$2\left(\frac{k}{2}\right)^2 C_{\frac{k}{2}}^2 = \left(\frac{k^2}{2}\right) C_{\frac{k}{2}}^2$$

such graphs and then

$$P(\mathbf{i},\mathbf{i}') - P(\mathbf{i})P(\mathbf{i}') = \int x^4 d\mu(x) - 1.$$

We hence obtain the contribution $(\frac{k^2}{2})C_{\frac{k}{2}}^2(\int x^4 d\mu(x) - 1)$ to the variance.

• $|\tilde{E}(\mathbf{i},\mathbf{i}')| = k$. In this case, the graph is no longer a tree and because $|\tilde{E}(\mathbf{i},\mathbf{i}')| - |\tilde{V}(\mathbf{i},\mathbf{i}')| = 1$, it contains exactly one cycle. This can be seen either by closer inspection of the arguments given after (1.1) or by using the formula that relates the genus of a graph and its number of vertices, faces and edges:

$$\sharp\text{vertices} + \sharp\text{faces} - \sharp\text{edges} = 2 - 2g \leq 2.$$

The faces are defined by following the boundary of the graph; each of these boundaries are exactly one cycle of the graph except one (since a graph has always one boundary) and therefore

$$\sharp\text{faces} = 1 + \sharp\text{cycles}.$$

So we get, for a connected graph with skeleton $(\tilde{V}, \tilde{E})$,

$$|\tilde{V}| \leq |\tilde{E}| + 1 - \sharp\text{cycles}. \tag{2.2}$$

In our case, $\sharp$vertices $=$ $\sharp$edges $= k$ and $\sharp$cycles ≥ 1 (since the graph is not a tree), so that the number of cycles must be exactly one. Counting the number of such graphs completes the proof of the convergence of $\mathbb{E}\left[(M_k^N)^2\right]$ to σ_k^2 (see [7] for more details).

- *Convergence to the Gaussian law.*
We next show that M_k^N is asymptotically Gaussian. This amounts to proving that $\lim_{N\to\infty} \mathbb{E}[(M_k^N)^{2l+1}] = 0$ whereas,

$$\lim_{N\to\infty} \mathbb{E}[(M_k^N)^{2l}] = \sharp\{\text{number of pair partitions of } 2l \text{ elements}\} \times \sigma_k^{2l}.$$

Again, we shall expand the expectation in terms of graphs and write for $l \in \mathbb{N}$,

$$\mathbb{E}[(M_k^N)^l] = \frac{1}{N^{\frac{kl}{2}}} \sum_{\mathbf{i}_1,\dots,\mathbf{i}_l} P(\mathbf{i}^1,\dots,\mathbf{i}^l)$$

with $P(\mathbf{i}^1,\dots,\mathbf{i}^l)$ given by

$$\mathbb{E}\Bigg[\Big(B_{i_1^1 i_2^1} \cdots B_{i_k^1 i_1^1} - \mathbb{E}[B_{i_1^1 i_2^1} \cdots B_{i_k^1 i_1^1}] \Big) \\ \cdots \Big(B_{i_1^l i_2^l} \cdots B_{i_k^l i_1^l} - \mathbb{E}[B_{i_1^l i_2^l} \cdots B_{i_k^l i_1^l}] \Big) \Bigg].$$

We denote by $G(\mathbf{i}^1,\dots,\mathbf{i}^l) = (V(\mathbf{i}^1,\dots,\mathbf{i}^l), E(\mathbf{i}^1,\dots,\mathbf{i}^l))$ the corresponding graph; $V(\mathbf{i}^1,\dots,\mathbf{i}^l) = \{i_n^j, 1 \le j \le l, 1 \le n \le k\}$ and $E(\mathbf{i}^1,\dots,\mathbf{i}^l) = \{(i_n^j, i_{n+1}^j), 1 \le j \le l, 1 \le n \le k\}$ with the convention $i_{l+1}^j = i_1^j$. As before, $P(\mathbf{i}^1,\dots,\mathbf{i}^l)$ equals zero unless each edge appears with multiplicity 2 at least. Also, because of the centering, it vanishes if there exists a $j \in \{1,\dots,l\}$ so that $E(\mathbf{i}^1,\dots,\mathbf{i}^l) \cap E(\mathbf{i}^j)$ does not intersect $E(\mathbf{i}^1,\dots,\mathbf{i}^{j-1},\mathbf{i}^{j+1},\dots,\mathbf{i}^l)$. Let us decompose $G(\mathbf{i}^1,\dots,\mathbf{i}^l)$ into its connected components $(G_1,\dots,G_c)$. We claim that

$$|V(\mathbf{i}^1,\dots,\mathbf{i}^l)| \le c - l + \left[\frac{l(k+1)}{2}\right]. \tag{2.3}$$

This type of bound is rather intuitive; if a connected component G_i contains $G(\mathbf{i}^{j_1}),\dots,G(\mathbf{i}^{j_p})$, each gluing of the $G(\mathbf{i}^{j_l})$ should create either a cycle or an edge with multiplicity 4, the total number of vertices decreasing at least by one in each gluing. Hence, $|V(\mathbf{i}^1,\dots,\mathbf{i}^l)|$ should grow linearly with the number of connected components. The proof is given in Appendix 20.3 for completeness (see [6] or [7]). With (2.3), we conclude that the only indices that will contribute are such that

$$c - l + \left[\frac{l(k+1)}{2}\right] \ge \frac{kl}{2}$$

with $c \le [\frac{l}{2}]$. This implies that

$$\frac{kl}{2} \le \left[\frac{l}{2}\right] - l + \left[\frac{l(k+1)}{2}\right] \le \frac{l}{2} - l + \frac{l(k+1)}{2} = \frac{kl}{2}$$

resulting in all inequalities being equalities. Thus, to get a first-order contribution we must have l even and $c = \frac{l}{2}$. In that case, we write $(s_j, r_j)_{1\le j\le l}$ the pairing so that $(G(\mathbf{i}_{s_j}), G(\mathbf{i}_{r_j}))_{1\le j\le l}$ are connected for all $1 \le j \le l$ (with the convention $s_j < r_j$). By independence of the entries, we have

$$P(\mathbf{i}_1,\dots,\mathbf{i}_{2l}) = \prod_{j=1}^{l} P(\mathbf{i}_{s_j}, \mathbf{i}_{r_j})$$

and so we have proved that

$$N^{-kl} \sum_{\mathbf{i}_1,\dots,\mathbf{i}_{2l}} P(\mathbf{i}_1,\dots,\mathbf{i}_{2l}) = \sum_{\substack{s_1<\cdots<s_l \\ r_j>s_j}} \left(N^{-k} \sum_{\mathbf{i}_1,\mathbf{i}_2} P(\mathbf{i}_1,\mathbf{i}_2) \right)^l + o(1)$$
$$= \sigma_k^{2l} \sum_{\substack{s_1<\cdots<s_l \\ r_j>s_j}} 1 + o(1)$$

which proves the claim since

$$\frac{1}{\sqrt{2\pi}} \int x^{2l} e^{-\frac{x^2}{2}} dx = \sum_{\substack{s_1<\cdots<s_l \\ r_j>s_j}} 1 = (2l-1)(2l-3)(2l-5)\cdots 1.$$

This completes the proof of the moments convergence.

□

Exercise 2.2. *Show that Theorem 2.1 implies that M_k^N converges weakly to the centered Gaussian variable with variance σ_k^2. Hint: control tails to approximate bounded continuous functions by polynomials.*

Bibliographical notes. Johansson [120] proved a rather general central limit theorem for the spectral measure of Gaussian random matrices (and more generally for particles interacting via a Coulomb gas potential). It was generalized to β-ensembles and Laguerre ensembles in [82] by using tri-diagonal representation of the classical ensembles [81]. The strategy of moments developed here follows an article of Anderson and Zeitouni [7] (see a generalization in [177]). Central limit theorems were also obtained in the case of Ginibre ensembles (with spectral measure converging to the so-called circular law) in [169].

We shall see in Part III that this kind of theorem generalizes to the multi-matrix setting that we shall introduce in the next chapter.

2.2 Estimates of the largest eigenvalue of Wigner matrices

In this section, we derive estimates on the largest eigenvalue of a Wigner matrix with real entries $A_{ij}^N = N^{-\frac{1}{2}} B_{ij}$ with $(B_{ij}, 1 \le i \le j \le N)$ independent equidistributed centered random variables with marginal distribution P. The idea is to improve the moments estimates of the previous chapter.

We shall assume that P is a symmetric law (see the recent article [166] for a relaxation of this hypothesis):

$$P(-x \in .) = P(x \in .).$$

We take the normalization $E[x^2] = 1$. Further, we assume that P has sub-Gaussian tail, i.e., that there exists a finite constant c such that for all $k \in \mathbb{N}$,

$$E[x^{2k}] \le (ck)^k.$$

We follow the article of S. Sinaï and A. Soshnikov [179] to prove the following result:

Theorem 2.3 (S. Sinaï–A. Soshnikov [179]). *For all $\epsilon > 0$, all $N \in \mathbb{N}$, there exists a finite function $o(s, N)$ such that* $\lim_{N\to\infty} \sup_{N^\epsilon \le s \le N^{\frac{1}{2}-\epsilon}} o(s, N) = 0$ *and*

$$\mathbb{E}[\mathrm{Tr}((A^N)^{2s})] = \frac{N 2^{2s}}{\sqrt{\pi s^3}}(1 + o(s, N)). \tag{2.4}$$

As a consequence, for all $\epsilon > 0$, if we let $\lambda_{max}(A^N)$ denote the spectral radius of A^N,

$$\lim_{N\to\infty} P(|\lambda_{\max}(A^N) - 2| \ge \epsilon) = 0.$$

A previous result of the same nature (but under weaker hypothesis (the symmetry hypothesis of the distribution of the entries being removed) under which the moments estimate (2.4) holds for a smaller range of s) was proved by Komlós and Füredi [93]. A later result of Soshnikov [180] improves the range of s under which (2.4) holds to s of order less than $n^{\frac{2}{3}}$, a result that captures the fluctuations of $\lambda_{\max}(A^N)$. We emphasize here that the proof below heavily depends on the assumption that the distribution of the entries is symmetric.

Proof. Let us first derive the convergence in probability from the moment estimates. First, note that

$$P(\lambda_{\max}(A^N) \le 2 - \epsilon) \le P\left(\int f(x) dL_{A^N} = 0\right)$$

for all functions f supported on $]2-\epsilon, \infty[$. Taking f bounded continuous, null on $]-\infty, 2-\epsilon]$ and strictly positive in $[2-\frac{\epsilon}{2}, 2]$, we see that $P(\int f(x) dL_{A^N} = 0)$ goes to zero by Theorem 1.15. For the upper bound on $\lambda_{\max}(A^N)$, we shall use Chebychev's inequality and the moment estimates (2.4) as follows:

$$\begin{aligned} P(\lambda_{\max}(A^N) \ge 2 + \epsilon) &\le \frac{1}{(2+\epsilon)^{2s}} \mathbb{E}[\lambda_{\max}(A^N)^{2s}] \le \frac{1}{(2+\epsilon)^{2s}} \mathbb{E}[\mathrm{Tr}((A^N)^{2s})] \\ &\le \frac{N 2^{2s}}{(2+\epsilon)^{2s}\sqrt{\pi s^3}}(1 + o(s, N)) \end{aligned}$$

where the right-hand side goes to zero with N when $s = N^\epsilon$ for some $\epsilon > 0$.

To prove the moment estimates we shall again expand the moments and count contributing paths, in particular estimate more precisely contributions from paths that are not trees. Yet, the central point of the proof is to show that these paths give a negligible contribution. We follow the presentation of [179].

1. *Moments expansion.* As usual, we write

$$\mathbb{E}[\mathrm{Tr}\left((\mathbf{A}^N)^{2s}\right)] = \frac{1}{N^s} \sum_{i_0,\dots,i_{2s-1}=1}^{N} \mathbb{E}[B_{i_0 i_1} \cdots B_{i_{2s-1},i_0}]. \qquad (2.5)$$

   We let E denote the set of edges of the graph, i.e., the undirected collection of couples $\{(i_p, i_{p+1}), p = 0, \dots, 2s-1\}$. Because we assumed the law of the B_{ij}'s symmetric, only indices such that each edge in E appears an even number of times will contribute. We call a *closed path* the sequence $P : i_0 \to i_1 \to \cdots \to i_{2s-1} \to i_0$. An *even path* is a closed path where each edge appears with even multiplicity; they are the only contributing paths.
2. *Descriptions of paths.* We will say that the ℓth step $i_{\ell-1} \to i_\ell$ of a path P is *marked* if during the first ℓ steps of P, the edge $\{i_{\ell-1}, i_\ell\}$ appears an odd number of times (note here that the ℓth step is counted, and so a step is marked iff the edge $\{i_{\ell-1}, i_\ell\}$ appears an even number of times in the previous step, in particular if it does not appear). The step is *unmarked* otherwise. For even paths, the number of marked and unmarked edges is equal to s. The complete set of vertices $\mathcal{V}$ is the collection $\{1, \dots, N\}$ of all possible values of the points $(i_k, 0 \le k \le 2s-1)$. We say that a vertex $i \in \mathcal{V}$ belongs to the subset $\mathcal{N}_k = \mathcal{N}_k(P)$ if the number of times we arrive at i via marked edges equals k. Note that no vertex of the path except i_0 can belong to $\mathcal{N}_0$. Moreover, $\mathcal{N}_p = 0$ for $p > s$ (since there are at most s edges). Note that if we let $n_k = \sharp\mathcal{N}_k$, since $(\mathcal{N}_0, \dots, \mathcal{N}_s)$ is a partition of $\mathcal{V}$, $\sum_{k=0}^{s} n_k = N$. Moreover, $(\mathcal{N}_0, \dots, \mathcal{N}_s)$ also induces a partition of the edges and hence

$$\sum_{k=0}^{s} k n_k = s.$$

   We say that P is of type $(n_0, n_1, \dots, n_s)$ if $n_k = \sharp\mathcal{N}_k = \sharp\mathcal{N}_k(P)$ for all $k \in \{0, \dots, s\}$. We finally say that a path is a *simple even path* if $i_0 \in \mathcal{N}_0$ and P is of type $(N-s, s, 0, \dots, 0)$. Observe that in a simple even path, each edge appears only twice (since there are at most s different edges in P and here exactly s since there are s different vertices in $\mathcal{N}_1$). Also, we see that the graph corresponding to P has exactly s vertices in $\mathcal{N}_1$ plus $i_0 \in \mathcal{N}_0$ and so exactly $s+1$ vertices. Hence, the skeleton $(V, \tilde{E})$ of the graph drawn by P satisfies the relation $|V| = |\tilde{E}| + 1$ and hence is a tree. The strategy of the proof will be to show that simple even paths dominate the expectation when $s = o(\sqrt{N})$.
3. *Contribution of simple even paths.* Considering (2.5), we see that for simple even paths, $\mathbb{E}[B_{i_0 i_1} \cdots B_{i_{2s-1} i_0}] = 1$. Moreover, given a simple even path, we have N possible choices for i_0, $N-1$ for the first new vertex encountered when following P, $N-2$ for the second new vertex encountered, etc. Since we have $C_s = (2s)!/s!(s+1)!$ simple even paths (see Property 1.10), we get the contribution

$$C_1^N = \frac{1}{N^s} N(N-1)\cdots(N-s)\frac{(2s)!}{s!(s+1)!} = \frac{2^{2s}N}{\sqrt{\pi s^3}}(1+o_1(s,N))$$

where we have used Stirling's formula and found

$$o_1(s,N) = -\frac{1}{N}\sum_{k=1}^{s} k + \frac{1}{s} \approx \frac{s^2}{2N} + \frac{1}{s}.$$

In the case where $i_0 \notin \mathcal{N}_0$ but $n_1 = s, n_2 = 0 \cdots, n_s = 0$, we must have $i_0 \in \mathcal{N}_1$. This means that we have one cycle and one different vertex less in the graph of an even path. Note that if we split the vertex i_0 into two vertices as in Figure 2.1, the new vertex being attached to the marked edge, then the old i_0 belongs to $\mathcal{N}_0$ and the new vertex to $\mathcal{N}_1$ and we are back to the case where $i_0 \in \mathcal{N}_0$.

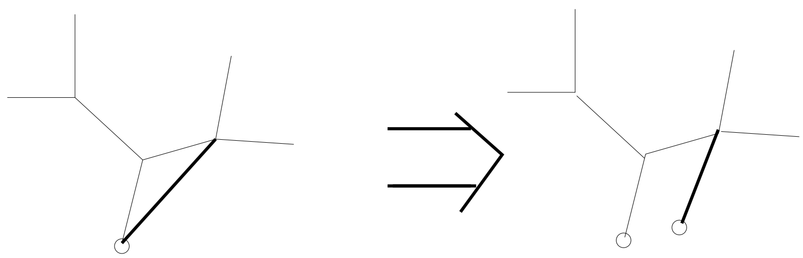


Fig. 2.1. Splitting of the graph

There are s possibilities for the position of the marked edge incoming in i_0, but we are losing $N-s$ possibilities to choose a different vertex. Hence, the contribution to this term is bounded by

$$C_2^N \leq \frac{s}{N-s} E[x^4] C_1^N$$

where the last term comes from the possibility that one edge attached to i_0 now has multiplicity 4.

4. *Contribution of paths that are not simple.* If a path is not as in the previous paragraph, there must be an $n_k \geq 1$ for $k \geq 2$. Let us count the number of these paths.
Given $n_0, n_1, \ldots, n_s$, we have $\frac{N!}{n_0! n_1! \cdots n_s!}$ ways to choose the values of the vertices. Then, among the n_0 vertices in $\mathcal{N}_0$, we have at most n_0 ways to choose the vertex corresponding to i_0 (if $i_0 \in \mathcal{N}_0$).
Being given the values of the vertices, a path is uniquely described if we know the order of appearance of the vertices at the marked steps, the times when the marked steps occur and the choice of end points of the unmarked steps. The moments of time when marked steps occur can be coded by

a Dick path by adding $+1$ when the step is marked and -1 otherwise. Hence, there are $C_s = (2s)!/s!(s+1)!$ choices for the times of marked steps. Once we are given this path, we have s marked steps. The marked steps are partitioned into s sets corresponding to the $\mathcal{N}_k$, $1 \le k \le s$, with cardinality $n_k k$ each. Hence, we have $\frac{s!}{\prod_{k=1}^{s}(n_k k)!}$ possibilities to assign the sets into which the end points of the marked steps are. Finally, we have $(n_k k)!/(k!)^{n_k}$ ways to partition the set $\mathcal{N}_k$ into k copies of the same point of $\mathcal{N}_k$. So far, we have prescribed uniquely the marked steps and the set to which they belong.

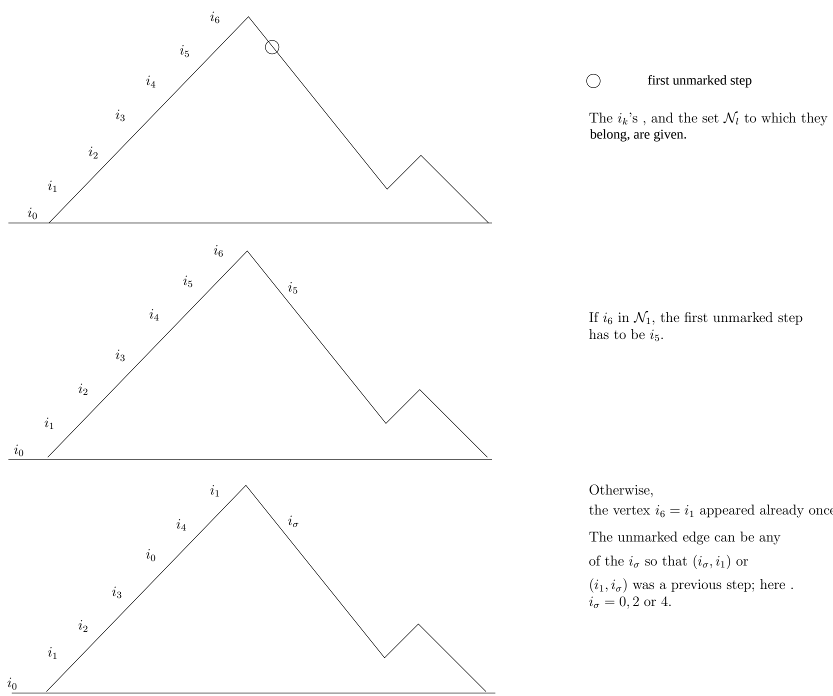


Fig. 2.2. Counting unmarked steps

To prescribe the unmarked steps, we still have an indeterminate. In fact, let us follow the Dick path of the marked steps till the first decreasing part corresponding to unmarked steps. Let i_ℓ be the vertex assigned to the last step. Then, if i_ℓ appeared only once in the past path (in the edge $(i_{\ell-1}, i_\ell)$), we have no choice and the next vertex in the path has to be $i_{\ell-1}$. This is the case in particular if $i_\ell \in \mathcal{N}_1$. If now $i_\ell \in \mathcal{N}_k$ for $k \ge 2$, the undirected step (i_p, i_ℓ) for some i_p may have occurred already at most $2k$ times (since it could occur either as a step (i_p, i_ℓ) or a step (i_ℓ, i_p), the later happening also less than k times since it requires that a marked

step arrived at i_ℓ before). We have thus at most $2k$ choices now for the next vertex; one of the i_p among the at most $2k$ vertices such that the step (i_p, i_ℓ) or (i_ℓ, i_p) were present in the past path. Once this choice has been made, we can proceed by induction since this choice comes with the prescription of the set $\mathcal{N}_l$ in which the vertex i_p belongs. Hence, since we have kn_k vertices in each set, we see that we have at most $\prod_{k=2}^s (2k)^{kn_k}$ choices for the end points of the unmarked steps.
Coming back to (2.5) we see that if the path is of type $(n_0, \ldots, n_s)$, entries appear at most n_k times with multiplicity $2k$ for $1 \le k \le s$. Thus Hölder's inequality gives

$$\mathbb{E}[B_{i_0 i_1} \cdots B_{i_{2s-1} i_0}] \le \prod_{k=1}^s \mathbb{E}[x^{2k}]^{n_k} \le \prod_{k=2}^s (ck)^{kn_k}$$

where we used that $\mathbb{E}[x^2] = 1$. This shows that the contribution of these paths can be bounded as follows.

$$\begin{aligned} E_{n_0,\ldots,n_s} &= \sum_{i_0,\cdots i_{2s-1}:P \text{ of type}(n_0,\ldots,n_s)} \mathbb{E}[B_{i_0 i_1} \cdots B_{i_{2s-1} i_0}] \\ &\le \frac{1}{N^s} n_0 \frac{N!}{n_0! n_1! \cdots n_s!} \frac{(2s)!}{s!(s+1)!} \frac{s!}{\prod_{k=1}^s (n_k k)!} \\ &\qquad \prod_{k=1}^s \frac{(n_k k)!}{(k!)^{n_k}} \prod_{k=2}^s (2k)^{kn_k} \prod_{k=2}^s (ck)^{kn_k} \\ &\le n_0 \frac{N(N-1)\cdots(n_0+1)}{N^s} \frac{(2s)!}{s!(s+1)!} \frac{1}{n_1! \cdots n_s!} \\ &\qquad \frac{s!}{\prod_{k=1}^s (ke^{-1})^{n_k k}} \prod_{k=1}^s (2ck^2)^{kn_k} \\ &\le N N^{N-n_0-s} \frac{(2s)!}{s!(s+1)!} \frac{s!}{n_1! \cdots n_s!} \prod_{k=2}^s (2cek)^{kn_k} \end{aligned}$$

where we have used that $(k!)^{n_k} \ge (ke^{-1})^{kn_k}$. Since $s = \sum_{k=1}^s kn_k$ and $N = \sum_k n_k$, we have $N - n_0 - s = \sum_{k=2}^s (1-k) n_k$. Using $s! \le (s)^s$, we obtain the bound

$$E_{n_0,\ldots,n_s} \le N \frac{(2s)!}{s!(s+1)!} \prod_{k=2}^s \frac{1}{n_k!} (N^{1-k} (2ceks)^k)^{n_k}.$$

We next sum over all $n_i \ge 0$ so that at least one $n_i \ge 1$ for $i \in \{2, \ldots, s\}$. This gives, with $\gamma_k := N^{1-k} (2ceks)^k$,

$$\sum_{n_0,\ldots,n_s:\max_{j\geq 2} n_j\geq 1} E_{n_0,\ldots,n_s} \leq N\frac{(2s)!}{s!(s+1)!}\sum_{k=2}^{s}(e^{\gamma_k}-1)\prod_{\ell\neq k}e^{\gamma_\ell}$$
$$\leq N\frac{(2s)!}{s!(s+1)!}e^{\sum_{\ell\geq 2}\gamma_\ell}(\sum_{\ell\geq 2}\gamma_\ell)$$

where we used that $e^x - 1 \leq xe^x$ for all $x \geq 0$. Note that in the range of s where $s^2 \leq N^{1-\epsilon}$, if we choose K big enough so that $K\epsilon \geq 1$,

$$\sum_{\ell}\gamma_\ell = \sum_{2\leq\ell\leq s} N^{1-\ell}(2ce\ell s)^\ell$$
$$\leq NK(2cesKN^{-1})^2 + N\sum_{K+1\leq\ell\leq s}(2ces^2N^{-1})^\ell$$
$$\leq \text{constant}(N^{-1}K^2s^2 + N(2ceN^{-\epsilon})^{K+1}) \leq \text{constant}N^{-\epsilon}$$

goes to zero as N goes to infinity. Thus, we conclude that

$$\sum_{n_0,\ldots,n_s} E_{n_0,\ldots,n_s} \leq CC_1^N N^{-\epsilon}.$$

Hence, in the regime s^2/N going to zero, the contribution of the indices $\{i_0,\ldots,i_{2s-1}\}$ associated with a path of type $(n_0,\ldots,n_s)$ with some $n_k \geq 1$ for some $k \geq 2$ is negligible compared to the contribution of simple even paths.

□

Exercise 2.4. *The extension of Theorem 2.3 to Hermitian Wigner matrices satisfying the same type of hypotheses is left to the reader as an exercise.*

Bibliographical notes. Soshnikov [181] elaborated on his combinatorial estimation of moments to prove that the largest eigenvalue fluctuations follow the Tracy–Widom law, by estimating moments of order $N^{\frac{2}{3}}$ when the entries are symmetrically distributed and have sub-Gaussian tails. By approximation, Ruzmaikina [174] could weaken the later hypothesis to the case where the entries have only the eighteenth (thirty-sixth according to [9]) moment finite. The case where the entries are not symmetrically distributed is still mysterious, despite recent progress by Péché and Soshnikov [166] who prove the universality of moments of order much larger than $\sqrt{N}$ (but still much smaller than $N^{\frac{2}{3}}$). A rather different result was proved by Johansson [121]; he showed the universality of the fluctuations of the largest eigenvalue for matrices whose entries are the convolution of a Gaussian law with a law with finite sixth moments. It is well known [15] that the largest eigenvalue of a Wigner matrix converges to 2 if and only if the entries have fourth moments. It is expected that the fluctuations follow the Tracy–Widom law when the fourth moment is finite. What happens when the entries have no finite moments is described in [9, 184]. Also, the case where one adds a finite rank perturbation to the matrix was studied in [16]; if the perturbation is sufficiently small the fluctuations still follows the Tracy–Widom law, whereas if it is large, they will be Gaussian.

Other classical ensembles were studied; for instance Wishart matrices [27, 183, 14, 190].

In the next chapter, we shall consider polynomials in several random matrices; it was shown in [111] that the spectral radius of polynomials in several independent matrices following the GUE converge to the expected limit (that is the edge of the support of the limiting spectral measure of this polynomial). This was generalized to the case of matrices interacting via a convex potential in [106].

3

Words in several independent Wigner matrices

In this chapter, we consider m independent Wigner $N \times N$ matrices $\{\mathbf{A}^{N,\ell}, 1 \leq \ell \leq m\}$ with real or complex entries. That is, the $\mathbf{A}^{N,\ell}$ are self-adjoint random matrices with independent entries $\left(A_{ij}^{N,\ell}, 1 \leq i \leq j \leq N\right)$ above the diagonal that are centered and with variance one. Moreover, the $\left(A_{ij}^{N,\ell}, 1 \leq i \leq j \leq N\right)_{1 \leq \ell \leq m}$ are independent. We shall generalize Theorem 3.3 to the case where one considers words in several matrices, that is show that $N^{-1}\mathrm{Tr}\left(\mathbf{A}^{N,\ell_1}\mathbf{A}^{N,\ell_2}\cdots\mathbf{A}^{N,\ell_k}\right)$ converges for all choices of $\ell_i \in \{1,\ldots,m\}$ and give a combinatorial interpretation of the limit. In Part VI, we describe the non-commutative framework proposed by D. Voiculescu to see the limit in the more natural framework of free probability. Here, we simply generalize Theorem 1.13 as a first step towards Part III. Let us first describe the combinatorial objects that we shall need.

3.1 Partitions of colored elements and stars

Because we now have m different matrices, the partitions that will naturally show up are partitions of elements with m different colors. In the following, each $\ell \in \{1,\ldots,m\}$ will be assigned to a different color. Also, because matrices do not commute, the order of the elements is important. This leads us to the following definition.

Definition 3.1. *Let $q(X_1,\ldots,X_m) = X_{\ell_1}X_{\ell_2}\cdots X_{\ell_k}$ be a monomial in m non-commutative indeterminates.*

We define the set $S(q)$ associated with the monomial q as the set of k colored points on the real line so that the first point has color ℓ_1, the second one has color ℓ_2, till the last one that has color ℓ_k.

$NP(q)$ is the set of non-crossing pair partitions of $S(q)$ such that two points of $S(q)$ cannot be in the same block if they have different colors.

Note that S defines a bijection between non-commutative monomials and the set of colored points on the real line.

Even though the language of non-crossing partitions is very much adapted to generalization in free probability (see the last part of these notes) where partitions can eventually be not pair partitions, it seems to us that it is more natural to consider the bijective point of view of stars when considering matrix models in Part III. The definition we give below is equivalent to the above definition according to Figure 1.4 (with colors).

Definition 3.2. *Let $q(X_1, \ldots, X_m) = X_{\ell_1} X_{\ell_2} \cdots X_{\ell_k}$ be a monomial in m non-commutative indeterminates.*

We define a star of type q as a vertex equipped with k colored half-edges, one marked half-edge and an orientation such that the marked half-edge is of color ℓ_1, the second (following the orientation) is of color ℓ_2, etc., until the last half-edge that is of color ℓ_k.

$PM(q)$ is the set of planar maps (see Definition 1.8) with one star of type q such that the half-edges can be glued only if they have the same color.

Equivalently, a star can be represented by an annulus with an orientation, colored dots and a marked dot (see Figure 3.1; color 1 is blue and color 2 is dashed).

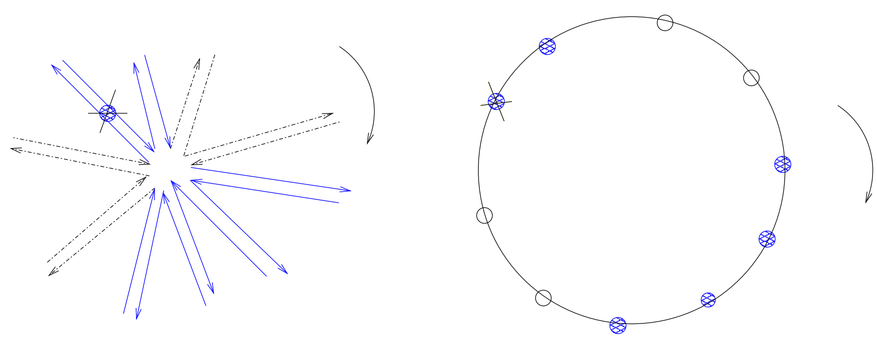


Fig. 3.1. The star of type $q(X) = X_1^2 X_2^2 X_1^4 X_2^2$

Remark 2. Planar maps with one colored star are also in bijection with trees with colored edges. However, when we deal with planar maps with several stars (see, e.g., Part III), the language of trees will become less transparent and we will no longer use it.

3.2 Voiculescu's theorem

The aim of this chapter is to prove the following:

Theorem 3.3 (Voiculescu [197]). *Assume that for all $k \in \mathbb{N}$,*

$$B_k := \sup_{1\le \ell \le m} \sup_{N\in\mathbb{N}} \sup_{ij\in\{1,\dots,N\}^2} \mathbb{E}[|\sqrt{N} A_{ij}^{N,\ell}|^k] < \infty \tag{3.1}$$

and

$$\max_{1\le i,j\le N} |\mathbb{E}[A_{ij}^{N,\ell}]| = 0, \ \lim_{N\to\infty} \max_{\ell} \frac{1}{N^2} \sum_{1\le i,j\le N} |N\mathbb{E}[|A_{ij}^{N,\ell}|^2] - 1| = 0.$$

Then, for any $\ell_j \in \{1,\dots,m\}, 1 \le j \le k$,

$$\lim_{N\to\infty} \frac{1}{N} \mathrm{Tr}\left(\mathbf{A}^{N,\ell_1}\mathbf{A}^{N,\ell_2}\cdots\mathbf{A}^{N,\ell_k}\right) = \sigma^m(X_{\ell_1}\cdots X_{\ell_k})$$

where the convergence holds in expectation and almost surely. $\sigma^m(X_{\ell_1}\cdots X_{\ell_k})$ is the number $|NP(X_{\ell_1}\cdots X_{\ell_k})| = |PM(X_{\ell_1}\cdots X_{\ell_k})|$ of planar maps with one star of type $X_{\ell_1}\cdots X_{\ell_k}$.

Remark 3. • Because a star has a marked edge and an orientation, each edge can equivalently be labeled. The counting is therefore performed for these labeled objects, regardless of possible symmetries.

• σ^m, once extended by linearity to all polynomials, is called the law of m free semi-circular variables since they satisfy the freeness property (17.1) and the moments of each variables are given by the moments of the semicircle law.

Proof. The proof is very close to that of Theorem 1.13.

1. *Expanding the expectation.*
   Setting $\mathbf{B}^N = \sqrt{N}\mathbf{A}^N$, we have

$$\mathbb{E}\left[\frac{1}{N}\mathrm{Tr}\left(\mathbf{A}^{N,\ell_1}\mathbf{A}^{N,\ell_2}\cdots\mathbf{A}^{N,\ell_k}\right)\right]$$
$$= \frac{1}{N^{\frac{k}{2}+1}} \sum_{i_1,\dots,i_k=1}^{N} \mathbb{E}[B_{i_1i_2}^{\ell_1}B_{i_2i_3}^{\ell_2}\cdots B_{i_ki_1}^{\ell_k}] \tag{3.2}$$

   where $B_{ij}^{\ell}, 1 \le i,j \le N$ denotes the entries of $\mathbf{B}^{N,\ell}$ (which may possibly depend on N). We denote by $\mathbf{i} = (i_1,\dots,i_k)$ and set

$$P(\mathbf{i},\ell) = \mathbb{E}[B_{i_1i_2}^{\ell_1}B_{i_2i_3}^{\ell_2}\cdots B_{i_ki_1}^{\ell_k}].$$

   By hypothesis, $P(\mathbf{i},\ell)$ is uniformly bounded by B_k. We let, as in the proof of Theorem 1.13, $V(\mathbf{i}) = \{i_1,\dots,i_k\}$ be the set of vertices, $E(\mathbf{i})$ the collection of the k half-edges $(e_p)_{p=1}^k = (i_p,i_{p+1})_{p=1}^k$ and consider the graph $G(\mathbf{i}) = (V(\mathbf{i}),E(\mathbf{i}))$. $G(\mathbf{i})$ is, as before, a rooted, oriented and connected graph.

$P(\mathbf{i},\ell)$ equals zero unless any edge has at least multiplicity two in $G(\mathbf{i})$. Therefore, by the same considerations as in the proof of Theorem 1.13, the indices that contribute to the first order in (3.2) are such that $G(\mathbf{i})$ is a rooted oriented tree. In particular, the limit equals zero if k is odd. This is equivalent to saying (see the bijection between trees and non-crossing partitions, Figure 1.3) that if we draw the points $i_1, \cdots i_k, i_1$ on the real line, we can draw a non-crossing pair-partition between the edges of $E(\mathbf{i})$. We write $E(\mathbf{i}) = \{(i_{s_l}, i_{s_{l+1}}); (i_{r_l}, i_{r_{l+1}})\}_{1\le l\le \frac{k}{2}}$ for the corresponding partition. Since $G(\mathbf{i})$ is a tree we have again that $(i_{s_l}, i_{s_{l+1}}) = (i_{r_{l+1}}, i_{r_l})$ for $l \in \{1, \dots, \frac{k}{2}\}$. Thus,

$$P(\mathbf{i},\ell) = \prod_{l=1}^{\frac{k}{2}} \mathbb{E}[B^{N,\ell_{s_l}}_{i_{s_l} i_{s_{l+1}}} B^{N,\ell_{r_l}}_{i_{r_l} i_{r_{l+1}}}].$$

By our hypothesis, we can replace $\mathbb{E}[B^{N,\ell_{s_l}}_{i_{s_l} i_{s_{l+1}}} B^{N,\ell_{r_l}}_{i_{r_l} i_{r_{l+1}}}]$ by $1_{\ell_{s_l}=\ell_{r_l}}$ up to a small error in the sum of the $P(\mathbf{i},\ell)$'s. Therefore, if the edges of $E(\mathbf{i})$ are colored according to which matrix they came from, the only contributing indices will come from a non-crossing pair-partition where only edges of the same color can belong to the same block. This proves that

$$\mathbb{E}\left[\frac{1}{N}\mathrm{Tr}\left(\mathbf{A}^{N,\ell_1}\mathbf{A}^{N,\ell_2}\cdots\mathbf{A}^{N,\ell_k}\right)\right] = |\sigma^m(X_{\ell_1}\cdots X_{\ell_k})| + o(1).$$

2. *Almost sure convergence.* To prove the almost sure convergence, we estimate the variance and then use the Borel–Cantelli lemma. The variance is given by

$$\begin{aligned}\mathrm{Var}(\mathbf{A}^{N,\ell_1}\mathbf{A}^{N,\ell_2}\cdots\mathbf{A}^{N,\ell_k}) &:= \mathbb{E}\left[\frac{1}{N^2}\mathrm{Tr}\left(\mathbf{A}^{N,\ell_1}\mathbf{A}^{N,\ell_2}\cdots\mathbf{A}^{N,\ell_k}\right)^2\right] \\ &\quad - \quad \mathbb{E}\left[\frac{1}{N}\mathrm{Tr}\left(\mathbf{A}^{N,\ell_1}\mathbf{A}^{N,\ell_2}\cdots\mathbf{A}^{N,\ell_k}\right)\right]^2 \\ &= \frac{1}{N^{2+k}} \sum_{\substack{i_1,\dots,i_k=1 \\ i'_1,\dots,i'_k=1}}^{N} [P(\mathbf{i},\mathbf{i}') - P(\mathbf{i})P(\mathbf{i}')]\end{aligned}$$

with $P(\mathbf{i})$ as before and

$$P(\mathbf{i},\mathbf{i}') := \mathbb{E}[B^{\ell_1}_{i_1 i_2} B^{\ell_2}_{i_2 i_3} \cdots B^{\ell_k}_{i_k i_1} B^{\ell_1}_{i'_1 i'_2} \cdots B^{\ell_k}_{i'_k i'_1}].$$

We denote by $G(\mathbf{i},\mathbf{i}')$ the graph with vertices $V(\mathbf{i},\mathbf{i}')$ given by $\{i_1,\dots,i_k,i'_1,\dots,i'_k\}$ and edges $E(\mathbf{i},\mathbf{i}')$ equal to $\{(i_p,i_{p+1})_{1\le p\le k}, (i'_p,i'_{p+1})_{1\le p\le k}\}$. For $\mathbf{i},\mathbf{i}'$ to contribute to the sum, $G(\mathbf{i},\mathbf{i}')$ must be connected (otherwise $P(\mathbf{i},\mathbf{i}') = P(\mathbf{i})P(\mathbf{i}')$), and so it is an oriented rooted connected graph. Again, each edge must appear twice and the walk on $G(\mathbf{i})$ begins and finishes at the root i_1. Therefore, exactly the same arguments that we used in the proof of Theorem 1.13 show that

$$|V(\mathbf{i}, \mathbf{i}')| \leq k.$$

By boundedness of $P(\mathbf{i}, \mathbf{i}') - P(\mathbf{i})P(\mathbf{i}')$ we conclude that

$$\mathrm{Var}(\mathbf{A}^{N,\ell_1}\mathbf{A}^{N,\ell_2}\cdots\mathbf{A}^{N,\ell_k}) \leq D_k N^{-2}$$

for some finite constant D_k. The proof is thus complete by a further use of the Borel–Cantelli lemma.

□

Exercise 3.4. *The next exercise concerns a special case of what is called "Asymptotic freeness" and was proved in greater generality by D. Voiculescu (see Theorem 17.5).*

Let $(A^N_{ij}, 1 \leq i \leq j \leq N)$ be independent real variables and consider $\mathbf{A}^N$ the self-adjoint matrix with these entries. Assume

$$\mathbb{E}[A^N_{ij}] = 0 \qquad \mathbb{E}[(\sqrt{N}A^N_{ij})^2] = 1 \quad \forall i \leq j.$$

Assume that for all $k \in \mathbb{N}$,

$$B_k = \sup_{N\in\mathbb{N}} \sup_{ij\in\{1,\ldots,N\}^2} \mathbb{E}[|\sqrt{N}A^N_{ij}|^k] < \infty.$$

Let D^N be a deterministic diagonal matrix such that

$$\sup_{N\in\mathbb{N}} \max_{i\leq j} |D^N_{ii}| < \infty \qquad \lim_{N\to\infty} \frac{1}{N}\mathrm{Tr}((D^N)^k) = m_k \text{ for all } k \in \mathbb{N}.$$

Show that:

1. *for any $k \in \mathbb{N}$,*

$$\lim_{N\to\infty} \mathbb{E}\left[\frac{1}{N}\mathrm{Tr}(D^N(\mathbf{A}^N)^k)\right] = C_{k/2}m_1,$$

2. *for any $k_1, k_2 \in \mathbb{N}$,*

$$\lim_{N\to\infty} \mathbb{E}\left[\frac{1}{N}\mathrm{Tr}((D^N)^{l_1}(\mathbf{A}^N)^{k_1}(D^N)^{l_2}(\mathbf{A}^N)^{k_2})\right]$$
$$= C_{k_1/2}C_{k_2/2}m_{l_1+l_2} + C_{(k_1+k_2)/2}m_{l_1}m_{l_2},$$

3. *for any $l_1, k_1, \cdots, l_p, k_p \in \mathbb{N}$,*

$$\lim_{N\to\infty} \mathbb{E}\Bigg[\left(\frac{1}{N}\mathrm{Tr}((D^N)^{l_1} - \frac{1}{N}\mathrm{Tr}(D^N)^{l_1}\right)\left((\mathbf{A}^N)^{k_1} - \mathbb{E}\left[\frac{1}{N}\mathrm{Tr}(\mathbf{A}^N)^{k_1}\right]\right)$$

$$\cdots\left((D^N)^{l_p}-\frac{1}{N}\mathrm{Tr}(D^N)^{l_p}\right)\left((\mathbf{A}^N)^{k_p}-\mathbb{E}\left[\frac{1}{N}\mathrm{Tr}(\mathbf{A}^N)^{k_p}\right]\right)\Bigg]$$

goes to zero as N goes to infinity for any integer numbers $l_1,\ldots,l_p$, $k_1,\ldots,k_p$.

Hint: Expand the trace in terms of a weighted sum over the indices and show that the main contribution comes from indices whose associated graph is a tree. Fixing the tree, average out the quantities in the D^N and conclude (be careful that the D^N's can come with the same indices but show then that the main contribution comes from independent entries of the $(A^N)^k_{ii})$'s because of the tree structure).

In [84, 6], the previous exercise is generalized to prove the convergence of any words in $\{\mathbf{A}_1^N,\ldots,\mathbf{A}_m^N\}$ and $\{D_1^N,\ldots,D_m^N\}$ when the trace of words in the deterministic matrices $\{D_1^N,\ldots,D_m^N\}$ are assumed to converge and $\{\mathbf{A}_1^N,\ldots,\mathbf{A}_m^N\}$ are independent Wigner matrices. This can also be deduced from Theorem 17.5 in the case of complex Gaussian matrices (by using their invariance under multiplication by unitary matrices).

Bibliographical notes. After the seminal article [197] of Voiculescu, Theorem 3.3 was generalized to non-Gaussian entries by Dykema [84]. Weaker hypotheses on the matrices $\mathbf{A}^N$ to generalize Exercise 3.4 are given in [6].

Part II

Wigner matrices and concentration inequalities

In the last twenty years, concentration inequalities have developed into a very powerful tool in probability theory. They provide a general framework to control the probability of deviations of smooth functions of random variables from their mean or their median. We begin this section by providing some general framework where concentration inequalities are known to hold. We first consider the case where the underlying measure satisfies a log-Sobolev inequality; we show how to prove this inequality in a simple situation and then how it implies concentration inequalities. We then review a few other situations where concentration inequalities hold. To apply these techniques to random matrices, we show that certain functions of the eigenvalues of matrices, such as $\int f(x)dL_{\mathbf{A}^N}(x)$ with f Lipschitz, are smooth functions of the entries of the matrix $\mathbf{A}^N$ so that concentration inequalities hold as soon as the joint law of the entries satisfies one of the conditions seen in the first two chapters of this part. Another useful *a priori* control is provided by Brascamp–Lieb inequalities; we shall apply them to the setting of random matrices at the end of this part.

To motivate the reader, let us state the type of result we want to obtain in this part.

To this end, we introduce some extra notations. Let us recall that if X is a symmetric (resp. Hermitian) matrix and f is a bounded measurable function, $f(X)$ is defined as the matrix with the same eigenvectors than X but with eigenvalues that are the image by f of those of X; namely, if e is an eigenvector of X with eigenvalue λ, $Xe = \lambda e$, $f(X)e := f(\lambda)e$. In terms of the spectral decomposition $X = UDU^*$ with U orthogonal (resp. unitary) and D diagonal real, one has $f(X) = Uf(D)U^*$ with $f(D)_{ii} = f(D_{ii})$. For $M \in \mathbb{N}$, we denote by $\langle \cdot, \cdot \rangle$ the Euclidean scalar product on $\mathbb{R}^M$ (resp. $\mathbb{C}^M$), $\langle x, y\rangle = \sum_{i=1}^M x_i y_i$ ($\langle x, y\rangle := \sum_{i=1}^M x_i y_i^*$), and by $||\cdot||_2$ the associated norm $||x||_2^2 := \langle x, x\rangle$.

Throughout this section, we denote the Lipschitz constant of a function $G : \mathbb{R}^M \to \mathbb{R}$ by

$$|G|_{\mathcal{L}} := \sup_{x \neq y \in \mathbb{R}^M} \frac{|G(x) - G(y)|}{\|x - y\|_2},$$

and call G a *Lipschitz function* if $|G|_{\mathcal{L}} < \infty$.

Lemma II.1. *Let $g : \mathbb{R}^N \to \mathbb{R}$ be Lipschitz with Lipschitz constant $|g|_{\mathcal{L}}$. Then, with $\mathbf{A}^N$ denoting the Hermitian (or symmetric) matrix with entries $(A^N_{ij})_{1\le i,j\le N}$, the map $\{A^N_{ij}\}_{1\le i\le j\le N} \mapsto \mathrm{Tr}(g(\mathbf{A}^N))$ is a Lipschitz function with constant $\sqrt{N}|g|_{\mathcal{L}}$. Therefore, if the joint law of $(A^N_{ij})_{1\le i\le j\le N}$ is "good", there exists $\alpha > 0$, constants $c > 0$ and $C < \infty$ so that for all $N \in \mathbb{N}$*

$$\mathbb{P}\left(\left|\mathrm{Tr}(g(\mathbf{A}^N)) - \mathbb{E}[\mathrm{Tr}(g(\mathbf{A}^N))]\right| > \delta |g|_{\mathcal{L}}\right) \le C e^{-c|\delta|^\alpha}.$$

"Good" here means for instance that the law satisfies a log-Sobolev inequality; an example is when the $\{A^N_{ij}\}_{1\le i\le j\le N}$ are independent Gaussian variables with uniformly bounded covariance (see Theorem 6.6).

The interest of results such as Lemma II.1 is that they provide bounds on deviations that do not depend on the dimension. They can be used to show laws of large numbers (reducing the proof of the almost sure convergence to the prove of the convergence in expectation) or to ease the proof of central limit theorems (indeed, when $\alpha = 2$ in Lemma II.1, $\mathrm{Tr}(g(\mathbf{A}^N)) - \mathbb{E}[\mathrm{Tr}(g(\mathbf{A}^N))]$ has a sub-Gaussian tail, providing tightness arguments for free).

We shall recall below the elements of the theory of concentration we shall need. In fact, we will mostly use concentration inequalities related to log-Sobolev inequalities; we shall therefore provide details on this point and give full proofs. We will then review other classical settings where concentration inequalities are known to apply. Finally, we will apply this theory to random matrices and provide for instance sufficient hypotheses so that Lemma II.1 holds.

4

Concentration inequalities and logarithmic Sobolev inequalities

We first derive concentration inequalities based on the logarithmic Sobolev inequality and then give some generic and classical examples of laws that satisfy this inequality. Since we shall use it in these notes for Wigner's matrices, we focus first on concentration for laws in $\mathbb{R}^N$. We then briefly generalize the results to compact Riemannian manifolds in order to state concentration inequalities for probability measures on the orthogonal or unitary group.

4.1 Concentration inequalities for laws satisfying logarithmic Sobolev inequalities

Throughout this section an integer number N will be fixed.

Definition 4.1. *A probability measure P on $\mathbb{R}^N$ is said to satisfy the logarithmic Sobolev inequality (LSI) with constant c if, for any differentiable function $f : \mathbb{R}^N \to \mathbb{R}$,*

$$\int f^2 \log \frac{f^2}{\int f^2 dP} dP \leq 2c \int \|\nabla f\|_2^2 dP. \tag{4.1}$$

Here, $\|\nabla f\|_2^2 = \sum_{i=1}^N (\partial_{x_i} f)^2$.

The interest in the logarithmic Sobolev inequality, in the context of concentration inequalities, lies in the following argument, that among other things, shows that LSI implies sub-Gaussian tails. This fact and a general study of logarithmic Sobolev inequalities may be found in [107],[171] or [138]. The Gaussian law, and any probability measure ν absolutely continuous with respect to the Lebesgue measure satisfying the Bobkov and Götze [38] condition (including $\nu(dx) = Z^{-1} e^{-|x|^\alpha} dx$ for $\alpha \geq 2$, where $Z = \int e^{-|x|^\alpha} dx$), as well as any distribution absolutely continuous with respect to such laws possessing a bounded above and below density, satisfies the LSI [138], [107, Property 4.6].

Lemma 4.2 (Herbst). *Assume that P satisfies the LSI on $\mathbb{R}^N$ with constant c. Let G be a Lipschitz function on $\mathbb{R}^N$, with Lipschitz constant $|G|_{\mathcal{L}}$. Then, for all $\lambda \in \mathbb{R}$, we have*

$$\int e^{\lambda(G - E_P(G))} dP \leq e^{c\lambda^2 |G|_{\mathcal{L}}^2/2}, \tag{4.2}$$

and so for all $\delta > 0$

$$P\left(|G - E_P(G)| \geq \delta\right) \leq 2e^{-\delta^2/2c|G|_{\mathcal{L}}^2}. \tag{4.3}$$

Note that Lemma 4.2 also implies that $E_P G$ is finite.

Proof of Lemma 4.2. We denote by E_P the expectation $E_P[f] = \int f dP$. Note first that (4.3) follows from (4.2). Indeed, by Chebychev's inequality, for any $\lambda > 0$,

$$\begin{aligned} P\left(|G - E_P G| \geq \delta\right) &\leq e^{-\lambda\delta} E_P[e^{\lambda|G - E_P G|}] \\ &\leq e^{-\lambda\delta}(E_P[e^{\lambda(G - E_P G)}] + E_P[e^{-\lambda(G - E_P G)}]) \\ &\leq 2e^{-\lambda\delta} e^{c|G|_{\mathcal{L}}^2 \lambda^2/2}. \end{aligned}$$

Optimizing with respect to λ (by taking $\lambda = \delta/c|G|_{\mathcal{L}}^2$) yields the bound (4.3).

Turning to the proof of (4.2), let us first assume that G is a bounded differentiable function such that

$$|| \, ||\nabla G||_2^2 ||_\infty := \sup_{x \in \mathbb{R}^N} \sum_{i=1}^N (\partial_{x_i} G(x))^2 < \infty.$$

Define

$$X_\lambda = \log E_P e^{2\lambda(G - E_P G)} \,.$$

Then, taking $f = e^{\lambda(G - E_P G)}$ in (4.1), some algebra reveals that for $\lambda > 0$,

$$\frac{d}{d\lambda}\left(\frac{X_\lambda}{\lambda}\right) \leq 2c|| \, ||\nabla G||_2^2 ||_\infty \,.$$

Now, because $G - E_P(G)$ is centered,

$$\lim_{\lambda \to 0^+} \frac{X_\lambda}{\lambda} = 0$$

and hence integrating with respect to λ yields

$$X_\lambda \leq 2c|| \, ||\nabla G||_2^2 ||_\infty \lambda^2 \,,$$

first for $\lambda \geq 0$ and then for any $\lambda \in \mathbb{R}$ by considering the function $-G$ instead of G. This completes the proof of (4.2) in the case where G is bounded and differentiable.

Let us now assume only that G is Lipschitz with $|G|_{\mathcal{L}} < \infty$. For $\epsilon > 0$, define $\bar{G}_\epsilon = G \wedge (-1/\epsilon) \vee (1/\epsilon)$, and note that $|\bar{G}_\epsilon|_{\mathcal{L}} \le |G|_{\mathcal{L}} < \infty$. Consider the regularization $G_\epsilon(x) = p_\epsilon * \bar{G}_\epsilon(x) = \int \bar{G}_\epsilon(y)p_\epsilon(x-y)dy$ with the Gaussian density $p_\epsilon(x) = e^{-|x|^2/2\epsilon}dx/\sqrt{(2\pi\epsilon)^N}$ such that $p_\epsilon(x)dx$ converges weakly to the atomic measure δ_0 as ϵ converges to 0. Since for any $x \in \mathbb{R}^N$,

$$|G_\epsilon(x) - \bar{G}_\epsilon(x)| \le |G|_{\mathcal{L}} \int ||y||_2 p_\epsilon(y)dy = |G|_{\mathcal{L}}\sqrt{\epsilon N},$$

G_ϵ converges pointwise to G. G_ϵ is also continuously differentiable and

$$\begin{aligned}
||||\nabla G_\epsilon||_2^2||_\infty &= \sup_{x\in\mathbb{R}^M} \sup_{u\in\mathbb{R}^M} \{2\langle \nabla G_\epsilon(x), u\rangle - ||u||_2^2\} \\
&\le \sup_{u,x\in\mathbb{R}^M} \sup_{\delta>0} \{2\delta^{-1}(G_\epsilon(x+\delta u) - G_\epsilon(x)) - ||u||_2^2\} \\
&\le \sup_{u\in\mathbb{R}^M} \{2|G|_{\mathcal{L}}||u||_2 - ||u||_2^2\} = |G|_{\mathcal{L}}^2. \qquad (4.4)
\end{aligned}$$

Thus, we can apply the previous result to find that for any $\epsilon > 0$ and all $\lambda \in \mathbb{R}$

$$E_P[e^{\lambda G_\epsilon}] \le e^{\lambda E_P G_\epsilon} e^{c\lambda^2 |G|_{\mathcal{L}}^2/2}. \qquad (4.5)$$

Therefore, by Fatou's lemma,

$$E_P[e^{\lambda G}] \le e^{\liminf_{\epsilon\to 0} \lambda E_P G_\epsilon} e^{c\lambda^2|G|_{\mathcal{L}}^2/2}. \qquad (4.6)$$

We next show that $\lim_{\epsilon\to 0} E_P G_\epsilon = E_P G$, which, in conjunction with (4.4), will conclude the proof. Indeed, (4.5) implies that

$$P\left(|G_\epsilon - E_P G_\epsilon| > \delta\right) \le 2e^{-\delta^2/2c|G|_{\mathcal{L}}^2}. \qquad (4.7)$$

Consequently,

$$\begin{aligned}
E[(G_\epsilon - E_P G_\epsilon)^2] &= 2\int_0^\infty x P\left(|G_\epsilon - E_P G_\epsilon| > x\right) dx \le 4\int_0^\infty x e^{-\frac{x^2}{2c|G|_{\mathcal{L}}^2}} dx \\
&= 4c|G|_{\mathcal{L}}^2 \qquad (4.8)
\end{aligned}$$

so that the sequence $(G_\epsilon - E_P G_\epsilon)_{\epsilon\ge 0}$ is uniformly integrable. Now, G_ϵ converges pointwise to G and therefore there exists a constant K, independent of ϵ, such that for $\epsilon < \epsilon_0$, $P(|G_\epsilon| \le K) \ge \frac{3}{4}$. On the other hand, (4.7) implies that $P(|G_\epsilon - E_P G_\epsilon| \le r) \ge \frac{3}{4}$ for some r independent of ϵ. Thus,

$$\{|G_\epsilon - E_P G_\epsilon| \le r\} \cap \{|G_\epsilon| \le K\} \subset \{|E_P G_\epsilon| \le K + r\}$$

is not empty, providing a uniform bound on $(E_P G_\epsilon)_{\epsilon<\epsilon_0}$. We thus deduce from (4.8) that $\sup_{\epsilon<\epsilon_0} E_P G_\epsilon^2$ is finite, and hence $(G_\epsilon)_{\epsilon<\epsilon_0}$ is uniformly integrable. In particular,

$$\lim_{\epsilon\to 0} E_P G_\epsilon = E_P G < \infty,$$

which finishes the proof. □

4.2 A few laws satisfying a log-Sobolev inequality

In the sequel, we shall be interested in laws of variables that are either independent or in interaction via a potential. We shall give sufficient conditions to ensure that a log-Sobolev inequality is satisfied.

- *Laws of independent variables.*
  One of the most important properties of the log-Sobolev inequality is the product property:

  Lemma 4.3. *Let $(\mu_i)_{i=1,2}$ be two probability measures on $\mathbb{R}^N$ and $\mathbb{R}^M$, respectively, satisfying the logarithmic Sobolev inequalities with coefficients $(c_i)_{i=1,2}$. Then, the product probability measure $\mu_1 \otimes \mu_2$ on $\mathbb{R}^{M+N}$ satisfies the logarithmic Sobolev inequality with coefficient* $\max(c_1, c_2)$.
  Consequently, if μ is a probability measure on $\mathbb{R}^M$ satisfying a logarithmic Sobolev inequality with a coefficient $c < \infty$, then the product probability measure $\mu^{\otimes n}$ satisfies the logarithmic Sobolev inequality with the same coefficient c for any integer n.

  Proof. Let f be a continuously differentiable function on $\mathbb{R}^N \times \mathbb{R}^M$. Then, using the logarithmic Sobolev inequality under the probability measure μ_1 applied to $f(., x_2)$ and under μ_2 applied to $\mu_1(f^2)(.) = \int f^2(x_1, .)d\mu(x_1)$, we obtain

$$\begin{aligned}
&\mu_1 \otimes \mu_2 \left(f^2 \log \frac{f^2}{\mu_1 \otimes \mu_2(f^2)} \right) \\
&= \mu_2 \left(\mu_1 (f^2 \log \frac{f^2}{\mu_1(f^2)}) + \mu_1(f^2) \log \frac{\mu_1(f^2)}{\mu_1 \otimes \mu_2(f^2)} \right) \\
&\le \mu_2 \left(2c_1 \mu_1 [\|\nabla_{x_1} f\|_2^2] \right) + 2c_2 \mu_2 \left(\|\nabla_{x_2} \sqrt{\mu_1(f^2)}\|_2^2 \right) \\
&\le \mu_2 \otimes \mu_1 \left(2c_1 \|\nabla_{x_1} f\|_2^2 + 2c_2 \|\nabla_{x_2} f\|_2^2 \right) \le 2\max(c_1, c_2) \mu_2 \otimes \mu_1 \left(\|\nabla f\|_2^2 \right)
\end{aligned}$$

  where we have used in the last line that $\|\nabla f\|_2^2 = \|\nabla_{x_1} f\|_2^2 + \|\nabla_{x_2} f\|_2^2$ and

$$\begin{aligned}
\|\nabla_{x_2} \sqrt{\mu_1(f^2)}\|_2^2 &= \sum_{i=1}^M (\partial_{x_2^i} (\int f(x_1, x_2)^2 d\mu_1(x_1))^{\frac{1}{2}})^2 \\
&= \sum_{i=1}^M \left(\frac{\int f(x_1, x_2) \partial_{x_2^i} f(x_1, x_2) d\mu_1(x_1)}{(\int f(x_1, x_2)^2 d\mu_1(x_1))^{\frac{1}{2}}} \right)^2 \\
&\le \sum_{i=1}^M \int (\partial_{x_2^i} f(x_1, x_2))^2 d\mu_1(x_1)
\end{aligned}$$

  by the Cauchy–Schwarz inequality. □

- *Log-Sobolev inequalities for variables in strictly convex interaction.* Below, we follow below [8, chapter 5] and [107, chapter 4], which we recommend for more details. We show that a log-Sobolev inequality holds if the so-called Bakry–Emery condition is satisfied. We then give sufficient conditions for the latter to be true. Let dx denote the Lebesgue measure on $\mathbb{R}$ and Φ be a smooth function (at least twice continuously differentiable) from $\mathbb{R}^N$ into $\mathbb{R}$ going to infinity fast enough so that the probability measure

$$\mu_\Phi(dx) := \frac{1}{Z} e^{-\Phi(x_1,\ldots,x_N)} dx_1 \cdots dx_N$$

is well defined. We consider the operator on the set $\mathcal{C}_b^2(\mathbb{R}^N)$ of twice continuously differentiable functions defined by

$$\mathcal{L}_\Phi = \Delta - \nabla\Phi.\nabla = \sum_{i=1}^N (\partial_i^2 - \partial_i\Phi\partial_i).$$

Here and below, we shall write for short $\partial_i = \partial_{x_i}$ for $i \in \{1,\ldots,N\}$. By integration by parts, one sees that $\mathcal{L}_\Phi$ is symmetric in $L^2(\mu_\Phi)$, i.e., for any functions $f, g \in \mathcal{C}_b^2(\mathbb{R}^N)$,

$$\mu_\Phi\left(f\mathcal{L}_\Phi g\right) = \mu_\Phi\left(g\mathcal{L}_\Phi f\right).$$

By the Hille–Yoshida theorem (see, e.g., [107, Chapter 1]), we can associate to the operator $\mathcal{L}_\Phi$ a Markov contractive semi-group $(P_t)_{t\geq 0}$, i.e. a family of linear operators on $\mathcal{C}_b^0(\mathbb{R}^N)$ such that $P_t : \mathcal{C}_b^0(\mathbb{R}^N) \to \mathcal{C}_b^0(\mathbb{R}^N)$ satisfies:
(1) $P_0 f = f$ for all $f \in \mathcal{C}_b^0(\mathbb{R}^N)$.
(2) The map $t \to P_t$ is continuous in the sense that for all $f \in \mathcal{C}_b^0(\mathbb{R}^N)$, $t \to P_t f$ is a continuous map from $\mathbb{R}^+$ into $\mathcal{C}_b^0(\mathbb{R}^N)$.
(3) For any $f \in \mathcal{C}_b^0(\mathbb{R}^N)$ and $(t,s) \in (\mathbb{R}^+)^2$,

$$P_{t+s}f = P_t P_s f.$$

(4) $P_t 1 = 1$ for all $t \geq 0$.
(5) P_t preserves positivity, i.e., for any $f \geq 0$, $P_t f \geq 0$. In particular, by (4), for all $t \geq 0$,

$$\|P_t f\|_\infty \leq \|f\|_\infty$$

(6)

$$\mathcal{L}_\Phi(f) = \lim_{t\downarrow 0} t^{-1}(P_t f - f)$$

for any function f for which this limit exists.

Exercise 4.4. *Let Φ be a twice continuously differentiable function, with uniformly bounded second derivatives. Then, by Theorem 20.16, there exists a unique solution to the stochastic differential equation*

$$dx_t^i = dB_t^i - \partial_i\Phi(x_t)dt$$

such that $x_0^i = z^i$ for $1 \leq i \leq N$. Denote by $x_\cdot^{i,z}$ this solution.

1. *Show that the law* P_t^z *of* x_t^z *obeys*

$$\partial_t \mathbb{E}[f(x_t^z)] = \mathbb{E}[\mathcal{L}_\Phi f(x_t^z)].$$

Hint: Use Itô's calculus.
2. *Let* $P_t f(z) := \mathbb{E}[f(x_t^z)]$. *Show that* P_t *satisfies conditions (1)–(6) above.*
3. *Assume* $Hess\, \Phi(x) \geq (1/c)I$ *for all* x *with some* $c > 0$ *and show that* $P_t f(x) - \mu_\Phi(f)$ *goes to zero exponentially fast for any* C^1 *function* f. *Hint: Write* $d(x_t^z - x_t^y) = -(\nabla\Phi(x_t^z) - \nabla\Phi(x_t^y))dt$ *and use that*

$$< \nabla\Phi(x_t^z) - \nabla\Phi(x_t^y), x_t^z - x_t^y >\geq \frac{1}{c}\|x_t^z - x_t^y\|_2^2$$

We define the operator "carré du champ" Γ_1 by

$$\Gamma_1(f,f) = \frac{1}{2}\left(\mathcal{L}_\Phi f^2 - 2f\mathcal{L}_\Phi f\right).$$

Simple algebra shows that $\Gamma_1(f,f) = \sum_{i=1}^N (\partial_i f)^2 = \|\nabla f\|_2^2$. We define $\Gamma_1(f,g)$ by bilinearity:

$$\Gamma_1(f,g) = \Gamma_1(g,f) = \frac{1}{2}\left(\Gamma_1(f+g,f+g) - \Gamma_1(f,f) - \Gamma_1(g,g)\right).$$

Note that because $\mathcal{L}_\Phi$ is symmetric in $L^2(\mu_\Phi)$, P_t is reversible in $L^2(\mu_\Phi)$, i.e.,

$$\mu_\Phi(fP_tg) = \mu_\Phi(gP_tf)$$

for any smooth functions f, g. In particular, since $P_t 1 = 1$, $\mu_\Phi P_t = \mu_\Phi$ and so μ_Φ is invariant under P_t. We expect that P_t is ergodic in the sense that for all $f \in \mathcal{C}_b^0(\mathbb{R}^N)$,

$$\lim_{t\to\infty} \mu_\Phi(P_t f - \mu_\Phi f)^2 = 0. \tag{4.9}$$

We shall not prove this point in the most general context here but only when the Bakry–Emery condition holds, see (4.12).
Finally, let us introduce the 'carré du champ itéré'

$$\begin{aligned}\Gamma_2(f,f) &= \frac{1}{2}\frac{d}{dt}\left(P_t(\Gamma_1(f,f)) - \Gamma_1(P_tf, P_tf)\right)|_{t=0} \\ &= \frac{1}{2}\{\mathcal{L}_\Phi\Gamma_1(f,f) - 2\Gamma_1(f,\mathcal{L}_\Phi f)\}.\end{aligned}$$

We define the Bakry–Emery condition as follows.

Definition 4.5. *We say that the Bakry–Emery condition* (*denoted* (BE)) *is satisfied if there exists a positive constant* $c > 0$ *such that*

$$\Gamma_2(f,f) \geq \frac{1}{c}\Gamma_1(f,f) \tag{4.10}$$

for any function f *for which* $\Gamma_1(f,f)$ *and* $\Gamma_2(f,f)$ *are well defined.*

In our case,

$$\Gamma_2(f,f) = \sum_{i,j=1}^{m} (\partial_i \partial_j f)^2 + \sum_{i,j=1}^{m} \partial_i f \mathrm{Hess}(\Phi)_{ij} \partial_j f$$

with $\mathrm{Hess}(\Phi)$ the Hessian of Φ; $\mathrm{Hess}(\Phi)_{ij} = \partial_i \partial_j \Phi$. Thus, (BE) is equivalent to $\mathrm{Hess}(\Phi)(x) \geq c^{-1} I$ (observe that the choice $f = \sum v_i x_i$ shows that (BE) implies the latter).

Theorem 4.6 (Bakry–Emery theorem). *Bakry–Emery condition implies that μ_Φ satisfies the logarithmic Sobolev inequality with constant c.*

Before going into the proof of this theorem, let us observe the following:

Corollary 4.7. *If for all $x \in \mathbb{R}^N$,*

$$\mathrm{Hess}(\Phi)(x) \geq \frac{1}{c} I$$

in the sense of the partial order on self-adjoint operators, then (BE) holds and μ_Φ satisfies the logarithmic Sobolev inequality with constant c.
In particular, if μ is the law of N independent Gaussian variables with variance bounded above by c, then μ satisfies the logarithmic Sobolev inequality with constant c.

Proof of Theorem 4.6. Let us first prove (4.9) when **(BE)** is satisfied. Let f be a continuously differentiable function such that $\|\nabla f\|_2$ is uniformly bounded. Fix $t > 0$ and consider, for $s \in [0,t]$, $\psi(s) = P_s \Gamma_1(P_{t-s}f, P_{t-s}f)$. We shall assume hereafter that $P_t f$ is sufficiently smooth so that $\Gamma_1(P_{t-s}f, P_{t-s}f)$ is in the domain of the generator. We refer to [6] or [171] for details about this assumption. Then, we find

$$\begin{aligned}\partial_s \psi(s) &= 2P_s \Gamma_2(P_{t-s}f, P_{t-s}f) \\ &\geq \frac{2}{c} P_s \Gamma_1(P_{t-s}f, P_{t-s}f) = \frac{2}{c}\psi(s)\end{aligned}$$

where we finally used **(BE)**. Thus, for all $t \geq 0$,

$$\Gamma_1(P_t f, P_t f) \leq e^{-\frac{2}{c}t} P_t \Gamma_1(f,f). \tag{4.11}$$

Since $\Gamma_1(f,f) = \|\nabla f\|_2^2$ is uniformly bounded, we deduce that $\Gamma_1(P_t f, P_t f) = \|\nabla P_t f\|_2^2$ goes to zero as t goes to infinity, ensuring that $P_t f$ converges almost surely to a constant. Indeed, for all x, y in $\mathbb{R}^N$, (4.11) implies that

$$
\begin{aligned}
|P_t f(x) - P_t f(y)| &= \left| \int_0^1 \langle \nabla P_t f(\alpha x + (1-\alpha)y), (x-y) \rangle d\alpha \right| \\
&\leq \max_{z \in \mathbb{R}^N} \|\nabla P_t f\|_2(z) \|x-y\|_2 \\
&\leq e^{-\frac{2}{c}t} \max_{z \in \mathbb{R}^N} P_t \|\nabla f\|_2(z) \|x-y\|_2 \\
&\leq e^{-\frac{2}{c}t} \| \|\nabla f\|_2 \|_\infty \|x-y\|_2
\end{aligned}
$$

where we used the fifth property of the Markov semi-group. Thus, for $f \in \mathcal{C}_b^1(\mathbb{R}^N)$,

$$\lim_{t\to\infty} P_t f = \lim_{t\to\infty} \mu_\Phi(P_t f) = \mu_\Phi(f) \quad \text{a.s.} \tag{4.12}$$

The convergence also holds in $L^2(\mu_\Phi)$ since $P_t f$ is uniformly bounded by property (5) of Markov processes, yielding (4.9).

Let f be a positive bounded continuous function so that $\mu_\Phi f = 1$. We set $f_t = P_t f$ and let

$$S_f(t) = \mu_\Phi(f_t \log f_t).$$

Since f_t converges to $\mu_\Phi f$ and $f_t \log f_t$ is uniformly bounded, we have

$$\lim_{t\to\infty} S_f(t) = \mu_\Phi(f) \log \mu_\Phi(f) = 0.$$

Hence,

$$S_f(0) = -\int_0^\infty dt \frac{d}{dt} S_f(t) = \int_0^\infty dt \mu_\Phi \Gamma_1(f_t, \log f_t). \tag{4.13}$$

Next using the fact that P_t is symmetric together with the Cauchy–Schwarz inequality, we get

$$
\begin{aligned}
\mu_\Phi \left[\Gamma_1(f_t, \log f_t)\right] &= \mu_\Phi \left[\Gamma_1\left(f, P_t(\log f_t)\right)\right] \qquad (4.14) \\
&\leq \left(\mu_\Phi \frac{\Gamma_1(f,f)}{f}\right)^{\frac{1}{2}} \left(\mu_\Phi \left[f \Gamma_1(P_t \log f_t, P_t \log f_t)\right]\right)^{\frac{1}{2}}.
\end{aligned}
$$

Applying (4.11) to the function $\log f_t$, we obtain

$$
\begin{aligned}
\left(\mu_\Phi(f\Gamma_1(P_t \log f_t, P_t \log f_t))\right)^{\frac{1}{2}} &\leq \left(\mu_\Phi(f e^{-\frac{2}{c}t} P_t \Gamma_1(\log f_t, \log f_t))\right)^{\frac{1}{2}} \qquad (4.15) \\
&= e^{-\frac{1}{c}t} \left(\mu_\Phi(f_t \Gamma_1(\log f_t, \log f_t))\right)^{\frac{1}{2}} \\
&= e^{-\frac{1}{c}t} \left(\mu_\Phi(\Gamma_1(f_t, \log f_t))\right)^{\frac{1}{2}}
\end{aligned}
$$

where in the last stage we have used symmetry of the semigroup and the fact that $\Gamma_1(f, \log f) = f\Gamma_1(\log f, \log f)$. The inequalities (4.14) and (4.15) imply the following bound:

$$\mu_\Phi \Gamma_1(f_t, \log f_t) \leq e^{-\frac{2}{c}t} \mu_\Phi \frac{\Gamma_1(f,f)}{f} = 4e^{-\frac{2}{c}t} \mu_\Phi \Gamma_1(f^{\frac{1}{2}}, f^{\frac{1}{2}}). \tag{4.16}$$

Plugging this into (4.13), one arrives at

$$S_f(0) \leq \int_0^\infty 4e^{-\frac{2t}{c}} dt \mu_\Phi \Gamma_1(f^{\frac{1}{2}}, f^{\frac{1}{2}})) = 2c\mu_\Phi \Gamma_1(f^{\frac{1}{2}}, f^{\frac{1}{2}})$$

which completes the proof. □

Bibliographical notes. The reader interested in the theory of concentration inequalities and log-Sobolev inequalities can find more material for instance in the articles [138, 107, 8, 140, 136]. The Bakry–Emery condition was introduced in [19, 18].

5

Generalizations

5.1 Concentration inequalities for laws satisfying weaker coercive inequalities

Concentration inequalities under log-Sobolev inequalities are optimal in the sense that they provide a Gaussian tail for statistics that are expected to satisfy a central limit theorem. However, for that very same reason, laws satisfying a log-Sobolev inequality must have a sub-Gaussian tail. One way to weaken this hypothesis is to weaken both requirements and hypotheses, for instance to assume a weaker coercivity inequality such as a Poincaré inequality. In this section, we keep the notations of the previous section. Let us recall that a probability measure μ on $\mathbb{R}^M$ satisfies Poincaré's inequality with coefficient $m > 0$ iff for any test function $f \in \mathcal{C}_b^2(\mathbb{R}^M)$

$$\mu_\Phi(\Gamma_1(f,f)) \geq m\mu_\Phi[(f - \mu(f))^2].$$

Exercise 5.1. *Show that Poincaré's inequality satisfies a product property similar to the product property of LSI that we saw in Lemma 4.3.*

We have:

Lemma 5.2. *[1, Theorem 2.5] Assume that μ_Φ satisfies Poincaré's inequality with constant m. Then, for any Lipschitz function f*

$$\mu_\Phi\left(\exp\{\sqrt{2m}\frac{f - \mu_\Phi(f)}{|f|_{\mathcal{L}}}\}\right) \leq K$$

with $K = 2\prod_1^\infty (1 - 4^{-m})^{-2^m}$. As a consequence, for all $\delta > 0$,

$$\mu_\Phi(|f - \mu_\Phi(f)| \geq \delta) \leq 2Ke^{-\sqrt{\frac{m}{2}}\frac{\delta}{|f|_{\mathcal{L}}}}.$$

Note that the lemma shows that measures satisfying Poincaré's inequality must have a sub-exponential tail.

Exercise 5.3. *Prove the above lemma by showing that for any $\lambda > 0$, any continuously differentiable function f,*

$$E[e^{\lambda f}] \le E[e^{\frac{\lambda}{2} f}]^2 + \frac{|f|_{\mathcal{L}}^2 \lambda^2}{4m} E[e^{\lambda f}].$$

Exercise 5.4. *Show that a log-Sobolev inequality with coefficient c implies a spectral gap inequality with coefficient bounded below by c^{-1}. Hint: Put $f = 1 + \epsilon g$ in (LSI) and let ϵ go to zero.*

5.2 Concentration inequalities by Talagrand's method

Talagrand's concentration inequality does not require that the underlying measure satisfies a coercive inequality. It holds for the law of independent equidistributed uniformly bounded variables. The price to pay is that one needs to assume that the test function is convex and also to consider concentration with respect to the median rather than the mean.

Let us recall that the median M_Y of a random variable Y is defined as the largest real number such that $P(Y \le x) \le 2^{-1}$. Then, let us state the following easy consequence of a theorem due to Talagrand [189, Theorem 6.6].

Theorem 5.5 (Talagrand). *Let K be a connected compact subset of $\mathbb{R}$ with diameter $|K| = \sup_{x,y\in K} |x - y|$. Consider a convex real-valued function f defined on K^N. Assume that f is Lipschitz on K^N, with constant $|f|_{\mathcal{L}}$. Let P be a probability measure on K and $X_1, \ldots, X_N$ be N independent copies with law P. Then, if M_f is the median of $f(X_1, \ldots, X_N)$, for all $\epsilon > 0$,*

$$P\left(|f(X_1, \ldots, X_N) - M_f| \ge \epsilon\right) \le 4e^{-\frac{\epsilon^2}{16|K|^2|f|_{\mathcal{L}}^2}}.$$

Theorem 6.6 of [189] deals with the case where $K \subset [-1, 1]$ (which easily generalizes in the above statement by rescaling) and functions f that can be Lipschitz only in a subset of K^N (in which case the above statement has to be corrected by the probability that $(X_1, \ldots, X_N)$ belongs to this subset).

Under the hypotheses of the above theorem,

$$\begin{aligned} E[|f(X_1, \ldots, X_N) - M_f|] &= \int_0^\infty P\left(|f(X_1, \ldots, X_N) - M_f| \ge t\right) dt \\ &\le 4 \int_0^\infty e^{-\frac{t^2}{16|K|^2|f|_{\mathcal{L}}^2}} dt = 16|K||f|_{\mathcal{L}}. \end{aligned}$$

Hence, we obtain as an immediate corollary to Theorem 5.5:

Corollary 5.6. *Under the hypotheses of Theorem 5.5, for all $t \in \mathbb{R}^+$,*

$$P\left(|f(X_1, \ldots, X_N) - E[f(X_1, \ldots, X_N)]| \ge (t + 16)|K||f|_{\mathcal{L}}\right) \le 4e^{-\frac{t^2}{16}}.$$

5.3 Concentration inequalities on compact Riemannian manifold with positive Ricci curvature

Let M be a compact connected manifold of dimension N equipped with a Riemannian metric g. g is a differentiable map on M such that for each $x \in M$, g_x is a scalar product on the tangent space of M at x and therefore can be identified with a positive $N \times N$ matrix $((g_x)_{ij})_{1 \le i,j \le N}$. We shall denote by μ the Lebesgue measure on (M, g), that is, the normalized volume measure; it is seen that locally

$$d\mu(x) = \sqrt{\det(g_x)}dx.$$

On (M, g), one can define the Laplace–Baltrami operator Δ (which generalizes the usual Laplace operator on $\mathbb{R}^N$) and a gradient ∇ such that for any smooth real-valued function f all $x \in M$, all y in the tangent space T_xM at x,

$$df_x(y) = g_x(\nabla f, y).$$

We let Φ be a smooth function on M and define

$$\mu_\Phi(dx) = \frac{1}{Z} e^{-\Phi(x)} d\mu(x)$$

as well as the operator $\mathcal{L}_\Phi$ such that for all smooth functions (h, f)

$$\mu_\Phi(f\mathcal{L}_\Phi h) = \mu_\Phi(h\mathcal{L}_\Phi f) = \mu_\Phi(g_x(\nabla f, \nabla h)).$$

By integration by parts, $\mathcal{L}_\Phi$ can be written in local coordinates:

$$\mathcal{L}_\Phi = \sum_{i,j=1}^{N} g_x^{ij} \partial_i \partial_j + \sum_{i=1}^{N} b_i^\Phi(x) \partial_i$$

with some b_i that can be explicitly computed in terms of Φ and g_x. We can define the “opérateurs carré du champ” as before. Simple algebra shows that

$$\begin{aligned}\Gamma_1(f,f)(x) &:= \left(\mathcal{L}_\Phi(f^2) - 2f\mathcal{L}_\Phi(f)\right)(x)\\ &= \sum_{i,j=1}^{N} (g_x)_{ij} \partial_i f(x) \partial_j f(x) = g_x^{-1}(\nabla f(x), \nabla f(x))\end{aligned}$$

and,

$$\begin{aligned}\Gamma_2(f,f)(x) &:= \left(\mathcal{L}_\Phi(\Gamma_1(f,f)) - 2\Gamma_1(\mathcal{L}_\Phi f, f)\right)(x)\\ &= (\mathrm{Hess}_x f, \mathrm{Hess}_x f)_{g_x} + (\mathrm{Ric}_x + \mathrm{Hess}_x \Phi)(\nabla f(x), \nabla f(x))\end{aligned}$$

Here, in local coordinates, the Hessian $(\mathrm{Hess} f)_{ij}$ of f at x is equal to $(\partial_{ij} - \Gamma_{ij}^k \partial_k) f$ where Γ_{ij}^k are the Christofell symbols,

$$(\mathrm{Hess} f, \mathrm{Hess} f)_g = \sum g_{ij} g_{jl} (\mathrm{Hess} f)_{ij} (\mathrm{Hess} f)_{jl},$$

$(\mathrm{Hess}\Phi)(\nabla f, \nabla f)$ is obtained by differentiating twice Φ in the direction ∇f and Ric denotes the Ricci tensor. An analytic definition of Ric is actually given above as the term due to the non-commutativity of derivatives on the manifold.

The arguments of Theorem 4.6 extend to the setting of a compact Riemannian manifold. Indeed, they were mainly based on the facts that g_x is positive definite and that ∇ obeys the Leibniz property

$$\nabla(h(f)) = \nabla f (\nabla h)(f)$$

for any differentiable functions $f, h : M \to M$. Since these properties still hold, the proof of Theorem 4.6 can be generalized to this setting yielding:

Corollary 5.7. *If for all $x \in M$ and $v \in T_x M$,*

$$(\mathrm{Ric}_x + \mathrm{Hess}\Phi_x)(v, v) \geq c^{-1} g_x^{-1} (\nabla f, \nabla f),$$

μ_Φ satisfies a log-Sobolev inequality with constant c, i.e., for any function $f : M \to \mathbb{R}$ we have

$$\mu_\Phi \left(f^2 \log \frac{f^2}{\mu_\Phi(f^2)} \right) \leq 2c \mu_\Phi(\Gamma_1(f, f)).$$

A straightforward generalization of the proof of Lemma 4.2 shows:

Corollary 5.8. *Assume that for all $x \in M$ and $v \in T_x M$,*

$$(\mathrm{Ric}_x + \mathrm{Hess}\Phi_x)(v, v) \geq c^{-1} g_x^{-1}(v, v).$$

Then, for any differentiable function f on M, if we set

$$\||\nabla f\||_2 := \sup_{x \in M} \Gamma_1(f, f)^{\frac{1}{2}}(x),$$

for all $\delta > 0$

$$\mu_\Phi \left(|f - \mu_\Phi(f)| \geq \delta \right) \leq 2 e^{-\frac{\delta^2}{2c\||\nabla f\||_2^2}}.$$

Exercise 1. Prove the corollary. *Hint: Prove and use Leibniz rule*

$$\Gamma_1(e^f, e^f) = e^{2f} \Gamma_1(f, f)$$

and follow the proof of Lemma 4.2.

5.4 Local concentration inequalities

In many instances, one may need to obtain concentration inequalities for functions that are only locally Lipschitz. To this end we state (and prove) the following lemma. Let (X, d) be a metric space and set for $f : X \to \mathbb{R}$

$$|f|_{\mathcal{L}} := \sup_{x,y \in X} \frac{|f(x) - f(y)|}{d(x, y)}.$$

Define, for a subset B of X, $d(x, B) = \inf_{y \in B} d(x, y)$. Then :

Lemma 5.9. *Assume that a probability measure μ on (X, d) satisfies a concentration inequality; for all $\delta > 0$, for all $f : X \to \mathbb{R}$,*

$$\mu(|f - \mu(f)| \geq \delta) \leq e^{-g(\frac{\delta}{|f|_{\mathcal{L}}})}$$

for some increasing function g on $\mathbb{R}^+$. Let B be a subset of X and let $f : B \to \mathbb{R}$ such that

$$|f|_{\mathcal{L}}^B := \sup_{x,y \in B} \frac{|f(x) - f(y)|}{d(x, y)}$$

is finite. Then, with $\delta(f) := \mu\left(1_{B^c}(\sup_{x \in B} |f(x)| + |f|_{\mathcal{L}}^B d(x, B))\right)$, we have

$$\mu(\{|f - \mu(f1_B)| \geq \delta + \delta(f)\} \cap B) \leq e^{-g(\frac{\delta}{|f|_{\mathcal{L}}^B})}.$$

Proof. It is enough to define a Lipschitz function $\tilde{f}$ on X, whose Lipschitz constant $|\tilde{f}|_{\mathcal{L}}$ is bounded above by $|f|_{\mathcal{L}}^B$ and so that $\tilde{f} = f$ on B. We set

$$\tilde{f}(x) = \sup_{y \in B} \{f(y) - |f|_{\mathcal{L}}^B d(x, y)\}.$$

Note that, if $x \in B$, since $f(y) - f(x) - |f|_{\mathcal{L}}^B d(x, y) \leq 0$, the above supremum is taken at $y = x$ and $\tilde{f}(x) = f(x)$. Moreover, using the triangle inequality, we get that for any $x, z \in X$,

$$\begin{aligned}\tilde{f}(x) &\geq \sup_{y \in B} \{f(y) - |f|_{\mathcal{L}}^B (d(x, z) + d(z, y))\} \\ &= -|f|_{\mathcal{L}}^B d(x, z) + \tilde{f}(z) \end{aligned} \tag{5.1}$$

and hence $\tilde{f}$ is Lipschitz, with constant $|f|_{\mathcal{L}}^B$. Therefore, we find that

$$\mu(\{|f - \mu(f1_B)| \geq \delta\} \cap B) \leq \mu(|\tilde{f} - \mu(\tilde{f})| \geq \delta + \mu(|1_B f - \tilde{f}|))$$

Note that $\mu(|1_B f - \tilde{f}|) = \mu(1_{B^c} |\tilde{f}|)$. (5.1) with $z \in B$ shows that

$$|\tilde{f}(x)| \leq |f(z)| + |f|_{\mathcal{L}}^B d(z, x)$$

and so optimizing over $z \in B$ gives

$$|\tilde{f}(x)| \leq \max_{z\in B}|f(z)| + |f|_{\mathcal{L}}^{B} d(B,x).$$

Hence,

$$\mu(|1_B f - \tilde{f}|) \leq \mu\left(1_{B^c}(\sup_{x\in B}|f(x)| + |f|_{\mathcal{L}}^{B} d(.,B))\right) =: \delta(f)$$

gives the desired estimate. $\square$

Bibliographical notes. Since the generalization of concentration inequalities to laws satisfying Poincaré's inequalities by Aida and Stroock [1], many recent results have considered the case where the decay at infinity is intermediate [94, 21] or even is very slow with heavy tails [22]. The generalization to Riemannian manifolds of the Bakry–Emery condition was already introduced in [18]. The case of discrete-valued random variables was considered by Talagrand [188].

6

Concentration inequalities for random matrices

In this chapter, we shall apply the previous general results on concentration inequalities to random matrix theory, in particular to the eigenvalues of random matrices. To this end, we shall first study the regularity of the eigenvalues of matrices as a function of its entries (since the idea will be to apply concentration inequalities to the entries of the random matrices and then see the eigenvalues as nice functions of these entries).

6.1 Smoothness and convexity of the eigenvalues of a matrix

We shall not follow [108] where smoothness and convexity were mainly proved by hand for smooth functions of the empirical measure and for the largest eigenvalue. We will rather, as in [6], rely on Weyl and Lidskii inequalities (see Theorems 19.1 and 19.4). We recall that we denote, for $\mathbf{B} \in \mathcal{M}_N(\mathbb{C})$, $\|\mathbf{B}\|_2$ its Euclidean norm:

$$\|\mathbf{B}\|_2 := \left(\sum_{i,j=1}^{N} |B_{ij}|^2 \right)^{\frac{1}{2}} .$$

From Weyl and Löwner Theorem 19.4, we will deduce that each eigenvalue of the matrix is a Lipschitz function of the entries of the matrix. We define $\mathcal{E}_N^{(1)} = \mathbb{R}^{N(N+1)/2}$ (resp. $\mathcal{E}_N^{(2)} = \mathbb{C}^{N(N-1)/2} \times \mathbb{R}^N$) and denote by $\mathbf{A}$ the symmetric (resp. Hermitian) $N \times N$ Wigner matrix such that $\mathbf{A} = \mathbf{A}^*; (\mathbf{A})_{ij} = A_{ij}, 1 \leq i \leq j \leq N$ for $(A_{ij})_{1 \leq i \leq j \leq N} \in \mathcal{E}_N^{(\beta)}$, $\beta = 1$ (resp. $\beta = 2$).

Lemma 6.1. *We denote by $\lambda_1(\mathbf{A}) \leq \lambda_2(\mathbf{A}) \leq \cdots \leq \lambda_N(\mathbf{A})$ the eigenvalues of $\mathbf{A} \in \mathcal{H}_N^{(2)}$. Then for all $k \in \{1, \ldots, N\}$, all $\mathbf{A}, \mathbf{B} \in \mathcal{H}_N^{(2)}$,*

$$|\lambda_k(\mathbf{A} + \mathbf{B}) - \lambda_k(\mathbf{A})| \leq \|\mathbf{B}\|_2.$$

In other words, for all $k \in \{1, \ldots, N\}$,

$$(A_{ij})_{1 \le i \le j \le N} \in \mathcal{E}_N^{(2)} \to \lambda_k(\mathbf{A})$$

is Lipschitz with constant one.

For all Lipschitz functions f *with Lipschitz constant* $|f|_{\mathcal{L}}$*, the function*

$$(A_{ij})_{1 \le i \le j \le N} \in \mathcal{E}_N^{(2)} \to \sum_{k=1}^{N} f(\lambda_k(\mathbf{A}))$$

is Lipschitz with respect to the Euclidean norm with a constant bounded above by $\sqrt{N}|f|_{\mathcal{L}}$*. When* f *is continuously differentiable we have*

$$\lim_{\epsilon \to 0} \epsilon^{-1} \left(\sum_{k=1}^{N} f(\lambda_k(\mathbf{A} + \epsilon \mathbf{B})) - \sum_{k=1}^{N} f(\lambda_k(\mathbf{A})) \right) = \mathrm{Tr}(f'(\mathbf{A})\mathbf{B}).$$

Proof. The first inequality is a direct consequence of Theorem 19.4 and entails the same control on $\lambda_{\max}(\mathbf{A})$. For the second we only need to use the Cauchy–Schwarz inequality:

$$\begin{aligned}
\left| \sum_{i=1}^{N} f(\lambda_i(\mathbf{A})) - \sum_{i=1}^{N} f(\lambda_i(\mathbf{A} + \mathbf{B})) \right| &\le |f|_{\mathcal{L}} \sum_{i=1}^{N} |\lambda_i(\mathbf{A}) - \lambda_i(\mathbf{A} + \mathbf{B})| \\
&\le \sqrt{N}|f|_{\mathcal{L}} \left(\sum_{i=1}^{N} |\lambda_i(\mathbf{A}) - \lambda_i(\mathbf{A} + \mathbf{B})|^2 \right)^{\frac{1}{2}} \\
&\le \sqrt{N}|f|_{\mathcal{L}} \|\mathbf{B}\|_2
\end{aligned}$$

where we used Theorem 19.4 in the last line. For the last point, we check it for $f(x) = x^k$ where the result is clear since

$$\mathrm{Tr}((\mathbf{A} + \epsilon \mathbf{B})^k) = \mathrm{Tr}(\mathbf{A}^k) + \epsilon k \mathrm{Tr}(\mathbf{A}^{k-1}\mathbf{B}) + O(\epsilon^2) \tag{6.1}$$

and complete the argument by density of the polynomials. $\square$

We can think of $\sum_{i=1}^{N} f(\lambda_i(\mathbf{A}))$ as $\mathrm{Tr}(f(\mathbf{A}))$. Then, the second part of the previous lemma can be extended to several matrices as follows.

Lemma 6.2. *Let* P *be a polynomial in* m *non-commutative indeterminates. For* $1 \le i \le m$*, we denote by* D_i *the cyclic derivative with respect to the* i*th variable given, if* P *is a monomial, by*

$$D_i P(X_1, \ldots, X_m) = \sum_{P = P_1 X_i P_2} P_2(X_1, \ldots, X_m) P_1(X_1, \ldots, X_m)$$

where the sum runs over all decompositions of P into $P_1 X_i P_2$ for some monomials P_1 and P_2. D_i is extended linearly to polynomials. Then, for all $(\mathbf{A}_1, \cdots, \mathbf{A}_m)$ and $(\mathbf{B}_1, \ldots, \mathbf{B}_m) \in \mathcal{H}_N^{(2)}$,

$$\lim_{\epsilon \to 0} \epsilon^{-1} \left(\mathrm{Tr}(P(\mathbf{A}_1 + \epsilon \mathbf{B}_1, \cdots, \mathbf{A}_m + \epsilon \mathbf{B}_m)) - \mathrm{Tr}(P(\mathbf{A}_1, \ldots, \mathbf{A}_m)) \right)$$

$$= \sum_{i=1}^m \mathrm{Tr}(D_i P(\mathbf{A}_1, \ldots, \mathbf{A}_m) \mathbf{B}_i).$$

In particular, if $(\mathbf{A}_1, \cdots, \mathbf{A}_m)$ belong to the subset Λ_M^N of elements of $\mathcal{H}_N^{(2)}$ with spectral radius bounded by $M < \infty$,

$$((A_k)_{ij})_{\substack{1 \le i \le j \le N \\ 1 \le k \le m}} \in \mathbb{C}^{N(N+1)m/2}, \mathbf{A}_k \in \mathcal{H}_N^{(2)} \cap \Lambda_M^N \to \mathrm{Tr}(P(\mathbf{A}_1, \ldots, \mathbf{A}_m))$$

is Lipschitz with a Lipschitz norm bounded by $\sqrt{N} C(P, M)$ for a constant $C(P, M)$ that depends only on M and P. If P is a monomial of degree d, one can take $C(P, M) = dM^{d-1}$.

Proof. We can assume without loss of generality that P is a monomial. The first equality is due to the simple expansion

$$\mathrm{Tr}(P(\mathbf{A}_1 + \epsilon \mathbf{B}_1, \cdots, \mathbf{A}_m + \epsilon \mathbf{B}_m)) - \mathrm{Tr}(P(\mathbf{A}_1, \ldots, \mathbf{A}_m))$$

$$= \epsilon \sum_{i=1}^m \sum_{P = P_1 X_i P_2} \mathrm{Tr}(P_1(\mathbf{A}_1, \ldots, \mathbf{A}_m) \mathbf{B}_i P_2(\mathbf{A}_1, \ldots, \mathbf{A}_m)) + O(\epsilon^2)$$

together with the trace property $\mathrm{Tr}(\mathbf{AB}) = \mathrm{Tr}(\mathbf{BA})$.

For the estimate on the Lipschitz norm, observe that if P is a monomial containing d_i times X_i, $\sum_{i=1}^m d_i = d$ and $D_i P$ is the sum of exactly d_i monomials of degree $d-1$. Hence, $D_i P(\mathbf{A}_1, \ldots, \mathbf{A}_m)$ has spectral radius bounded by $d_i M^{d-1}$ when $(\mathbf{A}_1, \ldots, \mathbf{A}_m)$ are Hermitian matrices in Λ_M^N. Hence, by Cauchy–Schwarz inequality, we obtain

$$\begin{aligned}
&\left| \sum_{i=1}^m \mathrm{Tr}(D_i P(\mathbf{A}_1, \ldots, \mathbf{A}_m) \mathbf{B}_i) \right| \\
&\qquad \le \left(\sum_{i=1}^m \mathrm{Tr}(|D_i P(\mathbf{A}_1, \ldots, \mathbf{A}_m)|^2) \right)^{\frac{1}{2}} \left(\sum_{i=1}^m \mathrm{Tr}(\mathbf{B}_i^2) \right)^{\frac{1}{2}} \\
&\qquad \le \left(N \sum_{i=1}^m d_i^2 M^{2(d-1)} \right)^{\frac{1}{2}} \left(\sum_{i=1}^m \|\mathbf{B}_i\|_2^2 \right)^{\frac{1}{2}} \\
&\qquad \le \sqrt{N} d M^{d-1} \left(\sum_{i=1}^m \|\mathbf{B}_i\|_2^2 \right)^{\frac{1}{2}}.
\end{aligned}$$

$\square$

Exercise 6.3. *Prove that when $m = 1$, $D_1 P(x) = P'(x)$.*

We now prove the following result originally due to Klein.

Lemma 6.4 (Klein's lemma). *Let $f : \mathbb{R} \to \mathbb{R}$ be a convex function. Then, if $\mathbf{A}$ is the $N \times N$ Hermitian matrix with entries $(A_{ij})_{1\le i\le j\le N}$ on and above the diagonal,*

$$\psi_f : (A_{ij})_{1\le i\le j\le N} \in \mathbb{C}^N \to \sum_{i=1}^{N} f(\lambda_i(\mathbf{A}))$$

is convex. Moreover, if f is twice continuously differentiable with $f''(x) \ge c$ for all x, ψ_f is twice continuously differentiable with Hessian bounded below by cI.

Proof. Let $X, Y \in \mathcal{H}_N^{(2)}$. We shall show that if f is a convex continuously differentiable function

$$\mathrm{Tr}\left(f(X) - f(Y)\right) \ge \mathrm{Tr}\left((X-Y)f'(Y)\right). \tag{6.2}$$

Taking $X = \mathbf{A}$ or $X = \mathbf{B}$ and $Y = 2^{-1}(\mathbf{A}+\mathbf{B})$ and summing the two resulting inequalities shows that for any couple $\mathbf{A}, \mathbf{B}$ of $N \times N$ Hermitian matrices,

$$\mathrm{Tr}\left(f\left(\frac{1}{2}\mathbf{A} + \frac{1}{2}\mathbf{B}\right)\right) \le \frac{1}{2}\mathrm{Tr}\left(f(\mathbf{A})\right) + \frac{1}{2}\mathrm{Tr}\left(f(\mathbf{B})\right)$$

which implies that $(A_{ij})_{1\le i\le j\le N} \to \mathrm{Tr}(f(\mathbf{A}))$ is convex. The result follows for general convex functions f by approximations.

To prove (6.2), let us denote by $\lambda_i(C)$ the eigenvalues of a Hermitian matrix C and by $\xi_i(C)$ the associated eigenvector and write

$$\begin{aligned}
&\langle \xi_i(X), (f(X) - f(Y))\, \xi_i(X)\rangle \\
&= f(\lambda_i(X)) - \sum_{j=1}^{N} |\langle \xi_i(X), \xi_j(Y)\rangle|^2 f(\lambda_j(Y)) \\
&= \sum_{j=1}^{N} |\langle \xi_i(X), \xi_j(Y)\rangle|^2 (f(\lambda_i(X)) - f(\lambda_j(Y))) \\
&\ge \sum_{j=1}^{N} |\langle \xi_i(X), \xi_j(Y)\rangle|^2 (\lambda_i(X) - \lambda_j(Y)) f'(\lambda_j(Y))
\end{aligned}$$

where we have used the convexity of f to write $f(x) - f(y) \ge (x-y)f'(y)$. The right-hand side of the last inequality is equal to $\langle \xi_i(X), ((X-Y)f'(Y))\, \xi_i(X)\rangle$ and therefore summing over i yields (6.2), which completes the first part of the proof of the lemma.

We give another proof below that also provides a lower bound of the Hessian of ψ_f. The smoothness of ψ_f is clear when f is a polynomial since then

$\psi_f((A_{ij})_{1\le i\le j\le N})$ is a polynomial function in the entries. Let us compute its second derivative when $f(x)=x^p$. Expanding (6.1) one step further gives

$$\begin{aligned}\mathrm{Tr}((\mathbf{A}+\epsilon\mathbf{B})^k) &= \mathrm{Tr}(\mathbf{A}^k)+\epsilon\sum_{k=0}^{p-1}\mathrm{Tr}(\mathbf{A}^k\mathbf{B}\mathbf{A}^{p-1-k})\\ &\quad+\epsilon^2\sum_{0\le k+l\le p-2}\mathrm{Tr}(\mathbf{A}^k\mathbf{B}A^l\mathbf{B}A^{p-2-k-l})+O(\epsilon^3)\\ &= \mathrm{Tr}(A^k)+\epsilon p\mathrm{Tr}(A^{p-1}\mathbf{B})\\ &\quad+\frac{\epsilon^2}{2}p\sum_{0\le l\le p-2}\mathrm{Tr}(A^l\mathbf{B}A^{p-2-l}\mathbf{B})+O(\epsilon^3). \end{aligned}\tag{6.3}$$

A compact way to write this formula is by defining, for two real numbers x,y,

$$K_f(x,y):=\frac{f'(x)-f'(y)}{x-y}$$

and setting for a matrix $\mathbf{A}$ with eigenvalues $\lambda_i(A)$ and eigenvector e_i, $1\le i\le N$,

$$K_f(\mathbf{A},\mathbf{A})=\sum_{i,j=1}^N K_f(\lambda_i(\mathbf{A}),\lambda_j(\mathbf{A}))e_ie_i^*\otimes e_je_j^*.$$

Since $K_{x^p}(x,y)=p\sum_{r=0}^{p-1}x^ry^{p-1-r}$, the last term in the r.h.s. of (6.3) reads

$$p\sum_{0\le l\le p-1}\mathrm{Tr}(\mathbf{A}^l\mathbf{B}\mathbf{A}^{p-2-l}\mathbf{B})=\langle K_{x^p}(\mathbf{A},\mathbf{A}),\mathbf{B}\otimes\mathbf{B}\rangle \tag{6.4}$$

where for $\mathbf{B},\mathbf{C},\mathbf{D},\mathbf{E}\in M_N(\mathbb{C})$, $\langle\mathbf{B}\otimes\mathbf{C},\mathbf{D}\otimes\mathbf{E}\rangle:=\langle\mathbf{B},\mathbf{D}\rangle_2\langle\mathbf{C},\mathbf{E}\rangle_2$ with $\langle\mathbf{B},\mathbf{D}\rangle_2=\sum_{i,j=1}^N B_{ij}\bar{D}_{ij}$. In particular, $\langle e_ie_i^*\otimes e_je_j^*,\mathbf{B}\otimes\mathbf{B}\rangle=|<e_i,Be_j>|^2$ with $<u,Bv>=\sum_{i,j=1}^N u_i\bar{v}_jB_{ij}$. By (6.3) and (6.4), for any Hermitian matrix $\mathbf{X}$,

$$\begin{aligned}\mathrm{Hess}(\mathrm{Tr}(\mathbf{A}^p))[X,X] &= \langle K_{x^p}(\mathbf{A},\mathbf{A}),X\otimes X\rangle\\ &=\sum_{r,m=1}^N K_{x^p}(\lambda_r(\mathbf{A}),\lambda_m(\mathbf{A}))|<e_r,Xe_m>|^2\end{aligned}$$

Now $K_f(\mathbf{A},\mathbf{A})$ makes sense for any twice continuously differentiable function f and by density of the polynomials in the set of twice continuously differentiable function f, we can conclude that ψ_f is twice continuously differentiable too. Moreover, for any twice continuously differentiable function f,

$$\mathrm{Hess}(\mathrm{Tr}(f(\mathbf{A})))[X,X]=\sum_{r,m=1}^N K_f(\lambda_r(\mathbf{A}),\lambda_m(\mathbf{A}))|<e_r,Xe_m>|^2.$$

Since K_f is pointwise bounded below by c when $f'' \geq c$ we finally deduce that

$$\mathrm{Hess}(\mathrm{Tr}(f(\mathbf{A})))[\mathbf{X},\mathbf{X}] \geq c\mathrm{Tr}(\mathbf{X}\mathbf{X}^*).$$

The proof is thus complete. □

Let us also notice the following:

Lemma 6.5. *Assume $\lambda_1(\mathbf{A}) \leq \lambda_2(\mathbf{A}) \cdots \leq \lambda_N(\mathbf{A})$. The functions*

$$\mathbf{A} \in \mathcal{H}_N^{(2)} \to \lambda_1(\mathbf{A}) \text{ and } \mathbf{A} \in \mathcal{H}_N^{(2)} \to \lambda_N(\mathbf{A})$$

are convex. For any norm $\|\cdot\|$ on $\mathcal{M}_N^{(2)}$, $(A_{ij})_{1\leq i,j\leq N} \to \|\mathbf{A}\|$ is convex.

Proof. The first result is clear since we have already seen that $\lambda_N(\mathbf{A}+\mathbf{B}) \leq \lambda_N(\mathbf{A}) + \lambda_N(\mathbf{B})$. Since for $\alpha \in \mathbb{R}$, $\lambda_i(\alpha\mathbf{A}) = \alpha\lambda_i(\mathbf{A})$, we conclude that $\mathbf{A} \to \lambda_N(\mathbf{A})$ is convex. The same result holds for λ_1 (by changing the sign $\mathbf{A} \to -\mathbf{A}$). The convexity of $(A_{ij})_{1\leq i,j\leq N} \to \|\mathbf{A}\|$ is due to the definition of the norm. □

6.2 Concentration inequalities for the eigenvalues of random matrices

We consider a Hermitian random matrix $\mathbf{A}$ whose real or complex entries have joint law μ^N that satisfies one of the two hypotheses below.

Either the entries of $\mathbf{A}$ are independent and satisfy for some $c > 0$ the following condition:

• (H1) $\mathbf{A} = \mathbf{X}^N/\sqrt{N} = (\mathbf{A})^*$ *with $(\mathbf{X}_{ij}^N, 1 \leq i \leq j \leq N)$ independent, with laws $(\mu_{ij}^N, 1 \leq i \leq j \leq N)$, that are probability measures on $\mathbb{C}$ or $\mathbb{R}$ satisfying a log-Sobolev inequality with constant $c < \infty$;*

or μ^N is a Gibbs measure with strictly convex potential, i.e., satisfies:

• (H2) *there exists a strictly convex twice continuously differentiable function $V : \mathbb{R} \to \mathbb{R}$, $V''(x) \geq \frac{1}{c} > 0$, so that*

$$\mu^N(d\mathbf{A}) = Z_N^{-1} e^{-N\mathrm{Tr}(V(\mathbf{A}))} d\mathbf{A}$$

with $d\mathbf{A} = \prod_{1\leq i\leq j\leq N} d\Re(A_{ij}) \prod_{1\leq i<j\leq N} d\Im(A_{ij})$ for complex entries or $d\mathbf{A} = \prod_{1\leq i\leq j\leq N} dA_{ij}$ for real entries.

Note that when $V = \frac{1}{2}x^2$, μ^N is the law of a Gaussian Wigner matrix but in any other case the entries of $\mathbf{A}$ with law μ^N are not independent.

We can now state the following theorem.

Theorem 6.6. *Suppose there exists $c > 0$ so that either (H1) or (H2) holds. Then:*

1. *For any Lipschitz function f on $\mathbb{R}$, for any $\delta > 0$,*

$$\mu^N\left(|L_{\mathbf{A}}(f) - \mu^N[L_{\mathbf{A}}(f)]| \geq \delta\right) \leq 2e^{-\frac{1}{4c|f|_{\mathcal{L}}^2}N^2\delta^2}.$$

2. *Let $\lambda_1(\mathbf{A}) \leq \lambda_2(\mathbf{A}) \cdots \leq \lambda_N(A)$ be the eigenvalues of a self-adjoint matrix $\mathbf{A}$. For any $k \in \{1, \ldots, N\}$,*

$$\mu^N\left(|\lambda_k(\mathbf{A}) - \mu^N(\lambda_k(\mathbf{A}))| \geq \delta\right) \leq 2e^{-\frac{1}{4c}N\delta^2}.$$

In particular, these results hold when the $\mathbf{X}_{ij}$ are independent Gaussian variables with uniformly bounded variances.

Proof of Theorem 6.6. For the first case, we use the product property of Lemma 4.3 which implies that $\otimes_{i\leq j}\mu_{ij}^N$ satisfies the log-Sobolev inequality with constant c. By rescaling, the law of the entries of $\mathbf{A}$ satisfies a log-Sobolev inequality with constant c/N. For the second case, the assumption $V''(x) \geq \frac{1}{c}$ implies, by Lemma 6.4, that $(A_{ij})_{1\leq i\leq j\leq N} \in \mathcal{E}_N^{(\beta)} \to N\mathrm{Tr}(V(\mathbf{A}))$ is twice continuously differentiable with Hessian bounded below by $\frac{N}{c}$. Therefore, by Corollary 4.7, μ^N satisfies a log-Sobolev inequality with constant c/N.

Thus, to complete the proof of the first result of the theorem, we only need to recall that by Lemma 6.1, $G(A_{ij}^N, 1 \leq i \leq j \leq N) = \mathrm{Tr}(f(\mathbf{A}))$ is Lipschitz with constant bounded by $\sqrt{N}|f|_{\mathcal{L}}$ whereas $A_{ij}^N, 1 \leq i \leq j \leq N \to \lambda_k(\mathbf{A})$ is Lipschitz with constant one. For the second, we use Lemma 6.5. $\square$

Exercise 6.7. *State the concentration result when the μ_{ij}^N only satisfy the Poincaré inequality.*

Exercise 6.8. *If $\mathbf{A}$ is not Hermitian but has all entries with a joint law of type μ^N as above, show that the law of the spectral radius of $\mathbf{A}$ satisfies a concentration of measure inequality.*

When the laws satisfy instead a Talagrand-type condition we state the induced concentration bounds:

Theorem 6.9. *Let $\mu^N(f(\mathbf{A})) = \int f(\mathbf{X}/\sqrt{N})\prod d\mu_{i,j}^N(X_{ij})$ with $(\mu_{i,j}^N, i \leq j)$ compactly supported probability measures on a connected compact subset K of $\mathbb{C}$. Fix $\delta_1 = 8|K|\sqrt{\pi}$. Then, for any $\delta \geq \delta_1 N^{-1}$, for any convex function f,*

$$\mu^N\left(|\mathrm{Tr}(f(\mathbf{A})) - \mu^N[\mathrm{Tr}(f(\mathbf{A}))]| \geq N\delta|f|_{\mathcal{L}}\right) \tag{6.5}$$
$$\leq \frac{32|K|}{\delta}\exp\left(-N^2\frac{1}{16|K|^2a^2}\frac{(\delta - \delta_1 N^{-1})^2}{16|K|}\right).$$

If $\lambda_{\max}(\mathbf{A})$ is the largest (or smallest) eigenvalue of $\mathbf{A}$, or the spectral radius of $\mathbf{A}$, for $\delta \geq \delta_1(N)$,

$$\mu^N\left(|\lambda_{\max}(\mathbf{A}) - E^N[\lambda_{\max}(\mathbf{A})]| \geq \delta N^{\frac{1}{2}}\right)$$
$$\leq \frac{32|K|}{\delta}\exp\left(-\frac{1}{16|K|^2a^2}\frac{(\delta - \delta_1 N^{-\frac{1}{2}})^2}{16|K|}\right).$$

Proof. Applying Corollary 5.6, Lemmas 6.1 and 6.4 with a function $f : \mathbf{A} \to \mathrm{Tr}(f(\mathbf{A}))$ that is Lipschitz with Lipschitz constant $|f|_{\mathcal{L}}$ provides the first bound. □

Observe that the speed of the concentration we obtained is optimal for $\mathrm{Tr}(f(\mathbf{X}^N))$ (since it agrees with the speed of the central limit theorem). It is also optimal in view of the large deviation principle we will prove in the next section. However, it does not capture the true scale of the fluctuations of $\lambda_{\max}(\mathbf{A})$ that are of order $N^{-\frac{1}{3}}$. Improvements of concentration inequalities in that direction were obtained by M. Ledoux [139].

We emphasize that Theorem 6.6 applies also when the variance of X_{ij}^N depends on i, j. For instance, it includes the case where $X_{ij}^N = a_{ij}^N Y_{ij}^N$ with Y_{ij}^N i.i.d. with law P satisfying the log-Sobolev inequality and a_{ij} uniformly bounded (since if P satisfies the log-Sobolev inequality with constant c, the law of ax under P satisfies it also with a constant bounded by $|a|^2 c$).

6.3 Concentration inequalities for traces of several random matrices

The previous theorems also extend to the setting of several random matrices. If we wish to consider polynomial functions of these matrices, we can use local concentration results (see Lemma 5.9). We do not need to assume the random matrices independent if they interact *via* a convex potential.

Definition 6.10. *Let V be a polynomial in m non-commutative variables. We say that V is convex iff for any $N \in \mathbb{N}$,*

$$\phi_V^N : ((A_k)_{ij})_{\substack{i \le j \\ 1 \le k \le m}} \in \mathcal{E}_N^{(2)} \to \mathrm{Tr} V(\mathbf{A}_1, \dots, \mathbf{A}_m)$$

is convex.

Exercise 6.11.
- *Define $X.Y = 2^{-1} \sum_{i=1}^m (X_i Y_i + Y_i X_i)$. Let $D = (D_1, \dots, D_m)$ with D_i the cyclic derivative with respect to the ith variable as defined in Lemma 6.2. Show that ϕ_V^N is convex if for any $\mathbf{X} = (X_i)_{1 \le i \le m}$ and $\mathbf{Y} = (Y_i)_{1 \le i \le m}$ in $\mathcal{H}_N^{(2)}(\mathbb{C})^m$, $V(\mathbf{X})^* = V(\mathbf{X})$ and*

$$(DV(\mathbf{X}) - DV(\mathbf{Y})).(X - Y)$$

is a non-negative matrix in $\mathcal{H}_N^{(2)}(\mathbb{C})$.
- *Show that ϕ_V^N is convex if $V(X_1, \dots, X_m) = \sum_{i=1}^k V_i(\sum_{j=1}^m \alpha_j^i X_j)$ when α_j^i are real variables and V_i are convex functions on $\mathbb{R}$. Hint: use Klein's Lemma 6.4.*

Let c be a positive real.

$$d\mu_V^{N,\beta}(\mathbf{A}_1,\ldots,\mathbf{A}_m) := \frac{1}{Z_V^N} e^{-N\mathrm{Tr}(V(\mathbf{A}_1,\ldots,\mathbf{A}_m))} d\mu_c^{N,\beta}(\mathbf{A}_1)\cdots d\mu_c^{N,\beta}(\mathbf{A}_m)$$

with $\mu_c^{N,\beta}$ the law of an $N\times N$ Wigner matrix with complex ($\beta = 2$) or real ($\beta = 1$) Gaussian entries with variance $1/cN$, that is, the law of the self-adjoint $N\times N$ matrix A with entries with law

$$\mu_c^{N,2}(dA) = \frac{1}{Z_N^c} e^{-\frac{cN}{2}\sum_{i,j=1}^N |A_{ij}|^2} \prod_{i\le j} d\Re A_{ij} \prod_{i\le j} d\Im A_{ij}$$

and

$$\mu^{N,1}(dA) = \frac{1}{Z_N^c} e^{-\frac{cN}{4}\sum_{i,j=1}^N A_{ij}^2} \prod_{i\le j} dA_{ij}.$$

We then have the following corollary.

Corollary 6.12. *Let $\mu_V^{N,\beta}$ be as above. Then:*

1. *For any Lipschitz function f of the entries of the matrices $A_i, 1\le i\le m$, for any $\delta > 0$,*

$$\mu_V^{N,\beta}(|f-\mu_V^{N,\beta}(f)| > \delta) \le 2e^{-\frac{Nc\delta}{2|f|_{\mathcal{L}}}}.$$

2. *Let M be a positive real, define $\Lambda_M^N = \{\mathbf{A}_i \in \mathcal{H}_N^{(2)}; \max_{1\le i\le m}\lambda_{\max}(A_i) \le M\}$ and let P be a monomial of degree $d\in\mathbb{N}$. Then, for any $\delta > 0$*

$$\begin{aligned} sabs\mu_V^{N,\beta}\Big(\{|\mathrm{Tr}(P(\mathbf{A}_1,\ldots,\mathbf{A}_m)) - \mu_V^{N,\beta}(\mathrm{Tr}(P(\mathbf{A}_1,\ldots,\mathbf{A}_m)1_{\Lambda_M^N}))| \\ > \delta + \delta(M,N)\} \cap \Lambda_M^N\Big) \le 2e^{-\frac{c\delta^2}{d^2M^{2(d-1)}}} \end{aligned}$$

with

$$\delta(M,N) \le M^d \mu_V^{N,\beta}\left((1+d\|\mathbf{A}\|_2)1_{(\Lambda_M^N)^c}\right).$$

Proof. By assumption, the law $\mu_V^{N,\beta}$ of the entries of $(\mathbf{A}_1,\ldots,\mathbf{A}_m)$ is absolutely continuous with respect to the Lebesgue measure. The Hessian of the logarithm of the density is bounded above by $-NcI$. Hence, by Corollary 4.7, $\mu_V^{N,\beta}$ satisfies a log-Sobolev inequality with constant $1/Nc$ and thus by Lemma 4.2 we find that $\mu_V^{N,\beta}$ satisfies the first statement of the corollary. We finally conclude by using Lemma 5.9 and the fact that $X_1,\ldots,X_m \to \mathrm{Tr}(P(X_1,\ldots,X_m))$ is locally Lipschitz by Lemma 6.2. $\square$

6.4 Concentration inequalities for the Haar measure on $O(N)$

Now, let us consider the Haar measure on the orthogonal group

$$O(N) = \{A \in \mathcal{M}_{N\times N}(\mathbb{R}); OO^T = I\}.$$

This is the unique non-negative regular Borel measure on the compact group $O(N)$ that is left-invariant (see [172, Theorem 5.14]) and with total mass one. Let us introduce

$$SO(N) = \{A \in O(N) : \det(O) = 1\}.$$

For any $A \in O(N)$, $\det(O) \in \{+1, -1\}$, so that $O(N)$ can be decomposed as two copies of $SO(N)$. One way to go from one copy of $SO(N)$ to the other is for instance to change the sign of one column vector of the matrix. Let T be such a transformation. Then, if m^N denotes the Haar measure on $O(N)$, M^N the Haar measure on $SO(N)$, and $T_\sharp M^N(.) = M^N(T.)$, we deduce that

$$m^N = \frac{1}{2}M^N + \frac{1}{2}T_\sharp M^N.$$

Note that concentration inequalities under the Haar measure on $O(N)$ do not hold in general by taking a function concentrated on only one of the copies of $SO(N)$. However, on $SO(N)$, concentration holds. Namely, endow $SO(N)$ with the Riemaniann metric given, for $M, M' \in SO(N)$, by

$$d(M, M') = \inf_{(M_t)_{t\in[0,1]}:M_0=M,M_1=M'} \int_0^1 \left(\frac{1}{2}\mathrm{Tr}[(\partial_t M_t)(\partial_t M_t)^*]\right)^{\frac{1}{2}} dt$$

where the infimum is taken over all differentiable paths $M_. : [0,1] \to SO(N)$. This metric is Riemannian. It is the invariant (under conjugation) metric on $SO(N)$ such that the circle consisting of the rotations around a fixed subspace $\mathbb{R}^{N-2}$ has length 2π. The later normalization can be checked by taking, if $(e_i)_{1\le i\le N}$ is an orthonormal basis of $\mathbb{R}^N$, M_t to be a rotation of angle θ in the vector space generated by e_1, e_2, the identity on $e_3, \ldots, e_N$). The Ricci curvature on $SO(N)$ for this metric has been computed in [155]:

Theorem 6.13 (Gromov). *[155, p. 129]*

$$\mathrm{Ric}(SO(N)) \ge \frac{N-2}{2} I.$$

This result extends to $SU(N)$ when one replaces $O(N)$ by $U(N)$; indeed, one has (see, e.g., [118] or [6])

$$\mathrm{Ric}(SU(N)) \ge N - 3/2. \tag{6.6}$$

To deduce concentration inequalities, let us compare the above metric on $SO(N)$ with the Euclidean metric on $\mathcal{M}_N(\mathbb{C})$. To do so, we observe that

$$\left(\frac{1}{2}\mathrm{Tr}((M-M')(M-M')^*)\right)^{\frac{1}{2}}$$
$$= \inf_{(M_t)_{t\in[0,1]}:M_0=M,M_1=M'} \int_0^1 \left(\frac{1}{2}\mathrm{Tr}[(\partial_t M_t)(\partial_t M_t)^*]\right)^{\frac{1}{2}} dt$$

if the infimum is taken on $\mathcal{M}_N(\mathbb{C})$ (just take $M_t = M + t(M'-M)$ to get $\geq$ and use the fact that $M \to \sqrt{\mathrm{Tr}(MM^*)}$ is convex (as a norm) with Jensen's inequality for the converse inequality). Hence, for all $M, M' \in SO(N)$,

$$\left(\frac{1}{2}\mathrm{Tr}((M-M')(M-M')^*)\right)^{\frac{1}{2}} \leq d(M,M').$$

Therefore, we deduce the following:

Corollary 6.14. *For any differentiable function $f : SO(N) \to \mathbb{R}$ such that, for any $X, Y \in SO(N)$, $|f(X)-f(Y)| \leq |f|_{\mathcal{L}}\|X-Y\|_2$, we have for all $t \geq 0$*

$$M^N\left(\left|f - \int_{SO(N)} f(O)dM^N(O)\right| \geq \delta\right) \leq 2e^{-\frac{N}{2^4|f|_{\mathcal{L}}^2}\delta^2}.$$

If f extends to $O(N)$ as a Lipschitz function on $SO(N)$ and $T(SO(N))$, we have

$$m^N\left(\left|f - \int_{O(N)} f(O)dm^N(O)\right| \geq \delta + |f|_{\mathcal{L}}\right) \leq 2e^{-\frac{N}{2^4|f|_{\mathcal{L}}^2}\delta^2}$$

Proof. The concentration result under M^N is a direct consequence of the lower bound on $\mathrm{Ric}(SO(N))$ and Corollary 5.8. Indeed, the previous comparison of the metrics shows that

$$|f(X)-f(Y)|^2 \leq |f|_{\mathcal{L}}^2 \mathrm{Tr}[(X-Y)(X-Y)^*] \leq 2|f|_{\mathcal{L}}^2 d(X,Y)^2.$$

Hence, if f is differentiable, $2|f|_{\mathcal{L}}^2 = \||\nabla f|\|_2^2$ and Corollary 5.8 allows us to conclude. Concentration under m^N is based on the fact that if T is a transformation of $SO(N)$ such as a change of sign of the first column vector, then

$$\mathrm{Tr}(X-TX)(X-TX)^* = 4\sum_{i=1}^N (O_{1,i})^2 = 4.$$

Therefore,

$$\left|\int_{SO(N)} f(O)dM^N(O) - \int_{SO(N)} f(TO)dM^N(O)\right| \leq 2|f|_{\mathcal{L}},$$

and so recalling that $m^N = 2^{-1}M^N + 2^{-1}T_\# M^N$,

$$\left| \int_{SO(N)} f(O) dM^N(O) - \int_{O(N)} f(O) dm^N(O) \right| \leq |f|_{\mathcal{L}}.$$

Hence, we find that

$$\begin{aligned} & m^N \left(\left| f - \int_{O(N)} f(O) dm^N(O) \right| \geq N\delta + |f|_{\mathcal{L}} \right) \\ & \leq \frac{1}{2} M^N \left(V : \left| f(TV) - \int_{SO(N)} f(TO) dM^N(O) \right| \geq \delta \right) \\ & \quad + \frac{1}{2} M^N \left(V : \left| f(V) - \int_{SO(N)} f(O) dM^N(O) \right| \geq \delta \right) \leq 2e^{-\frac{N-2}{8|f|_{\mathcal{L}}^2}\delta^2} \end{aligned}$$

which completes the proof. □

As an application, we have the following corollary.

Corollary 6.15. *Let F be a Lipschitz function on $\mathbb{R}$, and D and D' be fixed diagonal matrices (whose entries are real and uniformly bounded by $||D||_\infty$ and $\|D'||_\infty$ respectively). Then, for any $\delta > 0$,*

$$\begin{aligned} & M^N \left(|\mathrm{Tr}(F(D' + ODO^*)) - \mathbb{E}[\mathrm{Tr}(F(D' + ODO^*))]| \geq \delta N |F|_{\mathcal{L}} \right) \\ & \qquad \leq 2e^{-\frac{(N-2)N}{2^5 ||D||_\infty^2}\delta^2}. \end{aligned}$$

Proof. Put $f(O) = \mathrm{Tr}(F(D' + ODO^*))$ and note that, for any $O \in O(N)$, by Lemma 6.1,

$$|f(O) - f(\tilde{O})|^2 \leq N|F|_{\mathcal{L}}^2 \|ODO^* - \tilde{O}D\tilde{O}^*\|_2^2 \leq 4N|F|_{\mathcal{L}}^2 ||D||_\infty^2 \|O - \tilde{O}\|_2^2.$$

Plugging this estimate into the main result of Theorem 6.13 completes the proof. □

These concentration inequalities also extend to the Haar measure on $U(N)$ even though this time $U(N)$ decomposes as a continuum of copies of $SU(N)$ (namely $SU(N)$ times a rotation). This is, however, enough to get (see, e.g., [6]) the following theorem.

Theorem 6.16. *Let $(X_1^N, \ldots, X_m^N) \in \mathcal{H}_N^{(2)}$ be a sequence such that*

$$L := \sup_{1 \leq i \leq m} \sup_{N \in \mathbb{N}} \lambda_{\max}(X_i^N) < \infty$$

and denote m^N the Haar measure on $U(N)$. Then, for any polynomial P of $m + 2$ noncommutative variables $(X_1, \ldots, X_m, U, U^*)$, there exists $c = c(L, P) > 0$ such that for N large enough

$$m^N\left(\left|\frac{1}{N}\mathrm{Tr}(P(X_1,\ldots,X_m,U,U^*))\right.\right.$$
$$\left.\left.-\int\frac{1}{N}\mathrm{Tr}(P(X_1,\ldots,X_m,V,V^*))dm^N(V)\right|>\delta\right)\le e^{-cN^2\delta^2}.$$

Bibliographical notes. Klein's lemma can be found for instance in [173]. The idea to apply concentration of measures theory to the concentration of the spectral measure and the largest eigenvalue of random matrices started in [108], even though it was already obtained in particular cases in different papers by using for instance martingale expansions. It was generalized to the concentration of each eigenvalue around its median in [3], and to the concentration of the permanent of random matrices in [92]. In [151], the Lipschitz condition was generalized to norms different than the Euclidean norm. The applications of these ideas to the eigenvalues of Haar distributed random matrices was used in [103], based on a lower bound on the Ricci curvature of $SO(N)$ due to Gromov [155] and developed in [63] from the viewpoint of random walks on compact groups.

6.5 Brascamp–Lieb inequalities; applications to random matrices

We introduce first Brascamp–Lieb inequalities and show how they can be derived from results from optimal transport theory, following a proof of Hargé [114]. We then show how these inequalities can be used to obtain *a priori* controls for random-matrix quantities such as the spectral radius. Such controls will be particularly useful in the next chapter.

6.5.1 Brascamp–Lieb inequalities

The Brascamp–Lieb inequalities we shall be interested in allow us to compare the expectation of convex functions under a Gaussian law and under a law with a log-concave density with respect to this Gaussian law. It is stated as follows.

Theorem 6.17. (Brascamp–Lieb [47], Hargé [114, Theorem 1.1]) *Let $n\in\mathbb{N}$. Let g be a convex function on $\mathbb{R}^n$ and f a log-concave function on $\mathbb{R}^n$. Let γ be a Gaussian measure on $\mathbb{R}^n$. We suppose that all the following integrals are well defined, then:*

$$\int g(x+l-m)\frac{f(x)d\gamma(x)}{\int f d\gamma}\le\int g(x)d\gamma(x)$$

where

$$l=\int x d\gamma,\quad m=\int x\frac{f(x)d\gamma(x)}{\int f d\gamma}.$$

This theorem was proved by Brascamp and Lieb [47, Theorem 7] (case $g(x) = |x_1|^\alpha$), by Caffarelli [59, Corollary 6] (case $g(x) = g(x_1)$) and then for a general convex function g by Hargé [114]. Hargé followed the idea introduced by Caffarelli to use optimal transport of measure. Unfortunately we cannot develop the theory of optimal transport here but shall still provide Hargé's proof (which is based, as for the proof of log-Sobolev inequalities, on the use of a semi-group that interpolates between the two measures of interest) as well as the statement of the results in optimal transport theory that the proof requires. For more information on the latter, we refer the reader to the two survey books by Villani [195, 196].

We shall define $d\mu(x) = f(x)d\gamma(x)/\int f d\gamma$.

Brenier [48] (see also McCann [149]) has shown that there exists a convex function $\phi : \mathbb{R}^n \to \mathbb{R}$ such that

$$\int g(y)d\mu(y) = \int g(\nabla\phi(x))d\gamma(x).$$

In other words, μ can be realized as the image (or push forward) of μ by the map $\nabla\phi$.

Caffarelli [58, 57] then proved that if the density f is Hölder continuous with exponent $\alpha \in]0,1[$, ϕ is $\mathcal{C}^{2,\alpha}$ for any $\alpha \in]0,1[$ (i.e. twice continuously differentiable with a second derivative Hölder continuous with exponent α). Moreover, by Caffarelli [59, Theorem 11], we know (and here we need to have γ, μ as specified above to get the upper bound) that for any vector $e \in \mathbb{R}^n$,

$$0 \le \partial_{ee}\phi = \langle \mathrm{Hess}(\phi)e, e\rangle \le 1.$$

We now start the proof of Theorem 6.17. Observe first that we can assume without loss of generality that γ is the law of independent centered Gaussian variables with variance one (up to a linear transformation on the x's).

We let $\psi(x) = -\phi(x) + \frac{1}{2}\|x\|_2^2$ so that $0 \le \mathrm{Hess}(\psi) \le I$ (with I the identity matrix and where inequalities hold in the operator sense) and write

$$\int g(y)d\mu(y) = \int g(x - \nabla\psi(x))d\gamma(x).$$

The idea is then to consider the following interpolation

$$\theta(t) = \int g(x - P_t(\nabla\psi)(x))d\gamma(x)$$

with P_t the Ornstein–Uhlenbeck process given, for $h : \mathbb{R}^n \to \mathbb{R}$ by

$$P_t h(x) = \int h(e^{-\frac{t}{2}}x + \sqrt{1 - e^{-t}}y)d\gamma(y)$$

and $P_t(\nabla\psi) = (P_t(\nabla_1\psi), \ldots, P_t(\nabla_n\psi))$ with $\nabla_i\psi = \partial_{x_i}\psi$. Note that for a Lipschitz function h, for all $x \in \mathbb{R}^n$,

$$|P_t h(x) - h(x)| \le \int |h(e^{-\frac{t}{2}}x + \sqrt{1-e^{-t}}y) - h(x)| d\gamma(y)$$

$$\le |h|_{\mathcal{L}}(\sqrt{1-e^{-t}} + (1-e^{-\frac{t}{2}})) \int (\|x\|_2 + \|y\|_2) d\gamma(y)$$

goes to zero as t goes to zero (since $\int \|x\|_2 d\gamma(x) < \infty$). Similarly, for $t > 1$, there is a finite constant C such that

$$|P_t h(x) - \int h d\gamma| \le C|h|_{\mathcal{L}} e^{-\frac{t}{2}} \left(\|x\|_2 + \int \|y\|_2 d\gamma(y) \right)$$

which shows that $P_t h$ goes to $\int h d\gamma$ as t goes to infinity. Since ψ is twice continuously differentiable with Hessian bounded by one, each $\nabla_i \psi$, $1 \le i \le n$, has uniformly bounded derivatives (by one) and so is Lipschitz for the Euclidean norm (with norm bounded by $\sqrt{n}$). Hence, the above applies with $h = \nabla_i \psi$, $1 \le i \le n$.

Let us assume that g is smooth and ∇g is bounded. Then, we deduce from the above estimates that, again because $\int \|x\|_2 d\gamma(x)$ is finite,

$$\lim_{t \to 0} \theta(t) = \theta(0) = \int g(x - \nabla \psi(x)) d\gamma(x) = \int g(x) d\mu(x),$$

$$\lim_{t \to \infty} \theta(t) = \int g(x - \int \nabla \psi d\gamma) d\gamma(x).$$

Since

$$\int \nabla \psi d\gamma = \int (\nabla \psi - x) d\gamma + \int x d\gamma = \int x d\gamma - \int x d\mu$$

we see that Theorem 6.17 is equivalent to prove that $\theta(0) \le \theta(\infty)$ and so it is enough to show that θ is non-decreasing. But, $t \to \theta(t)$ is differentiable with derivative

$$\theta'(t) = -\int \langle \nabla g(x - P_t(\nabla \psi)(x)), \partial_t P_t(\nabla \psi)(x) \rangle \, d\gamma(x) \tag{6.7}$$

with

$$\begin{aligned}
&\partial_t P_t(h)(x) \\
&= \int \left\langle -\frac{1}{2} e^{-\frac{t}{2}} x + \frac{1}{2} e^{-t}(1-e^{-t})^{-\frac{1}{2}} y, \nabla h(e^{-\frac{t}{2}}x + \sqrt{1-e^{-t}}y) \right\rangle d\gamma(y) \\
&= -\frac{1}{2} e^{-\frac{t}{2}} \langle x, P_t(\nabla h)(x) \rangle + \frac{1}{2} e^{-t} \int \Delta h(e^{-\frac{t}{2}}x + \sqrt{1-e^{-t}}y) d\gamma(y) \\
&= -\frac{1}{2} \langle x, \nabla P_t h(x) \rangle + \frac{1}{2} \Delta(P_t h)(x) := L(P_t h)(x)
\end{aligned}$$

where in the second line we integrated by parts under the standard Gaussian law γ. Note also, again by integration by parts, that

$$\int h_1 L h_2 d\gamma(x) = -\frac{1}{2}\int \langle \nabla h_1, \nabla h_2\rangle d\gamma.$$

Hence, (6.7) implies

$$\begin{aligned}\theta'(t) &= -\int \sum_{i=1}^{n} (\partial_i g)(x - P_t(\nabla\psi)) L P_t(\partial_i \psi) d\gamma \\ &= \frac{1}{2}\sum_{i,j=1}^{n} \int \partial_j((\partial_i g)(x - P_t(\nabla\psi)))\partial_j(P_t(\partial_i\psi)))d\gamma \\ &= \frac{1}{2}\sum_{i,j,k=1}^{n} \int (1_{k=j} - \partial_j(P_t(\partial_k\psi)))(\partial_k\partial_i g)(x - P_t(\nabla\psi))P_t(\partial_i\psi))d\gamma.\end{aligned}$$

Thus, if we let

$$M_{ij}(x) := \partial_j(P_t(\partial_i\psi))(x), \text{ and } C_{ij}(x) = (\partial_j\partial_i g)(x - P_t(\nabla\psi)),$$

we have written, with $I_{ij} = 1_{i=j}$ the identity matrix,

$$\begin{aligned}\theta'(t) &= \frac{1}{2}\sum_{j=1}^{n}\sum_{i=1}^{n}\sum_{k=1}^{n} \int (I - M(x))_{kj} C_{ik}(x) M_{ij}(x) d\gamma(x) \\ &= \frac{1}{2}\int \mathrm{Tr}(C(x)(I - M(x))M^*(x))d\gamma(x) \geq 0\end{aligned}$$

since by Caffarelli we know that $0 \leq M(x) \leq I$ for all x, whereas $C \geq 0$ by hypothesis.

This completes the proof for smooth g with bounded gradient. The generalization to all convex functions g is easily done by approximation. The function can indeed be assumed as smooth as wished, since we can always restrict first the integral to a large ball $B(0, R)$, then on this large ball use the Stone–Weierstrass theorem to approximate g by a smooth function, and extend again the integral. We can assume the gradient of g bounded by approximating g by

$$g_R(x) = \sup_{y \in B(0,R)} \{g(y) + \langle \nabla g(y), x - y\rangle\}.$$

g_R is convex and with bounded gradient. Moreover, since $g(x) \geq g(y) + \langle \nabla g(y), x - y\rangle$ by convexity of g, $g_R = g$ on $B(0, R)$, while $g(0) + \langle \nabla g(0), x\rangle \leq g_R(x) \leq g(x)$ shows that $g_R, R \geq 0$ is uniformly integrable so that we can use the dominated convergence theorem to show that the expectation of g_R converges to that of g.

6.5.2 Applications of Brascamp–Lieb inequalities

We apply now Brascamp-Lieb inequalities to the setting of random matrices. To this end, we must restrict ourselves to random matrices with entries following a law that is absolutely continuous with respect to the Lebesgue measure and with strictly log-concave density. We restrict ourselves to the case of m $N \times N$ Hermitian (or symmetric) random matrices with entries following the law

$$d\mu_V^{N,\beta}(\mathbf{A}_1, \ldots, \mathbf{A}_m) := \frac{1}{Z_V^N} e^{-N\mathrm{Tr}(V(\mathbf{A}_1,\ldots,\mathbf{A}_m))} d\mu_c^{N,\beta}(\mathbf{A}_1) \cdots d\mu_c^{N,\beta}(\mathbf{A}_m)$$

with $\mu_c^{N,\beta}$ the law of an $N \times N$ Wigner matrix with complex ($\beta = 2$) or real ($\beta = 1$) Gaussian entries with covariance $1/cN$, that is, the law of the self-adjoint $N \times N$ matrix $\mathbf{A}$ with entries with law

$$\mu_c^{N,\beta}(d\mathbf{A}) = \frac{1}{Z_N^c} e^{-\frac{cN}{2}\mathrm{Tr}(\mathbf{A}^2)} d\mathbf{A}$$

with $d\mathbf{A} = \prod_{i \le j} d\Re(A_{ij}) \prod_{i \le j} d(\Im A_{ij})$ when $\beta = 2$ and $d\mathbf{A} = \prod_{i \le j} dA_{ij}$ if $\beta = 1$.

We assume that V is convex in the sense that for any $N \in \mathbb{N}$,

$$(A_{ij})_{1 \le i \le j \le N} \in \mathcal{E}_N^{(\beta)} \to \mathrm{Tr}(V(\mathbf{A}_1, \ldots, \mathbf{A}_m))$$

is real valued and convex, see sufficient conditions in Exercise 6.11.

Theorem 6.17 implies that for all convex functions g on $(\mathbb{R})^{\beta m N(N-1)/2 + mN}$,

$$\int g(\mathbf{A} - \mathbf{M}) d\mu_V^{N,\beta}(\mathbf{A}) \le \int g(\mathbf{A}) \prod_{i=1}^m d\mu_c^{N,\beta}(\mathbf{A}_i) \tag{6.8}$$

where $\mathbf{M} = \int \mathbf{A} d\mu_V^{N,\beta}(\mathbf{A})$ is the m-tuple of deterministic matrices $(\mathbf{M}_k)_{ij} = \int (\mathbf{A}_k)_{ij} d\mu_V^{N,\beta}(\mathbf{A})$. In (6.8), $g(\mathbf{A})$ is shorthand for a function of the (real and imaginary parts of the) entries of the matrices $\mathbf{A} = (\mathbf{A}_1, \ldots, \mathbf{A}_m)$.

By different choices of the function g we shall now obtain some a priori bounds on the random matrices $(\mathbf{A}_1, \ldots, \mathbf{A}_m)$ with law $\mu_c^{N,\beta}$.

Lemma 6.18. *Assume that the function V is convex and there exists $d > 0$ such that for some finite $c(V)$,*

$$V(X_1, \ldots, X_m) \le c(V) \left(1 + \sum_{i=1}^m X_i^{2d}\right).$$

For $c > 0$, there exists $C_0 = C_0(c, V(0), D_i V(0), c(V), d)$ finite such that for all $i \in \{1, \ldots, m\}$, all $n \in \mathbb{N}$,

$$\limsup_N \mu_V^{N,\beta}\left(\frac{1}{N}\mathrm{Tr}(\mathbf{A}_i^{2n})\right) \le C_0^n.$$

Moreover, C_0 depends continuously on $V(0), D_i V(0), c(V)$ and in particular is uniformly bounded when these quantities are.

Note that this lemma shows that, for $i \in \{1,\dots,m\}$, the spectral measure of $\mathbf{A}_i$ is asymptotically contained in the compact set $[-\sqrt{C_0}, \sqrt{C_0}]$.

Proof. Let k be in $\{1,\dots,m\}$. As $\mathbf{A} \to \mathrm{Tr}(\mathbf{A}_k^{4d})$ is convex by Klein's lemma 6.4, Brascamp–Lieb inequality (6.8) implies that

$$\mu_V^{N,\beta}\left(\frac{1}{N}\mathrm{Tr}(\mathbf{A}_k - \mathbf{M}_k)^{4d}\right) \le \mu_c^{N,\beta}\left(\frac{1}{N}\mathrm{Tr}(\mathbf{A}_k)^{4d}\right) = \mu_c^{N,\beta}(\mathbf{L}_{\mathbf{A}_k}(x^{4d})) \quad (6.9)$$

where $\mathbf{M}_k = \mu_V^{N,\beta}(\mathbf{A}_k)$ stands for the matrix with entries $\int (A_k)_{ij} d\mu_V^{N,\beta}(d\mathbf{A})$. Thus, since $\mu_c^{N,\beta}(\mathbf{L}_{\mathbf{A}_k}(x^{4d}))$ converges by Wigner's theorem 1.13 towards

$$c^{-2d}C_{2d} \le (c^{-1}4)^{2d}$$

with C_{2d} the Catalan number, we only need to control $\mathbf{M}_k$. First observe that for all k the law of A_k is invariant under the multiplication by unitary matrices so that for any unitary matrices U,

$$\mathbf{M}_k = \mu_V^{N,\beta}[\mathbf{A}_k] = U\mu_V^{N,\beta}[\mathbf{A}_k]U^* \Rightarrow \mathbf{M}_k = \mu_V^{N,\beta}\left(\frac{1}{N}\mathrm{Tr}(\mathbf{A}_k)\right)I. \qquad (6.10)$$

Let us bound $\mu_V^{N,\beta}(\frac{1}{N}\mathrm{Tr}(\mathbf{A}_k))$. Jensen's inequality implies

$$Z_N^V \ge e^{-N^2\mu_c^{N,\beta}(\frac{1}{N}\mathrm{Tr}(V))} \ge e^{-N^2 c(V)\mu_c^{N,\beta}\left(\frac{1}{N}\mathrm{Tr}(1+\sum X_i^{2d})\right)}$$

By Theorem 3.3, $\mu_c^{N,\beta}(\frac{1}{N}\mathrm{Tr}(X_i^{2d})$ converges to a finite constant and therefore we find a finite constant $C(V)$ such that $Z_N^V \ge e^{-N^2 C(V)}$.

We now use the convexity of V, to find that for all N,

$$\mathrm{Tr}\left(V(\mathbf{A})\right) \ge \mathrm{Tr}(V(0) + \sum_{i=1}^m D_i V(0)\mathbf{A}_i)$$

with D_i the cyclic derivative introduced in Lemma 6.2. By Chebyshev's inequality, we therefore obtain, for all $\lambda \ge 0$,

$$\begin{aligned}
&\mu_V^{N,\beta}(|\mathbf{L}_{\mathbf{A}_k}(x)| \ge y) \le \mu_V^{N,\beta}(\mathbf{L}_{\mathbf{A}_k}(x) \ge y) + \mu_V^{N,\beta}(-\mathbf{L}_{\mathbf{A}_k}(x) \ge y)\\
&\le e^{N^2(C(V)-V(0)-\lambda y)}\Big(\mu_c^{N,\beta}(e^{-N\mathrm{Tr}(\sum_{i=1}^m D_i V(0)\mathbf{A}_i - \lambda \mathbf{A}_k)})\\
&\qquad + \mu_c^{N,\beta}(e^{-N\mathrm{Tr}(\sum_{i=1}^m D_i V(0)\mathbf{A}_i + \lambda \mathbf{A}_k)})\Big)\\
&= e^{N^2(C(V)-V(0)-\lambda y)} e^{\frac{N}{2c}\sum_{\ell\neq k}\mathrm{Tr}(D_i V(0)^2)}\\
&\qquad \Big(e^{\frac{N}{2c}\mathrm{Tr}((D_k V(0)-\lambda)^2)} + e^{\frac{N}{2c}\mathrm{Tr}((D_k V(0)+\lambda)^2)}\Big).
\end{aligned}$$

Optimizing with respect to λ shows that there exists $B = B(V)$

$$\mu_V^N\left(|\mathbf{L}_{\mathbf{A}_k}(x)| \ge y\right) \le e^{BN^2 - \frac{N^2 c}{4}y^2}$$

so that for N large enough,

$$\mu_V^{N,\beta}(|\mathbf{L}_{\mathbf{A}_k}(x)|) = \int_0^\infty \mu_V^N\left(|\mathbf{L}_{\mathbf{A}_k}(x)|\geq y\right) dy$$
$$\leq 4\sqrt{c^{-1}B} + \int_{y\geq 4\sqrt{c^{-1}B}} e^{-\frac{N^2c}{4}(y^2-4\frac{B}{c})}dy \leq 8\sqrt{Bc^{-1}}. \quad (6.11)$$

This, with (6.9), completes the proof. □

Let us derive some other useful properties due to the Brascamp–Lieb inequality. We first obtain an estimate on the spectral radius $\lambda_{\max}^N(\mathbf{A})$, defined as the maximum of the spectral radius of $\mathbf{A}_1,\ldots,\mathbf{A}_m$ under the law $\mu_V^{N,\beta}$.

Lemma 6.19. *Under the hypothesis of Lemma 6.18, there exists $\alpha = \alpha(c) > 0$ and $M_0 = M_0(V) < \infty$ such that for all $M\geq M_0$ and all integer N,*

$$\mu_V^{N,\beta}(\lambda_{max}^N(\mathbf{A}) > M) \leq e^{-\alpha MN}.$$

Moreover, $M_0(V)$ is uniformly bounded when $V(0)$, $D_iV(0)$ and $c(V)$ are.

Proof. The spectral radius $\lambda_{\max}^N(\mathbf{A}) = \max_{1\leq i\leq m}\sup_{\|u\|_2=1} < u, \mathbf{A}_i\mathbf{A}_i^* u >^{\frac{1}{2}}$ is a convex function of the entries (see Lemma 6.5), so we can apply the Brascamp–Lieb inequality (6.8) to obtain that for all $s\in[0,\frac{c}{10}]$,

$$\int e^{sN\lambda_{\max}^N(\mathbf{A}-\mathbf{M})}d\mu_V^{N,\beta}(\mathbf{A}) \leq \int e^{sN\lambda_{\max}^N(\mathbf{A})}d\mu_c^{N,\beta}(\mathbf{A}).$$

But, by Theorem 6.6 applied with a quadratic potential V, we know that

$$\begin{aligned}
&\int e^{sN\lambda_{\max}^N(\mathbf{A})}d\mu_c^{N,\beta}(\mathbf{A}) \\
&\leq e^{sN\mu_c^{N,\beta}(\lambda_{\max}^N)}\int e^{sN(\lambda_{\max}^N-\mu_c^{N,\beta}(\lambda_{\max}^N))}d\mu_c^{N,\beta} \\
&= sNe^{sN\mu_c^{N,\beta}(\lambda_{\max}^N)}\int_{-\infty}^\infty e^{sNy}\mu_c^{N,\beta}\left(\lambda_{\max}^N-\mu_c^{N,\beta}(\lambda_{\max}^N)\geq y\right)dy \\
&\leq sNe^{sN\mu_c^{N,\beta}(\lambda_{\max}^N)}\left(1+2\int_0^\infty e^{sNy}e^{-\frac{Nc}{4}y^2}dy\right) \\
&\leq \sqrt{2\pi}sNe^{sN\mu_c^{N,\beta}(\lambda_{\max}^N)}(1+2e^{\frac{2s^2N}{c}})
\end{aligned}$$

Hence, since $\mu_c^{N,\beta}(\lambda_{\max}^N)$ is uniformly bounded by Theorem 2.3, we deduce that for all $s\geq 0$, there exists a finite constant $C(s)$ such that

$$\int e^{sN\lambda_{\max}^N(\mathbf{A}-\mathbf{M})}d\mu_V^{N,\beta}(\mathbf{A}) \leq C(s)^N.$$

By (6.10) and (6.11), we know that

$$\lambda^N_{\max}(\mathbf{A}) \le \lambda^N_{\max}(\mathbf{A}-\mathbf{M}) + \lambda^N_{\max}(\mathbf{M}) \le \lambda^N_{\max}(\mathbf{A}-\mathbf{M}) + 8\sqrt{Bc^{-1}}$$

from which we deduce that $\int e^{sN\lambda^N_{\max}(\mathbf{A})} d\mu_V^{N,\beta}(\mathbf{A}) \le C^N$ for a positive finite constant C. We conclude by a simple application of Chebyshev's inequality. $\square$

Lemma 6.20. *If $c > 0$, $\epsilon \in]0, \frac{1}{2}[$, then there exists $C = C(c,\epsilon) < \infty$ such that for all $d \le N^{\frac{1}{2}-\epsilon}$,*

$$\mu_V^{N,\beta}(|\lambda^N_{max}(\mathbf{A})|^d) \le C^d.$$

Note that this control could be generalized to $d \le N^{2/3-\epsilon}$, by using the refinements obtained by Soshnikov in [180, Theorem 2 p.17] but we shall not need it here.

Proof. Since $\mathbf{A} \to \lambda^N_{\max}(\mathbf{A})$ is convex, we can again use Brascamp–ieb inequalities to insure that

$$\mu_V^{N,\beta}\left(|\lambda^N_{\max}(\mathbf{A} - \mu_V^{N,\beta}(\mathbf{A}))|^d\right) \le \mu_c^{N,\beta}\left(|\lambda^N_{\max}(\mathbf{A} - \mu_c^{N,\beta}(\mathbf{A}))|^d\right).$$

Now, we have seen in the proof of Lemma 6.18 that $\mu_V^{N,\beta}(\mathbf{A})$ has a uniformly bounded spectral radius, say by x. Moreover, by Theorem 2.3, we find that

$$\mu_c^{N,\beta}\left(|\lambda^N_{\max}(\mathbf{A})|^{N^{\frac{1}{2}-\epsilon/2}}\right) \le c(\epsilon)\frac{N(2c^{-1})^{N^{\frac{1}{2}-\epsilon/2}}}{\sqrt{\pi N^{3(\frac{1}{2}-\epsilon/2)}}}.$$

Applying Jensen's inequality we therefore get, for $d \le N^{\frac{1}{2}-\epsilon}$,

$$\mu_c^{N,\beta}\left(|\lambda^N_{\max}(\mathbf{A})|^d\right) \le c'(\epsilon)(2c^{-1})^d.$$

Hence,

$$\mu_V^{N,\beta}\left(|\lambda^N_{\max}(\mathbf{A})|^d\right)^{\frac{1}{d}} \le x + c'(\epsilon)^{\frac{1}{d}} 2c^{-1}$$

which proves the claim. $\square$

6.5.3 Coupling concentration inequalities and Brascamp–Lieb inequalities

We next turn to concentration inequalities for the trace of polynomials on the set

$$\Lambda^N_M = \{\mathbf{A} \in \mathcal{H}^m_N : \lambda^N_{\max}(\mathbf{A}) = \max_{1\le i\le m}(\lambda^N_{\max}(\mathbf{A}_i)) \le M\} \subset \mathbb{R}^{N^2 m}.$$

We let

$$\tilde{\delta}^N(P) := \mathrm{Tr}(P(\mathbf{A}_1,\ldots,\mathbf{A}_m)) - \mu_V^{N,\beta}\left(\mathrm{Tr}(P(\mathbf{A}_1,\ldots,\mathbf{A}_m))\right).$$

Then, we have

Lemma 6.21. *For all N in $\mathbb{N}$, all $M > 0$, there exists a finite constant $C(P,M)$ and $\epsilon(P,M,N)$ such that for any $\epsilon > 0$,*

$$\mu_V^{N,\beta}\left(\{|\tilde{\delta}^N(P)| \geq \epsilon + \epsilon(P,M,N)\} \cap \Lambda_M^N\right) \leq 2e^{-\frac{c\epsilon^2}{2C(P,M)}}.$$

If P is a monomial of degree d we can choose

$$C(P,M) \leq d^2 M^{2(d-1)}$$

and there exists $M_0 < \infty$ so that for $M \geq M_0$, all $\epsilon \in]0, \frac{1}{2}[$, and all monomial P of degree smaller than $N^{1/2-\epsilon}$,

$$\epsilon(P,M,N) \leq 3dN(CM)^{d+1} e^{-\frac{\alpha}{2}NM}$$

with C the constant of Lemma 6.20.

Proof. It is enough to consider the case where P is a monomial. By Corollary 6.12, we only need to control $\epsilon(P,M,N)$.

$$\begin{aligned}
\epsilon(P,M,N) &\leq \mu_V^{N,\beta}\Bigg(1_{(\Lambda_M^N)^c}\Bigg(|\mathrm{Tr}(P)| + dM^{d-1}\sqrt{\sum_{i=1}^m \mathrm{Tr}(\mathbf{A}_i\mathbf{A}_i^*)} \\
&\qquad\qquad + \sup_{\mathbf{A}\in\Lambda_M^N} |\mathrm{Tr}(P(\mathbf{A}))|\Bigg)\Bigg) \\
&\leq N\mu_V^{N,\beta}\Big(1_{(\Lambda_M^N)^c}\Big(|\lambda_{\max}^N(\mathbf{A})|^d \\
&\qquad\qquad + \sqrt{C(P,M)}|\lambda_{\max}^N(\mathbf{A})|^2 + M^d\Big)\Big).
\end{aligned}$$

Now, by Lemmas 6.19 and 6.20, we find that

$$\mu_V^N\left(1_{(\Lambda_M^N)^c}|\lambda_{\max}^N(\mathbf{A})|^d\right) \leq \mu_V^N\left(1_{(\Lambda_M^N)^c}\right)^{\frac{1}{2}} \mu_V^N\left(|\lambda_{\max}^N(\mathbf{A})|^{2d}\right)^{\frac{1}{2}} \leq C^d e^{-\frac{\alpha}{2}NM}.$$

By the previous control on $C(P,M)$, we get, for $d \leq N^{\frac{1}{2}-\epsilon}$ and M large enough,

$$\epsilon(P,M,N) \leq 3dN(CM)^{d+1}e^{-\frac{\alpha}{2}NM},$$

which proves the claim. □

For later purposes, we have to find a control on the variance of $\mathbf{L}$.

Lemma 6.22. *For any $c > 0$ and $\epsilon \in]0, \frac{1}{2}[$, there exists $B, C, M_0 > 0$ such that for all $\mathbf{t} \in \mathbf{B}_{\eta,\mathbf{c}}$, all $M \geq M_0$, and monomial P of degree less than $N^{\frac{1}{2}-\epsilon}$,*

$$\mu_V^{N,\beta}\left((\tilde{\delta}^N(P))^2\right) \leq BC(P,M) + C^{2d}N^4 e^{-\frac{\alpha MN}{2}}. \tag{6.12}$$

Moreover, the constants C, M_0, B depend continuously on $V(0), D_iV(0)$ and $c(V)$.

Proof. If P is a monomial of degree d, we write

$$\mu_V^{N,\beta}((\tilde{\delta}^N(P))^2) \leq \mu_V^{N,\beta}(\mathbf{1}_{\Lambda_M^N}(\tilde{\delta}^N(P))^2) + \mu_V^{N,\beta}(\mathbf{1}_{(\Lambda_M^N)^c}(\tilde{\delta}^N(P))^2) = I_1 + I_2. \tag{6.13}$$

For I_1, the previous lemma implies that, for $d \leq N$,

$$\begin{aligned} I_1 &= 2\int_0^\infty x\mu_V^{N,\beta}\left(\{|\mathrm{Tr}(P) - \mu_V^{N,\beta}(\mathrm{Tr}(P))| \geq x\} \cap \Lambda_M^N\right) dx \\ &\leq \epsilon(P,N,M)^2 + 4\int_0^\infty x e^{-\frac{cx^2}{2C(P,M)}} dx \leq BC(P,M) \end{aligned}$$

with a constant B that depends only on c. For the second term, we take $M \geq M_0$ with M_0 as in Lemma 6.19 to get

$$I_2 \leq \mu_V^{N,\beta}[(\Lambda_M^N)^c]^{\frac{1}{2}} \mu_V^{N,\beta}((\tilde{\delta}^N(P))^4)^{\frac{1}{2}} \leq e^{-\frac{\alpha MN}{2}} \mu_V^{N,\beta}((\tilde{\delta}^N(P))^4)^{\frac{1}{2}}.$$

By the Cauchy–Schwartz inequality, we obtain the control

$$\mu_V^{N,\beta}[\tilde{\delta}^N(P)^4] \leq 2^4 \mu_V^{N,\beta}((\mathrm{Tr}(P))^4).$$

Now, by non-commutative Hölder's inequality Theorem 19.5,

$$[\mathrm{Tr}(P)]^4 \leq N^4 \max_{1 \leq i \leq m} \frac{1}{N}\mathrm{Tr}(A_i^{4d})$$

so that we obtain the bound

$$\mu_V^{N,\beta}[\tilde{\delta}^N(P)^4] \leq 2^4 N^4 \max_{1 \leq i \leq m} \mu_V^{N,\beta}[\frac{1}{N}\mathrm{Tr}(A_i^{4d})].$$

By Lemma 6.20, for $d \leq N^{\frac{1}{2}-\epsilon}$,

$$\mu_V^{N,\beta}\left[\frac{1}{N}\mathrm{Tr}(A_i^{4d})\right] \leq C^{2d}. \tag{6.14}$$

Plugging back this estimate into (6.13), we have proved that for N and M sufficiently large, all monomials P of degree $d \leq N^{\frac{1}{2}-\epsilon}$, all $\mathbf{t} \in \mathbf{B}_{\eta,\mathbf{c}}$

$$\mu_V^{N,\beta}\left((\hat{\delta}^N(P))^2\right) \leq BC(P,M) + C^{2d}N^4 e^{-\frac{\alpha MN}{2}}$$

with a finite constant C depending only on ϵ, c and M_0. $\square$

Bibliographical notes. Brascamp–Lieb inequalities were first introduced in [47]. The relation between FKG inequalities and optimal transportation was shown in [59], based on optimal bounds on the Hessian of the transport map. The application of this strategy to Brascamp–Lieb inequalities is due to Hargé [114]. It was used in the context of random matrices in [104, 105], following the lines that we shall develop in the next part.

Part III

Matrix models

In this part, we study matrix models, that is, the laws of interacting Hermitian matrices of the form

$$d\mu_V^{N,2}(\mathbf{A}_1,\ldots,\mathbf{A}_m) := \frac{1}{Z_V^N} e^{-N\mathrm{Tr}(V(\mathbf{A}_1,\ldots,\mathbf{A}_m))} d\mu^{N,2}(\mathbf{A}_1)\cdots d\mu^{N,2}(\mathbf{A}_m)$$

where Z_V^N is the normalizing constant given by the matrix integral

$$Z_V^N = \int e^{-N\mathrm{Tr}(V(\mathbf{A}_1,\ldots,\mathbf{A}_m))} d\mu^{N,2}(\mathbf{A}_1)\cdots d\mu^{N,2}(\mathbf{A}_m)$$

and V is a polynomial in m non-commutative variables:

$$V(X_1,\ldots,X_m) = \sum_{i=1}^n t_i q_i(X_1,\ldots,X_m)$$

with q_i non-commutative monomials:

$$q_i(X_1,\ldots,X_m) = X_{j_1^i}\cdots X_{j_{r_i}^i}$$

for some $j_l^k \in \{1,\ldots,m\}$, $r_i \geq 1$. Moreover, $d\mu^{N,2}(\mathbf{A})$ denotes the standard law of the **GUE**, i.e., under $d\mu^{N,2}(\mathbf{A})$, $\mathbf{A}$ is an $N\times N$ Hermitian matrix such that

$$A(k,l) = \bar{A}(l,k) = \frac{g_{kl} + i\tilde{g}_{kl}}{\sqrt{2N}},\ k<l, \quad A(k,k) = \frac{g_{kk}}{\sqrt{N}}$$

with independent centered standard Gaussian variables $(g_{kl},\tilde{g}_{kl})_{k\leq l}$. In other words

$$d\mu^{N,2}(\mathbf{A}) = Z_N^{-1} 1_{\mathbf{A}\in\mathcal{H}_N^{(2)}} e^{-\frac{N}{2}\mathrm{Tr}(\mathbf{A}^2)} \prod_{1\leq i\leq j\leq N} d\Re(A(i,j)) \prod_{1\leq i<j\leq N} d\Im(A(i,j)).$$

Since we restrict ourselves to Hermitian matrices in this part, we shall drop the subscript $\beta = 2$ and write for short $\mu^N = \mu^{N,2}$.

Let us define by $\mathbb{C}\langle X_1,\ldots,X_m\rangle$ the set of polynomials in m non-commutative variables and, for $P \in \mathbb{C}\langle X_1,\ldots,X_m\rangle$,

$$\hat{\mathbf{L}}^N(P) := \mathbf{L}_{\mathbf{A}_1,\ldots,\mathbf{A}_m}(P) = \frac{1}{N}\mathrm{Tr}\left(P(\mathbf{A}_1,\ldots,\mathbf{A}_m)\right).$$

When V vanishes, we have seen in Chapter 3 that for any polynomial function P, $\hat{\mathbf{L}}^N(P)$ converges as N goes to infinity. Moreover the limit $\sigma^m(P)$ is such that if P is a monomial, $\sigma^m(P)$ is the number of non-crossing pair partitions of a set of points with m colors, or equivalently the number of planar maps with one star of type P. In this part, we shall generalize such a type of result to the case where V does not vanish but is “small” and “nice” in a sense that we shall precise.

This part is motivated by a work of Brézin, Parisi, Itzykson and Zuber [50] and large developments that occurred thereafter in theoretical physics [78].

They specialized an idea of 't Hooft [187] to show that if $V = \sum_{i=1}^n t_i q_i$ with fixed monomials q_i of m non-commutative variables, and if we see $Z_V^N = Z_{\mathbf{t}}^N$ as a function of $\mathbf{t} = (t_1, \ldots, t_n)$,

$$\log Z_{\mathbf{t}}^N := \sum_{g \geq 0} N^{2-2g} F_g(\mathbf{t}), \tag{III.15}$$

where

$$F_g(\mathbf{t}) := \sum_{k_1,\ldots,k_n \in \mathbb{N}^k} \prod_{i=1}^k \frac{(-t_i)^{k_i}}{k_i!} \mathcal{M}_g((q_i, k_i)_{1 \leq i \leq k})$$

is a generating function of integer numbers $\mathcal{M}_g((q_i, k_i)_{1 \leq i \leq k})$ that count certain graphs called maps. A map is a connected oriented graph that is embedded into a surface. Its genus g is by definition the genus of a surface in which it can be embedded in such a way that edges do not cross and the faces of the graph (that are defined by following the boundary of the graph) are homeomorphic to a disk. The vertices of the maps we shall consider will have the structure of a star, that is a vertex with colored edges embedded into a surface (that is an order on the colored edges is specified). More precisely, a star of type q, for some monomial $q = X_{\ell_1} \cdots X_{\ell_k}$, is a vertex with degree $\deg(q)$ and oriented colored half-edges with one marked half edge of color ℓ_1, the second of color ℓ_2, etc., until the last one of color ℓ_k. $\mathcal{M}_g((q_i, k_i)_{1 \leq i \leq k})$ is then the number of maps with k_i stars of type q_i, $1 \leq i \leq n$.

Adding to V a term $t\,q$ for some monomial q and identifying the first-order derivative with respect to t at $t = 0$ we derive from (III.15)

$$\int \hat{\mathbf{L}}^N(q) d\mu_V^N = \sum_{g \geq 0} N^{-2g} \sum_{k_1,\ldots,k_n \in \mathbb{N}^k} \prod_{i=1}^k \frac{(-t_i)^{k_i}}{k_i!} \mathcal{M}_g((q_i, k_i)_{1 \leq i \leq k}, (q, 1)). \tag{III.16}$$

The equalities (III.15) and (III.16) derived in [50] are only formal, i.e., mean that all the derivatives on both sides of the equality coincide at $\mathbf{t} = 0$. They can thus be deduced from the Wick formula (which gives the expression of arbitrary moments of Gaussian variables) or equivalently by the use of Feynman diagrams.

Even though topological expansions such as (III.15) and (III.16) were first introduced by 't Hooft in the course of computing the integrals, the natural reverse question of computing the numbers $\mathcal{M}_g((q_i, k_i)_{1 \leq i \leq k})$ by studying the associated integrals over matrices encountered a large success in theoretical physics (see, e.g., the review papers [78, 70]). In the course of doing so, one would like for instance to compute $\lim_{N \to \infty} N^{-2} \log Z_{\mathbf{t}}^N$ and claim that this limit is equal to $F_0(\mathbf{t})$. There is here the belief that one can interchange derivatives and limit, a claim that needs to be justified.

We shall indeed prove that the formal limit can be strenghtened into a large N expansion in the sense that

$$\frac{1}{N^2}\log Z_{\mathbf{t}}^N = F_0(\mathbf{t}) + \frac{1}{N^2}F_1(\mathbf{t}) + o(N^{-2})$$

where $N^2 \times o(N^{-2})$ goes to zero as N goes to infinity. This asymptotic expansion holds when V is small and satisfies some convexity hypothesis (which insures that the partition function Z_V^N is finite and the support of the limiting spectral measures of $\mathbf{A}_i$, $1 \le i \le m$, under μ_V^N is connected, see [106]).

This part summarizes results from [104] and [105]. The full expansion (i.e., higher-order corrections) was obtained by E. Maurel Segala [148] in the multi-matrix setting. Such expansion in the one matrix case was already derived on a physical level of rigor in [4] and then made rigorous in [2, 86]. However, in the case of one matrix, techniques based on orthogonal polynomials can be used. In the multi-matrix case this approach fails in general (or at least has not yet been extended). [104, 105, 148] take a completely different route based on the free probability setting of limiting tracial states and of the so-called Master loop or Schwinger–Dyson equations.

We start this part by introducing the combinatorial objects we shall consider and their relations with non-commutive polynomials. Then, we prove the formal expansion of Brézin, Itzykson, Parisi and Zuber. The next two chapters consider the asymptotic expansion; we first obtain the convergence of the free energy towards the expected generating function for the enumeration of planar maps, and then study the first order correction to this limit, showing it is related with the enumeration of maps with genus one.

The techniques we shall present here have the advantage to be robust. We use them here to study partition functions of Hermitian matrices, but they can be generalized to orthogonal or symplectic matrices (in a work in progress of E. Maurel Segala) or to matrices following the Haar measure on the unitary group [66]. The last extension is particularly interesting since then Gaussian calculus and Feynman diagram techniques fail (since unitary matrices have no Gaussian entries) so that the diagrammatic representation of the limit is not straightforward even on a formal level (see [65] for a formal expansion with no diagrammatic interpretation).

7

Maps and Gaussian calculus

We start this chapter by introducing non-commutative polynomials and their relations with special vertices called stars. We then relate the enumeration of the maps buildt upon such vertices with the formal expansion of Gaussian matrix integrals.

7.1 Combinatorics of maps and non-commutative polynomials

In this section, we define non-commutative polynomials and non-commutative laws such as the "empirical distribution" of matrices $\mathbf{A}_1, \ldots, \mathbf{A}_m$ which generalize the notion of probability measures and empirical measures to the case of non-commutative variables. We will then describe precisely the combinatorial objects related with matrix integrals. Recalling the bijection between non-commutative monomials and graphical objects such as stars or ordered sets of colored points, we will show how operations such as derivatives on monomials have their graphical interpretation. This will be our basis to show that some differential equations for non-commutative laws can be interpreted in terms of some surgery on maps, as introduced by Tutte [194] to prove induction relations for map enumeration (see, e.g., Bender and Canfield [29] for generalizations).

7.2 Non-commutative polynomials

We denote by $\mathbb{C}\langle X_1, \ldots, X_m\rangle$ the set of complex polynomials in the non-commutative unknowns $X_1, \ldots, X_m$. Let $*$ denote the linear involution such that for all complex z and all monomials

$$(zX_{i_1} \cdots X_{i_p})^* = \overline{z} X_{i_p} \cdots X_{i_1}.$$

We will say that a polynomial P is self-adjoint if $P = P^*$ and denote by $\mathbb{C}\langle X_1, \dots, X_m\rangle_{sa}$ the set of self-adjoint elements of $\mathbb{C}\langle X_1, \dots, X_m\rangle$.

The potential V will later on be assumed to be self-adjoint. This means that

$$V(\mathbf{A}) = \sum_{j=1}^{n} t_j q_i = \sum_{j=1}^{n} \bar{t}_j q_j^* = \sum_{j=1}^{n} \Re(t_j) \frac{q_j + q_j^*}{2} + \sum_{j=1}^{n} \Im(t_j) \frac{q_j - q_j^*}{2i}.$$

Note that the parameters $(t_j = \Re(t_j) + i\Im(t_j), 1 \le j \le n)$ may a priori be complex. This hypothesis guarantees that $\mathrm{Tr}(V(\mathbf{A}))$ is real for all $\mathbf{A} = (\mathbf{A}_1, \dots, \mathbf{A}_m)$ in the set $\mathcal{H}_N$ of $N \times N$ Hermitian matrices.

In the sequel, the monomials $(q_i)_{1\le i\le n}$ will be fixed and we will consider $V = V_{\mathbf{t}} = \sum_{i=1}^{n} t_i q_i$ as the parameters t_i vary in such a way that V stays self-adjoint.

7.2.1 Convexity

We shall assume hereafter that V is convex, see Definition 6.10. While it may not be the optimal hypothesis, convexity provides many simple arguments. Note that as we add a Gaussian potential $\frac{1}{2}\sum_{i=1}^{m} X_i^2$ to V we can relax the hypothesis by the notion of c-convexity.

Definition 7.1. *We say that V is c-convex if $c > 0$ and $V + \frac{1-c}{2}\sum_1^m X_i^2$ is convex. In other words, the Hessian of*

$$\begin{array}{rccl} \phi_V^{N,c}: & \mathcal{E}_N^{(2)} & \longrightarrow & \mathbb{R} \\ & (\Re(A_k(i,j)), \Im(A_k(i,j)))_{1\le i\le j\le N}^{1\le k\le m} & \longrightarrow & \mathrm{Tr}(V(\mathbf{A}_1, \dots, \mathbf{A}_m) \\ & & & \quad + \frac{1-c}{2}\sum_{k=1}^{m} \mathbf{A}_i^2) \end{array}$$

is non-negative. Here, for $k \in \{1, \dots, m\}$, $\mathbf{A}_k$ is the Hermitian matrix with entries $\sqrt{2}^{-1}(A_k(p,q) + iA_k(q,p))$ above the diagonal and $A_k(i,i)$ on the diagonal.

Note that when V is c-convex, μ_V^N has a log-concave density with respect to the Lebesgue measure so that many results from the previous part will apply, in particular concentration inequalities and Brascamp–Lieb inequalities.

In the rest of this chapter, we assume that V is c-convex for some $c > 0$ fixed. Arbitrary potentials could be considered as far as first-order asymptotics are studied in [104], at the price of adding a cutoff. In fact, adding a cutoff and choosing the parameters t_i's small enough (depending eventually on this cutoff), forces the interaction to be convex so that most of the machinery we are going to describe will apply also in this context. We let $V = V_{\mathbf{t}} = \sum_{i=1}^{n} t_i q_i$ and define $U_c = \{\mathbf{t} \in \mathbb{C}^n : V_{\mathbf{t}} \text{ is } c\text{-convex}\} \subset \mathbb{C}^n$. Moreover, B_η will denote the open ball in $\mathbb{C}^n$ centered at the origin and with radius $\eta > 0$ (for instance for the metric $|\mathbf{t}| = \max_{1\le i\le n}|t_i|$).

7.2.2 Non-commutative derivatives

First, for $1 \leq i \leq m$, let us define the non-commutative derivatives ∂_i with respect to the variable X_i. They are linear maps from $\mathbb{C}\langle X_1, \ldots, X_m\rangle$ to $\mathbb{C}\langle X_1, \ldots, X_m\rangle^{\otimes 2}$ given by the Leibniz rule

$$\partial_i PQ = \partial_i P \times (1 \otimes Q) + (P \otimes 1) \times \partial_i Q \tag{7.1}$$

and $\partial_i X_j = \mathbf{1}_{i=j} 1 \otimes 1$. Here, $\times$ is the multiplication on $\mathbb{C}\langle X_1, \ldots, X_m\rangle^{\otimes 2}$; $P \otimes Q \times R \otimes S = PR \otimes QS$. So, for a monomial P, the following holds:

$$\partial_i P = \sum_{P = RX_iS} R \otimes S$$

where the sum runs over all possible monomials R, S so that P decomposes into RX_iS. We can iterate the non-commutative derivatives; for instance ∂_i^2 : $\mathbb{C}\langle X_1, \ldots, X_m\rangle \to \mathbb{C}\langle X_1, \ldots, X_m\rangle \otimes \mathbb{C}\langle X_1, \ldots, X_m\rangle \otimes \mathbb{C}\langle X_1, \ldots, X_m\rangle$ is given for a monomial function P by

$$\partial_i^2 P = 2 \sum_{P = RX_iSX_iQ} R \otimes S \otimes Q.$$

We denote by $\sharp : \mathbb{C}\langle X_1, \ldots, X_m\rangle^{\otimes 2} \times \mathbb{C}\langle X_1, \ldots, X_m\rangle \to \mathbb{C}\langle X_1, \ldots, X_m\rangle$ the map $P \otimes Q \sharp R = PRQ$ and generalize this notation to $P \otimes Q \otimes R \sharp (S, V) = PSQVR$. So $\partial_i P \sharp R$ corresponds to the derivative of P with respect to X_i in the direction R, and similarly $2^{-1}[D_i^2 P \sharp (R, S) + D_i^2 P \sharp (S, R)]$ the second derivative of P with respect to X_i in the directions R, S.

We also define the so-called cyclic derivative D_i. If m is the map $m(A \otimes B) = BA$, let us define $D_i = m \circ \partial_i$. For a monomial P, $D_i P$ can be expressed as

$$D_i P = \sum_{P = RX_iS} SR.$$

7.2.3 Non-commutative laws

For $(\mathbf{A}_1, \ldots, \mathbf{A}_m) \in \mathcal{H}_N^m$, let us define the linear form $\mathbf{L}_{\mathbf{A}_1, \ldots, \mathbf{A}_m}$ from $\mathbb{C}\langle X_1, \ldots, X_m\rangle$ into $\mathbb{C}$ by

$$\mathbf{L}_{\mathbf{A}_1, \ldots, \mathbf{A}_m}(P) = \frac{1}{N} \mathrm{Tr}\left(P(\mathbf{A}_1, \ldots, \mathbf{A}_m)\right)$$

where Tr is the standard trace $\mathrm{Tr}(A) = \sum_{i=1}^N A(i,i)$. $\mathbf{L}_{\mathbf{A}_1, \ldots, \mathbf{A}_m}$ will be called the empirical distribution of the matrices (note that in the case of one matrix, it is the empirical distribution of the eigenvalues of this matrix). When the matrices $\mathbf{A}_1, \ldots, \mathbf{A}_m$ are generic and distributed according to μ_V^N, we will drop the subscripts $\mathbf{A}_1, \ldots, \mathbf{A}_m$ and write for short $\hat{\mathbf{L}}^N = \mathbf{L}_{\mathbf{A}_1, \ldots, \mathbf{A}_m}$. We define, when $V = V_{\mathbf{t}} = \sum_{i=1}^n t_i q_i$,

$$\bar{\mathbf{L}}_t^N(P) := \mu_{\mathbf{V_t}}^N[\hat{\mathbf{L}}^N(P)].$$

$\hat{\mathbf{L}}^N, \bar{\mathbf{L}}_t^N$ will be seen as elements of the algebraic dual $\mathbb{C}\langle X_1, \ldots, X_m\rangle^{\mathcal{D}}$ of $\mathbb{C}\langle X_1, \ldots, X_m\rangle$ equipped with the involution $*$. $\mathbb{C}\langle X_1, \ldots, X_m\rangle^{\mathcal{D}}$ is equipped with its weak topology.

Definition 7.2. *A sequence $(\mu_n)_{n\in\mathbb{N}}$ in $\mathbb{C}\langle X_1, \ldots, X_m\rangle^{\mathcal{D}}$ converges weakly (or in moments) to $\mu \in \mathbb{C}\langle X_1, \ldots, X_m\rangle^{\mathcal{D}}$ iff for any $P \in \mathbb{C}\langle X_1, \ldots, X_m\rangle$,*

$$\lim_{n\to\infty} \mu_n(P) = \mu(P).$$

Lemma 7.3. *Let $C(\ell_1, \ldots, \ell_r), \ell_i \in \{1, \ldots, m\}, r \in \mathbb{N}$, be finite non-negative constants and*

$$K(C) = \{\mu \in \mathbb{C}\langle X_1, \ldots, X_m\rangle^{\mathcal{D}}; |\mu(X_{\ell_1} \cdots X_{\ell_r})| \le C(\ell_1, \ldots, \ell_r) \quad \forall \ell_i \in \{1, \ldots, m\}, r \in \mathbb{N}\}.$$

Then, any sequence $(\mu_n)_{n\in\mathbb{N}}$ in $K(C)$ is sequentially compact, i.e., has a subsequence $(\mu_{\phi(n)})_{n\in\mathbb{N}}$ that converges weakly (or in moments).

Proof. Since $\mu_n(X_{\ell_1} \cdots X_{\ell_r}) \in \mathbb{C}$ is uniformly bounded, it has converging subsequences. By a diagonalization procedure, since the set of monomials is countable, we can ensure that for a subsequence $(\phi(n), n \in \mathbb{N})$, the terms $\mu_{\phi(n)}(X_{\ell_1} \cdots X_{\ell_r}), \ell_i \in \{1, \ldots, m\}, r \in \mathbb{N}$ converge simultaneously. The limit defines an element of $\mathbb{C}\langle X_1, \ldots, X_m\rangle^{\mathcal{D}}$ by linearity. □

The following is a triviality, that however we recall since we will use it several times.

Corollary 7.4. *Let $C(\ell_1, \ldots, \ell_r), \ell_i \in \{1, \ldots, m\}, r \in \mathbb{N}$, be finite non negative constants and $(\mu_n)_{n\in\mathbb{N}}$ a sequence in $K(C)$ that has a unique limit point. Then $(\mu_n)_{n\in\mathbb{N}}$ converges weakly (or in moments) to this limit point.*

Proof. Otherwise we could choose a subsequence that stays at positive distance of this limit point, but extracting again a converging subsequence gives a contradiction. Note as well that any limit point will belong automatically to $\mathbb{C}\langle X_1, \ldots, X_m\rangle^{\mathcal{D}}$. □

Remark 7.5. *The laws $\hat{\mathbf{L}}^N, \bar{\mathbf{L}}_t^N$ are more than only linear forms on the space $\mathbb{C}\langle X_1, \ldots, X_m\rangle$; they satisfy also the properties*

$$\mu(PP^*) \ge 0, \quad \mu(PQ) = \mu(QP), \quad \mu(1) = 1 \tag{7.2}$$

for any polynomial functions P, Q. Since these conditions are closed for the weak topology, we see that any limit point of $\hat{\mathbf{L}}^N, \bar{\mathbf{L}}_t^N$ will also satisfy these properties. A linear functional on $\mathbb{C}\langle X_1, \ldots, X_m\rangle$ that satisfies such conditions are called tracial states, or non-commutative laws. This leads to the notion of C^*-algebras and representations of the laws as moments of non-commutative operators in C^*-algebras. However, we do not want to detail this point in these notes.

7.3 Maps and polynomials

In this section, we complete Section 3.1 to describe the graphs that will be enumerated by matrix models. Let $q(X_1, \ldots, X_m) = X_{\ell_1} X_{\ell_2} \cdots X_{\ell_k}$ be a monomial in m non-commutative variables.

Hereafter, monomials $(q_i)_{1 \le i \le n}$ will be fixed and we will write for short, for $\mathbf{k} = (k_1, \ldots, k_n)$,

$$\mathcal{M}^g_{\mathbf{k}} = \mathrm{card}\{ \text{ maps with genus } g$$

$$\text{and } k_i \text{ stars of type } q_i,\ 1 \le i \le n\},$$

and for a monomial P,

$$\mathcal{M}^g_{\mathbf{k}}(P) = \mathrm{card}\{ \text{ maps with genus } g$$

$$k_i \text{ stars of type } q_i,\ 1 \le i \le n \text{ and one of type } P\}.$$

7.3.1 Maps and polynomials

Because there is a one-to-one mapping between stars and monomials, the operations on monomials such as involution or derivatives have their graphical interpretation.

The involution comes to reverse the orientation and to shift the marked edge by one in the sense of the new orientation (see Figure 7.1). This is equivalent to considering the star in a mirror.

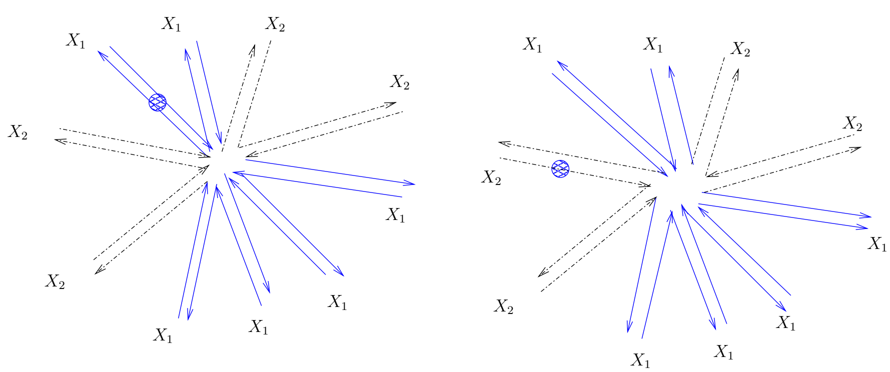


Fig. 7.1. A star of type q versus a star of type q^*

For derivations, the interpretation goes as follows.

Let q be a given monomial. The derivation ∂_i appears as a way to find out how to decompose a star of type q by pointing out a half-edge of color i: a star of type q can indeed be decomposed into one star of type q_1, one half-edge of color i and another star of type q_2, all sharing the same vertex, iff q can be written as $q = q_1 X_i q_2$. This is particularly useful to write induction relation on the number of maps. For instance, let us consider a planar map M and the event $A_M(X_i q)$ that, inside M, a star of type $X_i q$ is such that the first marked half-edge is glued to a half-edge of q. Then, if this happens, since the map is planar, it will be decomposed into two planar maps separated by the edge between these two X_i. Such a gluing can be done only with the edges X_i appearing in the decomposition of q as $q = q_1 X_i q_2$. Moreover, the two stars of type q_1 and q_2 will belong to two "independent" planar maps. So, we can symbolically write

$$1_{A_M(X_i q)} = \sum_{q=q_1 X_i q_2} 1_{q_1 \in M_1} \otimes_{M=M_1 \otimes_i M_2} 1_{q_2 \in M_2} \tag{7.3}$$

where $M = M_1 \otimes_i M_2$ means that M decomposes into two planar maps M_1 and M_2, M_2 being surrounded by a cycle of color i that separates it from M_1 (see Figure 7.2). Note here that we forgot in some sense that these three

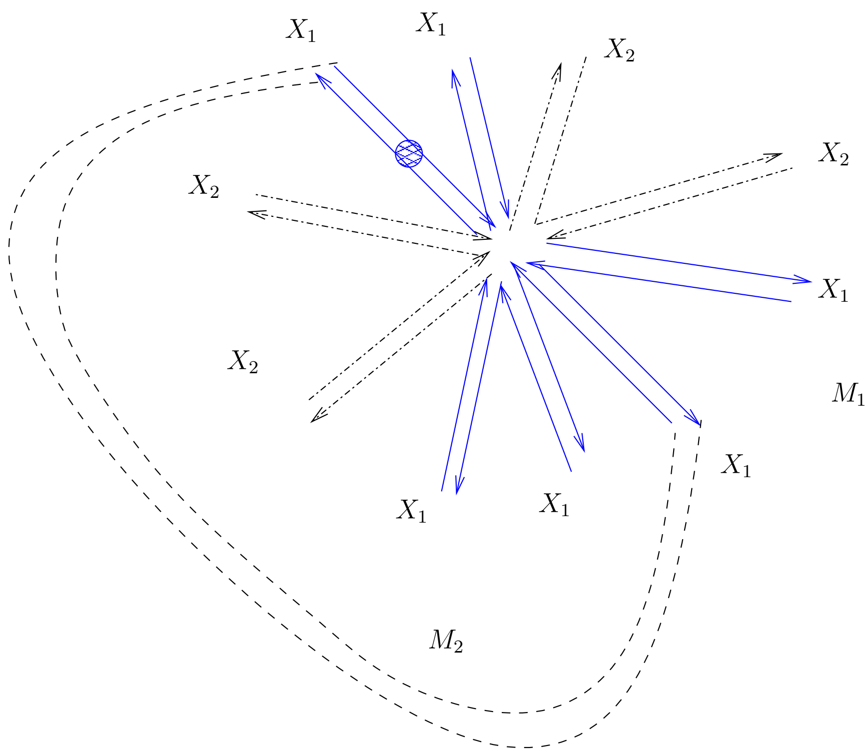


Fig. 7.2. A star of type $q = X_1^2 X_2^2 X_1^4 X_2^2$ decomposed into $X_1(X_1 X_2^2 X_1) X_1 (X_1^2 X_2^2)$

objects were sharing the same vertex; this is somehow irrelevant here since a vertex is finally nothing but the point of junction of several edges; as long as we are concerned with the combinatorial problem of enumerating these maps, we can safely split the map M into these three objects. (7.3) is very close to the derivation operation ∂_i.

Similarly, let us consider again a planar map M containing given stars of type $X_i q$ and q' and the event $B_M(X_i q, q')$ that, inside M, the star of type $X_i q$ is such that the first marked half-edge is glued to a half-edge of the star of type q'. Once we know that this happens, we can write

$$1_{B_M(X_i q, q')} = \sum_{q'=q_1 X_i q_2} 1_{q_2 q_1 \bullet_i q \in M}. \tag{7.4}$$

$q_2 q_1 \bullet_i q$ is a new star made of a star of type q and one of type $q_2 q_1$ with an edge of color i from one to the other just before the marked half-edges. Again, once we know that this edge of color i exists, from a combinatorial point of view, we can simply shorten it till the two stars merge into a bigger star of type $q_2 q_1 q$. This is the merging operation; it corresponds to the cyclic derivative D_i (see Figure 7.3).

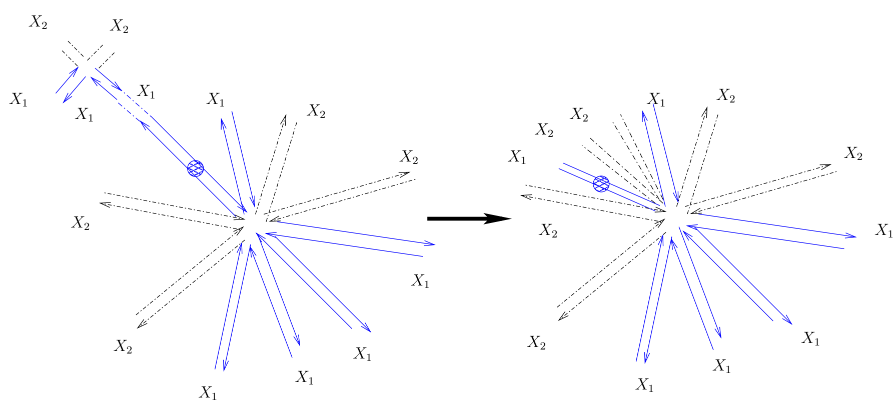


Fig. 7.3. Merging of a star of type $q = X_1^2 X_2^2 X_1^4 X_2^2$ and a star of type $X_1^2 X_2^2$

7.4 Formal expansion of matrix integrals

The expansion obtained by 't Hooft is based on Feynman diagrams, or equivalently on Wick's formula that can be stated as follows.

Lemma 7.6 (Wick's formula). *Let* $(G_1, \ldots, G_{2n})$ *be a Gaussian vector such that* $\mathbb{E}[G_i] = 0$ *for* $i \in \{1, \ldots, 2n\}$*. Then,*

$$\mathbb{E}[G_1 \cdots G_{2n}] = \sum_{\pi \in PP(2n)} \prod_{\substack{(b,b') \text{ block of } \pi, \\ b<b'}} \mathbb{E}[G_b G_{b'}]$$

where the sum runs over all pair-partitions of the ordered set $\{1, \ldots, 2n\}$.

Proof. Recall that if G is a standard Gaussian variable, for all $n \in \mathbb{N}$,

$$\mathbb{E}[G^{2n}] = 2n!! := \frac{(2n)!}{2^n n!}$$

is the number of pair-partitions of the ordered set $\{1, 2, \ldots, 2n\}$. Thus, for any real numbers $(\alpha_1, \ldots, \alpha_{2n})$, since $\sum_{i=1}^{2n} \alpha_i G_i$ is a centered Gaussian variable with covariance $\sigma^2 = \sum_{i,j=1}^{2n} \alpha_i \alpha_j \mathbb{E}[G_i G_j]$, and so $\sum_{i=1}^{2n} \alpha_i G_i$ has the same law that σ times a standard Gaussian variable,

$$\mathbb{E}\left[\left(\sum_{i=1}^{2n} \alpha_i G_i\right)^{2n}\right] = \left(\sum_{i,j=1}^{2n} \alpha_i \alpha_j \mathbb{E}[G_i G_j]\right)^n 2n!!.$$

Identifying on both sides the term corresponding to the coefficient $\alpha_1 \cdots \alpha_{2n}$, we obtain

$$(2n)!\mathbb{E}[G_1 \cdots G_{2n}] = 2n!! \sum_{\pi \in \Sigma} \prod_{(b,b') \in \pi} \mathbb{E}[G_b G_{b'}]$$

where Σ is the set of pairs of $2n$ elements. To compare this set with the collection of pairings of an ordered set, we have to order the elements of the pairs, and we have 2^n possible choices, and then order the pairs, which gives another $n!$ possible choices. Thus,

$$\sum_{\pi \in \Sigma} \prod_{(b,b') \in \pi} \mathbb{E}[G_b G_{b'}] = 2^n n! \sum_{\pi \in PP(2n)} \prod_{(i,j) \text{ block of } \pi} \mathbb{E}[G_i G_j]$$

completes the argument as $2^n n! 2n!! = (2n)!$. □

We now consider moments of traces of Gaussian Wigner's matrices. Since we shall consider the moments of products of several traces, we shall now use the language of stars. Let us recall that a star of type $q(X) = X_{\ell_1} \cdots X_{\ell_2}$ is a vertex equipped with k colored half-edges, one marked half-edge and an orientation such that the marked half-edge is of color ℓ_1, the second (following the orientation) of color ℓ_2, etc., till the last half-edge of color ℓ_k. The graphs we shall enumerate will be obtained by gluing pairwise the half-edges.

Definition 7.7. *Let* $r, m \in \mathbb{N}$. *Let* $q_1, \ldots, q_r$ *be* r *monomials in* m *non-commutative variables. A map of genus* g *with a star of type* q_i *for* $i \in \{1, \ldots, r\}$ *is a connected graph embedded into a surface of genus* g *with* r *vertices so that*

1. *For $1 \le i \le r$, one of the vertices has degree $deg(q_i)$, and this vertex is equipped with the structure of a star of type q_i (i.e., with the corresponding colored half-edges embedded into the surface in such a way that the orientation of the star and the orientation of the surface agree). The half-edges inherit the orientation of their stars, i.e., each side of each half-edge is endowed with an opposite orientation corresponding to the orientation of a path traveling around the star by following the orientation of the star.*
2. *The half-edges of the stars are glued pair-wise and two half-edges can be glued iff they have the same color and orientation; thus edges have only one color and one orientation.*
3. *A path traveling along the edges of the map following their orientation will make a loop. The surface inside this loop is homeomorphic to a disk and called a face (see Figure 7.4).*

Note that each star has a distinguished half-edge and so each half-edge of a star is labeled. Moreover, all stars are labeled. Hence, the enumeration problem we shall soon consider can be thought as the problem of matching the labeled half-edges of the stars and so we will distinguish all the maps where the gluings are not done between exactly the same set of half-edges, regardless of symmetries. This is important to make clear since we shall shortly consider enumeration issues. The genus of a map is defined as in Definition 1.8. Note that since at each vertex we imposed a cyclic orientation at the ends of the edges adjacent to this vertex, there is a unique way to embed the graph drawn with stars in a surface; we have to draw the stars so that their orientation agrees with the orientation of the surface.

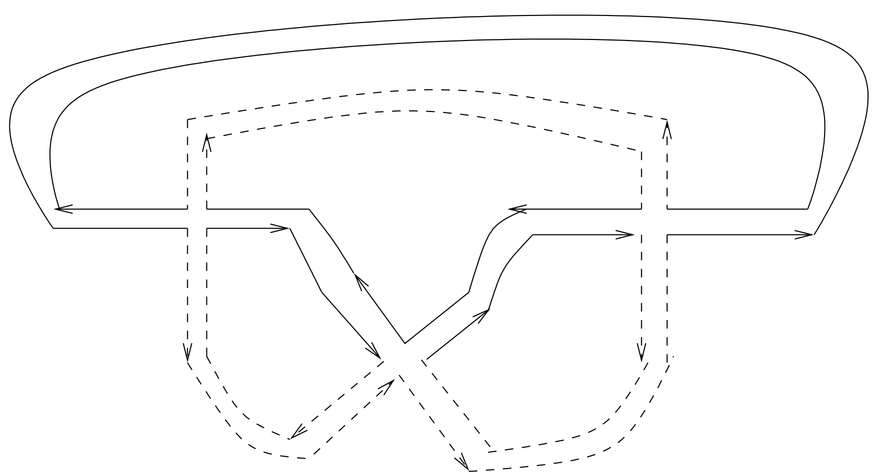


Fig. 7.4. A planar bi-colored map with stars of type $q_1 = X_1X_2X_1X_2$, $q_2 = X_1^2X_2^2$, $q_3 = X_1X_2X_1X_2$

There is a dual way to consider maps in the spirit of Figure 1.5; as in the figure in the center of Figure 1.5, we can replace a star of type $q(X) = X_{i_1} \cdots X_{i_p}$ by a polygon (of type q) with p faces, a boundary edge of the polygon replacing an edge of the star and taking the same color as the edge, and a marked boundary edge and an orientation. A map is then a covering of a surface (with the same genus as the map) by polygonals of type $q_1, \ldots, q_r$. The constraint on the colors becomes a constraint on the colors of the sides of the polygons of the covering.

Example 1. A triangulation (resp. a quadrangulation) of a surface of genus g by F faces (the number of triangles, resp. squares) is equivalent to a map of genus g with F stars of type $q(X) = X^3$ (resp. $q(X) = X^4$).

Exercise 7.8. *Draw the quadrangulation corresponding to Figure 7.4.*

We will define, for $\mathbf{k} = (k_1, \ldots, k_n)$,

$$\mathcal{M}_g((q_i, k_i), 1 \leq i \leq n) = \text{card}\{ \text{ maps with genus } g$$

$$\text{and } k_i \text{ stars of type } q_i, 1 \leq i \leq n\}.$$

Note here that the stars are labeled in the counting. Hence, the problem amounts to counting the possible matchings of the half-edges of the stars, all the half-edges being labeled.

In this section we shall first encounter eventually non-connected graphs; these graphs will then be (finite) unions of maps. We denote by $G_{g,c}((q_i, k_i), 1 \leq i \leq n)$ the set of graphs that can be described as a union of c maps, the total set of stars to construct these maps being k_i stars of type q_i, $1 \leq i \leq n$ and the genus of each connected components summing up to g. When counting these graphs, we will also assume that all half-edges are labeled. Moreover, we shall count these graphs up to homeomorphism, that is up to continuous deformation of the surface on which the graphs are embedded. Thus, our problem is to enumerate the number of possible pairings of the half-edges (of a given color) of the stars in such a way that the resulting graph has a given genus.

We now argue that

Lemma 7.9. *Let $q_1, \ldots, q_n$ be monomials. Then,*

$$\int \prod_{i=1}^{n} (N \text{Tr}(q_i(\mathbf{A}_1, \ldots, \mathbf{A}_m))) d\mu^N(\mathbf{A}_1) \cdots d\mu^N(\mathbf{A}_m)$$

$$= \sum_{g \in \mathbb{N}} \sum_{c \geq 1} \frac{1}{N^{2g-2c}} \sharp\{G_{g,c}((q_i, 1), 1 \leq i \leq n)\}.$$

Here, $G_{g,c}((q_i, 1), 1 \leq i \leq n)$ is the set of unions of c maps drawn on the stars of type $(q_i)_{1 \leq i \leq n}$, with the sum of the genera of each map equal to g. $\sharp\{G_{g,c}((q_i, 1), 1 \leq i \leq n)\}$ is the number of graphs of the set $G_{g,c}((q_i, 1), 1 \leq i \leq n)$.

As a warm-up, let us show the following:

Lemma 7.10. *Let q be a monomial. Then, we have the following expansion*

$$\int N^{-1}\mathrm{Tr}(q(\mathbf{A}_1,\ldots,\mathbf{A}_m))d\mu^N(\mathbf{A}_1)\cdots d\mu^N(\mathbf{A}_m) = \sum_{g\in\mathbb{N}} \frac{1}{N^{2g}}\sharp\{G_g((q,1))\}$$

where $\sharp\{G_0((q,1))\}$ equals $\sigma^m(q)$ as found by Voiculescu, Theorem 3.3.

Proof. As usual we expand the trace and write, if $q(X_1,\ldots,X_m) = X_{j_1}\cdots X_{j_k}$,

$$\begin{aligned}
&\int \mathrm{Tr}(q(\mathbf{A}_1,\ldots,\mathbf{A}_m))d\mu^N(\mathbf{A}_1)\cdots d\mu^N(\mathbf{A}_m) \\
&= \sum_{1\le r_1,\ldots,r_k\le N} \int A_{j_1}(r_1,r_2)\cdots A_{j_k}(r_k,r_1)d\mu^N(\mathbf{A}_1)\cdots d\mu^N(\mathbf{A}_m) \\
&= \sum_{r_1,\ldots,r_k}\sum_{\pi\in PP(k)} \prod_{\substack{(wv)\ \text{block of }\pi \\ w<v}} \mathbb{E}[A_{j_w}(r_w,r_{w+1})A_{j_v}(r_v,r_{v+1})]. \qquad (7.5)
\end{aligned}$$

Note that $\prod \mathbb{E}[A_{j_w}(r_w,r_{w+1})A_{j_v}(r_v.r_{v+1})]$ is either zero or $N^{-k/2}$. It is not zero only when $j_w = j_v$ and $r_w r_{w+1} = r_{v+1}r_v$ for all the blocks (v,w) of π. Hence, if we represent q by the star of type q, we see that all the graphs where the half-edges of the star are glued pairwise and colorwise will give a contribution. But how many indices will give the same graph ? To represent the indices on the star, we fatten the half-edges as double half-edges. Thinking that each random variable sits at the end of the half-edges, we can associate to each side of the fat half-edge one of the indices of the entry (see Figure 7.5). When the fattened half-edges meet at the vertex, observe that each side of the fattened half-edges meets one side of an adjacent half-edge on which sits the same index. Hence, we can say that the index stays constant over the broken line made of the union of the two sides of the fattened half-edges.

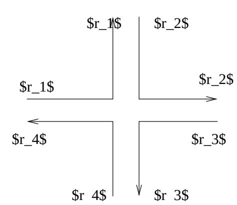


Fig. 7.5. Star of type X^4 with prescribed indices

When gluing pairwise the fattened half-edges we see that the condition $r_w r_{w+1} = r_{v+1}r_v$ means that the indices are the same on each side of the half-edge and hence stay constant on the resulting edge. The connected lines made with the sides of the fattened edges can be seen to be the boundaries of the faces of the correponding graphs. Therefore we have exactly N^F possible choices of indices for a graph with F faces. These graphs are otherwise connected, with one star of type q. (7.5) thus shows that

$$\int \mathrm{Tr}(q(\mathbf{A}_1, \ldots, \mathbf{A}_m)) d\mu^N(\mathbf{A}_1) \cdots d\mu^N(\mathbf{A}_m)$$
$$= \sum_{g \geq 0} \frac{N^F}{N^{\frac{k}{2}}} \sharp\{\text{maps with one star of type } q \text{ and } F \text{ faces}\}$$

Recalling that $2 - 2g = F + \sharp$ vertices $- \sharp$ edges $= F + 1 - k/2$ completes the proof. $\square$

Remark 7.11. *In the above it is important to take μ^N to be the law of the GUE (and not GOE for instance) to insure that $E[(A_k)_{ij}(A_k)_{ji}] = 1/N$ but $E[((A_k)_{ij})^2] = 0$. The GOE leads to the enumeration of other combinatorial objects (and in particular an expansion in N^{-1} rather than N^{-2}).*

Proof of Lemma 7.9. We let $q_i(X_1, \ldots, X_m) = X_{\ell_1^i} \cdots X_{\ell_{d_i}^i}$. As usual, we expand the traces:

$$\int \prod_{i=1}^n (N\mathrm{Tr}(q_i(\mathbf{A}_1, \ldots, \mathbf{A}_m))) d\mu^N(\mathbf{A}_1) \cdots d\mu^N(\mathbf{A}_m)$$
$$= N^n \sum_{\substack{i_1^k, \ldots, i_{d_k}^k \\ 1 \leq k \leq n}} \mathbb{E}[\prod_{1 \leq k \leq n} \mathbf{A}_{\ell_1^k}(i_1^k i_2^k) \cdots \mathbf{A}_{\ell_{d_k}^k}(i_{d_k}^k i_1^k)]$$
$$= N^n \sum_{\substack{i_1^k, \ldots, i_{d_k}^k \\ 1 \leq k \leq n}} \sum_{\pi \in PP(\sum d_i)} Z(\pi, \mathbf{i})$$

where in the last line we used Wick's formula, π is a pair partition of the edges
$\{(i_j^k, i_{j+1}^k)_{1 \leq j \leq d_k - 1}, (i_{d_k}^k, i_1^k), 1 \leq k \leq n\}$ and $Z(\pi, \mathbf{i})$ is the product of the variances over the corresponding blocks of the partition. A pictorial way to represent this sum over $PP(\sum d_i)$ is to represent $X_{\ell_1^k}(i_1^k i_2^k) \cdots X_{\ell_{d_k}^k}(i_{d_k}^k i_1^k)$ by its associated star of type q_k, for $1 \leq k \leq n$. Note that in the counting this star will be labeled (here by the number k). A partition π is represented by a pairwise gluing of the half-edges of the stars. $Z(\pi)$, as the product of the variances, vanishes unless each pairwise gluing is done in such a way that the indices written at the end of the glued half-edges coincides and the number of the variable (or color of the half-edges) coincide. Otherwise, each covariance being equal to N^{-1}, $Z(\pi, \mathbf{i}) = N^{-\sum_{i=1}^n k_i/2}$. Note also that once the gluing is

done, by construction the indices are fixed on the boundary of each face of the graph (this is due to the fact that $E[A_r(i,j)A_r(k,l)]$ is null unless $kl = ji$). Hence, there are exactly N^F possible choices of indices for a given graph, if F is the number of faces of this graph (note here that if the graph is disconnected, we count the number of faces of each connected part, including their external faces and sum the resulting numbers over all connected components). Thus, we find that

$$\sum_{\substack{i_1^k,\ldots,i_{d_k}^k \\ 1\le k\le n}} \sum_{\pi\in PP(\sum d_i)} Z(\pi,\mathbf{i}) = \sum_{F\ge 0} \sum_{G\in G_F((q_i,1),1\le i\le n)} N^{-\sum_{i=1}^n k_i/2} N^F$$

where G_F denotes the union of connected maps with a total number of faces equal to F (the external face of each map being counted). Note that for a connected graph, $2-2g = F - \sharp edges + \sharp vertices$. Because the total number of edges of the graphs is $\sharp edges = \sum_{i=1}^n k_i/2$ and the total number of vertices is $\sharp vertices = n$, we see that if $g_i, 1\le i\le c$, are the genera of each connected component of our graph, we must have

$$2c - 2\sum_{i=1}^c g_i = F - \sum_{i=1}^n k_i/2 - n.$$

This completes the proof. $\square$

We then claim that we find the topological expansion of Brézin, Itzykson, Parisi and Zuber [50]:

Lemma 7.12. *Let $q_1,\ldots,q_n$ be monomials. Then, we have the following formal expansion*

$$\log\left(\int e^{\sum_{i=1}^n t_i N \mathrm{Tr}(q_i(\mathbf{A}_1,\ldots,\mathbf{A}_m))} d\mu^N(\mathbf{A}_1)\cdots d\mu^N(\mathbf{A}_m)\right)$$
$$= \sum_{g\ge 0}\frac{1}{N^{2g-2}} \sum_{k_1,\ldots,k_n\in\mathbb{N}^n\backslash\{0\}} \prod_{i=1}^n \frac{(t_i)^{k_i}}{k_i!}\mathcal{M}_g((q_i,k_i),1\le i\le n)$$

where the equality means that derivatives of all orders at $t_i = 0, 1\le i\le n$, match.

Note here that the sum in the right-hand side is not absolutely convergent (in fact the left-hand side is in general infinite if the t_i's do not have the appropriate signs). However, we shall see in subsequent chapters that if we stop the expansion at $g \le G < \infty$ (but keep the summation over all k_i's) the expansion is absolutely converging for sufficiently small t_i's.

Proof of Lemma 7.12. The idea is to expand the exponential. Again, this has no meaning in terms of convergent series (and so we do not try to justify uses of Fubini's theorem, etc.) but can be made rigorous by the fact that we only wish to identify the derivatives at $t = 0$ (and so the formal expansion is only a way to compute these derivatives). So, we find that

$$
\begin{aligned}
L &:= \int e^{\sum_{i=1}^n t_i N \mathrm{Tr}(q_i(\mathbf{A}_1,\ldots,\mathbf{A}_m))} d\mu^N(\mathbf{A}_1)\cdots d\mu^N(\mathbf{A}_m) \\
&= \int \prod_{i=1}^n \left(e^{t_i N \mathrm{Tr}(q_i(\mathbf{A}_1,\ldots,\mathbf{A}_m))} \right) d\mu^N(\mathbf{A}_1)\cdots d\mu^N(\mathbf{A}_m) \\
&= \int \prod_{i=1}^n \left(\sum_{k_i \geq 0} \frac{(t_i)^{k_i}}{k_i!} (N\mathrm{Tr}(q_i(\mathbf{A}_1,\ldots,\mathbf{A}_m)))^{k_i} \right) d\mu^N(\mathbf{A}_1)\cdots d\mu^N(\mathbf{A}_m) \\
&= \sum_{k_1,\ldots,k_n \in \mathbb{N}} \frac{(t_1)^{k_1}\cdots(t_n)^{k_n}}{k_1!\cdots k_n!} \\
&\qquad\qquad \int \prod_{i=1}^n (N\mathrm{Tr}(q_i(\mathbf{A}_1,\ldots,\mathbf{A}_m)))^{k_i} d\mu^N(\mathbf{A}_1)\cdots d\mu^N(\mathbf{A}_m) \\
&= \sum_{k_1,\ldots,k_n \in \mathbb{N}} \frac{(t_1)^{k_1}\cdots(t_n)^{k_n}}{k_1!\cdots k_n!} \sum_{g\geq 0}\sum_{c \geq 0} \frac{1}{N^{2g-2c}} \sharp\{G_{g,c}((q_i,k_i), 1 \leq i \leq n)\}
\end{aligned}
\tag{7.6}
$$

where we finally used Lemma 7.9. Note that the case $c = 0$ is non-empty only when all the k_i's are null, and the resulting contribution is one. Now, we relate $\sharp\{G_{g,c}((q_i,k_i), 1 \leq i \leq n)\}$ with the number of maps. Since graphs in $G_{g,c}((q_i,k_i), 1 \leq i \leq n)$ can be decomposed into a union of disconnected maps, $\sharp\{G_{g,c}((q_i,k_i), 1 \leq i \leq n)\}$ is related with the ways to distribute the stars and the genus among the c maps, and the number of each of these maps. In other words, we have (since all stars are labeled)

$$
\begin{aligned}
&\sharp\{G_{g,c}((q_i,k_i), 1 \leq i \leq n)\} \\
&= \frac{1}{c!} \sum_{\substack{\sum_{i=1}^c g_i = g \\ g_i \geq 0}} \frac{g!}{g_1!\cdots g_c!} \sum_{\substack{\sum_{j=1}^c l_i^j = k_i \\ 1 \leq j \leq n}} \prod_{i=1}^n \frac{k_i!}{l_i^1!\cdots l_i^c!} \prod_{j=1}^c \mathcal{M}_g((q_i, l_i^j), 1 \leq i \leq n).
\end{aligned}
$$

Plugging this expression into (7.6) we get

$$
\begin{aligned}
L :=& \sum_{k_1,\dots,k_n\in\mathbb{N}} \frac{(t_1)^{k_1}\cdots(t_n)^{k_n}}{c!k_1!\cdots k_n!} \sum_{g\ge 0}\sum_{c\ge 0}\frac{1}{N^{2g-2c}} \sum_{\substack{\sum_{i=1}^c g_i=g\\ g_i\ge 0}} \frac{g!}{g_1!\cdots g_c!}\times \\
& \sum_{\substack{\sum_{j=1}^c l_i^j=k_i\\ 1\le j\le n}} \prod_{i=1}^n \frac{k_i!}{l_i^1!\cdots l_i^c!}\prod_{j=1}^c \mathcal{M}_g((q_i,l_i^j),1\le i\le n) \\
=& \sum_{c\ge 0}\frac{1}{c!}\sum_{g=\sum_{i=1}^c g_i}\frac{g!}{g_1!\cdots g_c!} \\
& \sum_{k_1,\dots,k_n\in\mathbb{N}} \sum_{\substack{\sum_{j=1}^c l_i^j=k_i\\ 1\le j\le n}} \prod_{j=1}^c\left(\frac{1}{N^{2g_j-2}}\prod_{i=1}^n \frac{(t_i)^{l_i^j}}{l_i^j!}\mathcal{M}_g((q_i,l_i^j),1\le i\le n)\right) \\
=& \sum_{c\ge 0}\frac{1}{c!}\left(\sum_{g\ge 0}\frac{1}{N^{2g-2}}\sum_{l_1,\cdots l_n\ge 0}\prod_{i=1}^n\frac{(t_i)^{l_i}}{l_i!}\mathcal{M}_g((q_i,l_i),1\le i\le n)\right)^c \\
=& \exp\left(\sum_{g\ge 0}\frac{1}{N^{2g-2}}\sum_{l_1,\cdots l_n\ge 0}\prod_{i=1}^n\frac{(t_i)^{l_i}}{l_i!}\mathcal{M}_g((q_i,l_i),1\le i\le n)\right)
\end{aligned}
$$

which completes the proof. □

The goal of subsequent chapters is to justify that this equality does not only hold formally but as a large N expansion. Instead of using Wick's formula, we shall base our analysis on differential calculus and its relations with Gaussian calculus (note here that Wick's formula might also have been proven by use of differential calculus). The point here will be that we can design an asymptotic framework for differential calculus, which will then encode the combinatorics of the first-order term in 't Hooft's expansion, that is, planar maps. To make this statement clear, we shall see that a nice set-up is when the potential $V=\sum t_i q_i$ possesses some convexity property.

Bibliographical notes. The formal relation between Gaussian matrix integrals and the enumeration of maps first appeared in the work of 't Hooft [187] in the context of quantum chromodynamics, and soon used in many situations [32, 50] in relation with 2D gravity [78, 99, 51, 207], and with string theory [41, 79]. It was used as well in mathematics [211, 133, 113, 132].

8

First-order expansion

At the end of this chapter (see Theorem 8.8) we will have proved that Lemma 7.12 holds as a first-order limit, i.e.,

$$\lim_{N\to\infty} \frac{1}{N^2} \log \int e^{\sum_{i=1}^n t_i N \mathrm{Tr}(q_i(\mathbf{A}_1,\ldots,\mathbf{A}_m))} d\mu^N(\mathbf{A}_1)\cdots d\mu^N(\mathbf{A}_m)$$

$$= \sum_{k_1,\ldots,k_n\in\mathbb{N}^n\setminus\{0\}} \prod_{i=1}^n \frac{(t_i)^{k_i}}{k_i!} \mathcal{M}_0((q_i,k_i), 1\le i\le n)$$

provided the parameters $\mathbf{t} = (t_i)_{1\le i\le n}$ are sufficiently small and such that the polynomial $V = \sum t_i q_i$ is strictly convex (i.e., belong to $U_c \cap B_\eta$ for some $c > 0$ and $\eta \le \eta(c)$ for some $\eta(c) > 0$). To prove this result we first show that, under the same assumptions, $\bar{\mathbf{L}}_t^N(q) = \mu^N_{\sum t_i q_i}(N^{-1}\mathrm{Tr}(q))$ converges as N goes to infinity to a limit that is as well related with map enumeration (see Theorem 8.4).

The central tool in our asymptotic analysis will be the so-called Schwinger–Dyson (or loop) equations. In finite dimension, they are simple emanation of the integration by parts formula (or, somewhat equivalently, of the symmetry of the Laplacian in $L^2(dx)$). As dimension goes to infinity, concentration inequalities show that $\bar{\mathbf{L}}_t^N$ approximately satisfies a closed equation that we will simply refer to as the Schwinger–Dyson equation. The limit points of $\bar{\mathbf{L}}_t^N$ will therefore satisfy this equation. We will then show that this equation has a unique solution in some small range of the parameters. As a consequence, $\bar{\mathbf{L}}_t^N$ will converge to this unique solution. Showing that an appropriate generating function of maps also satisfies the same equation will allow us to determine the limit of $\bar{\mathbf{L}}_t^N$.

8.1 Finite-dimensional Schwinger–Dyson equations

Property 8.1. *For all $P \in \mathbb{C}\langle X_1,\ldots,X_m\rangle$, all $i \in \{1,\ldots,m\}$,*

$$\mu_{V_{\mathbf{t}}}^N\left(\hat{\mathbf{L}}^N \otimes \hat{\mathbf{L}}^N(\partial_i P)\right) = \mu_{V_{\mathbf{t}}}^N\left(\hat{\mathbf{L}}^N((X_i + D_i V_{\mathbf{t}})P)\right).$$

Proof. A simple integration by part shows that for any differentiable function f on $\mathbb{R}$ such that $fe^{-N\frac{x^2}{2}}$ goes to zero at infinity,

$$N\int f(x)xe^{-N\frac{x^2}{2}}dx = \int f'(x)e^{-N\frac{x^2}{2}}dx.$$

Such a result generalizes to a complex Gaussian by the remark that

$$\begin{aligned} N(x+iy)e^{-N\frac{|x|^2}{2}-N\frac{|y|^2}{2}} &= -(\partial_x + i\partial_y)e^{-\frac{|x|^2}{2}-\frac{|y|^2}{2}} \\ &= -\partial_{x-iy}e^{-\frac{|x|^2}{2}-\frac{|y|^2}{2}}. \end{aligned}$$

As a consequence, applying such a remark to the entries of a Gaussian random matrix, we obtain for any differentiable function f of the entries, all $r, s \in \{1, \ldots, N\}^2$, all $r \in \{1, \ldots, m\}$,

$$N\int A_l(r,s)f(A_k(i,j), 1 \le i,j \le N, 1 \le k \le m)d\mu^N(\mathbf{A}_1)\cdots d\mu^N(\mathbf{A}_m) =$$

$$\int \partial_{A_l(s,r)}f(A_k(i,j), 1 \le i,j \le N, 1 \le k \le m)d\mu^N(\mathbf{A}_1)\cdots d\mu^N(\mathbf{A}_m).$$

Using repeatedly this equality, we arrive at

$$\begin{aligned} \int \frac{1}{N}\mathrm{Tr}(\mathbf{A}_k P)d\mu_V^N(\mathbf{A}) &= \frac{1}{2N^2}\sum_{i,j=1}^N \int \partial_{A_k(j,i)}(Pe^{-N\mathrm{Tr}(V)})_{ji}\prod d\mu^N(\mathbf{A}_i) \\ &= \frac{1}{2N^2}\sum_{i,j=1}^N \int \Bigg(\sum_{P=QX_kR} 2Q_{ii}R_{jj} \\ &\qquad -N\sum_{l=1}^n t_l \sum_{q_l=QX_kR}\sum_{h=1}^N 2P_{ji}Q_{hj}R_{ih}\Bigg) d\mu_V^N(\mathbf{A}) \\ &= \int \left(\frac{1}{N^2}(\mathrm{Tr}\otimes\mathrm{Tr})(\partial_k P) - \frac{1}{N}\mathrm{Tr}(D_k V P)\right) d\mu_V^N(\mathbf{A}) \end{aligned}$$

which yields

$$\int \left(\hat{\mathbf{L}}^N((X_k + D_k V)P) - \hat{\mathbf{L}}^N \otimes \hat{\mathbf{L}}^N(\partial_k P)\right) d\mu_V^N(\mathbf{A}) = 0. \tag{8.1}$$

$\square$

8.2 Tightness and limiting Schwinger–Dyson equations

We say that $\tau \in \mathbb{C}\langle X_1, \ldots, X_m \rangle^{\mathcal{D}}$ satisfies the Schwinger–Dyson equation with potential V, denoted for short by $\mathbf{SD[V]}$, if and only if for all $i \in \{1, \ldots, m\}$ and $P \in \mathbb{C}\langle X_1, \ldots, X_m \rangle$,

$$\tau(I) = 1, \quad \tau \otimes \tau(\partial_i P) = \tau((D_i V + X_i)P) \qquad \mathbf{SD}[\mathbf{V}].$$

We shall now prove the following:

Property 8.2. *Assume that $V_{\mathbf{t}}$ is c-convex. Then, $(\bar{\mathbf{L}}_t^N = \mu_{V_{\mathbf{t}}}^N(\hat{\mathbf{L}}^N), N \in \mathbb{N})$ is tight. Its limit points satisfy $\mathbf{SD}[V_{\mathbf{t}}]$ and*

$$|\tau(X_{\ell_1} \cdots X_{\ell_r})| \leq M_0^r \tag{8.2}$$

for all $\ell_1, \ldots, \ell_r \in \mathbb{N}$, all $r \in \mathbb{N}$, with an M_0 that only depends on c.

Proof. By Lemma 6.19, we find that for all $\ell_1, \ldots, \ell_r$,

$$\begin{aligned} |\bar{\mathbf{L}}_t^N(X_{\ell_1} \cdots X_{\ell_r})| &\leq \mu_{V_{\mathbf{t}}}^N(|\lambda_{max}(\mathbf{A})|^r) \\ &= \int_0^\infty r x^{r-1} \mu_{V_{\mathbf{t}}}^N(|\lambda_{max}(\mathbf{A})| \geq x) dx \qquad (8.3) \\ &\leq M_0^r + \int_{M_0}^\infty r x^{r-1} e^{-\alpha N x} dx \\ &= M_0^r + r(\alpha N)^{-r} \int_0^\infty r x^{r-1} e^{-x} dx. \qquad (8.4) \end{aligned}$$

Hence, if $K(C)$ denotes the compact set defined in Lemma 7.4, $\bar{\mathbf{L}}_t^N \in K(C)$ with $C(\ell_1, \ldots, \ell_r) = M_0^r + r\alpha^{-r} \int_0^\infty r x^{r-1} e^{-x} dx$. $(\bar{\mathbf{L}}_t^N, N \in \mathbb{N})$ is therefore tight. Let us consider now its limit points; let τ be such a limit point. By (8.4), we must have

$$|\tau(X_{\ell_1} \cdots X_{\ell_r})| \leq M_0^r. \tag{8.5}$$

Moreover, by concentration inequalities (see Lemma 6.22), we find that

$$\lim_{N\to\infty} \left| \int \hat{\mathbf{L}}_{\mathbf{A}}^N \otimes \hat{\mathbf{L}}_{\mathbf{A}}^N (\partial_k P) d\mu_V^N(\mathbf{A}) - \int \hat{\mathbf{L}}_{\mathbf{A}}^N d\mu_V^N(\mathbf{A}) \otimes \int \hat{\mathbf{L}}_{\mathbf{A}}^N d\mu_V^N(\mathbf{A})(\partial_k P) \right| = 0$$

so that Property 8.1 implies that

$$\limsup_{N\to\infty} \left| \bar{\mathbf{L}}_t^N \left((X_k + D_k V_{\mathbf{t}}) P) \right) - \bar{\mathbf{L}}_t^N \otimes \bar{\mathbf{L}}_t^N (\partial_k P) \right| = 0. \tag{8.6}$$

Hence, (8.1) shows that

$$\tau((X_k + D_k V)P) = \tau \otimes \tau(\partial_k P). \tag{8.7}$$

$\square$

8.2.1 Uniqueness of the solutions to Schwinger–Dyson's equations for small parameters

Let $R \in \mathbb{R}^+$ (we will always assume $R \geq 1$ in the sequel).

(CS(R)) *An element $\tau \in \mathbb{C}\langle X_1, \ldots, X_m\rangle^{\mathcal{D}}$ satisfies* **(CS(R))** *if and only if for all $k \in \mathbb{N}$,*

$$\max_{1 \leq i_1, \ldots, i_k \leq m} |\tau(X_{i_1} \cdots X_{i_k})| \leq R^k.$$

In the sequel, we denote by D the degree of V, that is the maximal degree of the $q_i's$; $q_i(X) = X_{j_1^i} \cdots X_{j_{d_i}^i}$ with, for $1 \leq i \leq n$, $\deg(q_i) =: d_i \leq D$ and equality holds for some i.

The main result of this paragraph is:

Theorem 8.3. *For all $R \geq 1$, there exists $\epsilon > 0$ so that for $|\mathbf{t}| = \max_{1 \leq i \leq n} |t_i| < \epsilon$, there exists at most one solution $\tau_{\mathbf{t}}$ to* $\mathbf{SD}[V_{\mathbf{t}}]$ *that satisfies* **(CS(R))**.

Remark: Note that if $V = 0$, our equation becomes

$$\tau(X_i P) = \tau \otimes \tau(\partial_i P).$$

Because if P is a monomial, $\tau \otimes \tau(\partial_i P) = \sum_{P = P_1 X_i P_2} \tau(P_1)\tau(P_2)$ with P_1 and P_2 with degree smaller than P, we see that the equation $\mathbf{SD[0]}$ allows us to define uniquely $\tau(P)$ for all P by induction. The solution can be seen to be exactly $\tau(P) = \sigma^m(P)$, σ^m the law of m free semi-circular variables found in Theorem 3.3. When V is not zero, such an argument does not hold a priori since the right-hand side will also depend on $\tau(D_i q_j P)$, with $D_i q_j P$ of degree strictly larger than $X_i P$. However, our compactness assumption **(CS(R))** gives uniqueness because it forces the solution to be in a small neighborhood of the law $\tau_0 = \sigma^m$ of m free semi-circular variables, so that perturbation analysis applies. We shall see in Theorem 8.5 that this solution is actually the generating function for the enumeration of maps.

Proof. Let us assume we have two solutions τ and τ'. Then, by the equation $\mathbf{SD[V]}$, for any monomial function P of degree $l-1$, for $i \in \{1, \ldots, m\}$,

$$(\tau - \tau')(X_i P) = ((\tau - \tau') \otimes \tau)(\partial_i P) + (\tau' \otimes (\tau - \tau'))(\partial_i P) - (\tau - \tau')(D_i V P)$$

Hence, if we let, for $l \in \mathbb{N}$,

$$\Delta_l(\tau, \tau') = \sup_{\text{monomial } P \text{ of degree l}} |\tau(P) - \tau'(P)|$$

we get, since if P is of degree $l-1$,

$$\partial_i P = \sum_{k=0}^{l-2} p_k^1 \otimes p_{l-2-k}^2$$

where p_k^i, $i = 1, 2$ are monomial of degree k or the null monomial, and $D_i V$ is a finite sum of monomials of degree smaller than $D - 1$,

$$\Delta_l(\tau, \tau') = \max_{P \text{ of degree } l-1} \max_{1 \le i \le m} \{|\tau(X_i P) - \tau'(X_i P)|\}$$

$$\le 2 \sum_{k=0}^{l-2} \Delta_k(\tau, \tau') R^{l-2-k} + C|t| \sum_{p=0}^{D-1} \Delta_{l+p-1}(\tau, \tau')$$

with a finite constant C (that depends on n only). For $\gamma > 0$, we set

$$d_\gamma(\tau, \tau') = \sum_{l \ge 0} \gamma^l \Delta_l(\tau, \tau').$$

Note that under **(CS(R))**, this sum is finite for $\gamma < (R)^{-1}$. Summing the two sides of the above inequality times γ^l we arrive at

$$d_\gamma(\tau, \tau') \le 2\gamma^2 (1 - \gamma R)^{-1} d_\gamma(\tau, \tau') + C|t| \sum_{p=0}^{D-1} \gamma^{-p+1} d_\gamma(\tau, \tau').$$

We finally conclude that if $(R, |t|)$ are small enough so that we can choose $\gamma \in (0, R^{-1})$ so that

$$2\gamma^2 (1 - \gamma R)^{-1} + C|t| \sum_{p=0}^{D-1} \gamma^{-p+1} < 1$$

then $d_\gamma(\tau, \tau') = 0$ and so $\tau = \tau'$ and we have at most one solution. Taking $\gamma = (2R)^{-1}$ shows that this is possible provided

$$\frac{1}{4R^2} + C|t| \sum_{p=0}^{D-1} (2R)^{p-1} < 1$$

so that when R is large, we see that we need $|t|$ to be at most of order $|R|^{-D+2}$.
□

8.3 Convergence of the empirical distribution

We are now in a position to state the main result of this part:

Theorem 8.4. *For all $c > 0$, there exists $\eta > 0$ and $M_0 \in \mathbb{R}^+$ (given in Lemma 6.19) so that for all $\mathbf{t} \in U_c \cap B_\eta$, $\hat{\mathbf{L}}^N$ (resp. $\bar{\mathbf{L}}_t^N$) converges almost surely (resp. everywhere) to the unique solution of* $\mathbf{SD}[V_\mathbf{t}]$ *such that*

$$|\tau(X_{\ell_1} \cdots X_{\ell_r})| \le M_0^r$$

for all choices of $\ell_1, \dots, \ell_r$.

Proof. By Property 8.2, the limit points of $\bar{\mathbf{L}}_t^N$ satisfy $\mathbf{CS}(M_0)$ and $\mathbf{SD}[V_{\mathbf{t}}]$. Since M_0 does not depend on $\mathbf{t}$, we can apply Theorem 8.3 to see that if $\mathbf{t}$ is small enough, there is only one such limit point. Thus, by Corollary 7.4 we can conclude that $(\bar{\mathbf{L}}_t^N, N \in \mathbb{N})$ converges to this limit point. From Lemma 6.22, we have that

$$\mu_V^N(|(\hat{\mathbf{L}}^N - \bar{\mathbf{L}}_t^N)(P)|^2) \le BC(P,M)N^{-2} + C^{2d}N^2e^{-\alpha MN/2}$$

insuring by Borel–Cantelli lemma that

$$\lim_{N\to\infty}(\hat{\mathbf{L}}^N - \bar{\mathbf{L}}_t^N)(P) = 0 \quad a.s$$

resulting with the almost sure convergence of $\hat{\mathbf{L}}^N$. □

8.4 Combinatorial interpretation of the limit

In this part, we are going to identify the unique solution $\tau_{\mathbf{t}}$ of Theorem 8.3 as a generating function for planar maps. Namely, for short, we write $\mathbf{k} = (k_1, \ldots, k_n) \in \mathbb{N}^n$ and denote by P a monomial in $\mathbb{C}\langle X_1, \ldots, X_m\rangle$,

$$\mathcal{M}_{\mathbf{k}}(P) = \text{card}\{ \text{ planar maps with } k_i \text{ labeled stars of type } q_i \text{ for } 1 \le i \le n$$
$$\text{and one of type } P\}.$$

This definition extends to $P \in \mathbb{C}\langle X_1, \ldots, X_m\rangle$ by linearity. By convention, $\mathcal{M}_{\mathbf{k}}(1) = 1_{\mathbf{k}=0}$. Then, we shall prove:

Theorem 8.5.

1. *The family $\{\mathcal{M}_{\mathbf{k}}(P), \mathbf{k} \in \mathbb{N}^n, P \in \mathbb{C}\langle X_1, \ldots, X_m\rangle\}$ satisfies the induction relation: for all $i \in \{1, \ldots, m\}$, all $P \in \mathbb{C}\langle X_1, \ldots, X_m\rangle$, all $\mathbf{k} \in \mathbb{N}^n$,*

$$\begin{aligned}\mathcal{M}_{\mathbf{k}}(X_iP) = &\sum_{\substack{0\le p_j\le k_j\\ 1\le j\le n}} \prod_{j=1}^n C_{k_j}^{p_j} \sum_{P=p_1X_ip_2} \mathcal{M}_{\mathbf{p}}(P_1)\mathcal{M}_{\mathbf{k}-\mathbf{p}}(P_2)\\ &+ \sum_{1\le j\le n} k_j \mathcal{M}_{\mathbf{k}-1_j}([D_iq_j]P)\end{aligned} \tag{8.8}$$

   where $1_j(i) = 1_{i=j}$ and $\mathcal{M}_{\mathbf{k}}(1) = 1_{\overline{k}=0}$. (8.8) defines uniquely the family $\{\mathcal{M}_{\mathbf{k}}(P), \mathbf{k} \in \mathbb{N}^n, P \in \mathbb{C}\langle X_1, \ldots, X_m\rangle\}$.
2. *There exists A, B finite constants so that for all $\mathbf{k} \in \mathbb{N}^n$, all monomial $P \in \mathbb{C}\langle X_1, \ldots, X_m\rangle$,*

$$|\mathcal{M}_{\mathbf{k}}(P)| \le \mathbf{k}! A^{\sum_{i=1}^n k_i} B^{deg(P)} \prod_{i=1}^n C_{k_i} C_{deg(P)} \tag{8.9}$$

   with $\mathbf{k}! := \prod_{i=1}^n k_i!$ and C_p the Catalan numbers.

3. For $\mathbf{t}$ *in* $B_{(4A)^{-1}}$,

$$\mathcal{M}_{\mathbf{t}}(P) = \sum_{\mathbf{k}\in\mathbb{N}^n} \prod_{i=1}^{n} \frac{(-t_i)^{k_i}}{k_i!} \mathcal{M}_{\mathbf{k}}(P)$$

is absolutely convergent. For $\mathbf{t}$ *small enough,* $\mathcal{M}_{\mathbf{t}}$ *is the unique solution of* $\mathbf{SD}[V_{\mathbf{t}}]$ *that satisfies* $\mathbf{CS(4B)}$.

By Theorem 8.3 and Theorem 8.4, we therefore readily obtain:

Corollary 8.6. *For all* $c > 0$*, there exists* $\eta > 0$ *so that for* $\mathbf{t} \in U_c \cap B_\eta$, $\hat{\mathbf{L}}^N$ *converges almost surely and in expectation to*

$$\tau_{\mathbf{t}}(P) = \mathcal{M}_{\mathbf{t}}(P) = \sum_{\mathbf{k}\in\mathbb{N}^n} \prod_{i=1}^{n} \frac{(-t_i)^{k_i}}{k_i!} \mathcal{M}_{\mathbf{k}}(P)$$

Let us remark that by definition of $\hat{\mathbf{L}}^N$, for all P, Q in $\mathbb{C}\langle X_1, \ldots, X_m\rangle$,

$$\hat{\mathbf{L}}^N(PP^*) \geq 0 \quad \text{and} \quad \hat{\mathbf{L}}^N(PQ) = \hat{\mathbf{L}}^N(QP).$$

These conditions are closed for the weak topology and hence we find:

Corollary 8.7. *There exists* $\eta > 0$ *(*$\eta \geq (4A)^{-1}$*) so that for* $\mathbf{t} \in B_\eta$, $\mathcal{M}_{\mathbf{t}}$ *is a linear form on* $\mathbb{C}\langle X_1, \ldots, X_m\rangle$ *such that for all* P, Q

$$\mathcal{M}_{\mathbf{t}}(PP^*) \geq 0 \quad \mathcal{M}_{\mathbf{t}}(PQ) = \mathcal{M}_{\mathbf{t}}(QP) \quad \mathcal{M}_{\mathbf{t}}(1) = 1.$$

Remark. This means that $\mathcal{M}_{\mathbf{t}}$ is a tracial state. The traciality property can easily be derived by symmetry properties of the maps. However, I do not know of any other way (and in particular any combinatorial way) to prove the positivity property $\mathcal{M}_{\mathbf{t}}(PP^*) \geq 0$ for all polynomial P, except by using matrix models. This property will be seen to be useful to actually solve the combinatorial problem (i.e., find an explicit formula for $\mathcal{M}_{\mathbf{t}}$), see Section 15.2.

Proof of Theorem 8.5.

1. *Proof of the induction relation* (8.8).
   - We first check them for $\mathbf{k} = \mathbf{0} = (0, \ldots, 0)$. By convention, there is only one planar map with no vertex, so $\mathcal{M}_{\mathbf{0}}(1) = 1$. We now check that

     $$\mathcal{M}_{\mathbf{0}}(X_i P) = \mathcal{M}_{\mathbf{0}} \otimes \mathcal{M}_{\mathbf{0}}(\partial_i P) = \sum_{P = p_1 X_i p_2} \mathcal{M}_{\mathbf{0}}(p_1)\mathcal{M}_{\mathbf{0}}(p_2).$$

     But this is clear from (7.3) since for any planar map with only one star of type $X_i P$, the half-edge corresponding to X_i has to be glued to another half-edge of P, hence the event $A_M(X_i P)$ must hold, and if X_i is glued to the half-edge X_i coming from the decomposition $P =$

$p_1X_ip_2$, the map is split into two (independent) planar maps with stars p_1 and p_2 respectively (note here that p_1 and p_2 inherits the structure of stars since they inherit the orientation from P as well as a marked half-edge corresponding to the first neighbor of the glued X_i.)

- We now proceed by induction over the $\mathbf{k}$'s and the degree of P; we assume that (8.8) is true for $\sum k_i \leq M$ and all monomials, and for $\sum k_i = M+1$ when $\deg(P) \leq L$. Note that $\mathcal{M}_{\mathbf{k}}(1) = 0$ for $|\mathbf{k}| \geq 1$ since we cannot glue a vertex with zero half-edges to any half-edge of another star. Hence, this induction can be started with $L = 0$. Now, consider $R = X_iP$ with P of degree less than L and the set of planar maps with a star of type X_iQ and k_j stars of type q_j, $1 \leq j \leq n$, with $|\mathbf{k}| = \sum k_i = M+1$. Then,

$\diamond$ either the half-edge corresponding to X_i is glued with an half-edge of P, say to the half-edge corresponding to the decomposition $P = p_1X_ip_2$; we then can use (7.3) to see that this cuts the map M into two disjoint planar maps M_1 (containing the star p_1) and M_2 (resp. p_2), the stars of type q_i being distributed either in one or the other of these two planar maps; there will be $r_i \leq k_i$ stars of type q_i in M_1, the rest in M_2. Since all stars all labeled, there will be $\prod C_{k_i}^{r_i}$ ways to assign these stars in M_1 and M_2.

Hence, the total number of planar maps with a star of type X_iQ and k_i stars of type q_i, such that the marked half-edge of X_iP is glued to a half-edge of P is

$$\sum_{P=p_1X_ip_2} \sum_{\substack{0\leq r_i\leq k_i\\ 1\leq i\leq n}} \prod_{i=1}^{n} C_{k_i}^{r_i} \mathcal{M}_{\mathbf{r}}(p_1)\mathcal{M}_{\mathbf{k}-\mathbf{r}}(p_2) \tag{8.10}$$

$\diamond$ Or the half-edge corresponding to X_i is glued to a half-edge of another star, say q_j; let's say to the edge coming from the decomposition of q_j into $q_j = q_1X_iq_2$. Then, we can use (7.4) to see that once we are given this gluing of the two edges, we can replace X_iP and q_j by q_2q_1P.

We have k_j ways to choose the star of type q_j and the total number of such maps is

$$\sum_{q_j=q_1X_iq_2} k_j\mathcal{M}_{\mathbf{k}-1_j}(q_2q_1P)$$

Summing over j, we obtain by linearity of $\mathcal{M}_{\mathbf{k}}$

$$\sum_{j=1}^{n} k_j\mathcal{M}_{\mathbf{k}-1_j}([D_iq_j]P) \tag{8.11}$$

(8.10) and (8.11) give (8.8). Moreover, it is clear that (8.8) defines uniquely $\mathcal{M}_{\mathbf{k}}(P)$ by induction.

2. *Proof of* (8.9). To prove the second point, we proceed also by induction over $\mathbf{k}$ and the degree of P. First, for $\mathbf{k} = \mathbf{0}$, $\mathcal{M}_{\mathbf{0}}(P)$ is the number of colored maps with one star of type P which is smaller than the number of planar maps with one star of type $x^{\text{deg P}}$ since colors only add constraints. Hence, we have, with C_k the Catalan numbers,

$$\mathcal{M}_{\mathbf{k}}(P) \leq C_{[\frac{\deg(P)}{2}]} \leq C_{\deg(P)}$$

showing that the induction relation is fine with $A = B = 1$ at this step. Hence, let us assume that (8.9) is true for $\sum k_i \leq M$ and all polynomials, and $\sum k_i = M+1$ for polynomials of degree less than L. Since $\mathcal{M}_{\mathbf{k}}(1) = 0$ for $\sum k_i \geq 1$ we can start this induction. Moreover, using (8.8), we get that, if we define $\mathbf{k}! = \prod_{i=1}^n k_i!$,

$$\begin{aligned}\frac{\mathcal{M}_{\mathbf{k}}(X_i P)}{\mathbf{k}!} &= \sum_{\substack{0 \leq p_i \leq k_i \\ 1 \leq j \leq n}} \sum_{P = P_1 X_i P_2} \frac{\mathcal{M}_{\mathbf{p}}(P_1)}{\mathbf{p}!} \frac{\mathcal{M}_{\mathbf{k}-\mathbf{p}}(P_2)}{(\mathbf{k}-\mathbf{p})!} \\ &\quad + \sum_{\substack{1 \leq j \leq n \\ k_j \neq 0}} \frac{\mathcal{M}_{\mathbf{k}-\mathbf{1_j}}((D_i q_j P)}{(\mathbf{k} - 1_j)!}.\end{aligned}$$

Hence, taking P of degree less or equal to L and using our induction hypothesis, we find that

$$\begin{aligned}&\left|\frac{\mathcal{M}_{\mathbf{k}}(X_i P)}{\mathbf{k}!}\right| \\ &\leq \sum_{\substack{0 \leq p_j \leq k_j \\ 1 \leq j \leq n}} \sum_{P = P_1 X_i P_2} A^{\sum k_i} B^{degP - 1} \prod_{i=1}^n C_{p_j} C_{k_j - p_j} C_{degP_1} C_{degP_2} \\ &\quad + 2 \sum_{1 \leq l \leq n} A^{\sum k_j - 1} \prod_j C_{k_j} B^{degP + degq_l - 1} C_{degP + degq_l - 1} \\ &\leq A^{\sum k_i} B^{degP+1} \prod_i C_{k_i} C_{degP+1} \left(\frac{4^n}{B^2} + 2 \frac{\sum_{1 \leq j \leq n} B^{degq_j - 2} 4^{degq_j - 2}}{A} \right)\end{aligned}$$

where we used Lemma 1.9 in the last line. It is now sufficient to choose A and B such that

$$\frac{4^n}{B^2} + 2 \frac{\sum_{1 \leq j \leq n} B^{degq_j - 2} 4^{degq_j - 2}}{A} \leq 1$$

(for instance $B = 2^{n+1}$ and $A = 4nB^{D-2} 4^{D-2}$ if D is the maximal degree of the q_j) to verify the induction hypothesis works for polynomials of all degrees (all L's).

3. *Properties of* $\mathcal{M}_{\mathbf{t}}$. From the previous considerations, we can of course define $\mathcal{M}_{\mathbf{t}}$ and the series is absolutely convergent for $|\mathbf{t}| \leq (4A)^{-1}$ since $C_k \leq 4^k$. Hence $\mathcal{M}_{\mathbf{t}}(P)$ depends analytically on $\mathbf{t} \in B_{(4A)^{-1}}$. Moreover, for all monomial P,

$$|\mathcal{M}_{\mathbf{t}}(P)| \leq \sum_{k \in \mathbb{N}^n} \prod_{i=1}^{n} (4t_i A)^{k_i} (4B)^{degP} \leq \prod_{i=1}^{n} (1 - 4At_i)^{-1} (4B)^{degP}.$$

so that for small t, $\mathcal{M}_{\mathbf{t}}$ satisfies **CS(4B)**.

4. $\mathcal{M}_{\mathbf{t}}$ *satisfies* $\mathbf{SD}[V_{\mathbf{t}}]$. This is derived by summing (8.8) written for all $\mathbf{k}$ and multiplied by the factor $\prod (t_i)^{k_i}/k_i!$. From this point and the previous one (note that B is independent from $\mathbf{t}$), we deduce from Theorem 8.3 that for sufficiently small $\mathbf{t}$, $\mathcal{M}_{\mathbf{t}}$ is the unique solution of $\mathbf{SD}[V_{\mathbf{t}}]$ that satisfies **CS(4B)**.

$\square$

8.5 Convergence of the free energy

Theorem 8.8. *Let $c > 0$. Then, for η small enough, for all $\mathbf{t} \in B_\eta \cap U_c$, the free energy converges towards a generating function of the numbers of certain planar maps:*

$$\lim_{N\to\infty} \frac{1}{N^2} \log \frac{Z_N^{V_{\mathbf{t}}}}{Z_N^0} = \sum_{\mathbf{k} \in \mathbb{N}^n \backslash (0,..,0)} \prod_{1 \leq i \leq n} \frac{(-t_i)^{k_i}}{k_i!} \mathcal{M}_{\mathbf{k}}.$$

Moreover, the limit depends analytically on $\mathbf{t}$ *in a neighborhood of the origin.*

Proof. We may assume without loss of generality that $c \in (0,1]$. For $\alpha \in [0,1]$, $V_{\alpha\mathbf{t}}$ is c-convex since

$$\begin{aligned} V_{\alpha\mathbf{t}} + \frac{1}{2}\sum_{i=1}^{m} X_i^2 = & \alpha (V_{\mathbf{t}}(X_1, \ldots, X_m) + \frac{1-c}{2} \sum_{i=1}^{m} X_i^2) \\ & + \frac{(1-\alpha)(1-c) + c}{2} \sum_{i=1}^{m} X_i^2 \end{aligned}$$

where all terms are convex (as we assumed $c \leq 1$), whereas the last one is c-convex. Set

$$F_N(\alpha) = \frac{1}{N^2} \log Z_N^{V_{\alpha\mathbf{t}}}.$$

Then, $\frac{1}{N^2} \log \frac{Z_N^{V_{\mathbf{t}}}}{Z_N^0} = F_N(1) - F_N(0)$. Moreover

$$\partial_\alpha F^N_{\gamma(\alpha)} = -\bar{\mathbf{L}}^N_{\alpha t}(V_{\mathbf{t}}). \tag{8.12}$$

By Theorem 8.4, we know that for all $\alpha \in [0,1]$ (since $V_{\alpha \mathbf{t}}$ is c-convex),

$$\lim_{N\to\infty} \bar{\mathbf{L}}^N_{\alpha t}(V_{\mathbf{t}}) = \tau_{\alpha \mathbf{t}}(V_{\mathbf{t}})$$

whereas by (8.4), we know that $\bar{\mathbf{L}}^N_{\alpha t}(V_{\mathbf{t}})$ stays uniformly bounded. Therefore, a simple use of dominated convergence theorem shows that

$$\lim_{N\to\infty} \frac{1}{N^2} \log \frac{Z_N^{V_{\mathbf{t}}}}{Z_N^0} = -\int_0^1 \tau_{\alpha \mathbf{t}}(V_{\mathbf{t}}) d\alpha = -\sum_{i=1}^n t_i \int_0^1 \tau_{\alpha \mathbf{t}}(q_i) d\alpha. \tag{8.13}$$

Now, observe that by Corollary 8.6,

$$\begin{aligned}
\tau_{\mathbf{t}}(q_i) &= \sum_{\mathbf{k}\in\mathbb{N}^n} \prod_{1\le j\le n} \frac{(-t_j)^{k_j}}{k_j!} \mathcal{M}_{\mathbf{k}+1_i} \\
&= -\partial_{t_i} \sum_{\mathbf{k}\in\mathbb{N}^n\backslash\{0,\dots,0\}} \prod_{1\le j\le n} \frac{(-t_j)^{k_j}}{k_j!} \mathcal{M}_{\mathbf{k}}
\end{aligned}$$

so that (8.13) implies that

$$\begin{aligned}
\lim_{N\to\infty} \frac{1}{N^2} \log \frac{Z_N^{V_{\mathbf{t}}}}{Z_N^0} &= -\int_0^1 \partial_\alpha [\sum_{\mathbf{k}\in\mathbb{N}^n\backslash\{0,\dots,0\}} \prod_{1\le j\le n} \frac{(-\alpha t_j)^{k_j}}{k_j!} \mathcal{M}_{\mathbf{k}}] d\alpha \\
&= -\sum_{\mathbf{k}\in\mathbb{N}^n\backslash\{0,\dots,0\}} \prod_{1\le j\le n} \frac{(-t_j)^{k_j}}{k_j!} \mathcal{M}_{\mathbf{k}}.
\end{aligned}$$

□

Bibliographical notes. The study of matrix models in mathematics is not new. The one matrix model was already studied by Pastur [164] who derived the limiting spectral density of such measures as well as the nearest-neighbor spacing distribution. The problem of the universality of the fluctuations of the largest eigenvalue was addressed in [163, 71]. Cases where the potential is not strictly convex were studied for instance in [72, 141]. Specific two-matrix models (mainly models with quadratic interaction) were studied by Mehta and coauthors [152, 154, 60] and, on less rigorous ground, for instance in [4, 87, 80, 125, 123]. In this case, large deviations techniques [109, 101] are also available (see [146] on a less rigorous ground), yielding a non-perturbative approach. Matrix models were considered in the generality presented in this section in [104].

9

Second-order expansion for the free energy

At the end of this chapter, we will have proved that Lemma 7.12 holds, up to the second-order correction in the large N limit, i.e., that

$$\frac{1}{N^2}\log\left(\int e^{\sum_{i=1}^n t_i N\mathrm{Tr}(q_i(X_1,\ldots,X_m))} d\mu^N(X_1)\cdots d\mu^N(X_m)\right)$$
$$=\sum_{g=0}^{1}\frac{1}{N^{2g-2}}\sum_{k_1,\ldots,k_n\in\mathbb{N}}\prod_{i=1}^{n}\frac{(t_i)^{k_i}}{k_i!}\mathcal{M}_g((q_i,k_i),1\le i\le n)+o\left(\frac{1}{N^2}\right)$$

for some parameters t_i small enough and such that $\sum t_i q_i$ is c-convex. As for the first order, we shall prove first a similar large N expansion for $\bar{\mathbf{L}}_t^N$. We will first refine the arguments of the proof of Theorem 8.3 to estimate $\bar{\mathbf{L}}_t^N-\tau_{\mathbf{t}}$. This will already prove that $(\bar{\mathbf{L}}_t^N-\tau_{\mathbf{t}})(P)$ is at most of order N^{-2} for any polynomial P. To get the limit of $N^2(\bar{\mathbf{L}}_t^N-\tau_{\mathbf{t}})(P)$, we will first obtain a central limit theorem for $\hat{\mathbf{L}}^N-\tau_{\mathbf{t}}$ which is of independent interest. The key argument in our approach, besides further uses of integration by parts-like arguments, will be the inversion of a differential operator acting on non-commutative polynomials which can be thought as a non-commutative analog of a Laplacian operator with a drift.

We shall now estimate differences of $\hat{\mathbf{L}}^N$ and its limit. So, we set

$$\begin{aligned}
\hat{\delta}_{\mathbf{t}}^N &= N(\hat{\mathbf{L}}^N-\tau_{\mathbf{t}})\\
\bar{\delta}^N &= \int \hat{\delta}^N d\mu_V^N = N(\bar{\mathbf{L}}_{\mathbf{t}}^N-\tau_{\mathbf{t}})\\
\tilde{\delta}_{\mathbf{t}}^N &= N(\hat{\mathbf{L}}^N-\bar{\mathbf{L}}_{\mathbf{t}}^N)=\hat{\delta}_{\mathbf{t}}^N-\bar{\delta}^N.
\end{aligned}$$

In order to simplify the notations, we will make $\mathbf{t}$ implicit and drop the subscript $\mathbf{t}$ in the rest of this chapter so that we will denote $\bar{\mathbf{L}}^N,\tau,\hat{\delta}^N,\bar{\delta}^N$ and $\tilde{\delta}^N$ in place of $\bar{\mathbf{L}}_{\mathbf{t}}^N,\tau_{\mathbf{t}},\hat{\delta}_{\mathbf{t}}^N,\bar{\delta}^N$ and $\tilde{\delta}_{\mathbf{t}}^N$, as well as V in place of $V_{\mathbf{t}}$.

9.1 Rough estimates on the size of the correction $\tilde{\delta}_{\mathbf{t}}^{N}$

In this section we improve on the perturbation analysis performed in Section 8.2.1 to get the order of

$$\bar{\delta}^N(P) = N(\bar{\mathbf{L}}^N(P) - \tau)(P)$$

for all monomial P.

Proposition 9.1. *For all $c > 0, \epsilon \in]0, \frac{1}{2}[$, there exists $\eta > 0, C < +\infty$, such that for all integer number N, all $\mathbf{t} \in B_\eta \cap U_c$, and all monomial function P of degree less than $N^{\frac{1}{2}-\epsilon}$,*

$$|\bar{\delta}^N(P)| \leq \frac{C^{deg\ (P)}}{N}.$$

Proof. The starting point is the finite dimensional Schwinger–Dyson equation of Property 8.1,

$$\mu_V^N(\hat{\mathbf{L}}^N[(X_i + D_i V)P]) = \mu_V^N\left(\hat{\mathbf{L}}^N \otimes \hat{\mathbf{L}}^N(\partial_i P)\right). \tag{9.1}$$

Therefore, since τ satisfies the Schwinger–Dyson equation $\mathbf{SD[V]}$, we get that for all polynomial P,

$$\bar{\delta}^N(X_i P) = -\bar{\delta}^N(D_i V P) + \bar{\delta}^N \otimes \bar{\mathbf{L}}^N(\partial_i P) + \tau \otimes \bar{\delta}^N(\partial_i P) + r(N,P) \tag{9.2}$$

with

$$r(N,P) := N^{-1}\mu_V^N\left(\tilde{\delta}^N \otimes \tilde{\delta}^N(\partial_i P)\right).$$

We take P as monomial of degree $d \leq N^{\frac{1}{2}-\epsilon}$ and see that

$$\begin{aligned}
|r(N,P)| &\leq \frac{1}{N} \sum_{P = P_1 X_i P_2} \mu_V^N\left(|\tilde{\delta}^N(P_1)|^2\right)^{\frac{1}{2}} \mu_V^N\left(|\tilde{\delta}^N(P_2)|^2\right)^{\frac{1}{2}} \\
&\leq \frac{C}{N} \sum_{l=0}^{d-1} (Bl^2 M^{2(l-1)} + C^l N^4 e^{-\frac{\alpha M N}{2}})^{\frac{1}{2}} \times \\
&\qquad (B(d-l-1)^2 M^{2(d-l-1)} + C^{(d-l-1)} N^4 e^{-\frac{\alpha M N}{2}})^{\frac{1}{2}} \\
&\leq \frac{C}{N} d(B(d-1)^2 M^{2(d-2)} + C^{(d-1)} N^4 e^{-\frac{\alpha M N}{2}}) := r(N,d,M)
\end{aligned}$$

where we used in the second line Lemma 6.22 and assumed $M \geq M_0$, and $d \leq N^{\frac{1}{2}-\epsilon}$. We set

$$\Delta_d^N := \max_{P \text{ monomial of degree } d} |\bar{\delta}^N(P)|.$$

Observe that by (6.14), for any monomial of degree d less than $N^{\frac{1}{2}-\epsilon}$,

$$|\bar{\mathbf{L}}_t^N(P)| \leq C(\epsilon)^d, \quad |\tau(P)| \leq C_0^d \leq C(\epsilon)^d.$$

Thus, by (9.2), writing $D_i V = \sum t_j D_i q_j$, we get that for $d < N^{\frac{1}{2}-\epsilon}$

$$\Delta_{d+1}^N \leq \max_{1\leq i\leq m} \sum_{j=1}^{n} |t_j| \Delta_{d+\deg(D_i q_j)}^N + 2\sum_{l=0}^{d-1} C(\epsilon)^{d-l-1} \Delta_l^N + r(N,d,M).$$

We next define for $\kappa \leq 1$

$$\Delta^N(\kappa,\epsilon) = \sum_{k=1}^{N^{\frac{1}{2}-\epsilon}} \kappa^k \Delta_k^N.$$

We obtain, if D is the maximal degree of V,

$$\begin{aligned}\Delta^N(\kappa,\epsilon) \leq {} & [C'|\mathbf{t}| + 2(1-C(\epsilon)\kappa)^{-1}\kappa^2]\Delta^N(\kappa,\epsilon) \\ & + C|\mathbf{t}| \sum_{k=N^{\frac{1}{2}-\epsilon}+1}^{N^{\frac{1}{2}-\epsilon}+D} \kappa^{k-D}\Delta_k^N + \sum_{k=1}^{N^{\frac{1}{2}-\epsilon}} \kappa^k r(N,k,M) \end{aligned} \tag{9.3}$$

where we choose κ small enough so that $C(\epsilon)\kappa < 1$. Moreover, since D is finite, bounding Δ_k^N by $2NC(\epsilon)^k$, we get

$$\sum_{k=N^{\frac{1}{2}-\epsilon}+1}^{N^{\frac{1}{2}-\epsilon}+D} \kappa^{k-D}\Delta_k^N \leq 2DN(\kappa C(\epsilon))^{N^{\frac{1}{2}-\epsilon}}\kappa^{-D}.$$

When $\kappa C(\epsilon) < 1$, as N goes to infinity, this term is negligible with respect to N^{-1} for all $\epsilon > 0$. The following estimate holds according to Lemma 6.22:

$$\sum_{k=1}^{N^{\frac{1}{2}-\epsilon}} \kappa^k r(N,k,M) \leq \frac{C}{N} \sum_{k=1}^{N^{\frac{1}{2}-\epsilon}} k\kappa^k (B(k-1)^2 M^{2(k-2)} + C^{(k-1)} N^4 e^{-\frac{\alpha NM}{2}}) \leq \frac{C''}{N}$$

if κ is small enough so that $M^2\kappa < 1$ and $C\kappa < 1$. We observed here that $N^4 e^{-\frac{\alpha NM}{2}}$ is uniformly bounded independently of $N \in \mathbb{N}$. Now, if $|\mathbf{t}|$ is small, we can choose κ so that

$$\zeta := 1 - [C'|\mathbf{t}| + 2(1-C(\epsilon)\kappa)^{-1}\kappa^2] > 0.$$

Plugging these controls into (9.3) shows that for all $\epsilon > 0$, and for $\kappa > 0$ small enough, there exists a finite constant $C(\kappa,\epsilon)$ so that

$$\Delta^N(\kappa,\epsilon) \leq C(\kappa,\epsilon) N^{-1}$$

and so for all monomial P of degree $d \leq N^{\frac{1}{2}-\epsilon}$,

$$|\bar{\delta}^N(P)| \leq C(\kappa,\epsilon)\kappa^{-d} N^{-1}.$$

$\square$

To get the precise evaluation of $N\bar{\delta}^N(P)$, we shall first obtain a central limit theorem under μ_V^N which in turn will allow us to estimate the limit of $Nr(N,P)$.

9.2 Central limit theorem

We shall here prove that

$$\hat{\delta}^N(P) = N(\hat{\mathbf{L}}^N - \tau)(P)$$

satisfies a central limit theorem for all polynomial P. By Proposition 9.1, it is equivalent to prove a central limit theorem for $\tilde{\delta}^N(P)$, $P \in \mathbb{C}\langle X_1, \dots, X_m\rangle$. We start by giving a weak form of a central limit theorem for Stieltjes-like functions. We then extend by density the result to polynomial functions in the image of some differential operator and finally to any polynomials by inverting this operator.

Until the end of this chapter, we will always assume the following hypothesis **(H)**.

(H): Let c be a positive real number. The parameter $\mathbf{t}$ is in $B_{\eta,c}$ with η sufficiently small such that we have the convergence to the solution of $\mathbf{SD}[V_{\mathbf{t}}]$ as well as the control given by Lemma 6.18, and Proposition 9.1.

Note that **(H)** implies also that the control of Lemma 6.18 is uniform, and that we can apply Lemma 6.19 and Lemma 6.21 with uniform constants.

9.2.1 Central limit theorem for Stieltjes test functions

One of the issues that one needs to address when working with polynomials is that they are not uniformly bounded. For that reason, we will prefer to work in this section with the complex vector space $\mathcal{C}_{st}^m(\mathbb{C})$ generated by the Stieltjes functionals

$$ST^m(\mathbb{C}) = \left\{ \prod_{1\le i\le p}^{\rightarrow} \left(z_i - \sum_{k=1}^m \alpha_i^k \mathbf{X}_k \right)^{-1} ; \quad z_i \in \mathbb{C}\backslash\mathbb{R}, \alpha_i^k \in \mathbb{R}, p \in \mathbb{N} \right\} \tag{9.4}$$

where $\prod^{\rightarrow}$ is the non-commutative product. We can also equip $ST^m(\mathbb{C})$ with an involution

$$\left(\prod_{1\le k\le p}^{\rightarrow} \left(z_k - \sum_{i=1}^m \alpha_i^k \mathbf{X}_i \right)^{-1} \right)^* = \prod_{1\le k\le p}^{\rightarrow} \left(\overline{z_{p-k}} - \sum_{i=1}^m \alpha_i^{p-k} \mathbf{X}_i \right)^{-1} .$$

We denote by $\mathcal{C}_{st}^m(\mathbb{C})_{sa}$ the set of self-adjoint elements of $\mathcal{C}_{st}^m(\mathbb{C})$. The derivative is defined by the Leibniz rule (7.1) (taken with P, Q which are Stieltjes functionals) and

$$\partial_i \left(z - \sum_{i=1}^m \alpha_i \mathbf{X}_i \right)^{-1} = \alpha_i \left(z - \sum_{i=1}^m \alpha_i \mathbf{X}_i \right)^{-1} \otimes \left(z - \sum_{i=1}^m \alpha_i \mathbf{X}_i \right)^{-1}.$$

We recall two notations; first $\sharp$ is the operator

$$(P \otimes Q)\sharp h = PhQ$$

and

$$(P \otimes Q \otimes R)\sharp(g,h) = PgQhR$$

so that for a monomial q

$$\partial_i \circ \partial_j q\#(h_i, h_j) = \sum_{q=q_0 X_i q_1 X_j q_2} q_0 h_i q_1 h_j q_2 + \sum_{q=q_0 X_j q_1 X_i q_2} q_0 h_j q_1 h_i q_2.$$

Lemma 9.2. *Assume* **(H)** *and let* $h_1, \ldots, h_m$ *be in* $\mathcal{C}_{st}^m(\mathbb{C})_{sa}$. *Then the random variable*

$$Y_N(h_1, \ldots, h_m) = N \sum_{k=1}^m \{\hat{\mathbf{L}}^N \otimes \hat{\mathbf{L}}^N(\partial_k h_k) - \hat{\mathbf{L}}^N[(X_k + D_k V)h_k]\}$$

converges in law to a real centered Gaussian variable with variance

$$C(h_1, \ldots, h_m) = \sum_{k,l=1}^m (\tau \otimes \tau[\partial_k h_l \times \partial_l h_k] + \tau(\partial_l \circ \partial_k V \sharp(h_k, h_l))) + \sum_{k=1}^m \tau(h_k^2).$$

Proof. Define $W = \frac{1}{2}\sum_i X_i^2 + V$. Notice that $Y_N(h_1, \ldots, h_m)$ is real-valued because the $h_k' s$ and W are self-adjoint. The proof of the lemma follows from a change of variable. We take $h_1, \ldots, h_m$ in $\mathcal{C}_{st}^m(\mathbb{C})_{sa}$, $\lambda \in \mathbb{R}$ and perform a change of variable $B_i = F(\mathbf{A})_i = \mathbf{A}_i + \frac{\lambda}{N} h_i(\mathbf{A})$ in Z_V^N. Note that since the h_i are $\mathcal{C}^\infty$ and uniformly bounded, this defines a bijection on $\mathcal{H}_N^m$ for N big enough. We shall compute the Jacobian of this change of variables up to its second-order correction. The Jacobian J may be seen as a matrix $(J_{i,j})_{1 \le i,j \le m}$ where the $J_{i,j}$ are in $\mathcal{L}(\mathcal{H}_N)$ the set of endomorphisms of $\mathcal{H}_N$, and we can write $J = I + \frac{\lambda}{N}\overline{J}$ with

$$\begin{aligned} \overline{J}_{i,j} : \mathcal{H}_N &\longrightarrow \quad \mathcal{H}_N \\ X &\longrightarrow \partial_i h_j \# X. \end{aligned}$$

Now, for $1 \le i,j \le m$, $X \longrightarrow \partial_i h_j \# X$ is bounded for the operator norm uniformly in N (since $h_j \in \mathcal{C}_{st}(\mathbb{C}), \partial_i h_j \in \mathcal{C}_{st}(\mathbb{C}) \otimes \mathcal{C}_{st}(\mathbb{C})$ is uniformly bounded) so that for sufficiently large N, the operator norm of $\frac{\lambda}{N}\overline{J}$ is less than 1. From this we deduce

$$\begin{aligned} |\det J| &= \left| \det \left(I + \frac{\lambda \overline{J}}{N} \right) \right| = \exp\left(\operatorname{Tr} \log \left(I + \frac{\lambda \overline{J}}{N} \right) \right) \\ &= \exp\left(\sum_{k \ge 1} \frac{(-1)^{k+1} \lambda^k}{k N^k} \operatorname{Tr}(\overline{J}^k) \right). \end{aligned}$$

Observe that as $\overline{J}$ is a matrix of size m^2N^2 and of uniformly bounded norm, the kth term $\frac{(-1)^{k+1}\lambda^k}{N^k} tr(\overline{J}^k)$ is of order $\frac{1}{N^{k-2}}$. Hence, only the two first terms in the expansion will contribute to the order 1. To compute them, we only have to remark that if ϕ an endomorphism of $\mathcal{H}_N$ is of the form $\phi(X) = \sum_l \mathbf{A}_l X \mathbf{B}_l$, with $N \times N$ matrices $\mathbf{A}_i$,$\mathbf{B}_i$ then $\mathrm{Tr}\phi = \sum_l \mathrm{Tr}\mathbf{A}_l \mathrm{Tr}\mathbf{B}_l$ (this can be checked by decomposing ϕ on the canonical basis of $\mathcal{H}_N$). Now,

$$\overline{J}_{ij}^k : X \longrightarrow \sum_{1 \le i_1, \ldots, i_{k-1} \le m} \partial_i h_{i_2} \sharp (\partial_{i_2} h_{i_3} \sharp (\cdots (\partial_{i_{k-1}} h_j \sharp X) \cdots)).$$

Thus, we get

$$\mathrm{Tr}(\overline{J}) = \sum_i \mathrm{Tr}\overline{J}_{ii} = \sum_{1 \le i_1, \ldots, i_k \le m} tr \otimes tr(\partial_i h_i)$$

and

$$\mathrm{Tr}(\overline{J}^2) = \sum_{ij} \mathrm{Tr}(\overline{J}_{ij}\overline{J}_{ji}) = \sum_{1 \le i,j \le m} tr \otimes tr(\partial_i h_j \times \partial_j h_i)$$

since $\overline{J}_{ij}\overline{J}_{ji}(X) = \partial_i h_j \sharp [\partial_j h_i \sharp X] = \partial_i h_j \times \partial_j h_i \sharp X$ where $X_i^1 \otimes Y_i^1 \times X_i^2 \otimes Y_i^2 = X_i^1 X_i^2 \otimes Y_i^2 Y_1^1$. We can now make the change of variable $\mathbf{A}_i \to \mathbf{A}_i + \frac{\lambda}{N} h_i(\mathbf{A})$ to find that

$$Z_V^N = \int e^{-N trV} d\mu^N = \int e^{-N\mathrm{Tr}(W(\mathbf{A}_i + \frac{\lambda}{N} h_i(\mathbf{X})) - W(\mathbf{A}_i))} e^{\frac{\lambda}{N}\sum_i \mathrm{Tr}\otimes\mathrm{Tr}(\partial_i h_i)} \times$$
$$e^{-\frac{\lambda^2}{2N^2}\sum_{i,j}\mathrm{Tr}\otimes\mathrm{Tr}(\partial_i h_j \partial_j h_i)} e^{O(\frac{1}{N})} d\mu_V^N \times Z_V^N$$

where $O(\frac{1}{N})$ is a function uniformly bounded by C/N for some finite $C = C(h)$.

The first term can be expanded into

$$W\left(\mathbf{A}_i + \frac{h_i(\mathbf{A})}{N}\right) - W(\mathbf{A}_i) = \frac{1}{N}\sum_i \partial_i W \sharp h_i + \frac{1}{N^2}\sum_{i,j} \partial_i \circ \partial_j W \# (h_i, h_j) + \frac{R_N}{N^3}$$

where R_N is a polynomial of degree less than the degree of $\mathcal{D}V$ whose coefficients are bounded by those of a fixed polynomial R. To sum up, the following equality holds:

$$\int e^{\lambda Y_N(h_1,\ldots,h_m) - \frac{\lambda^2}{2} C_N(h_1,\ldots h_m) + \frac{1}{N}\{O(\hat{\mathbf{L}}^N(R))\}} d\mu_V^N = 1$$

with

$$C_N(h_1, \ldots, h_m) := \hat{\mathbf{L}}^N \big(\sum_{i,j} \partial_i \circ \partial_j W \# (h_i, h_j) \big) + \hat{\mathbf{L}}^N \otimes \hat{\mathbf{L}}^N \big(\sum_{i,j} \partial_i h_j \partial_j h_i \big).$$

We can decompose the previous expectation in two terms E_1 and E_2 with

$$E_1 = \mathbb{E}_V \left[1_{\Lambda_M^N} e^{\lambda Y_N(h_1,\ldots,h_m) - \frac{\lambda^2}{2} C_N(h_1,\ldots,h_m) + \frac{O(\hat{\mathbf{L}}^N(R))}{N}} \right]$$

and

$$E_2 = \mathbb{E}_V \left[1_{(\Lambda_M^N)^c} e^{\lambda Y_N(h_1,\ldots,h_m) - \frac{\lambda^2}{2} C_N(h_1,\ldots,h_m) + \frac{O(\hat{\mathbf{L}}^N(R))}{N}} \right].$$

On $\Lambda_M^N = \{\mathbf{A} : \max_i(\lambda_{\max}^N(\mathbf{A}_i)) \le M\}$ all the quantities are Lipschitz bounded so that $\frac{O(\hat{\mathbf{L}}^N(R))}{N}$ goes uniformly to 0 and $Y_N(h_1,\ldots,h_m)$ is at most of order e^{cN}. Now, by concentration inequalities $C_N(h_1,\ldots,h_m)$ concentrates in the scale e^{-N^2} (see Lemma 6.21). Thus, in E_1, $\hat{\mathbf{L}}^N$ can be replaced by its expectation $\bar{\mathbf{L}}^N$ and then by its limit as $\bar{\mathbf{L}}^N$ converges to τ (see Theorem 8.6). This proves that we can replace C_N by C in E_1.

The aim is now to show that for M sufficiently large, E_2 vanishes when N goes to infinity. It would be an easy task if all the quantities were in $\mathcal{C}_{st}^m(\mathbb{C})$ but some derivatives of V appear so that there are polynomial terms in the exponential. The idea to pass this difficulty is to make the reverse change of variables. For N bigger than the norm of the h_i's, and with $B_i = \mathbf{A}_i + \frac{1}{N} h_i(\mathbf{A})$,

$$\begin{aligned} E_2 &= \mathbb{E}_V \left[1_{\{\mathbf{A}:\max_i(\lambda_{\max}^N(\mathbf{A}_i)) \ge M\}} e^{\lambda Y_N(h_1,\ldots,h_m) - \frac{\lambda^2}{2} C_N(h_1,\ldots,h_m) + \frac{O(\hat{\mathbf{L}}^N(R))}{N}} \right] \\ &= \mu_V^N \left(\mathbf{B} : \max_i(\lambda_{\max}^N(\mathbf{A}_i)) \ge M \right) \le \mu_V^N \left(\max_i(\lambda_{\max}^N(B_i)) \ge M - 1 \right). \end{aligned}$$

This last quantity goes exponentially fast to 0 for M sufficiently large by Lemma 6.19. Hence, we arrive, for M large enough, at

$$\lim_{N\to\infty} \int 1_{\Lambda_M^N} e^{\lambda Y_N(h_1,\ldots,h_m)} d\mu_V^N = e^{\frac{\lambda^2}{2} C(h_1,\ldots,h_m)}.$$

Because $\mu_V^N(\Lambda_M^N)$ goes to one as N goes to infinity by Lemma 6.19, we conclude that $Y_N(h_1,\ldots,h_m)$ converges in law under $1_{\Lambda_M^N} d\mu_V^N / \mu_V^N(\Lambda_M^N)$ to a centered Gaussian variable with covariance $C(h_1,\ldots,h_m)$ (since the convergence of the Laplace transforms to a Gaussian law ensures the weak convergence). But since $\mu_V^N(\Lambda_M^N)$ goes to one, this convergence in law also holds under μ_V^N (since for any bounded continuous function $\mu_V^N(f) - \int f 1_{\Lambda_M^N} d\mu_V^N / \mu_V^N(\Lambda_M^N)$ goes to zero as N goes to infinity). □

9.2.2 Central limit theorem for some polynomial functions

We now extend Lemma 9.2 to polynomial test functions.

Lemma 9.3. *Assume* **(H)**. *Then, for all* $P_1,\ldots,P_m$ *in* $\mathbb{C}\langle X_1,\ldots,X_m\rangle_{sa}$, *the variable*

$$Y_N(P_1,\ldots,P_m) = N \sum_{k=1}^{m} [\hat{\mathbf{L}}^N \otimes \hat{\mathbf{L}}^N(\partial_k P_k) - \hat{\mathbf{L}}^N[(X_k + D_k V)P_k]]$$

converges in law to a real centered Gaussian variable with variance

$$C(P_1,\cdots,P_m)=\sum_{k,l=1}^{m}(\tau_{\mathbf{t}}\otimes\tau_{\mathbf{t}}[\partial_k P_l\times\partial_l P_k]+\tau_{\mathbf{t}}(\partial_l\circ\partial_k V\sharp(P_k,P_l)))+\sum_{k=1}^{m}\tau_{\mathbf{t}}(P_k^2).$$

Proof. Let $P_1,\ldots,P_m$ be self-adjoint polynomials and $h_1^\epsilon,\ldots,h_m^\epsilon$ be Stieltjes functionals which approximate $P_1,\cdots,P_m$ such as

$$h_i^\epsilon(\mathbf{A})=P_i\left(\frac{\mathbf{A}_1}{1+\epsilon\mathbf{A}_1^2},\ldots,\frac{\mathbf{A}_m}{1+\epsilon\mathbf{A}_m^2}\right).$$

Since $E[Y_N(P_1,\ldots,P_m)]=0$ by (9.1),

$$Y_N(P_1,\ldots,P_m)=\tilde{\delta}^N(K_N(P_1,\ldots,P_m))$$

with

$$K_N(P_1,\ldots,P_m)=\sum_{k=1}^{m}\left(\hat{\mathbf{L}}^N\otimes I(\partial_k P_k)-(X_k+D_kV)P_k\right)$$

and the same for $Y_N(h_1^\epsilon,\ldots,h_m^\epsilon)$. It is not hard to see that on Λ_M^N,

$$K_N(h_1^\epsilon,\ldots,h_m^\epsilon)-K_N(P_1,\ldots,P_m)$$

is a Lipschitz function with a constant bounded by $\epsilon C(M)$ for some finite constant $C(M)$ which depends only on M. Hence, by Lemma 6.21, we have

$$\mu_V^N\left(|\hat{\delta}^N(K_N(h_1^\epsilon,\ldots,h_k^\epsilon)-K_N(P_1,\ldots,P_k))|\geq\delta\right)\leq e^{-\alpha MN}+e^{-\frac{\delta^2}{2c\epsilon^2C(M)^2}}$$

and so for any bounded continuous function $f:\mathbb{R}\to\mathbb{R}$, if ν_{σ^2} is the centered Gaussian law of variance σ^2, we deduce

$$\begin{aligned}&\lim_{N\to\infty}\mu_V^N\left(f(\tilde{\delta}^N(K_N(P_1,\ldots,P_k)))\right)\\&=\lim_{\epsilon\to0}\lim_{N\to\infty}\mu_V^N\left(f(\tilde{\delta}^N(K_N(h_1^\epsilon,\ldots,h_k^\epsilon))\right)\\&=\lim_{\epsilon\to0}\nu_{C(h_1^\epsilon,\ldots,h_m^\epsilon)}(f)=\nu_{C(P_1,\ldots,P_m)}(f)\end{aligned}$$

where we used in the second line Lemma 9.2 and in the last line Lemma 6.18 to obtain the convergence of $C(h_1^\epsilon,\ldots,h_m^\epsilon)$ to $C(P_1,\ldots,P_m)$. □

Y_N depends on $N\hat{\mathbf{L}}^N\otimes\hat{\mathbf{L}}^N$, in which one of the empirical distribution $\hat{\mathbf{L}}^N$ can be replaced by its deterministic limit. This is the content of the next lemma.

Lemma 9.4. *Assume* **(H)** *and let $P_1,\ldots,P_k$ be self-adjoint polynomial functions. Then, the variable*

$$Z_N(P_1,\ldots,P_m) = \hat{\delta}^N\left(\sum_{k=1}^m (X_k + D_kV)P_k - \sum_{k=1}^m (\tau_{\mathbf{t}}\otimes I + I\otimes\tau_{\mathbf{t}})(\partial_k P_k)\right)$$

converges in law to a centered Gaussian variable with variance

$$C(P_1,\cdots,P_m) = \sum_{k,l=1}^m (\tau_{\mathbf{t}}\otimes\tau_{\mathbf{t}}[\partial_k P_l\times\partial_l P_k]+\tau_{\mathbf{t}}(\partial_l\circ\partial_k V\sharp(P_k,P_l)))+\sum_{k=1}^m \tau_{\mathbf{t}}(P_k^2).$$

Proof. The only point is to notice that

$$Y_N(P_1,\cdots,P_m) = \sum_{k=1}^m \left(\hat{\delta}^N\otimes\tau_{\mathbf{t}} + \tau_{\mathbf{t}}\otimes\hat{\delta}^N\right)(\partial_k P_k) - \hat{\delta}^N((X_k+D_kV)P_k)+r_{N,P}$$

with $r_{N,P} = N^{-1}\sum_{k=1}^m \hat{\delta}^N\otimes\hat{\delta}^N(\partial_k P_k)$ of order N^{-1} with probability going to 1 by Lemma 6.21 and Property 9.1. Thus

$$\begin{aligned} &Y_N(P_1,\ldots,P_m)\\ &\quad= \tilde{\delta}^N\left(\sum_{k=1}^m (-(X_k+D_kV)P_k + (I\otimes\tau_{\mathbf{t}}+\tau_{\mathbf{t}}\otimes I)(\partial_k P_k))\right) + O\big(\frac{1}{N}\big)\\ &\quad= -Z_N(P_1,\ldots,P_m) + O\big(\frac{1}{N}\big). \end{aligned}$$

This, with the previous lemma, proves the claim. □

9.2.3 Central limit theorem for all polynomial functions

In the previous part, we obtained the central limit theorem only for the family of random variables $\hat{\delta}^N(Q)$ with Q in

$$\mathcal{F} = \left\{\sum_{k=1}^m (X_k+D_kV)P_k - \sum_{k=1}^m (\tau_{\mathbf{t}}\otimes I + I\otimes\tau_{\mathbf{t}})(\partial_k P_k), \forall i, P_i \text{ self-adjoint}\right\}.$$

In this section, we wish to extend it to $\hat{\delta}^N(Q)$ for any self-adjoint polynomial function Q, that is, prove the following theorem:

Theorem 9.5. *Let $\mathbf{t}\in U_c\cap B_\eta$. There exists $\eta_c>0$ so that for $\eta<\eta_c$, for all polynomials $P_1,\ldots,P_k\in\mathbb{C}\langle X_1,\ldots,X_m\rangle$, $(\mathrm{Tr}(P_i)-N\tau_{\mathbf{t}}(P_i))_{1\le i\le k}$ converges in law to a centered Gaussian vector with covariance $\{\sigma(P_i,P_j), 1\le i,j\le k\}$*

We shall describe σ in the course of the proof. Its interpretation as a generating function for maps is given in Section 9.3.

To prove the theorem, we have to show that the set $\mathcal{F}$ is dense for some convenient topology in $\mathbb{C}\langle X_1,\ldots,X_m\rangle$.

The strategy is to see $\mathcal{F}$ as the image of an operator that we will invert. The first operator that comes to mind is

$$\Psi : (P_1, \ldots, P_k) \to \sum_{k=1}^{m} (X_k + D_k V) P_k - \sum_{k=1}^{m} (\tau_{\mathbf{t}} \otimes I + I \otimes \tau_{\mathbf{t}})(\partial_k P_k)$$

as we immediately have $\mathcal{F} = \Psi(\mathbb{C}\langle X_1, \ldots, X_m\rangle_{sa}, \ldots, \mathbb{C}\langle X_1, \ldots, X_m\rangle_{sa})$.

In order to obtain an operator from $\mathbb{C}\langle X_1, \ldots, X_m\rangle$ to $\mathbb{C}\langle X_1, \ldots, X_m\rangle$ we will prefer to apply this with $P_k = D_k P$ for all k and for a given P; as we shall see later, $\Psi(D_1 P, \ldots, D_m P)$ is closely related with the projection on functions of the type $\mathrm{Tr}P$ of the operator on the entries $\mathcal{L} = \Delta - \nabla N tr(W).\nabla$ which is symmetric in $L^2(\mu_V^N)$ (here $W = V + \frac{1}{2}\sum \mathbf{A}_k^2$). The resulting operator is a differential operator. As such, it may be difficult to find a normed space stable for this operator (since the operator will deteriorate the smoothness of the functions) in which it is continuous and invertible.

To avoid this issue, we will first divide each monomials of P by its degree (which more or less amounts to integrate and then divide by x the function in the one variable case).

Then, we define a linear map Σ on $\mathbb{C}\langle X_1, \ldots, X_m\rangle$ such that for all monomials q of degree greater or equal to 1

$$\Sigma q = \frac{q}{\deg q}.$$

For later use, we set $\mathbb{C}\langle X_1, \ldots, X_m\rangle(0)$ to be the subset of polynomials P of $\mathbb{C}\langle X_1, \ldots, X_m\rangle_{sa}$ such that $P(0, \ldots, 0) = 0$. We let Π be the projection from $\mathbb{C}\langle X_1, \ldots, X_m\rangle_{sa}$ onto $\mathbb{C}\langle X_1, \ldots, X_m\rangle(0)$ (i.e., $\Pi(P) = P - P(0, \ldots, 0)$). We now define some operators on $\mathbb{C}\langle X_1, \ldots, X_m\rangle(0)$, i.e., from $\mathbb{C}\langle X_1, \ldots, X_m\rangle(0)$ into $\mathbb{C}\langle X_1, \ldots, X_m\rangle(0)$,

$$\Xi_1 : P \longrightarrow \Pi\left(\sum_{k=1}^{m} \partial_k \Sigma P \sharp D_k V\right)$$

$$\Xi_2 : P \longrightarrow \Pi\left(\sum_{k=1}^{m} (\mu \otimes I + I \otimes \mu)(\partial_k D_k \Sigma P)\right).$$

We define $\Xi_0 = I - \Xi_2$ and $\Xi = \Xi_0 + \Xi_1$, where I is the identity on $\mathbb{C}\langle X_1, \ldots, X_m\rangle(0)$. Note that the images of Ξ_i and Ξ are included in $\mathbb{C}\langle X_1, \ldots, X_m\rangle_{sa}$ since V is assumed self-adjoint. With these notations, Lemma 9.4, once applied to $P_i = D_i \Sigma P$, $1 \le i \le m$, reads:

Proposition 9.6. *For all P in $\mathbb{C}\langle X_1, \ldots, X_m\rangle(0)$, $\hat{\delta}^N(\Xi P)$ converges in law to a centered Gaussian variable with covariance*

$$\mathcal{C}(P) := C(D_1 \Sigma P, \ldots, D_m \Sigma P).$$

Proof. We have for all tracial states τ, $\tau(\partial_k P \sharp V) = \tau(D_k P V)$ and if P is in $\mathbb{C}\langle X_1, \ldots, X_m\rangle(0)$, we have the identity

$$P = \sum_k \partial_k \Sigma P \sharp X_k.$$

Then, as $\hat{\delta}^N$ is tracial (7.2) and vanishes on constant terms (so that the projection Π can be removed in the definition of Ξ), for all polynomial P,

$$\begin{aligned}\hat{\delta}^N(\Xi P) &= \hat{\delta}^N\left(P + \sum_{k=1}^m \partial_k \Sigma P \sharp D_k V - \sum_{k=1}^m (\mu \otimes I + I \otimes \mu)(\partial_k D_k \Sigma P)\right)\\ &= \hat{\delta}^N\left(\sum_{k=1}^m (X_k + D_k V) D_k \Sigma P - \sum_{k=1}^m (\mu \otimes I + I \otimes \mu)(\partial_k D_k \Sigma P)\right)\\ &= Z_N(D_1 \Sigma P, \ldots, D_m \Sigma P).\end{aligned}$$

We then use Lemma 9.4 to conclude. □

To generalize the central limit theorem to all polynomial functions, we need to show that the image of Ξ is dense and to control approximations. If P is a polynomial and q a non-constant monomial we will denote by $\lambda_q(P)$ the coefficient of q in the decomposition of P in monomials. We can then define a norm $\|.\|_A$ on $\mathbb{C}\langle X_1, \ldots, X_m\rangle(0)$ for $A > 1$ by

$$\|P\|_A = \sum_{\deg q \neq 0} |\lambda_q(P)| A^{\deg q}.$$

In the formula above, the sum is taken over all non-constant monomials. We also define the operator norm given, for T from $\mathbb{C}\langle X_1, \ldots, X_m\rangle(0)$ to $\mathbb{C}\langle X_1, \ldots, X_m\rangle(0)$, by

$$|||T|||_A = \sup_{\|P\|_A = 1} \|T(P)\|_A.$$

Finally, let $\mathbb{C}\langle X_1, \ldots, X_m\rangle(0)_A$ be the completion of $\mathbb{C}\langle X_1, \ldots, X_m\rangle(0)$ for $\|.\|_A$. We say that T is continuous on $\mathbb{C}\langle X_1, \ldots, X_m\rangle(0)_A$ if $|||T|||_A$ is finite. We shall prove that Ξ is continuous on $\mathbb{C}\langle X_1, \ldots, X_m\rangle(0)_A$ with continuous inverse when $\mathbf{t}$ is small.

Lemma 9.7. *With the previous notations:*

1. *The operator Ξ_0 is invertible on $\mathbb{C}\langle X_1, \ldots, X_m\rangle(0)$.*
2. *There exists $A_0 > 0$ such that for all $A > A_0$, the operators Ξ_2, Ξ_0 and Ξ_0^{-1} are continuous on $\mathbb{C}\langle X_1, \ldots, X_m\rangle(0)_A$ and their norm are uniformly bounded for $\mathbf{t}$ in B_η.*
3. *For all $\epsilon, A > 0$, there exists $\eta_\epsilon > 0$ such for $|\mathbf{t}| < \eta_\epsilon$, Ξ_1 is continuous on $\mathbb{C}\langle X_1, \ldots, X_m\rangle(0)_A$ and $|||\Xi_1|||_A \leq \epsilon$.*
4. *For all $A > A_0$, there exists $\eta > 0$ such that for $\mathbf{t} \in B_\eta$, Ξ is continuous, invertible with a continuous inverse on $\mathbb{C}\langle X_1, \ldots, X_m\rangle(0)_A$. Besides the norms of Ξ and Ξ^{-1} are uniformly bounded for $\mathbf{t}$ in B_η.*
5. *There exists $C > 0$ such that for all $A > C$, $\mathcal{C}$ is continuous from $\mathbb{C}\langle X_1, \ldots, X_m\rangle(0)_A$ into $\mathbb{R}$.*

Proof. 1. Observe that since Ξ_2 reduces the degree of a polynomial by at least 2,

$$P \to \sum_{n\geq 0} (\Xi_2)^n(P)$$

is well defined on $\mathbb{C}\langle X_1, \ldots, X_m\rangle(0)$ as the sum is finite for any polynomial P. This gives an inverse for $\Xi_0 = I - \Xi_2$.

2. First remark that a linear operator T has a norm less than C with respect to $\|.\|_A$ if and only if for all non-constant monomial q,

$$\|T(q)\|_A \leq C A^{\deg q}.$$

Recall that μ is uniformly compactly supported (see Lemma 6.18) and let $C_0 < +\infty$ be such that $|\mu(q)| \leq C_0^{\deg q}$ for all monomial q. Take a monomial $q = X_{i_1} \cdots X_{i_p}$, and assume that $A > 2C_0$,

$$\begin{aligned}
\left\| \Pi \left(\sum_k (I \otimes \mu)\partial_k D_k \Sigma q \right) \right\|_A &\leq p^{-1} \sum_{\substack{k, q=q_1 X_k q_2, \\ q_2 q_1 = r_1 X_k r_2}} \|r_1 \mu(r_2)\|_A \\
\leq p^{-1} \sum_{\substack{k, q=q_1 X_k q_2, \\ q_2 q_1 = r_1 X_k r_2}} A^{\deg r_1} C_0^{\deg r_2} &= \frac{1}{p} \sum_{n=0}^{p-1} \sum_{l=0}^{p-2} A^l C_0^{p-l-2} \\
\leq A^{p-2} \sum_{l=0}^{p-2} \left(\frac{C_0}{A} \right)^{p-2-l} &\leq 2A^{-2} \|q\|_A
\end{aligned}$$

where in the second line, we observed that once $\deg(q_1)$ is fixed, $q_2 q_1$ is uniquely determined and then r_1, r_2 are uniquely determined by the degree l of r_1. Thus, the factor $\frac{1}{p}$ is compensated by the number of possible decompositions of q, i.e., the choice of n, the degree of q_1. If $A > 2$, $P \to \Pi\left(\sum_k (I \otimes \mu)\partial_k D_k \Sigma P\right)$ is continuous of norm strictly less than $\frac{1}{2}$. And a similar calculus for $\Pi\left(\sum_k (\mu \otimes I)\partial_k D_k \Sigma\right)$ shows that Ξ_2 is continuous of norm strictly less than 1. It follows immediately that Ξ_0 is continuous. Since $\Xi_0^{-1} = \sum_{n\geq 0} \Xi_2^n$, Ξ_0^{-1} is continuous as soon as Ξ_2 is of norm strictly less than 1.

3. Let $q = X_{i_1} \cdots X_{i_p}$ be a monomial and let D be the degree of V and $B(\leq Dn)$ the sum of the maximum number of monomials in $D_k V$.

$$\begin{aligned}
\|\Xi_1(q)\|_A &\leq \frac{1}{p} \sum_{k, q=q_1 X_k q_2} \|q_1 D_k V q_2\|_A \leq \frac{1}{p} \sum_{k, q=q_1 X_k q_2} |\mathbf{t}| B A^{p-1+D-1} \\
&= |\mathbf{t}| B A^{D-2} \|q\|_A.
\end{aligned}$$

It is now sufficient to take $\eta_\epsilon < (BA^{D-2})^{-1}\epsilon$.

4. We choose $\eta < (BA^{D-2})^{-1} |||\Xi_0^{-1}|||_A^{-1}$ so that when $|\mathbf{t}| \leq \eta$,

$$|||\Xi_1|||_A |||\Xi_0^{-1}|||_A < 1.$$

By continuity, we can extend Ξ_0, Ξ_1, Ξ_2, Ξ and Ξ_0^{-1} on the space $\mathbb{C}\langle X_1,\ldots,X_m\rangle(0)_A$. The operator

$$P \to \sum_{n\geq 0}(-\Xi_0^{-1}\Xi_1)^n\Xi_0^{-1}$$

is well defined and continuous. This is an inverse of $\Xi = \Xi_0 + \Xi_1 = \Xi_0(I + \Xi_0^{-1}\Xi_1)$.

5. We finally prove that $\mathcal{C}$ is continuous from $\mathbb{C}\langle X_1,\ldots,X_m\rangle(0)_A$ into $\mathbb{R}$ where we recall that we assumed $A > C_0$. Let us consider the first term

$$\mathcal{C}_1(P) := \sum_{k,l=1}^m \mu\otimes\mu(\partial_k D_l \Sigma P \times \partial_l D_k \Sigma P).$$

Then, we obtain as in the second point of this proof

$$\begin{aligned}
|\mathcal{C}_1(P)| &\leq 4\sum_{k,l=1}^m \sum_{q,q'} \frac{|\lambda_q(P)||\lambda_{q'}(P)|}{\deg q \deg q'} \sum_{\substack{q=q_1X_kq_2,\,q'=q_1'X_lq_2'\\ q_2q_1=r_1X_lr_2,\,q_2'q_1'=r_1'X_kr_2'}} C_0^{\deg q+\deg q'-4}\\
&\leq 4\sum_{q,q'}|\lambda_q(P)||\lambda_{q'}(P)|\deg q\deg q' C_0^{\deg q+\deg q'-4}\\
&\leq 4(\sup_{\ell\geq 0}\ell C_0^{\ell-2}A^{-\ell})^2\|P\|_A^2.
\end{aligned}$$

We next turn to showing that

$$\mathcal{C}_2(P) := \sum_{k,l=1}^m \mu\left(\partial_k\circ\partial_l V\sharp(D_k\Sigma P, D_l\Sigma P)\right)$$

is also continuous for $\|.\|_A$. In fact, noting that we may assume $V \in \mathbb{C}\langle X_1,\ldots,X_m\rangle(0)$ without changing $\mathcal{C}_2$, we find

$$\begin{aligned}
|\mathcal{C}_2(P)| &\leq \sum_{p,q,q',k,l}|\lambda_p(V)| \sum_{\substack{q,q',p=p_1X_kp_2X_lp_3\\ q=q_1X_kq_2,\,q'=q_1'X_kq_2'}} \frac{|\lambda_q(P)||\lambda_{q'}(P)|C_0^{\deg p+\deg q+\deg q'-4}}{\deg q\deg q'}\\
&\leq n|\mathbf{t}|D^2\sum_{q,q'}|\lambda_q(P)||\lambda_{q'}(P)|C_0^{D+\deg q+\deg q'-4}\\
&\leq n|\mathbf{t}|D^2C_0^{D-4}\|P\|_A^2.
\end{aligned}$$

The continuity of the last term $\mathcal{C}_3(P) = \sum_{i=1}^m \mu\left((D_j\Sigma P)^2\right)$ is obtained similarly.

□

We can compare the norm $\|\cdot\|_A$ to a more intuitive norm, namely the Lipschitz norm $\|\cdot\|_{\mathcal{L}}^M$ defined by

$$\|P\|_{\mathcal{L}}^M = \sup_{N\in\mathbb{N}} \sup_{\substack{x_1,\dots,x_m\in\mathcal{H}_N^{(2)}\\ \forall i, \|x_i\|_\infty\leq M}} \sum_{k=1}^m (\|D_k P D_k P^*\|_{\mathcal{A}})^{\frac{1}{2}}.$$

We will say that a semi-norm $\mathcal{N}$ is weaker than a semi-norm $\mathcal{N}'$ if and only if there exists $C<+\infty$ such that for all P in $\mathbb{C}\langle X_1,\dots,X_m\rangle(0)$,

$$\mathcal{N}(P)\leq C\mathcal{N}'(P).$$

Lemma 9.8. *For $A > M$, the semi-norm $\|.\|_{\mathcal{L}}^M$ restricted to the space $\mathbb{C}\langle X_1,\dots,X_m\rangle(0)$ is weaker than the norm $\|.\|_A$.*

Proof. For all P in $\mathbb{C}\langle X_1,\dots,X_m\rangle(0)$, the following inequalities hold:

$$\|P\|_{\mathcal{L}}^M\leq\sum_q|\lambda_q(P)|\|q\|_{\mathcal{L}}^M\leq\sum_q|\lambda_q(P)|\deg q M^{\deg q}\leq\left(\sup_l l\left(\frac{M}{A}\right)^l\right)\|P\|_A.$$

□

To take into account the previous results, we define a new hypothesis **(H')** stronger than **(H)**.

(H'): **(H)** is satisfied, $A-1>\max(A_0,M_0,C)$ for the M_0 which appear in Lemma 6.19 and the C which appear in Proposition 9.1 Besides, $|\mathbf{t}|\leq\eta$ with η as in the fourth point of Lemma 9.7 in order that Ξ and Ξ^{-1} are continuous on $\mathbb{C}\langle X_1,\dots,X_m\rangle(0)_A$ and $\mathbb{C}\langle X_1,\dots,X_m\rangle(0)_{A-1}$, and that $\mathcal{C}$ is also continuous for these norms.

The two main additional consequences of this hypothesis are the continuity of Ξ for $\|\cdot\|_A$. The strange condition about the continuity of Ξ on $\mathbb{C}\langle X_1,\dots,X_m\rangle(0)_{A-1}$ is here assumed for a technical reason which will appear only in the last section on the interpretation of the first order correction to the free energy.

While $(\mathbf{H}')$ is full of conditions, the only important hypothesis is the c-convexity of V. Given such a V, we can always find constants A and η which satisfy the hypothesis. The only restriction will be then that $\mathbf{t}$ is sufficiently small.

We can now prove the general central limit theorem which is up to the identification of the variance equivalent to Theorem 9.5.

Theorem 9.9. *Assume* **(H')**. *For all P in $\mathbb{C}\langle X_1,\dots,X_m\rangle_{sa}$, $\hat{\delta}^N(P)$ converges in law to a centered Gaussian variable γ_P with variance*

$$\sigma^{(2)}(P) := \mathcal{C}(\Xi^{-1}\Pi(P)) = C(D_1\Sigma\Xi^{-1}\Pi(P),\cdots,D_m\Sigma\Xi^{-1}\Pi(P)).$$

If $P\in\mathbb{C}\langle X_1,\dots,X_m\rangle$, $\hat{\delta}^N(P)$ converges to the complex centered Gaussian variable $\gamma_{(P+P^*)/2}+i\gamma_{(P-P^*)/2i}$ (the covariance of $\gamma_{(P+P^*)/2}$ and $\gamma_{(P-P^*)/2i}$ being given by $\sigma^{(2)}((P+P^*)/2,(P-P^*)/2i)$ where $\sigma^{(2)}(\cdot,\cdot)$ is the bilinear form associated to the quadratic form $\sigma^{(2)}$).

Proof. As $\hat{\delta}^N(P)$ does not depend on constant terms, we can directly take $P = \Pi(P)$ in $\mathbb{C}\langle X_1,\ldots,X_m\rangle(0)$. Now, by Lemma 9.7. 4, we can find an element Q of $\mathbb{C}\langle X_1,\ldots,X_m\rangle(0)_A$ such that $\Xi Q = P$. But the space $\mathbb{C}\langle X_1,\ldots,X_m\rangle(0)$ is dense in $\mathbb{C}\langle X_1,\ldots,X_m\rangle(0)_A$ by construction. Thus, there exists a sequence Q_n in $\mathbb{C}\langle X_1,\ldots,X_m\rangle(0)$ such that

$$\lim_{n\to\infty} \|Q - Q_n\|_A = 0.$$

Let us define $R_n = P - \Xi Q_n$ in $\mathbb{C}\langle X_1,\ldots,X_m\rangle(0)$.

Now according to Property 9.6 for all n, $\hat{\delta}^N(\Xi Q_n)$ converges in law to a Gaussian variable γ_n of variance $\mathcal{C}(Q_n)$ with

$$\mathcal{C}(Q_n) = C(D_1 \Sigma Q_n, \ldots, D_m \Sigma Q_n).$$

As $\mathcal{C}$ is continuous by Lemma 9.7.4, it can be extended to the space $\mathbb{C}\langle X_1,\ldots,X_m\rangle(0)_A$ and $\sigma^{(2)}(P) = \mathcal{C}(\Xi^{-1}P) = \mathcal{C}(Q) = \lim_n \mathcal{C}(Q_n)$ is well defined. Hence, γ_n converges weakly to γ_∞, the centered Gaussian law with covariance $\mathcal{C}(Q)$, when n goes to $+\infty$. The last step is to prove the convergence in law of $\hat{\delta}^N(P)$ to γ_∞. We will use the Dudley distance d_D. Below, as a parameter of d_D, we, for short, write $\hat{\delta}^N(P)$ for the law of $\hat{\delta}^N(P)$. We make the following decomposition:

$$d_D(\hat{\delta}^N(P), \gamma_\infty) \le d_D(\hat{\delta}^N(P), \hat{\delta}^N(\Xi Q_n)) + d_D(\hat{\delta}^N(\Xi Q_n), \gamma_n) + d_D(\gamma_n, \gamma_\infty). \tag{9.5}$$

By the above remarks, $d_D(\hat{\delta}^N(\Xi Q_n), \gamma_n)$ goes to 0 when N goes to $+\infty$ and $d_D(\gamma_n, \gamma_\infty)$ goes to 0 when n goes to $+\infty$. We now use the bound on the Dudley distance

$$d_D(\hat{\delta}^N(P), \hat{\delta}^N(\Xi Q_n)) \le E[|\hat{\delta}^N(P) - \hat{\delta}^N(\Xi Q_n)| \wedge 1] = E[|\hat{\delta}^N(R_n)| \wedge 1].$$

We control the last term by Lemmas 6.21 and 6.19 so that for $M \ge M_0$,

$$E[|\hat{\delta}^N(R_n)| \wedge 1] \le e^{-\alpha NM} + 2\sqrt{\frac{2\pi}{c}} \|R_n\|_{\mathcal{L}}^M + \epsilon^N_{R_n,M} + |m^N_{R_n,M}|.$$

But we deduce from Lemma 9.8 that since we chose $M < A$, there exists a finite constant C such that

$$\|R_n\|_{\mathcal{L}}^M \le C\|R_n\|_A = C\|\Xi(Q - Q_n)\|_A \le C|||\Xi|||_A \|Q - Q_n\|_A$$

and so $\|R_n\|_{\mathcal{L}}^M$ goes to zero as n goes to infinity. And since $\|R_n\|_{\mathcal{L}}^M$ is finite, $\epsilon^N_{R_n,M}$ goes to zero. Similarly, using the bound of Lemma 6.21 on $m^N_{P,M}$ for P monomial, we find that

$$\begin{aligned}
|m^N_{R_n,M}| &\le N \sum_q |\lambda_q(R_n)| \deg(q)(3M^{\deg(q)} + \deg(q)^2) e^{-\alpha MN} \\
&\le N \sup_{\ell \ge 0} (\ell(3M^\ell + \ell^2) A^{-\ell}) \|R_n\|_A e^{-\alpha MN}
\end{aligned}$$

goes to zero as N goes to infinity. Thus, $E[|\hat{\delta}^N(R_n)| \wedge 1]$ goes to zero as n and N go to infinity. Putting things together, we obtain if we first let N go to $+\infty$ and then n, the desired convergence $\lim_N d_D(\hat{\delta}^N(P), \gamma_\infty) = 0$. □

Note that the convergence in law in Theorem 9.9 can be generalized to a convergence *in moments*:

Corollary 9.10. *Assume* **(H')**. *Let P be a self-adjoint polynomial, then $\hat{\delta}^N(P)$ converges in moments to a real centered Gaussian variable with variance $\sigma^{(2)}(P)$, i.e for all k in $\mathbb{N}$,*

$$\lim_{N\to\infty} \int (\hat{\delta}^N P)^k d\mu_V^N = \frac{1}{\sqrt{2\pi\sigma^{(2)}(P)}} \int x^k e^{-\frac{x^2}{2\sigma^{(2)}(P)}} dx.$$

Proof. Indeed, once again we decompose $\int (\hat{\delta}^N P)^k d\mu_V^N$ into $E_1^N + E_2^N$ with

$$E_1^N = \int \mathbf{1}_{\Lambda_M^N} (\hat{\delta}^N P)^k d\mu_V^N \quad E_2^N = \int \mathbf{1}_{(\Lambda_M^N)^c} (\hat{\delta}^N P)^k d\mu_V^N$$

with $M \geq M_0$. For E_1, we notice that the law of $\hat{\delta}^N P$ has a sub-Gaussian tail according to Lemma 6.21. Therefore, we can replace x^k by a bounded continuous function, producing an error independent of N. Applying Theorem 9.9 then shows that

$$\lim_{N\to\infty} \int \mathbf{1}_{\Lambda_M^N} (\hat{\delta}^N P)^k d\mu_V^N = \frac{1}{\sqrt{2\pi\sigma^{(2)}(P)}} \int x^k e^{-\frac{x^2}{2\sigma^{(2)}(P)}} dx.$$

For the second term, we use the trivial bound

$$\begin{aligned} |E_2^N| &\leq N^k \int \mathbf{1}_{(\Lambda_M^N)^c} (|\lambda_{max}(\mathbf{A})| + |\mu|(P))^k d\mu_V^N \\ &\leq k N^k \int_{\lambda \geq M} (\lambda + |\mu|(P))^{k-1} e^{-\alpha\lambda N} d\lambda \end{aligned}$$

which goes to zero as N goes to infinity for all finite k. □

9.3 Comments on the results

There is a natural interpretation of the operator Ξ in terms of the symmetric differential operator in $L^2(\mu_V^N)$ given by

$$\mathcal{L}_N = \sum_{k=1}^{m} \sum_{i,j=1}^{N} e^{N\mathrm{Tr}(V(\mathbf{X})+2^{-1}\sum_{i=1}^{m}(X^k)^2)} \partial_{X_{ij}^k} e^{-N\mathrm{Tr}(V(\mathbf{X})+2^{-1}\sum_{i=1}^{m}(X^k)^2)} \partial_{X_{ji}^k}$$
$$= \sum_{k=1}^{m} \sum_{i,j=1}^{N} \left(\partial_{X_{ij}^k} \partial_{X_{ji}^k} - N(D_k V + X_k)_{ji} \partial_{X_{ji}^k} \right).$$

One checks by integration by parts that for any pair of continuously differentiable functions f, g

$$\mu_V^N (f \mathcal{L}_N g) = \mu_V^N (g \mathcal{L}_N f) = - \sum_{k=1}^{m} \sum_{i,j=1}^{N} \mu_V^N \left(\partial_{X_{ij}^k} g \partial_{X_{ij}^k} f \right). \tag{9.6}$$

Moreover, for any polynomial P, with the notation of Lemma 9.3, we find that

$$\frac{1}{N} \mathcal{L}_N \mathrm{Tr}(\Sigma P(\mathbf{X})) = Y_N(D_1 \Sigma P, \ldots, D_m \Sigma P) = \mathrm{Tr}(\Xi P) + o(1)$$

according to Lemma 9.4. Applying (9.6) with $f = 1$ and $g = \mathrm{Tr}(\Sigma P(\mathbf{X}))$ shows that

$$\lim_{N \to \infty} \mu_V^N (\mathrm{Tr}(\Xi P)) = 0.$$

Furthermore, taking $f = \mathrm{Tr} P(\mathbf{X})$ and $g = \mathrm{Tr} Q(\mathbf{X})$ into (9.6) we deduce by (6.22) that

$$\begin{aligned}
\sigma^{(2)}(\Xi P, Q) &= \lim_{N \to \infty} \mu_V^N \left(\mathrm{Tr}(\Xi P(\mathbf{X})) \left(\mathrm{Tr} Q(\mathbf{X}) - \mu_V^N (\mathrm{Tr} Q(\mathbf{X})) \right) \right) \\
&= \lim_{N \to \infty} \frac{1}{N} \mu_V^N \left(\mathcal{L}_N \mathrm{Tr}(\Sigma P(\mathbf{X})) \left(\mathrm{Tr} Q(\mathbf{X}) - \mu_V^N (\mathrm{Tr} Q(\mathbf{X})) \right) \right) \\
&= \lim_{N \to \infty} \frac{1}{N} \sum_{k=1}^{m} \sum_{i,j=1}^{N} \mu_V^N \left((D_k \Sigma P(\mathbf{X}))_{ij} (D_k Q(\mathbf{X}))_{ij} \right) \\
&= \sum_{k=1}^{m} \tau_{\mathbf{t}} \left((D_k \Sigma P(\mathbf{X}))(D_k Q(\mathbf{X}))^* \right).
\end{aligned}$$

Thus, we have proved the following:

Lemma 9.11. *For all polynomial function P so that $P(0) = 0$,*

$$\sigma^{(2)}(\Xi P, Q) = \sum_{k=1}^{m} \tau_{\mathbf{t}} \left((D_k \Sigma P(\mathbf{X}))(D_k Q(\mathbf{X}))^* \right).$$

Let us define, for $\mathbf{k} = (k_1, \ldots, k_n)$,

$$\begin{aligned}
\mathcal{M}_{\mathbf{k}}(P, Q) = \sharp\{ &\text{ maps with } k_i \text{ stars of type } q_i, \\
&\text{one of type } P \text{ and one of type } Q\}.
\end{aligned}$$

and

$$\mathcal{M}_{\mathbf{t}}(P,Q) = \sum_{k_1,\ldots,k_n} \prod_{i=1}^{n} \frac{(-t_i)^{k_i}}{k_i!} \mathcal{M}_{k_1,\ldots,k_n}(P,Q).$$

We extend $\mathcal{M}_{\mathbf{t}}$ to polynomials by linearity. Then we claim that $\sigma^2(P,Q)$ and $\mathcal{M}(P,Q)$ satisfy the same kind of induction relation.

Proposition 9.12. *For all monomials P,Q and all k,*

$$\begin{aligned}
&\mathcal{M}_{k_1,\ldots,k_n}(X_kP,Q)\\
&= \sum_{0\le p_i\le k_i} \sum_{P=RX_kS} \prod_i C_{k_i}^{p_i} \mathcal{M}_{p_1,\ldots,p_n}(R,Q)\mathcal{M}_{k_1-p_1,\ldots,k_n-p_n}(S)\\
&+ \sum_{0\le p_i\le k_i} \sum_{P=RX_kS} \prod_i C_{k_i}^{p_i} \mathcal{M}_{p_1,\ldots,p_n}(S,Q)\mathcal{M}_{k_1-p_1,\ldots,k_n-p_n}(R)\\
&+ \sum_{0\le j\le n} k_j \mathcal{M}_{k_1,\ldots,k_j-1,\ldots,k_n}(D_kVP,Q)\\
&+ \mathcal{M}_{k_1,\ldots,k_n}(D_kQP)
\end{aligned}$$

and

$$\mathcal{M}_{\mathbf{t}}(X_kP,Q) = \mathcal{M}_{\mathbf{t}}((I\otimes\tau_{\mathbf{t}}+\tau_{\mathbf{t}}\otimes I)D_kP) - \mathcal{M}_{\mathbf{t}}(D_kVP,Q) + \tau_{\mathbf{t}}(D_kQP). \quad (9.7)$$

Besides there exists $\eta > 0$ so that there exists $R < +\infty$ such that for all monomials P and Q, all $\mathbf{t} \in B(0,\eta)$,

$$|\mathcal{M}_{\mathbf{t}}(P,Q)| \le R^{\deg P + \deg Q}. \quad (9.8)$$

Proof. The proof is very close to that of Theorem 8.5 which explains the decomposition of planar maps with one root. First we look for a relation on $\mathcal{M}_{k_1,\ldots,k_n}(X_kP,Q)$. We look at the first half-edge associated with X_k, then three cases may occur:

1. The first possibility is that the branch is glued to another branch of $P = RX_kS$. It cuts P into two: R and S and it occurs for all decomposition of P into $P = RX_kS$, which is exactly what D does. Then either the component R is linked to Q and to p_i stars of type q_i for each i, this leads to
$$\prod C_{k_i}^{p_i} \mathcal{M}_{p_1,\ldots,p_n}(R,Q)\mathcal{M}_{k_1-p_1,\ldots,k_n-p_n}(S)$$
possibilities or we have a symmetric case with S linked to Q in place of R.
2. The second case occurs when the branch is glued to a vertex of type q_j for a given j; first we have to choose between the k_j vertices of this type then we contract the edges arising from this gluing to form a vertex of type $D_iq_jP_1$, which creates

$$k_j \mathcal{M}_{k_1,\ldots,k_j-1,\ldots,k_n}(D_k q_j P, Q)$$

possibilities.

3. The last case is that the branch can be glued to the star associated to $Q = RX_iS$. We then only have to count planar graphs:

$$\mathcal{M}_{k_1,\ldots,k_n}(D_k QP).$$

We can now sum on the k's to obtain the relation on $\mathcal{M}$.

Finally, to show the last point of the proposition, we only have to prove that there exist $A > 0, B > 0$ such that for all k's, for all monomials P and Q,

$$\frac{\mathcal{M}_{k_1,\ldots,k_n}(P,Q)}{\prod_i k_i!} \leq A^{\sum_i k_i} B^{\deg P + \deg Q}.$$

This follows easily by induction over the degree of P with the previous relation on the $\mathcal{M}$ since we have proved such a control for $\mathcal{M}_{k_1,\ldots,k_n}(Q)$ in [104]. □

We can now prove the theorem:

Theorem 9.13. *Assume* **(H')** *with η small enough. Then, for all polynomials P, Q,*

$$\sigma^{(2)}(P,Q) = \mathcal{M}(P,Q)$$

Proof. First we transform the relation on $\mathcal{M}$. We use (9.7) with $P = D_k \Sigma R$ to deduce

$$\mathcal{M}(\Xi R, Q) = \sum_k \tau_{\mathbf{t}}(D_k Q D_k \Sigma R).$$

Let us define $\Delta = \sigma^{(2)} - \mathcal{M}$. Then according to the previous property, Δ is compactly supported and for all polynomials P and Q,

$$\Delta(\Xi P, Q) = 0.$$

Moreover, with $\mathcal{M}(1,Q) = 0 = \sigma^{(2)}(1,Q)$,

$$\Delta(1,Q) = 0.$$

To conclude we have to invert one more time the operator Ξ. For a polynomial P we take as in the proof of the central limit theorem, a sequence of polynomial S_n which goes to $S = \Xi^{-1}P$ in $\mathbb{C}\langle X_1,\ldots,X_m\rangle(0)_A$.

$$\Delta(P,Q) = \Delta(\Xi(S_n + S - S_n), Q) = \Delta(\Xi(S - S_n), Q).$$

But by continuity of Ξ, $\Xi(S - S_n)$ goes to 0 for the norm $\|.\|_A$. Moreover, because Δ is compactly supported, Δ is continuous for $\|.\|_R$, and so $\Delta(\Xi(S - S_n), Q)$ goes to zero when n goes to $+\infty$ provided $A \geq R$, which we can always assume if η is small enough. □

9.4 Second-order correction to the free energy

We now deduce from the central limit theorem the precise asymptotics of $N\bar{\delta}^N(P)$ and then compute the second-order correction to the free energy.

Let ϕ_0 and ϕ be the linear forms on $\mathbb{C}\langle X_1, \ldots, X_m\rangle$ which are given, if P is a monomial by

$$\phi_0(P) = \sum_{i=1}^{m} \sum_{P=P_1X_iP_2X_iP_3} \sigma^{(2)}(P_3P_1, P_2).$$

and $\phi = \phi_0 \circ \Sigma$.

Proposition 9.14. *Assume (**H'**). Then, for any polynomial P,*

$$\lim_{N\to\infty} N\bar{\delta}^N(P) = \phi(\Xi^{-1}P).$$

Proof. Again, we base our proof on the finite-dimensional Schwinger–Dyson equation (9.1) which, after centering, reads for $i \in \{1, \ldots, m\}$,

$$\begin{aligned} &N^2\mu_V^N\left((\hat{\mathbf{L}}^N - \tau_{\mathbf{t}})[(X_i + D_iV)P - (I \otimes \tau_{\mathbf{t}} + \tau_{\mathbf{t}} \otimes I)\partial_i P\right) \\ &\qquad = \mu_V^N\left(\hat{\delta}^N \otimes \hat{\delta}^N(\partial_i P)\right). \end{aligned} \tag{9.9}$$

Taking $P = D_i\Sigma P$ and summing over $i \in \{1, \ldots, m\}$, we thus have

$$N^2\mu_V^N\left((\hat{\mathbf{L}}^N - \tau_{\mathbf{t}})(\Xi P)\right) = \mu_V^N\left(\hat{\delta}^N \otimes \hat{\delta}^N\left(\sum_{i=1}^{m} \partial_i \circ D_i\Sigma P\right)\right) \tag{9.10}$$

By Theorem 9.9 and Lemma 9.10 we see that

$$\lim_{N\to\infty} \mu_V^N\left(\hat{\delta}^N \otimes \hat{\delta}^N(\sum_{i=1}^{m} \partial_i \circ D_iP)\right) = \phi(P)$$

which gives the asymptotics of $N\bar{\delta}^N(\Xi P)$ for all P.

To generalize the result to arbitrary P, we proceed as in the proof of the full central limit theorem. We take a sequence of polynomials Q_n which goes to $Q = \Xi^{-1}P$ when n goes to ∞ for the norm $\|.\|_A$. We define $R_n = P - \Xi Q_n = \Xi(Q - Q_n)$. Note that as P and Q_n are polynomials then R_n is also a polynomial.

$$N\bar{\delta}^N(P) = N\bar{\delta}^N(\Xi Q_n) + N\bar{\delta}^N(R_n).$$

According to Property 9.1, for any such monomial P of degree less than $N^{\frac{1}{2}-\epsilon}$,

$$|N\bar{\delta}^N(P)| \le C^{\deg(P)}.$$

So if we take the limit in N, for any monomial P,

$$\limsup_N |N\bar{\delta}^N(P)| \leq C^{\deg(P)}$$

and therefore $\limsup_N |N\bar{\delta}^N(P)| \leq \|P\|_C \leq \|P\|_A$. The last inequality comes from the hypothesis **(H')** which requires $C < A$. We now fix n and take the large N limit,

$$\limsup_N |N\bar{\delta}^N(P - \varXi Q_n)| \leq \limsup_N |N\bar{\delta}^N(R_n)| \leq \|R_n\|_A.$$

If we take the limit in n the right term vanishes and we are left with:

$$\lim_N N\bar{\delta}^N(P) = \lim_n \lim_N N\bar{\delta}^N(Q_n) = \lim_n \phi(Q_n).$$

It is now sufficient to show that ϕ is continuous for the norm $\|.\|_A$. But the map $P \to \sum_{i=1}^m \partial_i \circ D_i P$ is continuous from $\mathbb{C}\langle X_1, \ldots, X_m\rangle(0)_A$ to $\mathbb{C}\langle X_1, \ldots, X_m\rangle(0)_{A-1}$ and σ^2 is continuous for $\|.\|_{A-1}$ due to the technical hypothesis in **(H')**. This proves that ϕ is continuous and then can be extended on $\mathbb{C}\langle X_1, \ldots, X_m\rangle(0)_A$. Thus

$$\lim_N N\bar{\delta}^N(P) = \lim_n \phi(Q_n) = \phi(Q).$$

$\square$

Theorem 9.15. *Assume that $V_{\mathbf{t}}$ satisfies* **(H')** *with a given $c > 0$. Then*

$$\log \frac{Z_N^{V_{\mathbf{t}}}}{Z_N^0} = N^2 F_{\mathbf{t}} + F_{\mathbf{t}}^1 + o(1)$$

with

$$F_{\mathbf{t}} = \int_0^1 \tau_{\alpha\mathbf{t}}(V_{\mathbf{t}}) d\alpha$$

and

$$F_{\mathbf{t}}^1 = \int_0^1 \phi_{\alpha\mathbf{t}}(\varXi_{\alpha\mathbf{t}}^{-1} V_{\mathbf{t}}) d\alpha.$$

Proof. As in the proof of Theorem 8.8, we note that $\alpha V_{\mathbf{t}} = V_{\alpha\mathbf{t}}$ is c-convex for all $\alpha \in [0, 1]$ We use (8.12) to see that

$$\partial_\alpha \log Z_{V_{\alpha\mathbf{t}}}^N = \mu_{\alpha\mathbf{t}}^N(\hat{\mathbf{L}}^N(V_{\mathbf{t}}))$$

so that we can write

$$\begin{aligned} \log \frac{Z_{V_{\alpha\mathbf{t}}}^N}{Z_0^N} &= \int_0^1 \mu_{V_{\alpha\mathbf{t}}}^N(\hat{\mathbf{L}}^N(V_{\mathbf{t}})) d\alpha \\ &= N^2 F_{\mathbf{t}} + \int_0^1 [N\bar{\delta}_{\alpha\mathbf{t}}^N(V_{\mathbf{t}})] ds. \end{aligned} \tag{9.11}$$

Proposition 9.14 and (9.11) finish the proof of the theorem since by Proposition 9.1, all the $N\bar{\delta}^N(q_i)$ can be bounded independently of N and $t \in B_{\eta,c}$ so that dominated convergence theorem applies. $\square$

As for the combinatorial interpretation of the covariance we relate F^1 to a generating function of maps. This time, we will consider maps on a torus instead of a sphere. Such maps are said to be of genus 1.

$$\begin{aligned}\mathcal{M}^1_{k_1,\dots,k_n}(P) = \sharp\{ &\text{ maps of genus 1 with } k_i \text{ stars of type } q_i \text{ or } q_i^* \\ &\text{and one of type } P\}\end{aligned}$$

and

$$\mathcal{M}^1_{k_1,\dots,k_n} = \sharp\{ \text{ maps with } k_i \text{ stars of type } q_i \text{ or } q_i^*\}.$$

We also define the generating function:

$$\mathcal{M}^1_{\mathbf{t}}(P) = \sum_{k_1,\dots,k_n} \prod_{i=1}^n \frac{(-t_i)^{k_i}}{k_i!} \mathcal{M}^1_{k_1,\dots,k_n}(P).$$

Proposition 9.16. *For all monomials P and all k,*

$$\begin{aligned}&\mathcal{M}^1_{k_1,\dots,k_n}(X_k P)\\ &= \sum_{0\le p_i\le k_i} \sum_{P=RX_kS} \prod_i C_{k_i}^{p_i} \mathcal{M}^1_{p_1,\dots,p_n}(R)\mathcal{M}_{k_1-p_1,\dots,k_n-p_n}(S)\\ &+ \sum_{0\le p_i\le k_i} \sum_{P=RX_kS} \prod_i C_{k_i}^{p_i} \mathcal{M}_{p_1,\dots,p_n}(R)\mathcal{M}^1_{k_1-p_1,\dots,k_n-p_n}(S)\\ &+ \sum_{0\le j\le n} k_j \mathcal{M}^1_{k_1,\dots,k_j-1,\dots,k_n}(D_kVP,Q)\\ &+ \sum_{P=RX_kS} \mathcal{M}_{k_1,\dots,k_n}(R,S)\end{aligned}$$

and

$$\mathcal{M}^1_{\mathbf{t}}(X_kP) = \mathcal{M}^1_{\mathbf{t}}((I\otimes\tau_{\mathbf{t}}+\tau_{\mathbf{t}}\otimes I)\partial_k P) - \mathcal{M}^1_{\mathbf{t}}(D_kVP) + \mathcal{M}_{\mathbf{t}}\otimes\mathcal{M}_{\mathbf{t}}(\partial_k P). \quad (9.12)$$

Besides, for η small enough, there exists $R < +\infty$ such that for all monomials P, all $\mathbf{t} \in B(0,\eta)$,

$$|\mathcal{M}^1(P)| \le R^{\deg P}.$$

Proof. We proceed as for the combinatorial interpretation of the variance. We look at the first edge which comes out of the branch X_k, then two cases may occur:

1. The first possibility is that the branch is glued to another branch of $P = RX_kS$. It forms a loop starting from P. There are two cases.
   a) The loop can be retractible. It cuts P in two: R and S and it occurs for all decomposition of P into $P = RX_kS$ which is exactly what does D. Then either the component R or the component S is of genus 1 and the other component is planar. It produces either

$$\prod_i C_{k_i}^{p_i} \mathcal{M}^1_{p_1,\ldots,p_n}(R)\mathcal{M}_{k_1-p_1,\ldots,k_n-p_n}(S)$$

   possibilities ot the symmetric formula.
   b) The loop can also be non-trivial in the fundamental group of the surface. Then the surface is cut into two parts. We are left with a planar surface with two fixed stars R and S. This gives $\mathcal{M}_{k_1,\ldots,k_n}(R,S)$ possibilities.
2. The second possibility occurs when the branch is glued to a vertex of type q_j for a given j. First we have to choose between the k_j vertices of this type, then we contract the edges arising from this gluing to form a vertex of type $D_i q_j P_1$, which creates

$$k_j \mathcal{M}^1_{k_1,\ldots,k_j-1,\ldots,k_n}(D_k q_j P, Q)$$

   possibilities.

We can now sum on the k's to obtain the relation on $\mathcal{M}^1$.

Finally, to show that $\mathcal{M}^1$ is compactly supported we only have to prove that there exist $A > 0, B > 0$ such that for all k's, for all monomials P,

$$\frac{\mathcal{M}^1_{k_1,\ldots,k_n}(P)}{\prod_i k_i!} \le A^{\sum_i k_i} B^{\deg P}.$$

Again this follow easily by induction with the previous relation on the $\mathcal{M}^1$. □

Proposition 9.17. *Assume* **(H')**. *There exists* $\eta > 0$ *small enough so that for* $\mathbf{t} \in B_{\eta,c}$,

1. *For all monomials* P,
$$\phi(\Xi^{-1}P) = \mathcal{M}^1(P).$$
2.
$$F^1_{\mathbf{t}} = \sum_{\mathbf{k}\in\mathbb{N}^n\setminus\{0\}} \prod_{i=1}^n \frac{(-t_i)^{k_i}}{k_i!} \mathcal{M}^1_{k_1,\ldots,k_n}$$

Proof. We use the equation of the previous property on $\mathcal{M}^1$ with $P = D_k \Sigma P$ and we sum, then

$$\mathcal{M}^1(\Xi P) = \mathcal{M}(\sum_k \partial_k D_k P) = \sum_k \sigma^2(\partial_k D_k P) = \phi(\Xi P)$$

where we have used the combinatorial interpretation of the covariance (Theorem 9.13). As $\mathcal{M}^1$ and ϕ are continuous for $\|.\|_A$ when η is small enough, we can apply this to $\Xi^{-1}P$ and conclude.

Finally, for η sufficiently small the sum is absolutely convergent so that we can interchange the integral and the sum:

$$\begin{aligned} F^1_{\mathbf{t}} &= \int_0^1 \mathcal{M}^1_{\alpha\mathbf{t}}(V_{\mathbf{t}})d\alpha \\ &= \sum_{i=1}^n \sum_{k_i,\ldots,k_n} \int_0^1 (-t_i) \prod_j \frac{(-\alpha t_j)^{k_j}}{k_j!} \mathcal{M}^1_{\mathbf{k}}(q_i)d\alpha. \\ &= \int_0^1 \partial_\alpha \mathcal{M}^1_{\alpha\mathbf{t}} d\alpha = \mathcal{M}^1_{\mathbf{t}}. \end{aligned}$$

This proves the statement. □

Bibliographical notes. Central limit theorems for the trace of polynomials in independent Gaussian matrices were first considered in [54, 100], based on a dynamical approach. Similar questions were undertaken under a free probability perspective in [157] where the interpretation of the covariances appearing in the central limit theorem of independent Gaussian matrices in terms of planar diagrams is used to define a new type of freeness. The case where the potential is not convex and the limiting measure may have a disconnected support is addressed in [162]; in certain cases Pastur can compute the logarithm of the Laplace transform of linear statistics and show that it is not given by half the covariance, as it should if a central limit theorem would hold.

The asymptotic topological expansion of one-matrix integrals was studied in [2, 86], using orthogonal polynomials. The so-called Schwinger–Dyson (or loop, or Master loop) equations were already used to analyze these questions in many papers in physics, see, e.g., [87, 31, 88].

Part IV

Eigenvalues of Gaussian Wigner matrices and large deviations

In this part, we consider the case where the entries of the matrix $\mathbf{X}^{N,\beta}$ are the so-called Gaussian ensembles. Moreover, since the results depend upon the fact that the entries are real or complex, we now show the difference in the notations. We consider $N \times N$ self-adjoint random matrices with entries

$$X_{kl}^{N,\beta} = \frac{\sum_{i=1}^{\beta} g_{kl}^i e_\beta^i}{\sqrt{\beta N}}, \quad 1 \le k < l \le N, \quad X_{kk}^{N,\beta} = \sqrt{\frac{2}{\beta N}} g_{kk} e_\beta^1, \quad 1 \le k \le N$$

where $(e_\beta^i)_{1 \le i \le \beta}$ is a basis of $\mathbb{R}^\beta$, that is $e_1^1 = 1$, $e_2^1 = 1, e_2^2 = i$. This definition can be extended to the case $\beta = 4$, named the Gaussian symplectic ensemble, when N is even by choosing $\mathbf{X}^{N,\beta} = \left(X_{ij}^{N,\beta}\right)_{1 \le i,j \le \frac{N}{2}}$ with $X_{kl}^{N,\beta}$ a 2×2 matrix defined as above but with $(e_\beta^k)_{1 \le k \le 4}$ the Pauli matrices

$$e_4^1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad e_4^2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad e_4^3 = \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix}, \quad e_4^4 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}.$$

$(g_{kl}^i, k \le l, 1 \le i \le \beta)$ are independent equidistributed centered Gaussian variables with variance 1. $(\mathbf{X}^{N,2}, N \in \mathbb{N})$ is commonly referred to as the Gaussian Unitary Ensemble (**GUE**), $(\mathbf{X}^{N,1}, N \in \mathbb{N})$ as the Gaussian Orthogonal Ensemble (**GOE**) and $(\mathbf{X}^{N,4}, N \in \mathbb{N})$ as the Gaussian Symplectic Ensemble (**GSE**) since they can be characterized by the fact that their laws are invariant under the action of the unitary, orthogonal and symplectic group respectively (see [153]). We denote by $P_N^{(\beta)}$ the law of $\mathbf{X}^{N,\beta}$.

The main advantage of the Gaussian ensembles is that the law of the eigenvalues of these matrices is explicit and rather simple. Namely, we now discuss the following lemma.

Lemma IV.1. *Let $\mathbf{X} \in \mathcal{H}_N^{(\beta)}$ be random with law $P_N^{(\beta)}$. The joint distribution of the eigenvalues $\lambda_1(X) \le \cdots \le \lambda_N(X)$, has density proportional to*

$$1_{x_1 \le \cdots \le x_N} \prod_{1 \le i < j \le N} |x_i - x_j|^\beta \prod_{i=1}^N e^{-\beta x_i^2/4}. \tag{IV.13}$$

We shall prove this lemma later, when studying Dyson's Brownian motion, see Corollary 12.4. Let us, however, emphasize the ideas behind a direct proof in the case $\beta = 1$. It is simply to write the decomposition $X = UDU^*$, with the eigenvalues matrix D that is diagonal and with real entries, and with the eigenvectors matrix U (that is unitary). Suppose this map was a bijection (which it is not, at least at the matrices X that do not possess all distinct eigenvalues) and that one can parametrize the eigenvectors by $\beta N(N-1)/2$ parameters in a smooth way (which one cannot in general). Then, it is easy to deduce from the formula $X = UDU^*$ that the Jacobian of this change of variables depends polynomially on the entries of D and is of degree $\beta N(N-1)/2$ in these variables. Since the bijection must break down when $D_{ii} = D_{jj}$ for some $i \neq j$, the Jacobian must vanish on that set. When $\beta = 1$, this imposes that the polynomial must be proportional to $\prod_{1 \le i < j \le N}(x_i - x_j)$. Further degree and symmetry considerations allow us to generalize this to $\beta = 2$. We refer the reader to [6] for a full proof, that shows that the set of matrices for which the above manipulations are not permitted has Lebesgue measure zero.

10

Large deviations for the law of the spectral measure of Gaussian Wigner's matrices

In this section, we consider the law of N random variables $(\lambda_1, \ldots, \lambda_N)$ with law

$$P^N_{V,\beta}(d\lambda_1, \ldots, d\lambda_N) = (Z^N_{V,\beta})^{-1} |\Delta(\lambda)|^\beta e^{-N\sum_{i=1}^N V(\lambda_i)} \prod_{i=1}^N d\lambda_i, \tag{10.1}$$

for a continuous function $V : \mathbb{R} \to \mathbb{R}$ such that

$$\liminf_{|x|\to\infty} \frac{V(x)}{\beta \log |x|} > 1 \tag{10.2}$$

and a positive real number β. Here, $\Delta(\lambda) = \prod_{1\le i<j\le N}(\lambda_i - \lambda_j)$.

When $V(x) = 4^{-1}\beta x^2$, we have seen in Lemma IV that $P^N_{4^{-1}\beta x^2,\beta}$ is the law of the eigenvalues of an $N \times N$ GOE (resp. GUE, resp GSE) matrix when $\beta = 1$ (resp. $\beta = 2$, resp. $\beta = 4$). The case $\beta = 4$ corresponds to another matrix ensemble, namely the GSE. In view of these remarks and other applications discussed in Part III, we consider in this section the slightly more general model with a potential V. We emphasize, however, that the distribution (10.1) precludes us from considering random matrices with independent non-Gaussian entries.

We have proved already at the beginning of these notes that the empirical measure

$$L_N = \frac{1}{N}\sum_{i=1}^N \delta_{\lambda_i^N}$$

converges almost surely towards the semi-circular law. Moreover, we studied its fluctuations around its mean, both by the central limit theorem and by concentration inequalities. Such results did not depend much on the Gaussian nature of the entries.

We address here a different type of question. Namely, we study the probability that L_N takes a very unlikely value. This was already considered in our discussion of concentration inequalities (cf. Part II), where the emphasis was put on obtaining upper bounds on the probability of deviation. In contrast, the purpose of the analysis here is to exhibit a precise estimate on these probabilities, or at least on their logarithmic asymptotics. The appropriate tool for handling such questions is large deviation theory, and we present in Appendix 20.1 a concise introduction to that theory and related definitions and references.

Endow $\mathcal{P}(\mathbb{R})$ with the usual weak topology. Our goal is to estimate the probability $P^N_{V,\beta}(L_N \in A)$, for measurable sets $A \subset \mathcal{P}(\mathbb{R})$. Of particular interest is the case where A does not contain the limiting distribution of L_N.

Define the *non-commutative entropy* $\Sigma : \mathcal{P}(\mathbb{R}) \to [-\infty, \infty]$, as

$$\Sigma(\mu) = \int \int \log |x - y| d\mu(x) d\mu(y) . \tag{10.3}$$

Set next

$$I^V_\beta(\mu) = \begin{cases} \int V(x)d\mu(x) - \frac{\beta}{2}\Sigma(\mu) - c^V_\beta \,, \text{ if } \int V(x)d\mu(x) < \infty \\ \infty, \qquad\qquad\qquad\qquad\qquad\qquad \text{otherwise}\,, \end{cases} \tag{10.4}$$
with $c^V_\beta = \inf_{\nu \in \mathcal{P}(\mathbb{R})}\{\int V(x)d\nu(x) - \frac{\beta}{2}\Sigma(\nu)\}$.

Theorem 10.1. *Let $L_N = N^{-1}\sum_{i=1}^N \delta_{\lambda_i^N}$ be the empirical measure of the random variables $\{\lambda_i^N\}_{i=1}^N$ distributed according to the law $P^N_{V,\beta}$, see (10.1). Then, the family of random measures L_N satisfies, in $\mathcal{P}(\mathbb{R})$ equipped with the weak topology, a full large deviation principle with good rate function I^V_β in the scale N^2. That is, $I^V_\beta : \mathcal{P}(\mathbb{R}) \to [0, \infty]$ possesses compact level sets $\{\nu : I^V_\beta(\nu) \le M\}$ for all $M \in \mathbb{R}_+$, and*

$$\begin{aligned} &\text{For any open set } O \subset \mathcal{P}(\mathbb{R})\,, \\ &\liminf_{N\to\infty} \frac{1}{N^2} \log P^N_{\beta,V}(L_N \in O) \ge -\inf_O I^V_\beta\,, \end{aligned} \tag{10.5}$$

and

$$\begin{aligned} &\text{For any closed set } F \subset \mathcal{P}(\mathbb{R})\,, \\ &\limsup_{N\to\infty} \frac{1}{N^2} \log P^N_{\beta,V}(L_N \in F) \le -\inf_F I^V_\beta\,. \end{aligned} \tag{10.6}$$

The proof of Theorem 10.1 relies on the properties of the function I^V_β collected in Lemma 10.2 below.

Lemma 10.2.

(a) I^V_β *is well defined on $\mathcal{P}(\mathbb{R})$ and takes its values in $[0, +\infty]$.*
(b) I^V_β *is a good rate function.*
(c) I^V_β *is a strictly convex function on $\mathcal{P}(\mathbb{R})$.*
(d) I^V_β *achieves its minimum value at a unique probability measure σ^V_β on $\mathbb{R}$ characterized, if $C^V_\beta = \inf_{\nu\in\mathcal{P}(\mathbb{R})}\Big(\int (V(x) - \beta \int \log |x - y| d\sigma^V_\beta(y)) d\nu(x)\Big)$, by*

$$V(x) - \beta \int \log|y-x| d\sigma_\beta^V(y) = C_V^\beta, \ \sigma_\beta^V \ a.s., \tag{10.7}$$

and, for all x outside of the support of σ_β^V,

$$V(x) - \beta \int \log|y-x| d\sigma_\beta^V(y) \geq C_\beta^V. \tag{10.8}$$

As an immediate corollary of Theorem 10.1 and of part (d) of Lemma 10.2 we have the following.

Corollary 10.3 (Second proof of Wigner's theorem). *Under $P_{V,\beta}^N$, L_N converges almost surely towards σ_β^V.*

Proof of Lemma 10.2. If $I_\beta^V(\mu) < \infty$, since V is bounded below by assumption (10.2), $\Sigma(\mu) > -\infty$ and therefore also $\int V d\mu < \infty$. This proves that $I_\beta^V(\mu)$ is well defined (and by definition non-negative), yielding point (a).

Set

$$f(x,y) = \frac{1}{2}V(x) + \frac{1}{2}V(y) - \frac{\beta}{2}\log|x-y|. \tag{10.9}$$

Note that $f(x,y)$ goes to $+\infty$ when x, y do by (10.2). Indeed, $\log|x-y| \leq \log(|x|+1) + \log(|y|+1)$ implies

$$f(x,y) \geq \frac{1}{2}(V(x) - \beta\log(|x|+1)) + \frac{1}{2}(V(y) - \beta\log(|y|+1)) \tag{10.10}$$

as well as when x, y approach the diagonal $\{x = y\}$; for all $L > 0$, there exist constants $K(L)$ (going to infinity with L) such that

$$\{(x,y) : \ f(x,y) \geq K(L)\} \subset B_L,$$

$$B_L := \{(x,y) : \ |x-y| < L^{-1}\} \cup \{(x,y) : \ |x| > L\} \cup \{(x,y) : \ |y| > L\}. \tag{10.11}$$

Since f is continuous on the compact set B_L^c, we conclude that f is bounded below, and denote $b_f > -\infty$ a lower bound.

We now show that I_V^β is a good rate function, and first that its level sets $\{I_V^\beta \leq M\}$ are closed, that is, that I_V^β is lower semi-continuous. Indeed, by the monotone convergence theorem, we have the following:

$$\begin{aligned} I_\beta^V(\mu) &= \int\int f(x,y) d\mu(x) d\mu(y) - c_\beta^V \\ &= \sup_{M \geq 0} \int\int (f(x,y) \wedge M) d\mu(x) d\mu(y) - c_\beta^V. \end{aligned}$$

But $f^M = f \wedge M$ is bounded continuous and so for $M < \infty$,

$$I_\beta^{V,M}(\mu) = \int\int (f(x,y) \wedge M) d\mu(x) d\mu(y)$$

is bounded continuous on $\mathcal{P}(\mathbb{R})$. As a supremum of the continuous functions $I_\beta^{V,M}$, I_β^V is lower semi-continuous. Hence, by Theorem 20.11, to prove that $\{I_\beta^V \le L\}$ is compact, it is enough to show that $\{I_\beta^V \le L\}$ is included in a compact subset of $\mathcal{P}(\mathbb{R})$ of the form

$$K_\epsilon = \cap_{B\in\mathbb{N}}\{\mu \in \mathcal{P}(\mathbb{R}) : \mu([-B,B]^c) \le \epsilon(B)\}$$

with a sequence $\epsilon(B)$ going to zero as B goes to infinity.

Arguing as in (10.11), there exist constants $K'(L)$ going to infinity as L goes to infinity, such that

$$\{(x,y):\ |x|>L, |y|>L\} \subset \{(x,y):\ f(x,y) \ge K'(L)\}\,. \tag{10.12}$$

Hence, for any $L>0$ large,

$$\begin{aligned}
\mu\left(|x|>L\right)^2 &= \mu\otimes\mu\left(|x|>L, |y|>L\right)\\
&\le \mu\otimes\mu\left(f(x,y)\ge K'(L)\right)\\
&\le \frac{1}{K'(L)-b_f}\int\int (f(x,y)-b_f)d\mu(x)d\mu(y)\\
&= \frac{1}{K'(L)-b_f}(I_\beta^V(\mu)+c_\beta^V-b_f)
\end{aligned}$$

Hence, with $\epsilon(B) = [\sqrt{(M+c_\beta^V-b_f)_+}/\sqrt{(K'(B)-b_f)_+}]\wedge 1$ going to zero when B goes to infinity, one has that $\{I_\beta^V \le M\} \subset K_\epsilon$. This completes the proof of point (b).

Since I_β^V is a good rate function, it achieves its minimal value. Let σ_β^V be a minimizer. Then, for any signed measure $\bar\nu(dx) = \phi(x)\sigma_\beta^V(dx) + \psi(x)dx$ with two bounded measurable compactly supported functions (ϕ,ψ) such that $\psi\ge 0$ and $\bar\nu(\mathbb{R})=0$, for $\epsilon>0$ small enough, $\sigma_\beta^V+\epsilon\bar\nu$ is a probability measure so that

$$I_\beta^V(\sigma_\beta^V+\epsilon\bar\nu) \ge I_\beta^V(\sigma_\beta^V)$$

which gives

$$\int\left(V(x)-\beta\int\log|x-y|d\sigma_\beta^V(y)\right)d\bar\nu(x)\ge 0.$$

Taking $\psi=0$, we deduce by symmetry that there is a constant C_β^V such that

$$V(x)-\beta\int\log|x-y|d\sigma_\beta^V(y) = C_\beta^V\,,\quad \sigma_\beta^V \text{ a.s.,} \tag{10.13}$$

which implies that σ_β^V is compactly supported (as $V(x)-\beta\int\log|x-y|d\sigma_\beta^V(y)$ goes to infinity when x does). Taking $\phi(x) = -\int\psi(y)dy$, we then find that

$$V(x)-\beta\int\log|x-y|d\sigma_\beta^V(y) \ge C_\beta^V \tag{10.14}$$

Lebesgue almost surely, and then everywhere outside of the support of σ_β^V by continuity. By (10.13) and (10.14) we deduce that

$$C_\beta^V = \inf_{\nu \in \mathcal{P}(\mathbb{R})} \{ \int (V(x) - \beta \int \log |x-y| d\sigma_\beta^V(y)) d\nu(x) \} .$$

This completes the proof of (10.7) and (10.8). The claimed uniqueness of σ_β^V, and hence the completion of the proof of part (d), then follows from the strict convexity claim (point (c) of the lemma), which we turn to next.

Note first that we can rewrite I_β^V as

$$I_\beta^V(\mu) = -\frac{\beta}{2} \Sigma(\mu - \sigma_\beta^V) + \int \left(V - \beta \int \log |x-y| d\sigma_\beta^V(y) - C_\beta^V \right) d\mu(x).$$

The fact that I_β^V is strictly convex comes from the observation that Σ is strictly concave, as can be checked from the formula

$$\log |x-y| = \int_0^\infty \frac{1}{2t} \left(\exp\left\{ -\frac{1}{2t} \right\} - \exp\left\{ -\frac{|x-y|^2}{2t} \right\} \right) dt \tag{10.15}$$

which entails that for any $\mu \in \mathcal{P}(\mathbb{R})$,

$$\Sigma(\mu - \sigma_\beta^V) = -\int_0^\infty \frac{1}{2t} \left(\int \int \exp\{ -\frac{|x-y|^2}{2t} \} d(\mu - \sigma_\beta^V)(x) d(\mu - \sigma_\beta^V)(y) \right) dt.$$

Indeed, one may apply Fubini's theorem when μ_1, μ_2 are supported in $[-\frac{1}{2}, \frac{1}{2}]$ since then $\mu_1 \otimes \mu_2 (\exp\{-\frac{1}{2t}\} - \exp\{-\frac{|x-y|^2}{2t}\} \leq 0) = 1$. One then deduces the claim for any compactly supported probability measures by scaling and finally for all probability measures by approximations. The fact that for all $t \geq 0$,

$$\begin{aligned} &\int \int \exp\{ -\frac{|x-y|^2}{2t} \} d(\mu - \sigma_\beta^V)(x) d(\mu - \sigma_\beta^V)(y) \\ &= \sqrt{\frac{t}{2\pi}} \int_{-\infty}^{+\infty} \left| \int \exp\{i\lambda x\} d(\mu - \sigma_\beta^V)(x) \right|^2 \exp\{ -\frac{t\lambda^2}{2} \} d\lambda \end{aligned}$$

therefore entails that Σ is concave since $\mu \to \left| \int \exp\{i\lambda x\} d(\mu - \sigma_\beta^V)(x) \right|^2$ is convex for all $\lambda \in \mathbb{R}$. Strict convexity comes from Cauchy–Schwarz inequality, $\Sigma(\alpha\mu + (1-\alpha)\nu) = \alpha\Sigma(\mu) + (1-\alpha)\Sigma(\nu)$ if and only if $\Sigma(\nu - \mu) = 0$ which implies that all the Fourier transforms of $\nu - \mu$ are null, and hence $\mu = \nu$. This completes the proof of the lemma. □

Proof of Theorem 10.1. To begin, let us remark that with f as in (10.9),

$$P_{V,\beta}^N(d\lambda_1, \ldots, d\lambda_N) = (Z_N^{\beta,V})^{-1} e^{-N^2 \int_{x \neq y} f(x,y) dL_N(x) dL_N(y)} \prod_{i=1}^N e^{-V(\lambda_i)} d\lambda_i.$$

Hence, if $\mu \to \int_{x\neq y} f(x,y)d\mu(x)d\mu(y)$ was a bounded continuous function, the proof would follow from a standard Laplace method (see Theorem 20.8 in the appendix). The main point is therefore to overcome the singularity of this function, with the most delicate part being overcoming the singularity of the logarithm.

Following Appendix 20.1 (see Corollary 20.6 and Definition 13.10), a full large deviation principle can be proved by showing that exponential tightness holds, as well as estimating the probability of small balls. We follow these steps below.

• *Exponential tightness.* Observe that by Jensen's inequality,

$$\log Z_N^{\beta,V} \geq N \log \int e^{-V(x)}dx \\ - N^2 \int \left(\int_{x\neq y} f(x,y)dL_N(x)dL_N(y) \right) \prod_{i=1}^N \frac{e^{-V(\lambda_i)}d\lambda_i}{\int e^{-V(x)}dx} \geq -CN^2$$

with some finite constant C. Moreover, by (10.10) and (10.2), there exist constants $a>0$ and $c>-\infty$ so that

$$f(x,y) \geq a|V(x)| + a|V(y)| + c$$

from which one concludes that for all $M \geq 0$,

$$P_{V,\beta}^N \left(\int |V(x)|dL_N \geq M \right) \leq e^{-2aN^2M+(C-c)N^2} \left(\int e^{-V(x)}dx \right)^N .$$

Since V goes to infinity at infinity, $K_M = \{\mu \in \mathcal{P}(\mathbb{R}) : \int |V|d\mu \leq M\}$ is a compact set for all $M<\infty$, so that we have proved that the law of L_N under $P_{V,\beta}^N$ is exponentially tight.

•*Large deviation upper bound.* d denotes the Dudley metric, see (0.1). We prove here that for any $\mu \in \mathcal{P}(\mathbb{R})$, if we set $\bar{P}_{V,\beta}^N = Z_N^{\beta,V} P_{V,\beta}^N$

$$\lim_{\epsilon\to 0} \limsup_{N\to\infty} \frac{1}{N^2} \log \bar{P}_{V,\beta}^N \left(d(L_N,\mu) \leq \epsilon \right) \leq - \int f(x,y)d\mu(x)d\mu(y). \quad (10.16)$$

For any $M \geq 0$, the following bound holds:

$$\bar{P}_{V,\beta}^N \left(d(L_N,\mu) \leq \epsilon \right) \\ \leq \int_{d(L_N,\mu)\leq\epsilon} e^{-N^2 \int_{x\neq y} f(x,y)\wedge M dL_N(x)dL_N(y)} \prod_{i=1}^N e^{-V(\lambda_i)}d\lambda_i .$$

Since under the product Lebesgue measure, the λ_i's are almost surely distinct, it holds that $L_N \otimes L_N(x=y) = N^{-1}$, $\bar{P}_{V,\beta}^N$ almost surely. Thus, we deduce for all $M \geq 0$, with $f_M(x,y) = f(x,y) \wedge M$,

$$\int f_M(x,y)dL_N(x)dL_N(y) = \int_{x\neq y} f_M(x,y)dL_N(x)dL_N(y) + MN^{-1},$$

and so

$$\bar{P}^N_{V,\beta}\left(d(L_N,\mu)\le\epsilon\right)$$
$$\le e^{MN}\int_{d(L_N,\mu)\le\epsilon} e^{-N^2\int f_M(x,y)dL_N(x)dL_N(y)}\prod_{i=1}^N e^{-V(\lambda_i)}d\lambda_i.$$

Since $I^{V,M}_\beta(\nu)=\int f_M(x,y)d\nu(x)d\nu(y)$ is bounded continuous, we deduce that

$$\lim_{\epsilon\to0}\limsup_{N\to\infty}\frac{1}{N^2}\log\bar{P}^N_{V,\beta}\left(d(L_N,\mu)\le\epsilon\right)\le -I^{V,M}_\beta(\mu).$$

We finally let M go to infinity and conclude by the monotone convergence theorem. Note that the same argument shows that

$$\limsup_{N\to\infty}\frac{1}{N^2}\log Z^{\beta,V}_N\le-\inf_{\mu\in\mathcal{P}(\mathbb{R})}\int f(x,y)d\mu(x)d\mu(y). \tag{10.17}$$

- *Large deviation lower bound.* We prove here that for any $\mu\in\mathcal{P}(\mathbb{R})$

$$\lim_{\epsilon\to0}\liminf_{N\to\infty}\frac{1}{N^2}\log\bar{P}^N_{V,\beta}\left(d(L_N,\mu)\le\epsilon\right)\ge-\int f(x,y)d\mu(x)d\mu(y). \tag{10.18}$$

Note that we can assume without loss of generality that $I^V_\beta(\mu)<\infty$, since otherwise the bound in trivial, and so in particular, we may and shall assume that μ has no atoms. We can also assume that μ is compactly supported since if we consider $\mu_M=\mu([-M,M])^{-1}1_{|x|\le M}d\mu(x)$, clearly μ_M converges towards μ and by the monotone convergence theorem, one checks that, since f is bounded below,

$$\lim_{M\uparrow\infty}\int f(x,y)d\mu_M(x)d\mu_M(y)=\int f(x,y)d\mu(x)d\mu(y)$$

which insures that it is enough to prove the lower bound for $(\mu_M, M\in\mathbb{R}, I^V_\beta(\mu)<\infty)$, and so for compactly supported probability measures with finite entropy.

The idea is to localize the eigenvalues $(\lambda_i)_{1\le i\le N}$ in small sets and to take advantage of the fast speed N^2 of the large deviations to neglect the small volume of these sets. To do so, we first remark that for any $\nu\in\mathcal{P}(\mathbb{R})$ with no atoms, if we set

$$x^{1,N}=\inf\left\{x|\ \nu\left(]-\infty,x]\right)\ge\frac{1}{N+1}\right\}$$
$$x^{i+1,N}=\inf\left\{x\ge x^{i,N}|\ \nu\left(]x^{i,N},x]\right)\ge\frac{1}{N+1}\right\}\qquad 1\le i\le N-1,$$

for any real number η, there exists an integer number $N(\eta)$ such that, for any N larger than $N(\eta)$,

$$d\left(\nu, \frac{1}{N}\sum_{i=1}^{N} \delta_{x^{i,N}}\right) < \eta.$$

In particular, for $N \geq N(\frac{\delta}{2})$,

$$\left\{(\lambda_i)_{1\leq i\leq N} \mid |\lambda_i - x^{i,N}| < \frac{\delta}{2} \ \forall i \in [1,N]\right\} \subset \{(\lambda_i)_{1\leq i\leq N} \mid d(L_N,\nu) < \delta\}$$

so that we have the lower bound

$$\begin{aligned}
&\bar{P}^N_{V,\beta}\left(d(L_N,\mu)\leq \epsilon\right)\\
&\geq \int_{\cap_i\{|\lambda_i - x^{i,N}|<\frac{\delta}{2}\ \}} e^{-N^2\int_{x\neq y} f(x,y)dL_N(x)dL_N(y)} \prod_{i=1}^{N} e^{-V(\lambda_i)}d\lambda_i\\
&= \int_{\cap_i\{|\lambda_i|<\frac{\delta}{2}\}} \prod_{i<j} |x^{i,N} - x^{j,N} + \lambda_i - \lambda_j|^{\beta} e^{-N\sum_{i=1}^{N} V(x^{i,N}+\lambda_i)} \prod_{i=1}^{N} d\lambda_i\\
&\geq \left(\prod_{i+1<j} |x^{i,N} - x^{j,N}|^{\beta} \prod_i |x^{i,N} - x^{i+1,N}|^{\frac{\beta}{2}} e^{-N\sum_{i=1}^{N} V(x^{i,N})}\right)\\
&\quad\times\left(\int_{\substack{\cap_i\{|\lambda_i|<\frac{\delta}{2}\}\\ \lambda_i<\lambda_{i+1}}} \prod_i |\lambda_i - \lambda_{i+1}|^{\frac{\beta}{2}} e^{-N\sum_{i=1}^{N}[V(x^{i,N}+\lambda_i) - V(x^{i,N})]} \prod_{i=1}^{N} d\lambda_i\right)\\
&=: P_{N,1}\times P_{N,2}
\end{aligned} \tag{10.19}$$

where we used that $|x^{i,N} - x^{j,N} + \lambda_i - \lambda_j| \geq |x^{i,N} - x^{j,N}| \vee |\lambda_i - \lambda_j|$ when $\lambda_i \geq \lambda_j$ and $x^{i,N} \geq x^{j,N}$. To estimate $P_{N,2}$, note that since we assumed that μ is compactly supported, the $(x^{i,N}, 1 \leq i \leq N)_{N\in\mathbb{N}}$ are uniformly bounded and so by continuity of V

$$\lim_{N\to\infty} \sup_{N\in\mathbb{N}} \sup_{1\leq i\leq N} \sup_{|x|\leq\delta} |V(x^{i,N}+x) - V(x^{i,N})| = 0.$$

Moreover, writing $u_1 = \lambda_1$, $u_{i+1} = \lambda_{i+1} - \lambda_i$,

$$\begin{aligned}
\int_{\substack{|\lambda_i|<\frac{\delta}{2}\ \forall i\\ \lambda_i<\lambda_{i-1}}} \prod_i |\lambda_i - \lambda_{i+1}|^{\frac{\beta}{2}} \prod_{i=1}^{N} d\lambda_i &\geq \int_{0<u_i<\frac{\delta}{2N}} \prod_{i=2}^{N} u_i^{\frac{\beta}{2}} \prod_{i=1}^{N} du_i\\
&\geq \left(\frac{\delta}{(\beta+2)N}\right)^{N(\frac{\beta}{2}+1)}.
\end{aligned}$$

Therefore,

$$\lim_{\delta\to 0}\liminf_{N\to\infty}\frac{1}{N^2}\log P_{N,2}\geq 0. \tag{10.20}$$

To handle the term $P_{N,1}$, the uniform boundness of the $x^{i,N}$'s and the convergence of their empirical measure towards μ imply that

$$\lim_{N\to\infty}\frac{1}{N}\sum_{i=1}^{N}V(x^{i,N})=\int V(x)d\mu(x). \tag{10.21}$$

Finally, since $x\to\log(x)$ increases on $\mathbb{R}^+$, we notice that

$$\begin{aligned}
&\int_{x^{1,N}\leq x<y\leq x^{N,N}}\log(y-x)d\mu(x)d\mu(y)\\
&\leq\sum_{1\leq i\leq j\leq N-1}\log(x^{j+1,N}-x^{i,N})\int_{\substack{x\in[x^{i,N},x^{i+1,N}]\\ y\in[x^{j,N},x^{j+1,N}]}}1_{x<y}d\mu(x)d\mu(y)\\
&=\frac{1}{(N+1)^2}\sum_{i<j}\log|x^{i,N}-x^{j+1,N}|+\frac{1}{2(N+1)^2}\sum_{i=1}^{N-1}\log|x^{i+1,N}-x^{i,N}|.
\end{aligned}$$

Since $\log|x-y|$ is bounded when x,y are in the support of the compactly supported measure μ, the monotone convergence theorem implies that the left side in the last display converges towards $\int\int\log|x-y|d\mu(x)d\mu(x)$. Thus, with (10.21), we have proved

$$\liminf_{N\to\infty}\frac{1}{N^2}\log P_{N,1}\geq\int_{x<y}\log(y-x)d\mu(x)d\mu(y)-\int V(x)d\mu(x)$$

which concludes, with (10.19) and (10.20), the proof of (10.18).

• *Conclusion.* By (10.18), for all $\mu\in\mathcal{P}(\mathbb{R})$,

$$\begin{aligned}
\liminf_{N\to\infty}\frac{1}{N^2}\log Z^N_{\beta,V}&\geq\lim_{\epsilon\to 0}\liminf_{N\to\infty}\frac{1}{N^2}\log\bar{P}^N_{V,\beta}\left(d(L_N,\mu)\leq\epsilon\right)\\
&\geq-\int f(x,y)d\mu(x)d\mu(y)
\end{aligned}$$

and so optimizing with respect to $\mu\in\mathcal{P}(\mathbb{R})$ and with (10.17),

$$\lim_{N\to\infty}\frac{1}{N^2}\log Z^N_{\beta,V}=-\inf_{\mu\in\mathcal{P}(\mathbb{R})}\left\{\int f(x,y)d\mu(x)d\mu(y)\right\}=-c^V_\beta.$$

Thus, (10.18) and (10.16) imply the weak large deviation principle, i.e., that for all $\mu\in\mathcal{P}(\mathbb{R})$,

$$\begin{aligned}
&\lim_{\epsilon\to 0}\liminf_{N\to\infty}\frac{1}{N^2}\log P^N_{V,\beta}\left(d(L_N,\mu)\leq\epsilon\right)\\
&=\lim_{\epsilon\to 0}\limsup_{N\to\infty}\frac{1}{N^2}\log P^N_{V,\beta}\left(d(L_N,\mu)\leq\epsilon\right)=-I^V_\beta(\mu).
\end{aligned}$$

This, together with the exponential tightness property proved above completes the proof of the full large deviation principle stated in Theorem 10.1. □

Bibliographical Notes. The proof of Theorem 10.1 is a slight generalization of the techniques introduced in [25] to more general potentials. The ideas developed in this chapter were extended to the Ginibre ensembles in [28] and to diverse other situations, including Wishart matrices, in [117]. We discuss the generalization of large deviation principles to a multi-matrix setting in the last part of these notes.

11

Large deviations of the maximum eigenvalue

We here restrict ourselves to the case where $V(x) = \beta x^2/4$ and for short denote by P_β^N the law of the eigenvalues $(\lambda_i)_{1\le i\le N}$:

$$P_\beta^N(d\lambda_1,\ldots,d\lambda_N) = \frac{1}{Z_\beta^N}\prod_{1\le i<j\le N}|\lambda_i-\lambda_j|^\beta \prod_{1\le i\le N} e^{-\frac{\beta N\lambda_i^2}{4}}d\lambda_i$$

with

$$Z_\beta^N = \int \prod_{1\le i<j\le N}|\lambda_i-\lambda_j|^\beta \prod_{1\le i\le N} e^{-\frac{\beta N\lambda_i^2}{4}}d\lambda_i.$$

Selberg (cf. [153, Theorem 4.1.1] or [6]) found the explicit formula for Z_β^N for any $\beta\ge 0$:

$$Z_\beta^N = (2\pi)^{\frac{N}{2}}\left(\frac{\beta N}{2}\right)^{-\beta N(N-1)/4-\frac{N}{2}}\prod_{j=1}^N\frac{\Gamma(\frac{j\beta}{2})}{\Gamma(\frac{\beta}{2})}. \tag{11.1}$$

The knowledge of Z_β^N up to the second order is crucial below, reason why we restrict ourselves to quadratic potentials in the next theorem (see Exercise 11.4 for a slight extension).

Theorem 11.1. *[24] The law of the maximal eigenvalue $\lambda_N^* = \max_{i=1}^N\lambda_i$ under P_β^N, with $\beta\ge 0$, satisfies the LDP with speed N and the GRF*

$$I^*(x) = \begin{cases}\beta\int_2^x\sqrt{(z/2)^2-1}\,dz, & x\ge 2\,,\\ +\infty, & \textit{otherwise}\,.\end{cases} \tag{11.2}$$

The next estimate is key to the proof of Theorem 11.1.

Lemma 11.2. *For every M large enough and all N,*

$$P_\beta^N\left(\max_{i=1}^N|\lambda_i|\ge M\right)\le e^{-\beta N M^2/9}\,.$$

Proof. Observe that for any $|x| \geq M \geq 8$ and $\lambda_i \in \mathbb{R}$,

$$|x-\lambda_i|e^{-\frac{\lambda_i^2}{8}} \leq (|x|+|\lambda_i|)e^{-\frac{\lambda_i^2}{8}} \leq 2|x| \leq e^{\frac{x^2}{8}} .$$

Therefore, integrating with respect to λ_1 yields, for $M \geq 8$,

$$\begin{aligned}
&P_\beta^N(|\lambda_1| \geq M) \\
&= \frac{Z_\beta^{N-1}}{Z_\beta^N} \int_{|x|\geq M} dx e^{-\frac{\beta x^2}{4}\frac{(N+1)}{2}} \int \prod_{i=2}^N \left(|x-\lambda_i|e^{-\frac{\lambda_i^2}{4}-\frac{x^2}{8}}\right)^\beta dP_\beta^{N-1}(\lambda_j, j \geq 2) \\
&\leq e^{-\frac{\beta}{8}NM^2} \frac{Z_\beta^{N-1}}{Z_\beta^N} \int_{|x|\geq M} e^{-x^2/8} dx \int \prod_{i=2}^N (|x-\lambda_i|e^{-\lambda_i^2/4}e^{-x^2/8}) dP_\beta^{N-1} \\
&\leq e^{-\frac{\beta}{8}NM^2} \frac{Z_\beta^{N-1}}{Z_\beta^N} \int e^{-x^2/8} dx.
\end{aligned}$$

Further, following (11.1), we compute that

$$\lim_{N\to\infty} \frac{1}{N} \log \frac{Z_\beta^{N-1}}{Z_\beta^N} = -\frac{\beta}{4}. \tag{11.3}$$

Therefore, for any $M \geq 8$, for N large enough, we get

$$P_\beta^N(\max_{i=1}^N |\lambda_i| \geq M) \leq N P_\beta^N(|\lambda_1| \geq M) \leq e^{-\frac{\beta}{9}NM^2} ,$$

and the lemma follows. □

Proof of Theorem 11.1. $I^*(x)$ is a good rate function since it is a continuous function (except at $x = 2$ where it is lower semi-continuous) and it goes to infinity at infinity. Moreover, with $I^*(x)$ continuous and strictly increasing on $[2, \infty[$ it suffices to show that for any $x < 2$,

$$\limsup_{N\to\infty} \frac{1}{N} \log P_\beta^N (\lambda_N^* \leq x) = -\infty, \tag{11.4}$$

whereas for any $x > 2$

$$\lim_{N\to\infty} \frac{1}{N} \log P_\beta^N (\lambda_N^* \geq x) = -I^*(x). \tag{11.5}$$

In fact, from these two estimates and since I^* increases on $[2, \infty[$, we find that for all $x < y$,

$$\lim_{N\to\infty} \frac{1}{N} \log P_\beta^N (\lambda_N^* \in [x, y]) = -\inf_{z\in[x,y]} I^*(z),$$

the above right-hand side being equal to $-\infty$ if $y \leq 2$, to zero if $x \leq 2 \leq y$ and to $I^*(x)$ if $x \geq 2$. By continuity of I^*, we also deduce that we have the same limits if we take (x, y) instead of $[x, y]$. Since $\mathcal{A} = \{[x, y], (x, y), x < y\}$ is a basis for the topology on $\mathbb{R}$, we conclude by Theorem 20.5.

Starting with (11.4), fix $x < 2$ and $f \in \mathcal{C}_b(\mathbb{R})$ such that $f(y) = 0$ for all $y \leq x$, whereas $\int f d\sigma > 0$. Note that $\{\lambda_N^* \leq x\} \subseteq \{\int f dL_{\mathbf{X}^N} = 0\}$, so (11.4) follows by applying the upper bound of the large deviation principle of Theorem 10.1 for the closed set $F = \{\mu : \int f d\mu = 0\}$, such that $\sigma \notin F$. Turning to the upper bound in (11.5), fix $M \geq x > 2$, noting that

$$P_\beta^N\left(\lambda_N^* \geq x\right) = P_\beta^N(\max_{i=1}^N |\lambda_i| > M) + P_\beta^N\left(\lambda_N^* \geq x, \max_{i=1}^N |\lambda_i| \leq M\right) \tag{11.6}$$

By Lemma 11.2, the first term is exponentially negligible for all M large enough. To deal with the second term, let $P_N^{N-1}(\lambda \in \cdot) = P_\beta^{N-1}((1 - N^{-1})^{1/2}\lambda \in \cdot)$, $L_{N-1} = (N-1)^{-1}\sum_{i=2}^N \delta_{\lambda_i}$ and

$$C_N := \frac{Z_\beta^{N-1}}{Z_\beta^N}(1 - N^{-1})^{N(N-1)/4}.$$

Further, let $B(\sigma, \delta)$ denote an open ball in $\mathcal{P}(\mathbb{R})$ of radius $\delta > 0$ and center σ, and $B_M(\sigma, \delta)$ its intersection with $\mathcal{P}([-M, M])$. Observe that for any $z \in [-M, M]$ and $\mu \in \mathcal{P}([-M, M])$,

$$\Phi(z, \mu) := \beta \int \log|z - y| d\mu(y) - \frac{\beta}{4}z^2 \leq \beta \log(2M).$$

Thus, for the second term in (11.6),

$$\begin{aligned}
&P_\beta^N\left(\lambda_N^* \geq x, \max_{i=1}^N |\lambda_i| \leq M\right) \\
&\leq N C_N \int_x^M d\lambda_1 \int_{[-M,M]^{N-1}} e^{(N-1)\Phi(\lambda_1, L_{N-1})} dP_N^{N-1}(\lambda_j, j \geq 2) \\
&\leq N C_N \left(\int_x^M e^{(N-1)\sup_{\mu \in B_M(\sigma,\delta)} \Phi(z,\mu)} dz + (2M)^N P_N^{N-1}(L_{N-1} \notin B(\sigma, \delta))\right).
\end{aligned} \tag{11.7}$$

For any h of Lipschitz norm at most 1 and $N \geq 2$,

$$|(N-1)^{-1}\sum_{i=2}^N (h((1 - N^{-1})^{1/2}\lambda_i) - h(\lambda_i))| \leq 3N^{-1}\max_{i=2}^N |\lambda_i|.$$

Thus, by Lemma 11.2, the spectral measures L_{N-1} under σ^{N-1} are exponentially equivalent in $\mathcal{P}(\mathbb{R})$ to the spectral measures L_{N-1} under P_N^{N-1}, so Theorem 10.1 applies also for the latter (cf. Definition 20.9 and Lemma 20.10). In particular, the second term in (11.7) is exponentially negligible as $N \to \infty$ for any $\delta > 0$ and $M < \infty$ (since it behaves like $e^{-c(\delta)N^2}$). Therefore,

$$
\begin{aligned}
&\limsup_{N\to\infty} \frac{1}{N} \log P_\beta^N \left(\lambda_N^* \ge x, \max_{i=1}^N |\lambda_i| \le M\right) \\
&\le \limsup_{N\to\infty} \frac{1}{N} \log C_N + \lim_{\delta\downarrow 0} \sup_{\substack{z\in[x,M]\\ \mu\in B_M(\sigma,\delta)}} \Phi(z,\mu)
\end{aligned}
\tag{11.8}
$$

Note that $\Phi(z,\mu) = \inf_{\eta>0} \Phi_\eta(z,\mu)$ with $\Phi_\eta(z,\mu) := \beta \int \log(|z-y| \vee \eta) d\mu(y) - \frac{\beta}{4} z^2$ continuous on $[-M,M] \times \mathcal{P}([-M,M])$. Thus, $(z,\mu) \mapsto \Phi(z,\mu)$ is upper semi-continuous, which implies

$$
\lim_{\delta\downarrow 0} \sup_{\substack{z\in[x,M]\\ \mu\in B_M(\sigma,\delta)}} \Phi(z,\mu) = \sup_{z\in[x,M]} \Phi(z,\sigma) \tag{11.9}
$$

With σ supported on $[-2,2]$, $D(z) := \frac{d}{dz}\Phi(z,\sigma)$ exists for $z \ge 2$. Moreover, $D(z) = -\beta\sqrt{(z/2)^2-1} \le 0$. It is shown in [25, Lemma 2.7] that $\Phi(2,\sigma) = -\beta/2$. Hence, for $x > 2$,

$$
\sup_{z\ge x} \Phi(z,\sigma) = \Phi(x,\sigma) = -\frac{1}{2} - I^*(x)\,. \tag{11.10}
$$

By (11.3), we deduce that

$$
\lim_{N\to\infty} N^{-1} \log C_N = \frac{\beta}{2}.
$$

Combining this with (11.8)–(11.10) completes the proof of the upper bound for (11.5). To prove the complementary lower bound, fix $y > x > r > 2$ and $\delta > 0$, noting that for all N,

$$
\begin{aligned}
P_\beta^N &(\lambda_N^* \ge x) \\
&\ge P_\beta^N \left(\lambda_1 \in [x,y], \max_{i=2}^N |\lambda_i| \le r\right) \\
&= C_N \int_x^y e^{-\lambda_1^2/4} d\lambda_1 \int_{[-r,r]^{N-1}} e^{(N-1)\Phi(\lambda_1, L_{N-1})} dP_N^{N-1}(\lambda_j, j\ge 2) \\
&\ge k C_N \exp\Big((N-1) \inf_{\substack{z\in[x,y]\\ \mu\in B_r(\sigma,\delta)}} \Phi(z,\mu)\Big) P_N^{N-1}\left(L_{N-1} \in B_r(\sigma,\delta)\right)
\end{aligned}
$$

with $k = k(x,y) > 0$. Recall that the large deviation principle with speed N^2 and good rate function $I(\cdot)$ applies for the measures L_{N-1} under P_N^{N-1}. It follows by this LDP's upper bound that $P_N^{N-1}(L_{N-1} \notin B(\sigma,\delta)) \to 0$, whereas by the symmetry of $P_\beta^N(\cdot)$ and the upper bound of (11.5),

$$
P_N^{N-1}\left(L_{N-1} \notin \mathcal{P}([-r,r])\right) \le 2P_\beta^{N-1}(\lambda_N^* \ge r) \to 0
$$

as $N \to \infty$. Consequently,

$$\liminf_{N\to\infty} \frac{1}{N} \log P_\beta^N \left(\lambda_N^* \geq x\right) \geq \frac{1}{2} + \beta \inf_{\substack{z\in[x,y]\\ \mu\in B_r(\sigma,\delta)}} \Phi(z,\mu)$$

Observe that $(z,\mu) \mapsto \Phi(z,\mu)$ is continuous on $[x,y]\times\mathcal{P}([-r,r])$, for $y > x > r > 2$. Hence, considering $\delta \downarrow 0$ followed by $y \downarrow x$ results in the required lower bound

$$\liminf_{N\to\infty} \frac{1}{N} \log P_\beta^N(\lambda_N^* \geq x) \geq \frac{\beta}{2} + \beta\Phi(x,\sigma) \ .$$

$\square$

Exercise 11.3 (suggested by B. Collins). *Generalize the proof to obtain the large deviation principle for the joint law of the kth largest eigenvalues (k finite) with good rate function given by*

$$I^*(x_1,\ldots,x_k) = \sum_{l=1}^{k} I^*(x_k) - \beta \sum_{1\leq \ell\leq p\leq k} \log(x_\ell - x_k) + \mathit{constant}.$$

if $x_1 \geq x_2 \cdots \geq x_k \geq 2$ *and* $+\infty$ *otherwise.*

Exercise 11.4. *Consider*

$$P_{\alpha V}^N(d\lambda_1,\ldots,d\lambda_N) = e^{-N\alpha\sum_{i=1}^N V(\lambda_i)} dP_2^N(d\lambda_1,\ldots,d\lambda_N)/Z_{\alpha V}^N$$

with V *a polynomial such that* $V''(x) \geq 0$ *for* $|x|$ *large enough. Show that for* α *positive small enough, the law of* λ_N^* *under* $P_{\alpha V}^N$ *satisfies a large deviation principle with rate function*

$$I_{\alpha V}^*(x) = \begin{cases} \Phi(\mu_{\alpha V}, x) - \inf_y \Phi(\mu_{\alpha V}, y), & x \geq x_V \ , \\ +\infty, & \mathit{otherwise}\,. \end{cases}$$

with $\Phi(\mu,x) = 2\int \log|x-y| d\mu(y) - \frac{1}{2}x^2 - \alpha V(x)$ *and* μ_V *the unique solution of the Schwinger–Dyson equation of Theorem 8.3.*

Hint: Observe that we are in the situation of Part III so that we know that $\frac{1}{N}\sum \delta_{\lambda_i}$ *converges almost surely to* $\mu_{\alpha V}$ *and* $Z_{\alpha V}^N = e^{N^2 I_{\alpha V}} C_{\alpha V}(1+o(1))$. *Then, show that the proof of Theorem 11.1 extends.*

Bibliographical notes. This proof is taken from [24]. It was generalized to the case of a deformed Gaussian ensemble in [143].

Part V

Stochastic calculus

We shall now study the Hermitian Brownian motion. It is a matrix-valued process $(H_t^N)_{t\geq 0}$ constructed as Gaussian Wigner matrices but with Brownian motion entries instead of Gaussians. We shall describe below the symmetric and the Hermitian Brownian motions, leaving the generalization to the symplectic Brownian motions as exercises. We define the symmetric (resp. Hermitian) Brownian motion $H^{N,\beta}$ for $\beta = 1$ (resp. $\beta = 2$) as a process with values in the set of $N \times N$ symmetric (resp. Hermitian) matrices with entries $\left\{H_{i,j}^{N,\beta}(t), t \geq 0, i \leq j\right\}$ constructed via independent real-valued Brownian motions $(B_{i,j}, \tilde{B}_{i,j}, 1 \leq i \leq j \leq N)$ by

$$H_{k,l}^{N,\beta}(t) = \begin{cases} \frac{1}{\sqrt{\beta N}}(B_{k,l}(t) + i(\beta - 1)\tilde{B}_{k,l}(t)), & \text{if } k < l \\ \frac{\sqrt{2}}{\sqrt{\beta N}} B_{l,l}(t), & \text{if } k = l. \end{cases} \tag{V.1}$$

Considering the matrix-valued processes, and the associated dynamics, has the advantage to allow us not only to consider one Gaussian Wigner matrix $X^N = H^{N,\beta}(1)$ but also, if $X^N(0)$ is some Hermitian Wigner matrix, the sum $X^N(1) = H^{N,\beta}(1) + X^N(0)$ seen as the matrix at time one of the matrix-valued process $X^N(t) = H^{N,\beta}(t) + X^N(0)$. Studying the evolution of the eigenvalues of $X^N(t)$ allows us to prove the law of large numbers for the spectral measure of $X^N(1)$ (see Lemma 12.5) as well as large deviation principles (see Theorem 13.1). The latter large deviations estimates result in the asymptotics for the spherical or Itzykson–Zuber–Harich–Chandra integrals (see Theorem 14.1) that in turn will give us the value of free energies for diverse two matrices matrix models (see Theorem 15.1) as well as estimates on Schur functions (see Corollary 14.2).

12

Stochastic analysis for random matrices

12.1 Dyson's Brownian motion

Let $X^N(0)$ be a symmetric (resp. Hermitian) matrix with eigenvalues $(\lambda_N^1(0), \dots, \lambda_N^N(0))$. Let, for $t \geq 0$, $\lambda_N(t) = (\lambda_N^1(t), \dots, \lambda_N^N(t))$ denote the (real) eigenvalues of $X^N(t) = X^N(0) + H^{N,\beta}(t)$ for $t \geq 0$. We shall prove that $(\lambda_N(t))_{t\geq 0}$ is a semi-martingale with respect to the filtration $\mathcal{F}_t = \sigma(B_{i,j}(s), \tilde{B}_{ij}(s), 1 \leq i,j \leq N, s \leq t)$ whose evolution is described by a stochastic differential system. This result was first stated by Dyson [85], and $(\lambda_N(t))_{t\geq 0}$, when $X^N(0) = 0$, has since then been called Dyson's Brownian motion. To begin with, let us describe the stochastic differential system that governs the evolution of $(\lambda_N(t))_{t\geq 0}$ and show that it is well defined.

Lemma 12.1. *Let $(W^1, \dots, W^N)$ be an N-dimensional Brownian motion in a probability space (Ω, P) equipped with a filtration $\mathcal{F} = \{\mathcal{F}_t, t \geq 0\}$. Let Δ_N be the simplex $\Delta_N = \{(x_i)_{1\leq i\leq N} \in \mathbb{R}^N : x_1 < x_2 < \cdots < x_{N-1} < x_N\}$ and take $\lambda_N(0) = (\lambda_N^1(0), \dots, \lambda_N^N(0)) \in \Delta_N$. Let $\beta \geq 1$. Let $T \in \mathbb{R}^+$. There exists a unique strong solution (see Definition 20.13) to the stochastic differential system*

$$d\lambda_N^i(t) = \frac{\sqrt{2}}{\sqrt{\beta N}} dW_t^i + \frac{1}{N} \sum_{j\neq i} \frac{1}{\lambda_N^i(t) - \lambda_N^j(t)} dt \tag{12.1}$$

with initial condition $\lambda_N(0)$ such that $\lambda_N(t) \in \Delta_N$ for all $t \geq 0$. We denote by $P^N_{T,\lambda_N(0)}$ its law in $\mathcal{P}(\mathcal{C}([0,T], \Delta_N))$. It is called Dyson's Brownian motion. This weak solution (see Definition 20.14) is as well unique.

For any $\beta \geq 1$, the Dyson Brownian motion can be defined from general initial conditions, thus extending Lemma 12.1 to $\lambda_N(0) \in \overline{\Delta}_N$ (cf. [6]). We shall take this generalization for granted in the sequel.

Proof. To prove the claim, let us introduce, for $R > 0$, the auxiliary system

$$d\lambda^i_{N,R}(t) = \sqrt{\frac{2}{\beta N}} dW^i_t + \frac{1}{N}\sum_{j\neq i}\phi_R(\lambda^i_{N,R}(t) - \lambda^j_{N,R}(t))dt, \tag{12.2}$$

with $\phi_R(x) = x^{-1}$ if $|x| \geq R^{-1}$ and $\phi_R(x) = R\mathrm{sgn}(x)$ if $|x| < (R)^{-1}$. We take $\lambda^i_{N,R}(0) = \lambda^i_N(0)$ for $i \in \{1, \ldots, N\}$. Since ϕ_R is uniformly Lipschitz, it is known (cf. Theorem 20.16) that this system admits a unique strong solution as well as a unique weak solution $P^{N,R}_{T,\lambda_N(0)}$, that is a probability measure on $\mathcal{C}([0,T],\mathbb{R}^N)$. Moreover, this strong solution is adapted to the filtration $\mathcal{F}$. We can now construct a solution to (12.1) by putting $\lambda_N(t) = \lambda_{N,R}(t)$ on $|\lambda^i_N(s) - \lambda^j_N(s)| \geq R^{-1}$ for all $s \leq t$ and all $i \neq j$. To prove that this construction is possible, we need to show that almost surely $\lambda_{N,R}(t) = \lambda_{N,R'}(t)$ for $R > R'$ for some (random) R' and for all t's in a compact set. To do so, we want to prove that for all times $t \geq 0$, $(\lambda^1_N(t), \ldots, \lambda^N_N(t))$ stay sufficiently apart. To prove this fact, let us consider the Lyapounov function

$$f(x_1, \ldots, x_N) = \frac{1}{N}\sum_{i=1}^N x_i^2 - \frac{1}{N^2}\sum_{i\neq j}\log|x_i - x_j|$$

and for $M > 0$ set

$$T_M = \inf\{t \geq 0 : f(\lambda_N(t)) \geq M\}.$$

Since f is $\mathcal{C}^\infty(\Delta_N, \mathbb{R})$ on sets where it is uniformly bounded (note here that f is bounded below uniformly), we also deduce that $\{T_M > T\}$ is in $\mathcal{F}_T$ for all $T \geq 0$. Thus, T_M is a stopping time. Moreover, using that $\log|x-y| \leq \log(|x|+1) + \log(|y|+1)$ and $x^2 - 2\log(|x|+1) \geq c$ for some finite constant, we find that for all $i \neq j$,

$$-\frac{1}{N^2}\log|x_i - x_j| \leq f(x_1, \ldots, x_N) - c.$$

Thus, on $\{T_M > T\}$, for all $t \leq T$,

$$|\lambda_i(t) - \lambda_j(t)| \geq e^{N^2(-M+c)} =: R^{-1}$$

so that λ_N coincides with $\lambda_{N,R}$ and is therefore adapted. Itô's calculus (see Theorem 20.18) gives

$$\begin{aligned}
&df(\lambda_N(t)) \\
&= \frac{2}{N}\sum_{i=1}^N \left(\lambda^i_N(t) - \frac{1}{N}\sum_{k\neq i}\frac{1}{\lambda^i_N(t) - \lambda^k_N(t)}\right)\frac{1}{N}\sum_{l\neq i}\frac{1}{\lambda^i_N(t) - \lambda^l_N(t)}dt \\
&\quad + \frac{2}{\beta N}\sum_{i=1}^N\left(1 + \frac{1}{N}\sum_{k\neq i}\frac{1}{(\lambda^i_N(t) - \lambda^k_N(t))^2}\right)dt + dM_N(t)
\end{aligned}$$

with M_N the local martingale

$$dM_N(t) = \frac{2^{\frac{3}{2}}}{\beta^{\frac{1}{2}} N^{\frac{3}{2}}} \sum_{i=1}^{N} \left(\lambda_N^i(t) - \frac{1}{N} \sum_{k \neq i} \frac{1}{\lambda_N^i(t) - \lambda_N^k(t)} \right) dW_t^i.$$

Observing that for all $i \in \{1, \ldots, N\}$,

$$\begin{aligned}
&\sum_{i=1}^{N} \left(\left(\sum_{k \neq i} \frac{1}{\lambda_N^i(t) - \lambda_N^k(t)} \right) \left(\sum_{l \neq i} \frac{1}{\lambda_N^i(t) - \lambda_N^l(t)} \right) - \sum_{k \neq i} \frac{1}{(\lambda_N^i(t) - \lambda_N^k(t))^2} \right) \\
&= \sum_{\substack{k \neq i, l \neq i \\ k \neq l}} \frac{1}{\lambda_N^i(t) - \lambda_N^k(t)} \frac{1}{\lambda_N^i(t) - \lambda_N^l(t)} \\
&= \sum_{\substack{k \neq i, l \neq i \\ k \neq l}} \frac{1}{\lambda_N^k(t) - \lambda_N^l(t)} \left(\frac{1}{\lambda_N^i(t) - \lambda_N^k(t)} - \frac{1}{\lambda_N^i(t) - \lambda_N^l(t)} \right) \\
&= -2 \sum_{\substack{k \neq i, l \neq i \\ k \neq l}} \frac{1}{\lambda_N^i(t) - \lambda_N^k(t)} \frac{1}{\lambda_N^i(t) - \lambda_N^l(t)} = 0
\end{aligned}$$

and

$$\sum_{i=1}^{N} \lambda_N^i(t) \sum_{k \neq i} \frac{1}{\lambda_N^i(t) - \lambda_N^k(t)} = \frac{N(N-1)}{2},$$

we obtain that

$$df(\lambda_N(t)) = \frac{\beta(N+1)}{\beta N} dt + \frac{(2-2\beta)}{\beta N^2} \sum_{k,i,k \neq i} \frac{1}{(\lambda_N^i(t) - \lambda_N^k(t))^2} dt + dM_N(t).$$

Thus, for all $\beta \geq 1$, for all $M < \infty$, since $(M_N(t \wedge T_M), t \geq 0)$ is a martingale with vanishing expectation,

$$\mathbb{E}[f(\lambda_N(t \wedge T_M))] \leq 3\mathbb{E}[t \wedge T_M] + f(\lambda_N(0)).$$

Therefore, if $c = -\inf\{f(x_1, \ldots, x_N); (x_i)_{1 \leq i \leq N} \in \mathbb{R}^N\}$,

$$\begin{aligned}
(M+c)\mathbb{P}(t \geq T_M) &\leq \mathbb{E}[(f(\lambda_N(t \wedge T_M)) + c)\, 1_{t \geq T_M}] \\
&\leq \mathbb{E}[f(\lambda_N(t \wedge T_M)) + c] \\
&\leq 3\mathbb{E}[t \wedge T_M] + c + f(\lambda_N(0)) \\
&\leq 3t + c + f(\lambda_N(0))
\end{aligned}$$

which proves that for $M + c > 0$,

$$\mathbb{P}\left(t \geq T_M\right) \leq \frac{3t + c + f(\lambda_N(0))}{M + c}.$$

Hence, the Borel–Cantelli lemma implies that for all $t \in \mathbb{R}^+$,

$$\mathbb{P}\left(\cup_{M_0} \cap_{M \geq M_0} \{T_{M^2} \geq t\}\right) = 1,$$

and so in particular, $\sum_{i \neq j} \log |\lambda_N^i(t) - \lambda_N^j(t)| > -\infty$ almost surely for all times. Since $(\lambda_N(t \wedge T_M))_{t \geq 0}$ are continuous for all $M < \infty$ (as bounded perturbations of Brownian motions), we conclude that $\lambda_N(t \wedge T_M) \in \Delta_N$ for all $t \geq 0$ and all $M > 0$. As T_M goes to infinity almost surely as M goes to infinity, we conclude that $\lambda_N(t) \in \Delta_N$ for all $t \geq 0$.

Now, to prove uniqueness of the weak solution, let us consider, for $R \in \mathbb{R}^+$, the auxiliary system (12.2) with strong solution $(\lambda_{N,R}^j(t), 1 \leq j \leq N)_{t \geq 0}$. Remark that for all R, there exists $M = M(R, N) < \infty$ so that

$$\{T \leq T_M\} \subset \cap_{t \leq T} \cap_{i \neq j} \{|\lambda_N^i(t) - \lambda_N^j(t)| \geq R^{-1}\}.$$

Hence, on $\{T \leq T_M\}$, $(\lambda_N(t))_{0 \leq t \leq T}$ satisfies (12.2) and therefore $\lambda_N^i(t) = \lambda_{N,R}^i(t)$ is uniquely determined for all $i \in \{1, \ldots, N\}$ and all $t \leq T$. Since we have seen that T_{M^2} goes to infinity almost surely, we conclude that there exists a unique strong solution $(\lambda_N(t))_{0 \leq t \leq T}$ to (12.1); it coincides with $(\lambda_{N,R}^j(t), 1 \leq j \leq N)_{0 \leq t \leq T}$ for some R sufficiently large (and random). Its weak solution $P_{T,\lambda_N(0)}^N$ is also unique since its restriction to $T \leq T_M$ is uniquely determined for all $M < \infty$. $\square$

Let $\beta = 1$ or 2 and $X^{N,\beta}(0) \in \mathcal{H}_N^{(\beta)}$ with eigenvalues $\lambda_N(0) \in \mathbb{R}^N$ and set

$$X^{N,\beta}(t) = X^{N,\beta}(0) + H^{N,\beta}(t).$$

Theorem 12.2 (Dyson). *[85] Let $\beta = 1$ or 2 and $\lambda_N(0)$ be in Δ_N. Then, the eigenvalues $(\lambda_N(t))_{t \geq 0}$ of $\left(X^{N,\beta}(t)\right)_{t \geq 0}$ are semi-martingales. Their joint law is the weak solution to (12.1).*

The proof we present goes "backward" by proposing a way to construct the matrix $X^N(t)$ from the solution of (12.1) and a Brownian motion on the orthogonal group. Its advantage with respect to a "forward" proof is that we do not need to care about justifying that certain quantities defined from X^N are semi-martingales so that Itô's calculus applies.

Proof of Theorem 12.2.. We present the proof in the case $\beta = 1$ and leave the generalization to $\beta = 2$ as an exercise.

We can assume without loss of generality that $X^N(0)$ is the diagonal matrix $\text{diag}(\lambda_N^1(0), \ldots, \lambda_N^N(0))$ since otherwise if O is an orthogonal matrix so that $X^N(0) = ODO^T$,

$$X^N(t) = ODO^* + H^N(t) = O(D + \tilde{H}^N(t))O^T = O\tilde{X}^N(t)O^T$$

with $\tilde{H}^N(t) = O^T H^N(t) O^T$ another Hermitian Brownian motion, independent from O. Since $\tilde{X}^N(t)$ has the same eigenvalues than $X^N(t)$ and the same law, we can assume without loss of generality that $X^N(0)$ is diagonal.

Let $M > 0$ be fixed. We consider the strong solution of (12.1) till the random time T_M. We let $w_{ij}, 1 \leq i < j \leq N$ be independent Brownian motions. Hereafter, all solutions will be equipped with the natural filtration $\mathcal{F}_t = \sigma((w_{ij}(s), W_i(s), s \leq t \wedge T_M)$ with W_i the Brownian motions of (12.1), independent of $w_{ij}, 1 \leq i < j \leq N$. We set for $i < j$

$$dR_{ij}^N(t) = \frac{1}{\sqrt{N}} \frac{1}{\lambda_N^j(t) - \lambda_N^i(t)} dw_{ij}(t).$$

We let $R^N(t)$ be the skew-symmetric matrix (i.e., $R^N(t) = -R^N(t)^T$) with such entries above the diagonal and set O^N to be the strong solution of

$$dO^N(t) = O^N(t) dR^N(t) - \frac{1}{2} O^N(t) d\langle (R^N)^T R^N \rangle_t \tag{12.3}$$

with $O^N(0) = I$. Here, for semi-martingales A, B with values in $\mathcal{M}_N(\mathbb{R})$, $\langle A, B \rangle_t = (\sum_{k=1}^N \langle A_{ik}, B_{kj} \rangle_t)_{1 \leq i,j \leq N}$ is the martingale bracket of A and B and $\langle A \rangle_t$ is the finite variation part of A at time t. Existence and uniqueness of strong solutions of (12.3) for the filtration $\mathcal{F}_t$ are given till the random time T_M since (12.3) has bounded Lipschitz coefficients (see Theorem 20.16). Note that since the martingale bracket of a semi-martingale is given by the bracket of its martingale part,

$$d\langle (O^N)^T O^N \rangle_t = \langle [d(R^N)^T](O^N)^T, O^N dR^N \rangle_t = d\langle (R^N)^T R^N \rangle_t.$$

Hence

$$dO^N(t)^T O^N(t) = O^N(t)^T dO^N(t) + (dO^N(t)^T) O^N(t) + d\langle (O^N)^T O^N \rangle_t.$$

If $O^N(t)^T O^N(t) = I$, we deduce that $dO^N(t)^T O^N(t)$ is equal to $dR^N(t) + dR^N(t)^T = 0$ as $R^N(t)$ is skew-symmetric, from which it can be guessed that $O^N(t)^T O^N(t) = I$ at all times. This can be in fact proved, see [6], by showing that $dO^N(t)^T O^N(t)$ is linear in $O^N(t)^T O^N(t) - I$ with uniformly bounded coefficients on $\{T_M \geq t\}$. Thus, $(O^N)^T(t) O^N(t) = I$ at all times. We now show that $Y^N(t) := O^N(t)^T D(\lambda_N(t)) O^N(t)$ has the same law than $X^N(t)$ which will prove the claim. By construction, $Y^N(0) := \mathrm{diag}(\lambda_N(0)) = X^N(0)$. Moreover,

$$\begin{aligned} dY^N(t) = {} & dO^N(t) D(\lambda_N(t)) O^N(t)^T + O^N(t) D(\lambda_N(t)) dO^N(t)^T \\ & + O^N(t) dD(\lambda_N(t)) O^N(t)^T + d\langle O^N D(\lambda_N) (O^N)^T \rangle(t) \end{aligned} \tag{12.4}$$

where for all $i, j \in \{1, \ldots, N\}$, we adopted the notation

$$\begin{aligned}
&\big(d\langle O^N D(\lambda_N)(O^N)^T\rangle_t\big)_{ij} \\
&\qquad = \sum_{k=1}^N \bigg(\frac{1}{2}O_{ik}^N(t)d\langle \lambda_N^k, O_{jk}^N\rangle_t + \lambda_N^k(t)d\langle O_{ik}^N, O_{jk}^N\rangle_t \\
&\qquad\qquad + \frac{1}{2}O_{jk}^N(t)d\langle \lambda_N^k, O_{ik}^N\rangle_t\bigg) \\
&\qquad = \sum_{k=1}^N \lambda_N^k(t)d\langle O_{ik}^N, O_{jk}^N\rangle_t.
\end{aligned}$$

The last equality is due to the independence of $(W_i)_{1\le i\le N}$ and $(w_{ij})_{1\le i<j\le N}$ which results in $\langle \lambda_N^k, O_{ik}^N\rangle_t \equiv 0$. By left multiplication by $(O^N(t))^T$ and right multiplication by $O^N(t)$ of (12.4) we arrive at

$$\begin{aligned}
dW_N(t) = (O^N(t))^T dO^N(t)D(\lambda_N(t)) + D(\lambda_N(t))dO^N(t)^T O^N(t) \quad (12.5)\\
+ dD(\lambda_N(t)) + (O^N(t))^T d\langle O^N D(\lambda_N)(O^N)^T\rangle(t)O^N(t)
\end{aligned}$$

with $dW_N(t) = (O^N(t))^T dY^N(t)O^N(t)$. Let us compute the last term in the right-hand side of (12.5). For all $i,j \in \{1,\dots,N\}^2$, we have

$$\begin{aligned}
d\langle O^N D(\lambda_N)(O^N)^T\rangle_t^{ij} &= \sum_{k=1}^N \lambda_N^k(t)d\langle O_{ik}^N, O_{jk}^N\rangle_t \\
&= \sum_{k,l,m=1}^N \lambda_N^k(t)O_{il}^N(t)O_{jm}^N(t)d\langle R_{lk}^N, R_{mk}^N\rangle_t \\
&= \frac{1}{N}\sum_{k\neq l} \frac{\lambda_N^k(t)}{(\lambda_N^k(t)-\lambda_N^l(t))^2}O_{il}^N(t)O_{jl}^N(t)dt
\end{aligned}$$

where we finally used the definition of R^N to compute the martingale brackets that gives

$$d\langle R_{lk}^N, R_{mk}^N\rangle_t = 1_{l=m\neq k}\frac{1}{N(\lambda_N^k(t)-\lambda_N^l(t))^2}dt.$$

Hence, for all $i,j \in \{1,\dots,N\}^2$, we get

$$[(O^N)^T(t)d\langle O^N D(\lambda_N)(O^N)^T\rangle_t O^N(t)]_{ij} = 1_{i=j}\sum_{k\neq i}^N \frac{\lambda_N^k(t)}{N(\lambda_N^i(t)-\lambda_N^k(t))^2}dt.$$

Similarly, recall that

$$(O^N)^T(t)dO^N(t) = dR^N(t) - 2^{-1}d\langle (R^N)^T R^N\rangle_t$$

with for all $i, j \in \{1, \dots, N\}^2$,

$$\begin{aligned} d\langle (R^N)^T R^N \rangle_t^{ij} &= \sum_{k=1}^{N} d\langle R^N_{ki}, R^N_{kj} \rangle_t \\ &= \frac{1_{i=j}}{N} \sum_{k \neq i} (\lambda_N^k(t) - \lambda_N^i(t))^{-2} dt. \end{aligned}$$

Therefore, identifying the terms on the diagonal in (12.5) and recalling that R^N is null on the diagonal, we find that

$$dW_N^{ii}(t) = \sqrt{\frac{2}{N}} dW_t^i.$$

Outside the diagonal, for $i \neq j$, we get

$$\begin{aligned} dW_N^{ij}(t) &= [dR^N(t) D(\lambda_N(t)) + D(\lambda_N(t)) dR^N(t)^T]_{ij} \\ &= \frac{1}{\sqrt{N}} dw_{ij}(t). \end{aligned}$$

Hence, $W_N(t)$ has the law of a symmetric Brownian motion. Thus, since $(O^N(t), t \geq 0)$ is adapted, $dY^N(t) = O^N(t) dW_N(t) (O^N(t))^T$ is a continuous matrix-valued martingale whose quadratic variation is given by

$$\langle Y_{ij}^N, Y_{kl}^N \rangle_t = 1_{ij=kl \text{ or } lk} N^{-1} t \quad i \neq j, \langle Y_{ii}^N, Y_{kl}^N \rangle_t = 1_{i=k=l} 2N^{-1} t.$$

Therefore, by Levy's theorem (cf. [122, p. 157]), $(Y^N(t) - Y^N(0), t \geq 0)$ is a symmetric Brownian motion, and so $(Y^N(t), t \geq 0)$ has the same law than $(X^N(t), t \geq 0)$ since $X^N(0) = Y^N(0)$. □

Corollary 12.3 (Dyson). *Let $\beta = 1$ or 2 and $\lambda_N(0)$ in $\overline{\Delta}_N$. Then, the eigenvalues $(\lambda_N(t))_{t \geq 0}$ of $\left(X^{N,\beta}(t)\right)_{t \geq 0}$ are continuous semi-martingales with values in Δ_N for all $t > 0$. The joint law of $(\lambda_N(t))_{t \geq \epsilon}$ is the weak solution to (12.1) starting from $\lambda_N(\epsilon) \in \Delta_N$ for any $\epsilon > 0$. $\lambda_N(\epsilon)$ converges to $\lambda_N(0)$ as ϵ goes to zero.*

Proof. To remove the hypothesis that the eigenvalues of $X_N(0)$ belong to Δ_N, note that for all $t > 0$, $\lambda_N(t)$ belongs to Δ_N almost surely. Indeed, the set of symmetric matrices with at least one double eigenvalue can be characterized by the fact that the discriminant of their characteristic polynomial vanishes, and so the entries of such matrices belong to a submanifold with codimension greater than one. Since the law of the entries of $X^N(t)$ is absolutely continuous with respect to the Lebesgue measure, this set has measure zero. Therefore, we can represent the eigenvalues of $(X^N(t), t \geq \epsilon)$ as solution of (12.1) for any $\epsilon > 0$. By using Lemma 1.16, we see that for all $s, t \in \mathbb{R}$

$$\sum_{i=1}^{N}(\lambda_N^i(t)-\lambda_N^i(s))^2 \le \frac{1}{N}\sum_{i,j=1}^{N}(B_{ij}(t)-B_{ij}(s))^2$$

so that the continuity of the Brownian motions paths results in the continuity of $t \to \lambda_N(t)$ for any given N. Hence, the eigenvalues of $(X^N(t))_{t\le T}$ are strong solutions of (12.1) for all $t>0$ and are continuous at the origin. □

Exercise 2. Let $X^{N,4} = \left(X_{ij}^{N,4}\right)$ be a $2N \times 2N$ complex matrix defined as the $N \times N$ self-adjoint random matrices with entries

$$X_{kl}^{N,4} = \frac{\sum_{i=1}^{\beta} g_{kl}^i e_4^i}{\sqrt{\beta N}}, \quad 1 \le k < l \le N, \quad X_{kk}^{N,4} = \sqrt{\frac{2}{\beta N}} g_{kk} e_4^1, \quad 1 \le k \le N$$

where $(e_4^i)_{1\le i\le 4}$ are the Pauli matrices

$$e_4^1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \; e_4^2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \; e_4^3 = \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix}, \; e_4^4 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}.$$

Define $H^{N,4}$ similarly by replacing the Gaussian entries by Brownian motions. Show that if $X^N(0)$ a Hermitian matrix with eigenvalues $\lambda_{2N}(0))$, the eigenvalues $\lambda_{2N}(t))$ of $X^N(0) + H^{N,4}$ satisfy the stochastic differential system

$$d\lambda_{2N}^i(t) = \frac{1}{\sqrt{4N}} dW_t^i + \frac{1}{2N}\sum_{j\ne i} \frac{1}{\lambda_N^i(t)-\lambda_N^j(t)} dt. \tag{12.6}$$

Corollary 12.4. *For $\beta = 1$ or 2, the law of the eigenvalues of the Gaussian Wigner matrix $X^{N,\beta}$ is given by*

$$P_\beta^N(dx_1,\ldots,dx_N) = \frac{1}{Z_N} 1_{x_1\le\cdots\le x_N} \prod_{1\le i<j\le N} |x_i - x_j|^\beta \prod_{i=1}^{N} e^{-\beta x_i^2/4}. \tag{12.7}$$

Proof. In Dyson's theorem, we can easily replace the Hermitian Brownian motion $(H^N(t))_{t\in\mathbb{R}^+}$ by the Hermitian Ornstein–Uhlenbeck process $(\tilde{H}^N(t))_{t\in\mathbb{R}^+}$ whose entries are solutions of

$$d\tilde{H}_{k,l}(t) = dH_{k,l}(t) - \frac{1}{2}\tilde{H}_{k,l}(t)dt.$$

$\tilde{H}_{k,l}(t)$ converges as t goes to infinity towards a centered Gaussian variable with covariance N^{-1}, independently of the initial condition $X^N(0)$. Hence, Wigner matrices appear as the large time limit of $\tilde{H}$ and in particular their law is invariant for the dynamics of the Ornstein–Uhlenbeck process. On the other hand, a slight modification of the proof of Dyson's Theorem 12.2 (we can here assume that $X^N(0)$ has eigenvalues in the simplex) shows that the eigenvalues of $\tilde{H}$ follow the SDE

$$d\lambda_N^i(t) = \frac{\sqrt{2}}{\sqrt{\beta N}} dW_t^i - \frac{1}{2}\lambda_N^i(t)dt + \frac{1}{N}\sum_{j\neq i} \frac{1}{\lambda_N^i(t) - \lambda_N^j(t)} dt.$$

Hence, the law $P_N^{(\beta)}$, as the large time limit of the law of $\lambda_N(t)$, must be invariant under the above dynamics. Itô's calculus shows that the infinitesimal generator of these dynamics is

$$\mathcal{L} = \frac{1}{\beta N}\sum_{i=1}^N \partial_i^2 + \sum_{i=1}^N \left(\frac{1}{N}\sum_{j\neq i}\frac{1}{\lambda^i - \lambda^j} - \frac{1}{2}\lambda_i\right)\partial_i$$

and therefore we must have, for any twice continuously differentiable function f on $\mathbb{R}^N$,

$$\int \mathcal{L} f(\lambda_1, \ldots, \lambda_N) dP_N^{(\beta)}(\lambda_1, \ldots, \lambda_N) = 0.$$

Some elementary algebra shows that the choice proposed in (12.7) fulfills this requirement. Furthermore it is the unique such probability measure on the simplex since if there was another invariant distribution Q_N for $\mathcal{L}$, we could follow the proof of Theorem 12.2 to reconstruct a Hermitian Ornstein-Uhlenbeck process $\tilde{H}^N(t)$ and a matrix $X^N(0)$ whose eigenvalues would follow Q_N so that $\tilde{H}_{k,l}(0) = X^N(0)_{k,l}$ and

$$d\tilde{H}_{k,l}(t) = dH_{k,l}(t) - \frac{1}{2}\tilde{H}_{k,l}(t)dt.$$

But this gives a contradiction since as time goes to infinity, the law of $\tilde{H}_{k,l}$ is a Gaussian law, independently of the law Q_N. $\square$

12.2 Itô's calculus

Let $(W^1, \ldots, W^N)$ be independent Brownian motions and $(\lambda_N^1(0), \ldots, \lambda_N^N(0))$ be real numbers. Let β be a real number greater than one and let $(\lambda_N(t))_{t\geq 0}$ be the unique strong solution to (12.1). We denote by

$$L_N(t, dx) := \frac{1}{N}\sum_{i=1}^N \delta_{\lambda_N^i(t)} \in \mathcal{P}(\mathbb{R})$$

the empirical measure of $\lambda_N(t)$. We shall sometimes use the short notation for bounded measurable functions f on $\mathbb{R}$,

$$\int f dL_N(t) := \int f(x) L_N(t, dx) = \frac{1}{N}\sum_{i=1}^N f(\lambda_N^i(t)).$$

Then, by Itô's calculus Theorem 20.18, we know that for all $f \in \mathcal{C}^2([0,T] \times \mathbb{R}, \mathbb{R})$,

$$\int f(t,x)L_N(t,dx) = \int f(0,x)L_N(0,dx) + \int_0^t \int \partial_s f(s,x)L_N(s,dx)ds$$
$$+\frac{1}{2}\int_0^t \int \frac{\partial_x f(s,x) - \partial_x f(s,y)}{x-y} L_N(s,dx)L_N(s,dy)ds \tag{12.8}$$
$$+\left(\frac{2}{\beta}-1\right)\frac{1}{2N}\int_0^t \int \partial_x^2 f(s,x)L_N(s,dx)ds + M_N^f(s)$$

with M_N^f the martingale given for $s \le T$ by

$$M_N^f(t) = \frac{\sqrt{2}}{\sqrt{\beta}N^{\frac{3}{2}}}\sum_{i=1}^N \int_0^t \partial_x f(s, \lambda_N^i(s))dB_s^i.$$

Note that M_N^f is a martingale with bracket

$$\langle M_N^f \rangle_t = \frac{2}{\beta N^2}\int_0^t \int (\partial_x f(s,x))^2 L_N(s,dx)du \le \frac{2\|\partial_x f\|_\infty^2 t}{\beta N^2}.$$

12.3 A dynamical proof of Wigner's Theorem 1.13

In this section, we shall give a dynamical proof of Theorem 1.13; it is restricted to Gaussian entries but generalized in the sense that we can study the asymptotic behavior of the spectral measure of any sum of two independent symmetric matrices, one being a Gaussian Wigner matrix, the second being deterministic with a converging spectral distribution. Moreover, our proof only relies on (12.1) and thus our result generalizes to any $\beta \ge 1$, and in particular to the Hermitian and the symplectic case too (that corresponds to $\beta = 2$ and 4).

For $T > 0$, we denote by $\mathcal{C}([0,T], \mathcal{P}(\mathbb{R}))$ the space of continuous processes from $[0,T]$ into $\mathcal{P}(\mathbb{R})$ equipped with its weak topology. We have

Lemma 12.5. *Let $\lambda_N(0) \in \mathbb{R}^N$ so that $L_N(0) = \frac{1}{N}\sum_{i=1}^N \delta_{\lambda_N^k(0)}$ converges as N goes to infinity towards $\mu \in \mathcal{P}(\mathbb{R})$. We assume*

$$C_0 := \sup_{N \ge 0} \int \log(x^2+1)dL_N(0)(x) < \infty. \tag{12.9}$$

Let $(\lambda_N^1(t), \ldots, \lambda_N^N(t))_{t\ge 0}$ be the solution to (12.1) *and set*

$$L_N(t) = \frac{1}{N}\sum_{i=1}^N \delta_{\lambda_N^i(t)}.$$

Then, for any finite time T, $(L_N(t), t \in [0,T])$ converges almost surely in $\mathcal{C}([0,T], \mathcal{P}(\mathbb{R}))$. Its limit is the unique measure-valued process $(\mu_t, t \in [0,T])$ so that $\mu_0 = \mu$ and for all $z \in \mathbb{C}\backslash\mathbb{R}$

$$G_t(z) = \int (z-x)^{-1} d\mu_t(x)$$

satisfies the complex Burgers equation

$$G_t(z) = G_0(z) - \int_0^t G_s(z)\partial_z G_s(z) ds. \tag{12.10}$$

with given initial condition G_0.

We begin the proof by showing that $(L_N(t), t \in [0,T])$ is almost surely tight in $\mathcal{C}([0,T], \mathcal{P}(\mathbb{R}))$ and then show that it has a unique limit point characterized by (12.10).

We first describe compact sets of $\mathcal{C}([0,T], \mathcal{P}(\mathbb{R}))$; they are of the form

$$\mathcal{K} = \{\forall t \in [0,T], \mu_t \in K\} \cap_{i \geq 0} \{t \to \mu_t(f_i) \in C_i\} \tag{12.11}$$

where

• K is a compact set of $\mathcal{P}(\mathbb{R})$ such as

$$K_{\epsilon,m} := \cap_{m \geq 0} \{\mu([-m,m]) \leq \epsilon_m\} \tag{12.12}$$

for a sequence $\{\epsilon_m, m \geq 0\}$ of positive real numbers going to zero as m goes to infinity.

• $(f_i)_{i \geq 0}$ is a sequence of bounded continuous functions dense in $\mathcal{C}_0(\mathbb{R})$ and C_i are compact sets of $\mathcal{C}([0,T], \mathbb{R})$. By the Arzela–Ascoli theorem, it is known that the latter are of the form

$$C_{\epsilon,M} := \{g : [0,T] \to \mathbb{R}, \sup_{\substack{t,s \in [0,T] \\ |t-s| \leq \eta_n}} |g(t) - g(s)| \leq \epsilon_n, \sup_{t \in [0,T]} |g(t)| \leq M\} \tag{12.13}$$

with sequences $\{\epsilon_n, n \geq 0\}$ and $\{\eta_n, n \geq 0\}$ of positive real numbers going to zero as n goes to infinity.

In fact, if we take a sequence μ^n in $\mathcal{K}$, for all $i \in \mathbb{N}$, we can find a subsequence such that $\mu_{\cdot}^{\phi_i(n)}(f_i)$ converges as a bounded continuous function on $[0,T]$. By a diagonalization procedure, we can find ϕ so that $\mu_{\cdot}^{\phi(n)}(f_i)$ converges simultaneously towards some $\mu_{\cdot}(f_i)$ for all $i \in \mathbb{N}$. Note at this point that since the f_i have compact support, the limit $\mu_{\cdot}$ might not have mass one. This is dealt with by the second condition. Indeed, since the time marginals of $\mu_t^{\phi(n)}$ are tight for all t we can, again up to take another subsequence, insure that $\mu_t \in \mathcal{P}(\mathbb{R})$, at least for a countable number of times. The continuity of $t \to \mu_t(f_i)$ and the density of the family f_i then shows that $\mu_t \in \mathcal{P}(\mathbb{R})$ for all t. Hence, we have proved that μ^n is sequentially compact. Further, the limit μ also belongs to $\mathcal{K}$, which finishes to show that $\mathcal{K}$ is compact.

We shall prove below that $\hat{\mathbf{L}}^N$ is almost surely tight. For later purposes (namely the study of large deviation properties), we next prove a slightly stronger result.

Lemma 12.6. *Let $T \in \mathbb{R}^+$. Assume (12.9). Then, there exists $a = a(T) > 0$ and $M(T) < \infty$ so that:*

1. *For $M \geq M(T)$*

$$P\left(\sup_{t\in[0,T]} \int \log(x^2+1)L_N(t,dx) \geq M\right) \leq e^{-a(T)MN^2}.$$

2. *For any $\delta > 0$ and $M > 0$, for any twice continuously differentiable function f so that $\|f''\|_\infty \leq 2^{-3}M\delta^{-\frac{3}{4}}$,*

$$P\Bigg(\sup_{\substack{t,s\in[0,T]\\|t-s|\leq\delta}}\left|\int f(x)L_N(t,dx) - \int f(x)L_N(s,dx)\right| \geq M\delta^{\frac{1}{4}}\Bigg) \leq 2(T\delta^{-1}+1)e^{-\frac{\beta N^2M^2}{2^8\|f'\|_\infty^2\delta^{\frac{1}{2}}}}.$$

3. *For all $T \in \mathbb{R}^+$, all $L \in \mathbb{N}$, there exists a compact set $\mathcal{K}(L)$ of the set $\mathcal{C}([0,T],\mathcal{P}(\mathbb{R}))$ of continuous probability measures-valued processes so that*

$$P\left(L_N(.) \in \mathcal{K}(L)^c\right) \leq e^{-N^2L}.$$

In particular, the law of $(L_N(s), s \in [0,T])$ is almost surely tight in $\mathcal{C}([0,T],\mathcal{P}(\mathbb{R}))$.

Proof. We base our proof on (12.1). Using Section 12.2 with $f(x) = \log(x^2+1)$, we get since

$$\left|\frac{f'(x)-f'(y)}{x-y}\right| = \left|\int_0^1 f''(\alpha x + (1-\alpha)y)d\alpha\right| \leq \|f''\|_\infty < \infty,$$

for all $s \geq 0$,

$$\left|\int \log(x^2+1)dL_N(s)\right| \leq \left|\int \log(x^2+1)dL_N(0)\right| + 2\|f''\|_\infty s + |M_s^N| \quad (12.14)$$

with M_s^N the martingale

$$M_s^N = \frac{2\sqrt{2}}{\sqrt{\beta}N^{\frac{3}{2}}}\sum_{i=1}^N \int_0^s \frac{\lambda_N^i(u)}{\lambda_N^i(u)^2+1}dW_u^i.$$

Note that

$$\langle M^N \rangle_s = \frac{8}{\beta N^3} \sum_{i=1}^{N} \int_0^s \frac{\lambda_N^i(u)^2}{(\lambda_N^i(u)^2+1)^2} du \le \frac{8s}{\beta N^2}.$$

Hence, we can use Corollary 20.23 to obtain for all $L \ge 0$

$$P\left(\sup_{s\le T} |M_s^N| \ge L\right) \le 2e^{-\frac{\beta N^2 L^2}{16T}}.$$

Thus, (12.14) shows that for $M \ge C_0 + 2\|f''\|_\infty T$,

$$P\left(\sup_{s\le T}\left|\int \log(x^2+1) dL_N(s)\right| \ge M\right) \le 2e^{-\frac{\beta N^2(M-C_0+2\|f''\|_\infty T)^2}{16T}} \tag{12.15}$$

which proves the first point. For the second point we proceed similarly by first noticing that if $t_i = i\delta$ for $i \in [0, [T/\delta]+1]$,

$$\left\{\sup_{\substack{t,s\in[0,T]\\|t-s|\le\delta}}\left|\int f dL_N(t) - \int f dL_N(s)\right| \ge M\delta^{\frac14}\right\}$$

$$\subset \cup_{1\le i\le [T/\delta]+1}\left\{\sup_{t_i\le s\le t_{i+1}}\left|\int f dL_N(s) - \int f dL_N(t_i)\right| \ge 2^{-1}M\delta^{\frac14}\right\}.$$

Now, for $s \in [t_i, t_{i+1}]$, we write by a further use of Section 12.2,

$$\left|\int f dL_N(s) - \int f dL_N(t_i)\right| \le 2\|f''\|_\infty \delta + |M_N^f(s)|$$

with

$$\langle M_N^f \rangle_s = \frac{8}{\beta N^3}\sum_{i=1}^N \int_{t_i}^s (f'(\lambda_N^i(u)))^2 du \le \frac{8\delta\|f'\|_\infty^2}{\beta N^2}.$$

Thus Corollary 20.23 shows that

$$\sup_{t_i\le s\le t_{i+1}}\left|\int f dL_N(s) - \int f dL_N(t_i)\right| \le 2\|f''\|_\infty\delta + \epsilon$$

with probability greater than $1 - 2e^{-\frac{\beta N^2(\epsilon)^2}{16\delta\|f'\|_\infty^2}}$. As a conclusion, for $\epsilon = 2^{-1}M\delta^{\frac14} - 2\|f''\|_\infty\delta \ge 2^{-2}M\delta^{\frac14}$ we have proved

$$P\left(\sup_{\substack{t,s\in[0,T]\\|t-s|\le\delta}}\left|\int f dL_N(t) - \int f dL_N(s)\right| \ge M\delta^{\frac14}\right) \le \sum_{i=1}^{[T/\delta]+1} 2e^{-\frac{\beta N^2 M^2}{2^8\|f'\|_\infty^2\delta^{\frac12}}}$$

which proves the second claim.

To conclude our proof, let us notice that:

- The set

$$K_M = \{\mu \in \mathcal{P}(\mathbb{R}) : \int \log(1+x^2) d\mu(x) \le M\}$$

is compact and by Borel–Cantelli lemma, we deduce from point 1 that

$$P\left(\{L_N(t) \in K_M \ \forall\, t \in [0,T]\}^c\right) \le e^{-a(T)MN^2}.$$

• Take f twice continuously differentiable and consider the compact subset of $\mathcal{C}([0,T], \mathcal{P}(\mathbb{R})$

$$C_T(f, M) := \{\mu \in \mathcal{C}([0,T], \mathcal{P}(\mathbb{R})) : \sup_{|t-s|\le n^{-2}} |\mu_t(f) - \mu_s(f)| \le M\sqrt{n}^{-1} \quad \forall n \in \mathbb{N}^*\}.$$

The previous estimates imply that if $\|f''\|_\infty \le Mn$ for all n

$$P\left(L_N \in C_T(f,M)^c\right) \le \sum_{n\ge 1} 2(Tn^2+1) e^{-\frac{\beta N^2 n M^2}{2^{16}\|f'\|_\infty^2}} \le C(T) e^{-\frac{\beta N^2 M^2}{2^{16}\|f'\|_\infty^2}}$$

with some finite constant $C(T)$ that only depends on T. Choosing a countable family f_i of twice continuously differentiable functions dense in $\mathcal{C}_0(\mathbb{R})$ and such that $\|f''\|_\infty \le i$ and $\|f'\|_\infty \le \sqrt{i}$ for all i, we obtain

$$P\left(\{L_N \in (\cap_{i\ge 0} C_T(f_i, Mi))^c\}\right) \le C(T) \sum_{i\ge 1} e^{-\frac{\beta N^2 i^2 M^2}{2^{16}\|f_i'\|_\infty^2}} \le C'(T) e^{-\frac{\beta N^2 M^2}{2^{16}}}.$$

• Hence, we conclude that the compact set

$$\mathcal{K}(M) = K_M \cap \cap_{i\ge 0} C_T(f_i, Mi)$$

of $\mathcal{C}([0,T], \mathcal{P}(\mathbb{R}))$ is such that $P(L_N \in \mathcal{K}(M)^c) \le C(T) e^{-\frac{\beta N^2 M^2}{2^{16}}}$ and thus by Borel–Cantelli lemma

$$P(\cup_{N_0} \cap_{N\ge N_0} \{L_N \in \mathcal{K}(M)\}) = 1.$$

□

To characterize the limit points of L_N, let us use also Itô's calculus of Section 12.2 with $f(t,x) = f(x) = (z-x)^{-1}$ for $z \in \mathbb{C}\backslash\mathbb{R}$ (or separately for its real and imaginary parts). Again by Corollary 20.23, M_f^N goes almost surely to zero and therefore, any limit point $(\mu_t, t \in [0,T])$ of $(L_N(t), t \in [0,T])$ satisfies the equation

$$\int f(x) d\mu_t(x) = \int f(x) d\mu_0(x) + \int_0^t \int \partial_s f(x) d\mu_s(x) ds \qquad (12.16)$$
$$+ \frac{1}{2} \int_0^t \int \frac{\partial_x f(x) - \partial_x f(y)}{x-y} d\mu_s(x) d\mu_s(y) ds.$$

Thus, $G_t(z) = \int (z-x)^{-1} d\mu_t(x)$ satisfies (12.10).

To conclude our proof, we show that (12.10) has a unique solution.

Lemma 12.7. *For any $t > 0$ and z with modulus $|z|$ large enough, $z + tG_0(z)$ is invertible with inverse H_t. The solution of* (12.10) *is the unique analytic function on $\mathbb{C}\backslash\mathbb{R}$ such that for any $t > 0$ and z with large enough modulus*

$$G_t(z) = G_0\left(H_t(z)\right).$$

Comments. This result is a particular case of free convolution (see Section 17.3.2) and of the notion of subordination (cf. [35]).

Exercise 3. Take $\mu = \delta_0$ and prove that μ_t is the semicircular law with variance t. *Hint.* Use that by scaling property $G_t(z) = t^{-\frac{1}{2}} G_1(t^{-\frac{1}{2}} z)$ for all $t > 0$ and deduce a formula for G_1.

Proof. We use the characteristic method. Let us associate to z the solution $\{z_t, t \geq 0\}$ of the equation

$$\partial_t z_t = G_t(z_t),\ z_0 = z.$$

Such a solution exists at least up to time $(\Im z)^2/2$ since if $\Im(z) > 0$, $\Im(G_t(z)) \in [-\frac{1}{\Im(z)}, 0]$ implies that we can construct a unique solution z_t with $\Im(z_t) > 0$ up to that time, a domain on which G_t is Lipschitz. Now, $\partial_t G_t(z_t) = 0$ implies

$$z_t = tG_0(z) + z,\ \ G_t(z + tG_0(z)) = G_0(z)$$

from which the conclusion follows. $\square$

Bibliographical notes. The previous arguments on Dyson's Brownian motion are inspired by [150, p.123], where the density of the eigenvalues of symmetric Brownian motions are discussed, [170] where the stochastic differential equation (12.1) is studied, [160] where Brownian motions of ellipsoids is considered as well as [165] where decompositions of Brownian motions on certain manifolds of matrices are analyzed. Theorem 12.2 is also stated in Mehta's book [153, Theorem 8.2.1](see also Chan [61]). Similar results can be obtained for Wishart processes, see [52]. The interpretation of Dyson's Brownian motion as a Brownian motion in a Weyl chamber was used in [40, 36].

13

Large deviation principle for the law of the spectral measure of shifted Wigner matrices

The goal of this section is to prove the following theorem.

Theorem 13.1. *Assume that D_N is uniformly bounded with spectral measure converging to μ_D. Let $X^{\beta,N}$ be a Gaussian symmetric (resp. Hermitian) Wigner matrix when $\beta = 1$ (resp. $\beta = 2$). Then the law of the spectral measure $L_{Y^{N,\beta}}$ of the Wigner matrix $Y^{N,\beta} = D_N + X^{\beta,N}$ satisfies a large deviation principle in the scale N^2 with a certain good rate function $J_\beta(\mu_D, .)$.*

We shall base our approach on Bryc's theorem (6.13), that says that the above large deviation principle statement is equivalent to the fact that for any bounded continuous function f on $\mathcal{P}(\mathbb{R})$,

$$\Lambda(f) = \lim_{N\to\infty} \frac{1}{N^2} \log \int e^{N^2 f(L_{Y^{N,\beta}})} d\mathbb{P}$$

exists and is given by $-\inf\{J_\beta(\mu_D, \nu) - f(\nu)\}$. It is not clear how one could a priori study such limits, except for very trivial functions f. However, if we consider the matrix-valued process $Y^{N,\beta}(t) = D^N + H^{N,\beta}(t)$ with Brownian motion $H^{N,\beta}$ described in (V.1) and its spectral measure process

$$L_{N,\beta}(t) := L_{Y^{N,\beta}(t)} = \frac{1}{N} \sum_{i=1}^{N} \delta_{\lambda_i(Y^{N,\beta}(t))} \in \mathcal{P}(\mathbb{R}),$$

we may construct martingales by use of Itô's calculus. Indeed, continuous martingales lead to exponential martingales, which have constant expectation, and therefore allow one to compute the exponential moments of a whole family of functionals of $L_N(t)$. This idea gives easily a large deviation upper bound for the law of $(L_{N,\beta}(t), t \in [0,1])$, and therefore for the law of $L_{Y^{N,\beta}}$, that is, the law of $L_{N,\beta}(1)$. The difficult point here is to check that this bound is sharp, i.e., it is enough to compute the exponential moments of this family of functionals in order to obtain the large deviation lower bound.

An alternative tempting way to prove this large deviation lower bound would be, as for the proof of Theorem 10.1, to force the paths of the eigenvalues to be in small tubes around the quantiles of their limiting law. However, these tubes would need to be very small, with width of order $\delta \approx N^{-1}$ and the probability $P(\sup_{0\le s\le 1}|B_s|\le\delta)\approx e^{-\frac{c}{\delta^2}}$ is now giving a contribution on the scale e^{N^2}.

Let us now state more precisely our result. We shall consider $\{L_{N,\beta}(t), t\in[0,1]\}$ as an element of the set $\mathcal{C}([0,1],\mathcal{P}(\mathbb{R}))$ of continuous processes with values in $\mathcal{P}(\mathbb{R})$. The rate function for these deviations shall be given as follows. For any $f,g\in\mathcal{C}_b^{2,1}(\mathbb{R}\times[0,1])$, any $s\le t\in[0,1]$, and any $\nu_.\in\mathcal{C}([0,1],\mathcal{P}(\mathbb{R}))$, we let

$$\begin{aligned}S^{s,t}(\nu,f) &= \int f(x,t)d\nu_t(x) - \int f(x,s)d\nu_s(x)\\ &\quad-\int_s^t\int \partial_u f(x,u)d\nu_u(x)du\\ &\quad-\frac{1}{2}\int_s^t\int\int\frac{\partial_x f(x,u)-\partial_x f(y,u)}{x-y}d\nu_u(x)d\nu_u(y)du,\end{aligned}\qquad(13.1)$$

$$\langle f,g\rangle_\nu^{s,t} = \int_s^t\int \partial_x f(x,u)\partial_x g(x,u)d\nu_u(x)du\,,\qquad(13.2)$$

and

$$\bar{S}_\beta^{s,t}(\nu,f) = S^{s,t}(\nu,f)-\frac{1}{\beta}\langle f,f\rangle_{s,t}^\nu\,.\qquad(13.3)$$

Set, for any probability measure $\mu\in\mathcal{P}(\mathbb{R})$,

$$S_{\mu,\beta}(\nu) := \begin{cases}+\infty\,, & \text{if } \nu_0\neq\mu,\\ S_\beta^{0,1}(\nu) := \sup_{f\in\mathcal{C}_b^{2,1}(\mathbb{R}\times[0,1])}\sup_{0\le s\le t\le 1}\bar{S}_\beta^{s,t}(\nu,f)\,, & \text{otherwise.}\end{cases}\qquad(13.4)$$

Then, the main theorem of this section is the following:

Theorem 13.2. *Let $\beta=1$ or 2. (1) For any $\mu\in\mathcal{P}(\mathbb{R})$, $S_{\mu,\beta}$ is a good rate function on $\mathcal{C}([0,1],\mathcal{P}(\mathbb{R}))$, i.e. $\{\nu\in\mathcal{C}([0,1],\mathcal{P}(\mathbb{R})); S_\mu(\nu)\le M\}$ is compact for any $M\in\mathbb{R}^+$.*

(2) *Assume that*

$$\sup_N L_{D_N}(|x|^4)<\infty,\quad L_{D_N}\text{ converges to }\mu_D,\qquad(13.5)$$

then the law of $(L_{N,\beta}(t), t\in[0,1])$ satisfies a large deviation upper-bound in the scale N^2 with good rate function $S_{\mu_D,\beta}$. The large deviation lower bound holds around any measure-valued path with uniformly bounded fourth moment, i.e., for all $\mu_.$ so that $\sup_{t\in[0,1]}\mu_t(x^4)$ is finite:

$$\liminf_{\delta\to 0}\liminf_{N\to\infty}\frac{1}{N^2}\log P\left(\sup_{t\in[0,1]} d(L_{N,\beta}(t),\mu_t)<\delta\right)\geq -S_{\mu_D,\beta}(\mu_.).$$

In [110, Theorem 3.3] O. Zeitouni and I proved the following:

Theorem 13.3. *Take $\beta = 1$ or 2 and take*

$$\mathcal{A} = \{\mu \in \mathcal{P}(\mathbb{R}) : \text{ there exists } \epsilon > 0, \int |x|^{5+\epsilon} d\mu_D(x) < \infty\}.$$

Then, for any ν such that $S_{\mu_D}(\nu) < \infty$, there exists a sequence $\nu^n : [0,1] \to \mathcal{A}$ of measure-valued paths such that

$$\lim_{n\to\infty}\nu_n = \nu \qquad \lim_{n\to\infty} S_{\mu_D,\beta}(\nu_n) = S_{\mu_D,\beta}(\nu).$$

This result could be extended by replacing $5+\epsilon$ by $4+\epsilon$ (which we needed first, since Theorem 13.2 was originally obtained under these 5^+ moment conditions) but we decided not to enter into this proof here since it is purely analytical. We shall, however, provide here a complete proof of Theorem 13.2 that slightly simplifies that given in [109].

Theorems 13.2 and 13.3 imply the following:

Theorem 13.4. *Assume that* (13.5) *holds and $\mu_D \in \mathcal{A}$. Then the law of $(L_N(t), t \in [0,1])$ satisfies a large deviation principle in the scale N^2 with good rate function S_{μ_D}.*

Note that the application $(\mu_t, t \in [0,1]) \to \mu_1$ is continuous from $\mathcal{C}([0,1],\mathcal{P}(\mathbb{R}))$ into $\mathcal{P}(\mathbb{R})$, so that Theorem 13.2 and the contraction principle Theorem 20.7 imply that the law of $L_N(1)$ satisfies a large deviation principle.

Theorem 13.5. *Under assumption* (13.5) *and $\mu_D \in \mathcal{A}$, Theorem 13.1 is true with*

$$J_\beta(\mu_D,\mu_E) = \frac{\beta}{2}\inf\{S_{\mu_D}(\nu_.);\nu_1 = \mu\}.$$

Remark 4. Remark that without using Theorem 13.3, we could still get a large deviation lower bound for the law of $L_N(1)$ around measure with fourth moments; this is due to the fact that the optimal paths in the above infimum have fourth moments (as can be guessed from the remark that optimal paths have to be Brownian bridges, cf., e.g., [55]).

In [101], the infimum in Theorem 13.5 was studied. It was shown that it is achieved and that, if $\int \log|x-y|d\mu_D(x)d\mu_D(y) > -\infty$, the minimizer $\mu^*_. \in (C([0,1],\mathcal{P}(\mathbb{R}))$ is such that $\mu^*_t(dx) = \rho^*_t(x)dx$ is absolutely continuous with respect to Lebesgue measure for all $t \in (0,1)$ and there exists a measurable function $u_.$ such that $f_t(x) = u_t(x) + i\pi\rho_t(x)$ is solution (at least in a weak sense) of the complex Burgers equation

$$\partial_t f_t(x) = -\frac{1}{2}\partial_x f_t(x)^2$$

with boundary conditions given by the imaginary part of f at $t = 0$ and $t = 1$.

This result was stated first by Matytsin [146]. Interestingly, the complex Burgers equation also describes limit shapes of plane partitions and dimers, see [127].

The main point to prove Theorem 13.2 is to observe that the evolution of L_N is described, thanks to Itô's calculus 12.2, by an autonomous differential equation. This is the starting point to use the ideas of Kipnis–Olla–Varadhan papers [131, 130]. These papers concern the case where the diffusive term is not vanishing (βN is of order one). The large deviations for the law of the empirical measure of the particles following (12.1) in such a scaling have been studied by Fontbona [91] in the context of McKean–Vlasov diffusion with singular interaction. We shall first recall for the reader the techniques of [131, 130] applied to the empirical measures of independent Brownian motions as presented in [130]. We will then describe the necessary changes to adapt this strategy to our setting.

13.1 Large deviations from the hydrodynamical limit for a system of independent Brownian particles

Note that the deviations of the law of the empirical measure of independent Brownian motions on path space

$$L_N = \frac{1}{N}\sum_{i=1}^{N} \delta_{B^i_{[0,1]}} \in \mathcal{P}(\mathcal{C}([0,1],\mathbb{R}))$$

are well known by Sanov's theorem which yields (cf. [74, Section 6.2]):

Theorem 13.6. *Let $\mathcal{W}$ be the Wiener law. Then, the law $(L_N)_\#\mathcal{W}^{\otimes N}$ of L_N under $\mathcal{W}^{\otimes N}$ satisfies a large deviation principle in the scale N with rate function given, for $\mu \in \mathcal{P}(\mathcal{C}([0,1],\mathbb{R}))$, by $I(\mu|\mathcal{W})$ that is infinite if μ is not absolutely continuous with respect to Wiener measure and otherwise given by*

$$I(\mu|\mathcal{W}) = \int \log\frac{d\mu}{d\mathcal{W}} \log\frac{d\mu}{d\mathcal{W}} d\mathcal{W}.$$

Thus, if we consider

$$L_N(t) = \frac{1}{N}\sum_{i=1}^{N} \delta_{B^i_t}, \quad t \in [0,1],$$

since $L_N \to (L_N(t), t \in [0,1])$ is continuous from $\mathcal{P}(\mathcal{C}([0,1],\mathbb{R}))$ into $\mathcal{C}([0,1],\mathcal{P}(\mathbb{R}))$, the law of $(L_N(t), t \in [0,1])$ under $\mathcal{W}^{\otimes N}$ satisfies a large deviation principle by the contraction principle Theorem 20.7. Its rate function is given, for $p \in \mathcal{C}([0,1],\mathcal{P}(\mathbb{R}))$, by

$$S(p) = \inf\{I(\mu|\mathcal{W}) \quad : \quad (x_t)_\# \mu = p_t \quad \forall t \in [0,1]\}.$$

Here, $(x_t)_\#\mu$ denotes the law of x_t under μ. It was shown by Föllmer [90] that in fact $S(p)$ is infinite unless there exists $k \in L^2(p_t(dx)dt)$ such that

$$\inf_{f \in \mathcal{C}^{1,1}(\mathbb{R}\times[0,1])} \int_0^1 \int (\partial_x f(x,t) - k(x,t))^2 p_t(dx)dt = 0, \tag{13.6}$$

and for all $f \in \mathcal{C}^{2,1}(\mathbb{R}\times[0,1])$,

$$\partial_t p_t(f_t) = p_t(\partial_t f_t) + \frac{1}{2} p_t(\partial_x^2 f_t) + p_t(\partial_x f_t k_t).$$

Moreover, we then have

$$S(p) = \frac{1}{2}\int_0^1 p_t(k_t^2)dt. \tag{13.7}$$

Kipnis and Olla [130] proposed a direct approach to obtain this result based on exponential martingales. Its advantage is to be much more robust and to adapt to many complicated settings encountered in hydrodynamics (cf. [129]). Let us now summarize it. It follows the following scheme:

- *Exponential tightness and study of the rate function S.* Since the rate function S is the contraction of the relative entropy $I(.|\mathcal{W})$, it is clearly a good rate function. This can be proved directly from formula (13.7) as we shall detail it in the context of the eigenvalues of large random matrices. Similarly, we shall not detail here the proof that $L_{N\#}\mathcal{W}^{\otimes N}$ is exponentially tight, a property that reduces the proof of the large deviation principle to the proof of a weak large deviation principle and thus to estimate the probability of deviations into small open balls (cf. Theorem 20.4). We will now concentrate on this last point.
- *Itô's calculus.* Itô's calculus (cf. Theorem 20.18) implies that for any function F in $\mathcal{C}_b^{2,1}(\mathbb{R}^N\times[0,1])$, any $t \in [0,1]$

$$\begin{aligned} F(B_t^1,\ldots,B_t^N,t) =& F(0,\ldots,0) + \int_0^t \partial_s F(B_s^1,\ldots,B_s^N,s)ds \\ &+ \sum_{i=1}^N \int_0^t \partial_{x_i} F(B_s^1,\cdots,B_s^N,s)dB_s^i \\ &+ \frac{1}{2}\sum_{1\le i,j\le N}\int_0^t \partial_{x_i}\partial_{x_j} F(B_s^1,\ldots,B_s^N,s)ds. \end{aligned}$$

  Moreover, $M_t^F = \sum_{i=1}^N \int_0^t \partial_{x_i} F(B_s^1,\cdots,B_s^N,s)dB_s^i$ is a martingale with respect to the filtration of the Brownian motion, with bracket

$$\langle M^F\rangle_t = \sum_{i=1}^N \int_0^t [\partial_{x_i} F(B_s^1,\cdots,B_s^N,s)]^2 ds.$$

Taking $F(x^1, \ldots, x^N, t) = N^{-1} \sum_{i=1}^N f(B_t^i, t) = \int f(x,t) L_N(t, dx) = \int f_t dL_N(t)$, we deduce that for any $f \in \mathcal{C}_b^{2,1}(\mathbb{R} \times [0,1])$,

$$M_f^N(t) = \int f_t dL_N(t) - \int f_0 dL_N(0) - \int_0^t \int \partial_s f_s dL_{N\,s} ds \\ - \int_0^t \frac{1}{2} \int \partial_x^2 f_s dL_N(s) ds$$

is a martingale with bracket

$$\langle M_f^N \rangle_t = \frac{1}{N} \int_0^t \int (\partial_x f_s)^2 dL_N(s) ds.$$

The last ingredient of stochastic calculus we want to use is that (cf. Theorem 20.20) for any bounded continuous martingale m_t with bracket $\langle m \rangle_t$, any $\lambda \in \mathbb{R}$,

$$\left\{ \exp(\lambda m_t - \frac{\lambda^2}{2} \langle m \rangle_t), t \in [0,1] \right\}$$

is a martingale. In particular, it has constant expectation. Thus, we deduce that for all $f \in \mathcal{C}_b^{2,1}(\mathbb{R} \times [0,1])$, all $t \in [0,1]$,

$$\mathbb{E}[\exp\{N(M_f^N(t) - \frac{1}{2} \langle M_f^N \rangle_t)\}] = 1. \tag{13.8}$$

- *Weak large deviation upper bound.*
  We equip $\mathcal{C}([0,1], \mathcal{P}(\mathbb{R}))$ with the weak topology on $\mathcal{P}(\mathbb{R})$ and the uniform topology on the time variable. It is then a Polish space. A distance compatible with such a topology is given, for any $\mu, \nu \in \mathcal{C}([0,1], \mathcal{P}(\mathbb{R}))$, by

$$D(\mu, \nu) = \sup_{t \in [0,1]} d(\mu_t, \nu_t)$$

with a distance d on $\mathcal{P}(\mathbb{R})$ compatible with the weak topology such as the Dudley distance (0.1).

Lemma 13.7. *For any $p \in \mathcal{C}([0,1], \mathcal{P}(\mathbb{R}))$,*

$$\limsup_{\delta \to 0} \limsup_{N \to \infty} \frac{1}{N} \log \mathcal{W}^{\otimes N} \left(D(L_N, p) \le \delta \right) \le -S(p).$$

Proof. Let $p \in \mathcal{C}([0,1], \mathcal{P}(\mathbb{R}))$. Observe first that if $p_0 \neq \delta_0$, since $L_N(0) = \delta_0$ almost surely,

$$\limsup_{\delta \to 0} \limsup_{N \to \infty} \frac{1}{N} \log \mathcal{W}^{\otimes N} \left(\sup_{t \in [0,1]} d(L_N(t), p_t) \le \delta \right) = -\infty.$$

Therefore, let us assume that $p_0 = \delta_0$. We set

$$B(p,\delta) = \{\mu \in \mathcal{C}([0,1],\mathcal{P}(\mathbb{R})) : D(\mu,p) \le \delta\}.$$

Let us define, for $f,g \in \mathcal{C}_b^{2,1}(\mathbb{R}\times[0,1])$, $\mu \in \mathcal{C}([0,1],\mathcal{P}(\mathbb{R}))$, $0 \le t \le 1$,

$$T^{0,t}(f,\mu) = \mu_t(f_t) - \mu_0(f_0) - \int_0^t \mu_s(\partial_s f_s)ds - \int_0^t \mu_s\left(\frac{1}{2}\partial_x^2 f_s\right)ds$$

and

$$\langle f,g\rangle_\mu^{0,t} := \int_0^t \mu_s(\partial_x f_s \partial_x g_s)ds.$$

Then, by (13.8), for any $t \le 1$,

$$\mathbb{E}\left[\exp\left\{N\left(T^{0,t}(f,L_N) - \frac{1}{2}\langle f,f\rangle_{L_N}^{0,t}\right)\right\}\right] = 1.$$

Therefore, if we write for short $T(f,\mu) = T^{0,1}(f,\mu) - \frac{1}{2}\langle f,f\rangle_\mu^{0,1}$,

$$\begin{aligned}
&\mathcal{W}^{\otimes N}\left(D(L_N,p)\le\delta\right)\\
&\quad= \mathcal{W}^{\otimes N}\left(1_{D(L_N,p)\le\delta}\frac{e^{NT(f,L_N)}}{e^{NT(f,L_N)}}\right)\\
&\quad\le \exp\{-N\inf_{B(p,\delta)} T(f,.)\}\mathcal{W}^{\otimes N}\left(1_{D(L_N,p)\le\delta}e^{NT(f,L_N)}\right)\\
&\quad\le \exp\{-N\inf_{B(p,\delta)} T(f,.)\}\mathcal{W}^{\otimes N}\left(e^{NT(f,L_N)}\right) \qquad (13.9)\\
&\quad= \exp\{-N\inf_{\mu\in B(p,\delta)} T(f,\mu)\}.
\end{aligned}$$

Since $\mu \to T(f,\mu)$ is continuous when $f \in \mathcal{C}_b^{2,1}(\mathbb{R}\times[0,1])$, we arrive at

$$\limsup_{\delta\to 0}\limsup_{N\to\infty}\frac{1}{N}\log\mathcal{W}^{\otimes N}\left(\sup_{t\in[0,1]} d(L_N(t),p_t)\le\delta\right) \le -T(f,p).$$

We now optimize over f to obtain a weak large deviation upper bound with rate function

$$\begin{aligned}
S(p) &= \sup_{f\in\mathcal{C}_b^{2,1}(\mathbb{R}\times[0,1])}\left(T^{0,1}(f,p) - \frac{1}{2}\langle f,f\rangle_p^{0,1}\right)\\
&= \sup_{f\in\mathcal{C}_b^{2,1}(\mathbb{R}\times[0,1])}\sup_{\lambda\in\mathbb{R}}(\lambda T^{0,1}(f,p) - \frac{\lambda^2}{2}\langle f,f\rangle_p^{0,1})\\
&= \frac{1}{2}\sup_{f\in\mathcal{C}_b^{2,1}(\mathbb{R}\times[0,1])}\frac{T^{0,1}(f,p)^2}{\langle f,f\rangle_p^{0,1}} \qquad (13.10)
\end{aligned}$$

From the last formula, one sees that any p such that $S(p) < \infty$ is such that $f \to T_f(p)$ is a linear map that is continuous with respect to the norm $||f||_p^{0,1} = (\langle f,f\rangle_p^{0,1})^{\frac{1}{2}}$. Hence, Riesz's theorem asserts that there exists a function k verifying (13.6, 13.7).

- *Large deviation lower bound.* The derivation of the large deviation upper bound was thus fairly easy. The lower bound is a bit more sophisticated and relies on the proof of the following points:
(a) The solutions to the heat equations with a smooth drift are unique.
(b) The set described by these solutions is dense in $\mathcal{C}([0,1],\mathcal{P}(\mathbb{R}))$.
(c) The entropy behaves continuously with respect to some approximation by elements of this dense set.
We now describe more precisely these ideas. In the previous section (see (13.9)), we have merely obtained the large deviation upper bound from the observation that for all $\nu \in \mathcal{C}([0,1],\mathcal{P}(\mathbb{R}))$, all $\delta > 0$ and any $f \in \mathcal{C}_b^{2,1}([0,1],\mathbb{R})$,
$$\mathbb{E}\left[1_{L_N \in B(\nu,\delta)} \exp\left(N\left(T^{0,1}(L_N,f) - \frac{1}{2}\langle f,f\rangle_{L_N}^{0,1}\right)\right)\right]$$
$$\leq \mathbb{E}\left[\exp\left(N\left(T^{0,1}(L_N,f) - \frac{1}{2}\langle f,f\rangle_{L_N}^{0,1}\right)\right)\right] = 1.$$
To make sure that this upper bound is sharp, we need to check that for any $\nu \in \mathcal{C}([0,1],\mathcal{P}(\mathbb{R}))$ and $\delta > 0$, this inequality is almost an equality for some function $f = k$, i.e., there exists $k \in \mathcal{C}_b^{2,1}([0,1],\mathbb{R})$,
$$\liminf_{N\to\infty} \frac{1}{N} \log \frac{\mathbb{E}\left[1_{L_N \in B(\nu,\delta)} \exp\left(N(T^{0,1}(L_N,k) - \frac{1}{2}\langle k,k\rangle_{L_N}^{0,1})\right)\right]}{\mathbb{E}\left[\exp\left(N(T^{0,1}(L_N,k) - \frac{1}{2}\langle k,k\rangle_{L_N}^{0,1})\right)\right]} \geq 0.$$
In other words that we can find a k such that the probability that $L_N(.)$ belongs to a small neighborhood of ν under the shifted probability measure
$$\mathbb{P}^{N,k} = \frac{\exp\left(N(T^{0,1}(L_N,k) - \frac{1}{2}\langle k,k\rangle_{L_N}^{0,1})\right)}{\mathbb{E}[\exp\left(N(T^{0,1}(L_N,k) - \frac{1}{2}\langle k,k\rangle_{L_N}^{0,1})\right)]}$$
is not too small. In fact, we shall prove that for good processes ν, we can find k such that this probability goes to one by the following argument.
Take $k \in \mathcal{C}_b^{2,1}(\mathbb{R}\times[0,1])$. Under the shifted probability measure $\mathbb{P}^{N,k}$, it is not hard to see that $L_N(.)$ is exponentially tight (indeed, for $k \in \mathcal{C}_b^{2,1}(\mathbb{R}\times[0,1])$, the density of $\mathbb{P}^{N,k}$ with respect to $\mathbb{P}$ is uniformly bounded by $e^{C(k)N}$ with a finite constant $C(k)$ so that $\mathbb{P}^{N,k}\circ(L_N(.))^{-1}$ is exponentially tight since $\mathbb{P}\circ(L_N(.))^{-1}$ is). As a consequence, $L_N(.)$ is almost surely tight. We let $\mu_.$ be a limit point. Now, by Itô's calculus, for any $f \in \mathcal{C}_b^{2,1}(\mathbb{R}\times[0,1])$, any $0 \leq t \leq 1$,
$$T^{0,t}(L_N,f) = \int_0^t \int \partial_x f_u(x)\partial_x k_u(x) dL_N(u)(x)du + M_t^N(f)$$
with a martingale $(M_t^N(f), t \in [0,1])$ with bracket

$$\left(N^{-2}\int_0^t\int(\partial_x f(x))^2 dL_N(s)(x)ds, t\in[0,1]\right).$$

Since the bracket of $M_t^N(f)$ goes to zero, the martingale $(M_t^N(f), t\in[0,1])$ goes to zero uniformly almost surely. Hence, any limit point $\mu_.$ of $L_N(.)$ under $\mathbb{P}^{N,k}$ must satisfy

$$T^{0,1}(\mu,f)=\int_0^1\int \partial_x f_u(x)\partial_x k_u(x)d\mu_u(x)du \tag{13.11}$$

for any $f\in\mathcal{C}_b^{2,1}(\mathbb{R}\times[0,1])$.
When (μ,k) satisfies (13.11) for all $f\in\mathcal{C}_b^{2,1}(\mathbb{R}\times[0,1])$, we say that k is the field associated with μ.
Therefore, if we can prove that there exists a unique solution $\nu_.$ to (13.11), we see that $L_N(.)$ converges almost surely under $\mathbb{P}^{N,k}$ to this solution. This proves the lower bound at any measure-valued path $\nu_.$ that is the unique solution of (13.11), namely for any $k\in\mathcal{C}_b^{2,1}(\mathbb{R}\times[0,1])$ such that there exists a unique solution ν_k to (13.11),

$$\begin{aligned}
&\liminf_{\delta\to0}\liminf_{N\to\infty}\frac{1}{N}\log\mathcal{W}^{\otimes N}\left(\sup_{t\in[0,1]}d(L_N(t),\nu_k)<\delta\right)\\
&\quad=\liminf_{\delta\to0}\liminf_{N\to\infty}\frac{1}{N}\log\mathbb{P}^{N,k}\left(1_{\sup_{t\in[0,1]}d(L_N(t),\nu_k)<\delta}e^{-NT(k,L_N)}\right)\\
&\quad\ge -T(k,\nu_k)+\liminf_{\delta\to0}\liminf_{N\to\infty}\frac{1}{N}\log\mathbb{P}^{N,k}\left(\sup_{t\in[0,1]}d(L_N(t),\nu_k)<\delta\right)\\
&\quad\ge -S(\nu_k).
\end{aligned}\tag{13.12}$$

where we used in the second line the continuity of $\mu\to T(\mu,k)$ due to our assumption that $k\in\mathcal{C}_b^{2,1}(\mathbb{R}\times[0,1])$ and the fact that

$$\mathbb{P}^{N,k}\left(\sup_{t\in[0,1]}d(L_N(t),\nu_k)<\delta\right)$$

goes to one in the third line. Hence, the question boils down to uniqueness of the weak solutions of the heat equation with a drift. This problem is not too difficult to solve here and one can see that for instance for fields k that are analytic within a neighborhood of the real line, there is at most one solution to this equation. To generalize (13.12) to any $\nu\in\{S<\infty\}$, it is not hard to see that it is enough to find, for any such ν, a sequence ν_{k_n} for which (13.12) holds and such that

$$\lim_{n\to\infty}\nu_{k_n}=\nu,\quad \lim_{n\to\infty}S(\nu_{k_n})=S(\nu). \tag{13.13}$$

Now, observe that S is a convex function so that for any probability measure p_ϵ,

$$S(\mu * p_\epsilon) \leq \int S((.-x)_\# \mu) p_\epsilon(dx) = S(\mu) \tag{13.14}$$

where in the last inequality we neglected the condition at the initial time to say that $S((.-x)_\#\mu) = S(\mu)$ for all x. Hence, since S is also lower semicontinuous, one sees that $S(\mu * p_\epsilon)$ will converge to $S(\mu)$ for any μ with finite entropy S. Performing also a regularization with respect to time and taking care of the initial conditions allows us to construct a sequence ν_n with analytic fields satisfying (13.13). This point is quite technical but still manageable in this context. Since it will be done quite explicitly in the case we are interested in, we shall not detail it here.

13.2 Large deviations for the law of the spectral measure of a non-centered large dimensional matrix-valued Brownian motion

To prove a large deviation principle for the law of the spectral measure of Hermitian Brownian motions, the first natural idea would be, following (12.1), to prove a large deviation principle for the law of the spectral measure of $\tilde{L}^N : t \to N^{-1} \sum_{i=1}^N \delta_{\sqrt{N}^{-1} B^i(t)}$, to use Girsanov's theorem to show that the law we are considering is absolutely continuous with respect to the law of independent Brownian motions, with a density that only depends on $\tilde{L}^N$ and conclude by Laplace's method (cf. Theorem 20.8). However, this approach presents difficulties due to the singularity of the interacting potential, and thus of the density. Here, the techniques developed in [130] will, however, be very efficient because they only rely on smooth functions of the empirical measure since the empirical measures are taken as distributions so that the interacting potential is smoothed by the test functions. (Note, however, that this strategy would not have worked with more singular potentials.) According to (12.1), we can in fact follow the very same approach.

Itô's calculus

With the notations of (13.1) and (13.2), we have by Section 12.2:

Theorem 13.8. *For all $\beta \geq 1$, for any $N \in \mathbb{N}$, any $f \in \mathcal{C}_b^{2,1}(\mathbb{R} \times [0,1])$ and any $s \in [0,1)$, $\left(S^{s,t}(L_{N,\beta}, f) + \frac{\beta/2-1}{2N} \int_s^t \int \partial_x^2 f(y,s) dL_{N,\beta}(s)(y) ds, s \leq t \leq 1\right)$ is a bounded martingale with quadratic variation*

$$\langle S^{s,\cdot}(L_{N,\beta}, f)\rangle_t = \frac{2}{\beta N^2} \langle f, f\rangle^{s,t}_{L_{N,\beta}}.$$

Remark 5. Observe that if the entries were not Brownian motions but diffusions described for instance as solution of a stochastic differential equation

$$dx_t = dB_t + U(x_t)dt,$$

then the evolution of the spectral measure of the matrix would no longer be autonomous. In fact, our strategy is strongly based on the fact that the variations of the spectral measure under small changes of time only depends on the spectral measure, allowing us to construct exponential martingales that are functions of the process of the spectral measure only. It is easy to see that if the entries of the matrix are not Gaussian, the variations of the spectral measures will depend on much more general functions of the entries than those of the spectral measure.

However, this strategy can also be used to study the spectral measure of other Gaussian matrices as emphasized in [55, 101].

From now on, we shall consider the case where $\beta = 2$ and drop the subscript β in $\mathbf{H}^{N,\beta}$, $L_{N,\beta}$, etc. This case is slightly easier to write down since there are no error terms in Itô's formula, but everything extends readily to the cases $\beta \geq 1$. One also needs to notice that

$$S_\beta(\mu) = \sup_{\substack{f\in\mathcal{C}_\beta^{2,1}(\mathbb{R}\times[0,1])\\ 0\le s\le t\le 1}} \left\{ S^{s,t}(\mu,f) - \frac{1}{\beta}\langle f,f\rangle_\mu^{s,t} \right\} = \frac{\beta}{2} S_2(\mu)$$

where the last equality is obtained by changing f into $2^{-1}\beta f$.

Large deviation upper bound

From the previous Itô's formula, one can deduce by following the ideas of [131] (see Section 13.1) a large deviation upper bound for the measure-valued process $L_N(.) \in \mathcal{C}([0,1],\mathcal{P}(\mathbb{R}))$). To this end, we shall make the following assumption on the initial condition D_N:

$$\text{(H)} \qquad C_D := \sup_{N\in\mathbb{N}} L_{D_N}(\log(1+|x|^2)) < \infty,$$

implying that $(L_{D_N}, N \in \mathbb{N})$ *is tight. Moreover,* L_{D_N} *converges weakly, as* N *goes to infinity, to a probability measure* μ_D.

Then, we shall prove, with the notations of (13.1)–(13.3), the following:

Theorem 13.9. *Assume (H). Then:*
(1) S_{μ_D} *is a good rate function on* $\mathcal{C}([0,1],\mathcal{P}(\mathbb{R}))$.
(2) For any closed set F *of* $\mathcal{C}([0,1],\mathcal{P}(\mathbb{R}))$,

$$\limsup_{N\to\infty} \frac{1}{N^2} \log \mathbb{P}\left(L_N(.) \in F\right) \leq -\inf_{\nu\in F} S_{\mu_D}(\nu).$$

We first prove that S_{μ_D} is a good rate function. Then, we show that exponential tightness holds and finally obtain a weak large deviation upper bound, these two arguments yielding (2) (cf. Theorem 20.4).

(a) Let us first observe that $S_{\mu_D}(\nu)$ is also given, when $\nu_0 = \mu_D$, by

$$S_{\mu_D}(\nu) = \frac{1}{2} \sup_{f \in \mathcal{C}_b^{2,1}(\mathbb{R}\times[0,1])} \sup_{0 \le s \le t \le 1} \frac{S^{s,t}(\nu, f)^2}{\langle f, f \rangle_\nu^{s,t}}. \tag{13.15}$$

Consequently, S_{μ_D} is non-negative. Moreover, S_{μ_D} is obviously lower semicontinuous as a supremum of continuous functions.

Hence, we merely need to check that its level sets are contained in relatively compact sets. By (12.11), it is enough to show that, for any $M > 0$,

(1) For any integer m, there is a positive real number L_m^M so that for any $\nu \in \{S_{\mu_D} \le M\}$,

$$\sup_{0 \le s \le 1} \nu_s(|x| \ge L_m^M) \le \frac{1}{m}, \tag{13.16}$$

proving that $\nu_s \in K_{\frac{1}{m}, L_m^M}$ defined in (12.12) for all $s \in [0,1]$.

(2) For any integer m and $f \in \mathcal{C}_b^2(\mathbb{R})$, there exists a positive real number δ_m^M so that for any $\nu \in \{S_{\mu_D} \le M\}$,

$$\sup_{|t-s| \le \delta_m^M} |\nu_t(f) - \nu_s(f)| \le \frac{1}{m}, \tag{13.17}$$

showing that $s \to \nu_s(f)$ belongs to the compact set $C_{\delta^M, ||f||_\infty}$ as defined in (12.13).

To prove (13.16), we consider, for $\delta > 0$, $f_\delta(x) = \log\left(x^2(1+\delta x^2)^{-1} + 1\right) \in \mathcal{C}_b^{2,1}(\mathbb{R}\times[0,1])$. We observe that

$$C := \sup_{0<\delta\le 1} ||\partial_x f_\delta||_\infty + \sup_{0<\delta\le 1} ||\partial_x^2 f_\delta||_\infty$$

is finite and, for $\delta \in (0,1]$,

$$\left| \frac{\partial_x f_\delta(x) - \partial_x f_\delta(y)}{x - y} \right| \le C.$$

Hence, (13.15) implies, by taking $f = f_\delta$ in the supremum, that for any $\delta \in (0,1]$, any $t \in [0,1]$, any $\mu_. \in \{S_{\mu_D} \le M\}$,

$$\mu_t(f_\delta) \le \mu_0(f_\delta) + 2Ct + 2C\sqrt{Mt}.$$

Consequently, we deduce by the monotone convergence theorem and letting δ decrease to zero that for any $\mu_. \in \{S_{\mu_D} \le M\}$,

$$\sup_{t\in[0,1]} \mu_t(\log(x^2+1)) \le \mu_D(\log(x^2+1)) + 2C(1+\sqrt{M}).$$

Chebycheff's inequality and hypothesis (H) thus imply that for any $\mu_. \in \{S_{\mu_D} \le M\}$ and any $K \in \mathbb{R}^+$,

$$\sup_{t\in[0,1]} \mu_t(|x| \ge K) \le \frac{C_D + 2C(1+\sqrt{M})}{\log(K^2+1)}$$

which finishes the proof of (13.16).

The proof of (13.17) again relies on (13.15), which implies that for any $f \in \mathcal{C}_b^2(\mathbb{R})$, any $\mu_. \in \{S_{\mu_D} \le M\}$ and any $0 \le s \le t \le 1$,

$$|\mu_t(f) - \mu_s(f)| \le ||\partial_x^2 f||_\infty |t-s| + 2||\partial_x f||_\infty \sqrt{M}\sqrt{|t-s|}. \tag{13.18}$$

(b) Exponential tightness. By Lemma 12.6, we have:

Lemma 13.10. *For any integer number L, there exists a finite integer number $N_0 \in \mathbb{N}$ and a compact set $\mathcal{K}_L$ in $\mathcal{C}([0,1],\mathcal{P}(\mathbb{R}))$ such that $\forall N \ge N_0$,*

$$\mathbb{P}(L_N \in \mathcal{K}_L^c) \le \exp\{-LN^2\}\,.$$

(c) Weak large deviation upper bound. Following the arguments of Section 13.1, we readily get:

Lemma 13.11. *For every process ν in $\mathcal{C}([0,1],\mathcal{P}(\mathbb{R}))$, if $B_\delta(\nu)$ denotes the open ball with center ν and radius δ for the distance D, then*

$$\lim_{\delta\to 0}\limsup_{N\to\infty} \frac{1}{N^2}\log \mathbb{P}\big(L_N \in B_\delta(\nu)\big) \le -S_{\mu_D}(\nu).$$

Moreover, processes with finite entropy are characterized as follows.

Lemma 13.12. *For any $\mu \in \{S_{\mu_D} < \infty\}$, there exists a measurable function k such that for any $f \in \mathcal{C}_b^2([0,1]\times\mathbb{R},\mathbb{R})$*

$$S^{s,t}(\mu, f) = \int_s^t \int k_u(x)\partial_x f(x,u) d\mu_u(x) du. \tag{13.19}$$

Moreover,

$$S_{\mu_D}(\mu_.) = \frac{1}{2}\int_0^1 \int k_u(x)^2 d\mu_u(x) du.$$

Proof. By (13.15), if $S_{\mu_D}(\mu_.) < \infty$, for any $f \in \mathcal{C}_b^2([0,1]\times\mathbb{R},\mathbb{R})$, all $s \le t$

$$\left|\int_s^t \int k_u(x)\partial_x f(x,u) d\mu_u(x) du\right| \le 2S_{\mu_D}(\mu_.)^{\frac{1}{2}} \left(\int_0^1 \int f(x,u)^2 d\mu_u(x) du\right)^{\frac{1}{2}}.$$

Hence, $f \to \int_0^1 \int k_u(x)\partial_x f(x,u) d\mu_u(x)$ is linear, bounded in the Hilbert space obtained by completing and separating $\mathcal{C}_b^2([0,1]\times\mathbb{R},\mathbb{R})$ for the norm

$$\|f\|_2 = \left(\int_0^1 \int f(x,u)^2 d\mu_u(x) du\right)^{\frac{1}{2}}.$$

Riesz's theorem (cf. [172]) allows us to conclude for $s = 0, t = 1$. We get the general case by taking $f = 0$ outside $[s-\epsilon, t+\epsilon]$ and a smooth interpolation in the time variable in $[s-\epsilon, t+\epsilon]\backslash[s,t]$. $\square$

Large deviation lower bound

We shall prove at the end of this section the following:

Lemma 13.13. *Let*

$$\mathcal{MF}_\infty = \Big\{ h \in \mathcal{C}_b^\infty(\mathbb{R} \times [0,1]); \exists \epsilon > 0, C_. \in L^2([0,1], dt) \text{ so that}$$

$$h_t(x) = C_t + \int e^{i\lambda x} \hat{h}_t(\lambda) d\lambda \text{ with } \int_0^1 \max_{\lambda \in \mathbb{R}} (e^{2\epsilon|\lambda|} |\hat{h}_t(\lambda)|^2) dt < \infty. \Big\}$$

For any field k in $\mathcal{MF}_\infty$, there exists a unique solution ν_k to

$$S^{s,t}(f, \nu) = \langle f, k \rangle_\nu^{s,t} \tag{13.20}$$

for any $f \in \mathcal{C}_b^{2,1}(\mathbb{R} \times [0,1])$. We set $\mathcal{MC}([0,1], \mathcal{P}(\mathbb{R}))$ to be the subset of $\mathcal{C}([0,1], \mathcal{P}(\mathbb{R}))$ consisting of such solutions.

Note that h belongs to $\mathcal{MF}_\infty$ implies that it can be extended analytically to $\{z : |\Im(z)| < \epsilon\}$ for almost all $t \in [0,1]$.

As a consequence of Lemma 13.13, if we take $\nu \in \mathcal{MC}([0,1], \mathcal{P}(\mathbb{R}))$ associated with a field k, and if we define

$$\mathbb{P}^{N,k} = \exp\left\{ N^2 \left(S^{0,1}(L_N(.), k) - \frac{1}{2} \langle k, k \rangle_{L_N(.)}^{0,1} \right) \right\} \mathbb{P}^N_{T, \lambda_N(0)},$$

the limit points of $L_N(.)$ under $\mathbb{P}^{N,k}$ coincide with ν (see the classical analog (13.11)). Thus, for any open subset $O \in \mathcal{C}([0,1], \mathcal{P}(\mathbb{R}))$, any $\nu \in O \cap \mathcal{MC}([0,1], \mathcal{P}(\mathbb{R}))$,

$$\begin{aligned}
\mathbb{P}(L_N(.) \in O) &\geq \mathbb{P}(d(L_N(.), \nu) < \delta) \\
&= \mathbb{P}^{N,k}\left(1_{d(L_N(.), \nu) < \delta} e^{-N^2 (S^{0,1}(L_N, k) - \frac{1}{2} \langle k, k \rangle_{L_N}^{0,1})} \right) \\
&\geq e^{-N^2 (S^{0,1}(\nu, k) - \frac{1}{2} \langle k, k \rangle_\nu^{0,1}) - g(\delta) N^2} \mathbb{P}^{N,k}(d(L_N(.), \nu) < \delta)
\end{aligned}$$

with a function g vanishing at the origin. Hence, for any $\nu \in O \cap \mathcal{MC}([0,1], \mathcal{P}(\mathbb{R}))$

$$\liminf_{N \to \infty} \frac{1}{N^2} \log \mathbb{P}(L_N(.) \in O) \geq -(S^{0,1}(\nu, k) - \frac{1}{2} \langle k, k \rangle_\nu^{0,1}) = -S_{\mu_D}(\nu)$$

and therefore

$$\liminf_{N \to \infty} \frac{1}{N^2} \log \mathbb{P}(L_N(.) \in O) \geq - \inf_{O \cap \mathcal{MC}([0,1], \mathcal{P}(\mathbb{R}))} S_{\mu_D}. \tag{13.21}$$

To complete the lower bound, it is therefore sufficient to prove that for any $\nu \in \mathcal{C}([0,1], \mathcal{P}(\mathbb{R}))$ with uniformly bounded fourth moment, there exists a sequence $\nu^n \in \mathcal{MC}([0,1], \mathcal{P}(\mathbb{R}))$ such that

$$\lim_{n\to\infty} \nu^n = \nu \text{ and } \lim_{n\to\infty} S_{\mu_D}(\nu^n) = S_{\mu_D}(\nu). \tag{13.22}$$

The rate function S_{μ_D} is not convex a priori since it is the supremum of **quadratic** functions of the measure-valued path ν so that there is no reason why it should be reduced by standard convolution as in the classical setting (cf. Section 13.1). Thus, it is now unclear how we can construct the sequence ν^n satisfying (13.22). Further, we begin with a degenerate rate function that is infinite unless $\nu_0 = \mu_D$.

To overcome the lack of convexity, we shall remember the origin of the problem; in fact, we have been considering the spectral measure of matrices and should not forget the special features of operators due to the matrices structure. By definition, the differential equation satisfied by a Hermitian Brownian motion should be invariant if we translate the entries, that is, translate the Hermitian Brownian motion by a self-adjoint matrix. The natural limiting framework of large random matrices is free probability, and the limiting spectral measure of the sum of a Hermitian Brownian motion and a deterministic self-adjoint matrix converges to the free convolution of their respective limiting spectral measure. Intuitively, we shall therefore expect (and in fact we will show in the specific case of Cauchy laws) that the rate function $S^{0,1}$ decreases by free convolution, generalizing the fact that standard convolution was decreasing the Brownian motion rate function (cf. (13.14)). However, because free convolution by a Cauchy law is equal to the standard convolution by a Cauchy law, we shall regularize our laws by convolution by Cauchy laws. We now prove the large deviation lower bound of Theorem 13.2.

•*Regularization by Cauchy laws.* We prove below that convolution by Cauchy laws reduces the entropy, a point analogous to (13.14). This result is a special case of Theorem 4.1 in [56] where it is shown that any free convolution reduces the entropy. This generalization was in fact used with the free convolution with respect to the semi-circular in [110] to prove Theorem 13.3.

Lemma 13.14. *Let $\mu_.$ satisfy* (13.19) *for some measurable function $k \in L^2(d\mu_t(x)dt)$. Let p_ϵ be the Cauchy law*

$$dp_\epsilon(c) = \frac{\epsilon}{\pi(\epsilon^2 + c^2)} dc.$$

Then, $(\mu_t * p_\epsilon)_{t\in[0,1]}$ satisfies (13.19) *with field k^ϵ given by*

$$k_t^\epsilon(c) = \frac{\int \frac{k_t(x)}{\epsilon^2+(x-c)^2} d\mu_t(x)}{\int \frac{1}{\epsilon^2+(x-c)^2} d\mu_t(x)}.$$

Moreover

$$S_{\mu_D}(\mu_.) \geq S_{\mu_D * p_\epsilon}(\mu_. * p_\epsilon).$$

Proof. We verify (13.19) for Stieltjes transform, i.e., functions of the form $f(x) = (z-x)^{-1}$ and any $[s,t]$. Note that if we do that, we can then use the density of these functions in $\mathcal{C}_b^2(\mathbb{R},\mathbb{R})$ (again based on the Weierstrass theorem) and then play on the parameters s,t to get the equation for all $f \in \mathcal{C}_b^2([0,1]\times\mathbb{R},\mathbb{R})$. Now, for $f=(z-x)^{-1}$ we find that

$$
\begin{aligned}
p_\epsilon * \mu_t((z-x)^{-1}) \\
&= \int \mu_t((z-c-x)^{-1})dp_\epsilon(c) \\
&= p_\epsilon * \mu_s((z-x)^{-1}) \\
&\quad + \int\int_s^t \mu_u((z-c-x)^{-1})\mu_u((z-c-x)^{-2})du dp_\epsilon(c) \\
&\quad + \int\int_s^t \mu_u((z-c-x)^{-2}k_u(x))du dp_\epsilon(c)
\end{aligned}
$$

where we have used Fubini and equation (13.19) for $\mu_.$. Observe that if $z \in \mathbb{C}^+ = \{z : \Im(z) > 0\}$, as $c \to (z-c-x)^{-1}(z-c-x')^{-2}$ is analytic on $\mathbb{C}^-$, the residue theorem implies that, since $dp_\epsilon(c)/dc$ has a unique pole at $-i\epsilon$ in $\mathbb{C}^-$,

$$
\begin{aligned}
\int (z-c-x)^{-1}(z-c-x')^{-2}dp_\epsilon(c) \\
&= (z+i\epsilon-x)^{-1}(z+i\epsilon-x')^{-2} \\
&= \int (z-c-x)^{-1}dp_\epsilon(c)\int (z-c-x')^{-2}dp_\epsilon(c)
\end{aligned}
$$

and therefore we deduce

$$
\begin{aligned}
p_\epsilon * \mu_t((z-x)^{-1}) &= p_\epsilon * \mu_s((z-x)^{-1}) \\
&\quad + \int_s^t p_\epsilon * \mu_u((z-x)^{-1})p_\epsilon * \mu_u((z-x)^{-2})du \\
&\quad + \int_s^t p_\epsilon * \mu_u((z-x)^{-2}k_u^\epsilon(x))du
\end{aligned}
$$

where we finally used

$$\begin{aligned}
\int \mu_u((z-c-x)^{-2}k_u(x))dp_\epsilon(c) \\
&= \int (z-c)^{-2}\left(\int \frac{\epsilon k_u(x)}{\pi((x-c)^2+\epsilon^2)}d\mu_u(x)\right)dc \\
&= \int (z-c)^{-2}k_u^\epsilon(c)dp_\epsilon * \mu_u(c).
\end{aligned}$$

This is (13.20) with the dense set of functions $f = (z-x)^{-1}$. For the last point notice that in fact

$$k_u^\epsilon(x) = E[k_u(x)|x+C_\epsilon]$$

when C_ϵ is a Cauchy variable independent of x and x has law μ_u. Hence,

$$\begin{aligned}
S_{\mu_D * p_\epsilon}(\mu_. * p_\epsilon) &= \frac{1}{2}\int_0^1 \int E[(E[k_u(x_u)|x_u+C_\epsilon])^2]du \\
&\le \frac{1}{2}\int_0^1 \int E[k_u(x_u)^2]du = S_{\mu_D}(\mu_.).
\end{aligned}$$

□

Thus, we find that convolution by Cauchy laws $(p_\epsilon)_{\epsilon>0}$ decreases the entropy. Since the entropy is lower semicontinuous (as a supremum of continuous functions), we deduce that

$$\lim_{\epsilon\to 0} S_{\mu_D * p_\epsilon}(\mu_. * p_\epsilon) = S_{\mu_D}(\mu_.). \tag{13.23}$$

Note that $x \to k_u^\epsilon(x)$ is analytic in the strip $|\Im(z)| \le \epsilon$ so that there is some good chance that $p_\epsilon * \mu_.$ will be in $\mathcal{MC}([0,1],\mathcal{P}(\mathbb{R}))$. We shall see this point below for $\mu_. \in \mathcal{A}$ with

$$\mathcal{A} = \{\mu_. : [0,1] \to \mathcal{P}(\mathbb{R}) : \sup_{t\in[0,1]} \mu_t(x^4) < \infty\}.$$

Namely, we prove:

Lemma 13.15. *For $\mu_. \in \mathcal{A} \cap \{S_{\mu_D} < \infty\}$, for all $\epsilon > 0$, $p_\epsilon * \mu_.$ belongs to $\mathcal{MC}([0,1],\mathcal{P}(\mathbb{R}))$.*

Proof. The strategy is to show that $k_t^\epsilon \in L^2(dx) \cap L^1(dx)$ in order to use Plancherel representation. Since $k_t^\epsilon(x)$ goes to $\int k_t(y)d\mu_t(y)$ as x goes to infinity, we need to substract this quantity to make sure this can happen. For further use, we need to consider $k_t^\epsilon(x+i\delta)$ for $\delta < \epsilon$, say $\delta = \epsilon/2$. We then write

$$k_t^\epsilon(x+i\delta) = \frac{\int \frac{k_t(y)}{(x+i\delta-y)^2+\epsilon^2}d\mu_t(y)}{\int \frac{1}{(x+i\delta-y)^2+\epsilon^2}d\mu_t(y)}$$

Note that since $\mu_t(x^2)$ is uniformly bounded by say C, $\mu_t([-M,M]^c) \le CM^{-2}$ for all t by Chebyshev inequality. Thus, for $x \in [-M, M]$, with M sufficiently large

$$\left| \int \frac{1}{(x+i\delta-y)^2+\epsilon^2} d\mu_t(y) \right| \ge \frac{1}{(2M+\delta)^2+\epsilon^2}$$

and therefore there exists a finite constant $C = C(\epsilon, M)$ so that

$$|k_t^\epsilon(x+i\delta)| \le C \left(\int k_t(y)^2 d\mu_t(y) \right)^{\frac{1}{2}} .$$

We choose below M large. For $x \in [-M,M]^c$, we have that

$$\begin{aligned} &\int_{[-2|x|,2|x|]} \frac{((x+i\delta)^2+\epsilon^2)k_t(y)}{(x+i\delta-y)^2+\epsilon^2} d\mu_t(y) \\ &\qquad = \int_{[-2|x|,2|x|]} k_t(y) \Bigg(1 + 2\frac{(x+i\delta)y}{(x+i\delta)^2+\epsilon^2} \\ &\qquad\qquad + O\left(\frac{y^2}{(x+i\delta)^2+\epsilon^2} \right) \Bigg) d\mu_t(y) \end{aligned}$$

$$\begin{aligned} &\left| \int_{[-2|x|,2|x|]^c} \frac{((x+i\delta)^2+\epsilon^2)k_t(y)}{(x+i\delta-y)^2+\epsilon^2} d\mu_t(y) \right| \\ &\quad \le \frac{1}{\epsilon^2-\delta^2} \int_{[-2|x|,2|x|]^c} |k_t(y)| d\mu_t(y) \\ &\quad \le \frac{1}{\epsilon^2-\delta^2} \left(\int k_t(y)^2 d\mu_t(y) \right)^{\frac{1}{2}} \left(\int \frac{y^4}{x^4} d\mu_t(y) \right)^{\frac{1}{2}} . \end{aligned}$$

Letting $C_t^1(k) = \int k_t(y) d\mu_t(y)$, $C_t^2(k) = \int y k_t(y) d\mu_t(y)$ and $C_t^3(k) = \left(\int k_t(y)^2 d\mu_t(y) \right)^{\frac{1}{2}}$, we get

$$\begin{aligned} &\int \frac{k_t(y)}{(x+i\delta-y)^2+\epsilon^2} d\mu_t(y) \\ &\quad = C_t^1(k) + (2C_t^2(k))(x+i\delta)((x+i\delta)^2+\epsilon^2)^{-1}) + h_t^\epsilon(x+i\delta) \end{aligned}$$

with some function h_t^ϵ so that

$$|h_t^\epsilon(x+i\delta)| \le c C_t^3(k) \frac{1}{|x|^2+1}$$

with a constant c that only depends on $\sup_t \mu_t(y^4)$. Doing the same for the denominator, we conclude that

$$k_t^\epsilon(x+i\delta) = C_t^1 + 2[C_t^2(k) - C_t^1(k)C_t^2(1)](x+i\delta)((x+i\delta)^2+\epsilon^2)^{-1} + \bar{k}_t^\epsilon(x+i\delta) \tag{13.24}$$

with some bounded function $\bar{k}_t^\epsilon(x+i\delta)$ such that

$$|\bar{k}_t^\epsilon(x+i\delta)| \le cC_t^3(k)\frac{x^2}{(x^2-\delta^2+\epsilon^2)^2}.$$

Applying this result to $\delta = 0$, we see that $\bar{k}_t^\epsilon(x) \in L^1(dx) \cap L^2(dx)$ extends to $|\Im(x)| \le \delta$ analytically while staying in L^1. This implies in particular that for $\lambda > 0$

$$\begin{aligned}|\hat{k}_t^\epsilon(\lambda)| &= \pi^{-1}\left|\int e^{i\lambda x}\bar{k}_t^\epsilon(x)dx\right| = \pi^{-1}\left|\int e^{i\lambda(x+i\delta)}\bar{k}_t^\epsilon(x+i\delta)dx\right| \\ &\le e^{-\delta\lambda}\int |\bar{k}_t^\epsilon(x+i\delta)|dx \le c'C_t^3(k)e^{-\delta\lambda}\end{aligned}$$

and the same bound for $\lambda < 0$. Hence, by the Plancherel formula

$$\bar{k}_t^\epsilon(x) = \int e^{i\lambda x}\hat{k}_t^\epsilon(\lambda)d\lambda,$$

with $\hat{k}_t^\epsilon(\lambda)$ satisfying the required property that

$$|\hat{k}_t^\epsilon(\lambda)| \le c'C_t^3(k)e^{-\delta|\lambda|}$$

with $C_t^3(k)$ in $L^2([0,1],dt)$ since μ has finite entropy. Moreover, Note that

$$\frac{x}{x^2+\epsilon^2} = \Re(x+i\epsilon)^{-1}$$

can be written for $\epsilon > 0$ as

$$\frac{x}{x^2+\epsilon^2} = \Re\left(-i\int_0^\infty e^{i\xi(x+i\epsilon)}d\xi\right) = \int e^{i\xi x}\left(\sin(\xi)e^{-\epsilon|\xi|}\right)d\xi. \tag{13.25}$$

Hence, we have written by (13.24)

$$k_t^\epsilon(x+i\delta) = C_t^1 + \int e^{i\lambda x}\hat{k}_t^{\epsilon,2}(\lambda)d\lambda \tag{13.26}$$

with

$$|\hat{k}_t^{\epsilon,2}(\lambda)e^{\delta|\lambda|}| \le 2[C_t^2(k) + C_t^1(k)C_t^2(1)] + c'C_t^3(k)$$

for all $\lambda \in \mathbb{R}$. The proof of the lemma is thus complete since $C_t^2(k)$, $C_t^2(1)$, $C_t^3(k)$ are in $L^2([0,1],dt)$, whereas $C_t^2(1) = \int y d\mu_t(x)$ is uniformly bounded as $\int y^2 d\mu_t(y)$ is. $\square$

$\bullet$*Large deviations lower bound for processes regularized by Cauchy distribution.* Everything looks nice except that we modified the initial condition from μ_D into $\mu_D * p_\epsilon$, so that in fact $S_{\mu_D}(\nu^{\epsilon,\Delta}) = +\infty$! and moreover, the empirical measure-valued process cannot deviate toward processes of the form $\nu^{\epsilon,\Delta}$ even after some time because these processes do not have a finite second moment (it can indeed be checked that if $\mu_D(x^2) < \infty$, $S_{\mu_D}(\mu_.) < \infty$ implies that $\sup_t \mu_t(x^2) < \infty$). To overcome this problem, we first note that this result will still give us a large deviation lower bound if we change the initial data of our matrices. Namely, let, for $\epsilon > 0$, C_ϵ^N be an $N \times N$ diagonal matrix with spectral measure converging to the Cauchy law p_ϵ and consider the matrix-valued process

$$\mathbf{X}_t^{N,\epsilon} = \mathbf{U}_N C_\epsilon^N \mathbf{U}_N^* + D_N + \mathbf{H}^N(t)$$

with $\mathbf{U}_N$ a $N \times N$ unitary measure following the Haar measure m_2^N on $U(N)$. Then, it is well known [30] that the spectral distribution of $\mathbf{U}_N C_\epsilon^N \mathbf{U}_N^* + D_N$ converges to $= p_\epsilon * \mu_D$. We choose C_ϵ^N satisfying (H).

Hence, we can proceed as before to obtain the following large deviation estimates on the law of the spectral measure $L_{N,\epsilon}(t)t = L_{X_t^{N,\epsilon}}$. We define

$$\mathcal{A}^p = \{\mu \in \mathcal{C}([0,1], \mathcal{P}(\mathbb{R})); \mu_t \in \mathcal{A} \quad \forall t \in [0,1]\}$$

Corollary 13.16. *Assume (H). For any $\epsilon > 0$, for any closed subset F of $\mathcal{C}([0,1], \mathcal{P}(\mathbb{R}))$,*

$$\limsup_{N\to\infty} \frac{1}{N^2} \log \mathbb{P}\left(L_{N,\epsilon}(.) \in F\right) \le -\inf\{S_{P_\epsilon * \mu_D}(\nu), \nu \in F\}.$$

Further, for any open set O of $\mathcal{C}([0,1], \mathcal{P}(\mathbb{R}))$,

$$\begin{aligned}\liminf_{N\to\infty} \frac{1}{N^2} \log \mathbb{P}\left(L_{N,\epsilon}(.) \in O\right) \\ \ge -\inf\{S_{P_\epsilon * \mu_D}(\nu), \nu \in O, \nu = P_\epsilon * \mu, \mu \in \mathcal{A}^p \cap \{S_{\mu_D} < \infty\}\}.\end{aligned}$$

The only point is to prove the lower bound (see Theorem 13.11 for the upper bound). In [109] we proceed by an extra regularization in time. We will bypass this argument here.

1. *Time discretization of $k^\epsilon \in \mathcal{MF}_\infty$.* We note that since $\int_0^1 \int k_t^\epsilon(x)^2 dp_\epsilon * \mu_t(x)dt$ is finite, we can approximate $k_t^\epsilon(x)$ by

$$k_t^{\epsilon,p}(x) = k_{t_n}^\epsilon(x) + \frac{t - t_n}{t_{n+1} - t_n}(k_{t_{n+1}}^\epsilon(x) - k_{t_n}^\epsilon(x)) \quad \text{for } t \in [t_n, t_{n+1}[$$

for some time discretization $0 = t_0 < t_1 < \cdots t_p = 1$ in such a way that

$$\lim_{p\to\infty} \int_0^1 \int \left(k_t^\epsilon(x) - k_t^{\epsilon,p}(x)\right)^2 dp_\epsilon * \mu_t(x)dt = 0.$$

Indeed, this is clearly true if $t \to k_t^\epsilon(x)$ is continuous (since $k_t^\epsilon(x)$ is uniformly bounded, see Lemma 13.15 so that bounded convergence theorem applies) and then generalizes to all uniformly bounded field by density of continuous functions in L^2.

2. *Change of measure and convergence under the shifted probability measure.* We let $\mathbb{P}^{N,k,\epsilon}$ be the law with density

$$\Lambda^{N,k} = \exp\left\{\sqrt{N}\sum_{i=1}^N \int_0^T k_t^\epsilon(\lambda_N^i(t))dW_t^i - \frac{N}{2}\sum_{i=1}^N \int_0^T k_t^\epsilon(\lambda_N^i(t))^2 dt\right\}$$

with respect to the law $P^N_{T,\lambda_N^\epsilon(0)}$ of the eigenvalues of $X^{N,\epsilon}(t), t \in [0,1]$ (W being the d dimensional Brownian motion appearing in (12.1)) . By Girsanov 's theorem 20.21, since $k_t^\epsilon(x)$ is uniformly bounded, $\mathbb{P}^{N,k,\epsilon}$ is the law of

$$d\lambda_N^i(t) = \frac{1}{N}\sum_{j=1}^N \frac{1}{\lambda_N^i(t) - \lambda_N^j(t)}dt + k_t^\epsilon(\lambda_N^i(t))dt + \frac{1}{\sqrt{N}}d\tilde{W}_t^i$$

with an N-dimensional Brownian motion $\tilde{W}$ under $\mathbb{P}^{N,k,\epsilon}$. Applying Itô's calculus and exactly the same argument than in Lemma 12.5 (we leave this as an exercise, and note that it is important that k_t^ϵ is uniformly bounded and continuous for all $t \in [0,1]$), we find that the limit point of $L_N(.)$ under $\mathbb{P}^{N,k,\epsilon}$ are solution of (13.20) with $k = k^\epsilon$ and $\mu_0 = \mu_D * p_\epsilon$. Hence, by Lemma 13.13,

$$\lim_{N\to\infty} \mathbb{P}^{N,k,\epsilon}\left(L_N(.) \in B(p_\epsilon * \mu_., \delta)\right) = 1 \tag{13.27}$$

for all $\delta > 0$.

3. *Approximating the shifted law.* Because we did not regularize k^ϵ in time, $\frac{1}{N^2}\log \Lambda^{N,k}$ is not necessarily a continuous function of $L_N(.)$; indeed, we cannot use Itô's calculus to transform $\sum_{i=1}^N \int_0^T k_t^\epsilon(\lambda_N^i(t))dW_t^i$ into an integral over dt since $t \to k_t^\epsilon$ may not be differentiable. To circumvent this problem, we consider the law $\mathbb{P}^{N,k^p,\epsilon}$ corresponding to the discretized field $k^{p,\epsilon}$. Then, since $k^{\epsilon,p}$ is continuously differentiable in time, if Λ^{N,k^p} is the density of $\mathbb{P}^{N,k^p,\epsilon}$ with respect to $\mathbb{P}^N_{T,\lambda_N(0)}$, we have

$$\begin{aligned}\frac{1}{N^2}\log \Lambda^{N,k^{p,\epsilon}} &= N^2\left(S^{0,1}(L_N(.), K^{p,\epsilon}) - \frac{1}{2}\langle K^{p,\epsilon}, K^{p,\epsilon}\rangle_{L_N}^{0,1}\right)\\ &= N^2 \bar{S}^{0,1}(L_N(.), K^{p,\epsilon})\end{aligned}$$

with $K_t^{p,\epsilon} = \int_{-\infty}^x k_t^{p,\epsilon}(y)dy$. Hence, $\frac{1}{N^2}\log \Lambda^{N,k^{p,\epsilon}}$ is a smooth function of $L_N(.)$ since $K_t^{p,\epsilon}$ and its time derivative are $\mathcal{C}^\infty$ with uniformly bounded derivatives. Therefore, for a fixed δ, we find a $\kappa(\epsilon, p)$ vanishing as δ goes to zero for any $p \in \mathbb{N}$, such that

$$\mathbb{P}^{N,\epsilon}\left(B(p_\epsilon * \mu_., \delta)\right) \geq e^{-N^2(\bar{S}^{0,1}(p_\epsilon*\mu_., K^p)+\kappa(\epsilon,p))}\mathbb{P}^{N,k^p,\epsilon}\left(B(p_\epsilon * \mu_., \delta)\right) \tag{13.28}$$

To replace $\mathbb{P}^{N,k^p,\epsilon}$ by $\mathbb{P}^{N,k,\epsilon}$ and conclude, note that by Girsanov's theorem 20.21, if $\Delta k_t^{\epsilon,p} = (k_t^{\epsilon,p} - k_t^\epsilon)$,

$$\Lambda_N := \log \frac{d\mathbb{P}^{N,k^p,\epsilon}}{d\mathbb{P}^{N,k,\epsilon}} = M_1^N - \frac{1}{2}\langle M^N\rangle_1$$

with the martingale $(M_s^N)_{0\leq s\leq 1}$ given by

$$M_s^N = \sqrt{N}\sum_{i=1}^N \int_0^s \Delta k_t^{\epsilon,p}(\lambda_N^i(t))dW_t^i$$

for an N-dimensional Brownian motion $(W^i)_{1\leq i\leq N}$ under $\mathbb{P}^{N,k,\epsilon}$. We have

$$\begin{aligned}\mathbb{P}^{N,k^p,\epsilon}\left(B(p_\epsilon * \mu_., \delta)\right) &\geq e^{-\kappa N^2}\mathbb{P}^{N,k,\epsilon}\left(\{\Lambda_N \geq e^{-\kappa N^2}\} \cap B(p_\epsilon * \mu_., \delta)\right)\\ &\geq e^{-\kappa N^2}\Big(\mathbb{P}^{N,k,\epsilon}(B(p_\epsilon * \mu_., \delta))\\ &\quad - \mathbb{P}^{N,k,\epsilon}(\{\Lambda_N \leq e^{-\kappa N^2}\} \cap B(p_\epsilon * \mu_., \delta))\Big).\end{aligned} \tag{13.29}$$

Since $x \to k_t^\epsilon(x)$ is uniformly Lipschitz, we find that on $B(p_\epsilon * \mu_., \delta)$,

$$\begin{aligned}\frac{1}{N^2}\langle M^N\rangle_1 &= \frac{1}{N}\sum_{i=1}^N \int_0^1 (\Delta k_t^{\epsilon,p}(\lambda_N^i(t)))^2 dt\\ &= \int_0^1 \int (\Delta k_t^{\epsilon,p}(x))^2 dp_\epsilon * \mu_t(x)dt + O(\delta) = o(p,\delta)\end{aligned}$$

by our choice of $k^{\epsilon,p}$, and with $o(p,\delta) = o(p) + o(\delta)$ going to zero as p goes to infinity and δ to zero. Thus, we obtain that

$$\begin{aligned}&\mathbb{P}^{N,k,\epsilon}\left(\{\Lambda_N \leq e^{-\kappa N^2}\} \cap B(p_\epsilon * \mu_., \delta)\right)\\ &\leq \mathbb{P}^{N,k,\epsilon}\left(\{M_1^N \leq -(\kappa - o(\delta,p))N^2\} \cap B(p_\epsilon * \mu_., \delta)\right)\\ &\leq \mathbb{P}^{N,k,\epsilon}\left(e^{-\lambda M_1^N - \frac{\lambda^2}{2}\langle M^N\rangle_1} \geq e^{[\lambda(\kappa - o(\delta,p)) - \frac{\lambda^2}{2}o(\delta,p)]N^2}\right)\\ &\leq e^{-[\lambda(\kappa - o(\delta,p)) - \frac{\lambda^2}{2}o(\delta,p)]N^2}\end{aligned}$$

for any $\lambda > 0$. Hence, taking $\lambda = (\kappa - o(\delta,p))/o(\delta,p)$ we conclude that for any $\kappa > 0$,

$$\limsup_{\substack{\delta\to 0\\ p\to\infty}} \limsup_{N\to\infty} \frac{1}{N^2}\log \mathbb{P}^{N,k,\epsilon}\left(\{\Lambda_N \leq e^{-\kappa N^2}\} \cap B(p_\epsilon * \mu_., \delta)\right) = -\infty. \tag{13.30}$$

By (13.28), (13.29) and (13.27) we thus conclude that for $\kappa > 0$, δ small enough and p large enough

$$\liminf_{N\to\infty} \frac{1}{N^2} \log \mathbb{P}^{N,\epsilon} \left(B(p_\epsilon * \mu_., \delta) \right) \geq -\bar{S}^{0,1}(p_\epsilon * \mu_., K^p) - \kappa + o(\epsilon, p, \delta) \tag{13.31}$$

with $o(\epsilon, p, \delta)$ going to zero as p goes to infinity and δ to zero. Finally, our choice of $k^{p,\epsilon}$ shows that

$$\lim_{p\to\infty} \bar{S}^{0,1}(p_\epsilon * \mu_., K^p) = \bar{S}^{0,1}(p_\epsilon * \mu_., K)$$

completing the proof of the lower bound by letting δ going to zero, p going to infinity and then κ to zero.

• *Large deviation lower bound for processes in $\mathcal{A}^p$.* To deduce our result for the case $\epsilon = 0$, we proceed by exponential approximation. In fact, we have the following lemma:

Lemma 13.17. *Consider, for $L \in \mathbb{R}^+$, the compact set K_L of $\mathcal{P}(\mathbb{R})$ given by*

$$K_L = \{\mu \in \mathcal{P}(\mathbb{R}); \mu(\log(x^2+1)) \leq L\}.$$

Then, on $\mathcal{K}_\epsilon^N(K_L) := \bigcap_{t\in[0,1]} \left\{ \{L_{N,\epsilon_t} \in K_L\} \cap \{L_N(t) \in K_L\} \right\}$, with d the Duddley distance (0.1),

$$d(L_{N,\epsilon}(.), L_N(.)) \leq f(N, \epsilon)$$

where

$$\limsup_{\epsilon\to 0} \limsup_{N\to\infty} f(N, \epsilon) = 0.$$

Proof. *Step 1: compactly supported measure approximation.* Write $C_\epsilon^N = \sqrt{\epsilon} C_N$ with a matrix C_N whose spectral measure converges to a standard Cauchy law. Denote by $(c_i)_{1\leq i\leq N}$ the eigenvalues of C_N. For $M > 0$, we set

$$B_M := \{i : |c_i| > M\} := \left\{ j_1, \cdots, j_{|B_M|} \right\}.$$

Define

$$C_{N,M}(i,i) = \begin{cases} c_i & \text{if } i \notin B_M \\ 0 & \text{otherwise.} \end{cases}$$

Let

$$X_N^{\epsilon,M}(t) = H_N(t) + D_N + \sqrt{\epsilon} U_N C_{N,M} U_N^*$$

and denote by $L_{N,\epsilon,M}(t)$ its spectral measure. Then,

$$D(L_{N,\epsilon,M}(.), L_N(.)) \leq \epsilon M.$$

In fact, for any continuously differentiable function f, any $t \in [0, 1]$,

$$
\begin{aligned}
&|\hat{\mu}_t^{N,\epsilon,M}(f) - L_{N\,t}(f)| \\
&= \epsilon \left| \int_0^1 \frac{1}{N} \mathrm{Tr}\left(f'(X_N(t) + \alpha\sqrt{\epsilon} U_N C_{N,M} U_N^*) U_N C_{N,M} U_N \right) d\alpha \right| \\
&\le \frac{\epsilon}{N} \sum_{i \in B_M} |c_i| \int_0^1 \left| \langle e_i, f'(X_N(t) + \alpha\sqrt{\epsilon} U_N C_{N,M} U_N^*) e_i \rangle \right| d\alpha \\
&\le \epsilon M \int_0^1 \left(\frac{1}{N} \mathrm{Tr}(f'(X_N(t) + \alpha\epsilon C_{N,M})^2) \right)^{\frac{1}{2}} d\alpha.
\end{aligned}
$$

Extending this inequality to Lipschitz functions, we deduce that

$$
|\int f dL_{N,\epsilon,M}(t) - \int f dL_N(t)| \le ||f||_{\mathcal{L}} \epsilon M
$$

which gives the desired estimate on $D(\hat{\mu}_{\cdot}^{N,\epsilon,M}, L_{N\,\cdot})$.

Step 2: small rank perturbation approximation. On the compact set K_L, the Duddley distance is equivalent to the distance

$$
d_1(\mu,\nu) = \sup_{||f||_{\mathcal{L}} \le 1, f\uparrow} \left| \int f d\nu - \int f d\mu \right|.
$$

Write

$$
X_N^\epsilon(t) = X_N^{\epsilon,M}(t) + \epsilon \sum_{i=1}^{|B_M|} c_{j_i} e_{j_i} e_{j_i}^T.
$$

Following Lidskii's theorem (see (1.18)), we find that

$$
d_1(L_{N,\epsilon,M}(t), L_{N,\epsilon}(t)) \le \frac{4|B_M|}{N}. \tag{13.32}
$$

But Chebycheff's inequality yields

$$
\begin{aligned}
\frac{|B_M|}{N} &= \int 1_{|x| \ge M} dL_{C_N}(x) \\
&\le \frac{1}{(\log(M^2+1))^2} \sup_N \int (\log(x^2+1))^2 dL_{C_N}(x)
\end{aligned}
$$

giving finally, according to condition (13.32), a finite constant C such that

$$
d_1(L_{N,\epsilon,M}(t), L_{N,\epsilon}(t)) \le \frac{CL}{(\log(M^2+1))^2}.
$$

Now, since d_1 and d are equivalent on K_L, the proof of the lemma is complete. □

We then can prove the following:

Theorem 13.18. *Assume that L_{D_N} converges to μ_D while*

$$\sup_{N\in\mathbb{N}} L_{D_N}(x^4) < \infty.$$

Then, for any $\mu_. \in \mathcal{A}$

$$\lim_{\delta\to 0}\liminf_{N\to\infty}\frac{1}{N^2}\log\mathbb{P}\left(D(L_N(.),\mu_.)\le\delta\right)\ge -S_{\mu_D}(\mu_.).$$

so that for any open subset $O\in\mathcal{C}([0,1],\mathcal{P}(\mathcal{P}(\mathbb{R})))$,

$$\liminf_{N\to\infty}\frac{1}{N^2}\log\mathbb{P}\left(L_N(.)\in O\right)\ge -\inf_{O\cap\mathcal{A}} S_{\mu_D}$$

Proof of Theorem 13.18. Following Lemma 13.10, we deduce that for any $M\in\mathbb{R}^+$, we can find $L_M\in\mathbb{R}^+$ such that for any $L\ge L_M$,

$$\sup_{0\le\epsilon\le 1}\mathbb{P}(\mathcal{K}_\epsilon^N(K_L)^c)\le e^{-MN^2}. \tag{13.33}$$

Fix $M > S_{\mu_D}(\mu)+1$ and $L\ge L_M$. Let $\delta>0$ be given. Next, observe that $P_\epsilon * \mu_.$ converges weakly to $\mu_.$ as ϵ goes to zero and choose consequently ϵ small enough so that $D(P_\epsilon*\mu_.,\mu_.)<\frac{\delta}{3}$. Then, write

$$\begin{aligned}
&\mathbb{P}\left(L_N(.)\in B(\mu_.,\delta)\right)\\
&\ge\mathbb{P}\left(D(L_N(.),\mu_.)<\frac{\delta}{3}, L_{N,\epsilon_.}\in B(P_\epsilon*\mu_.,\frac{\delta}{3}),\mathcal{K}_\epsilon^N(K_L)\right)\\
&\ge\mathbb{P}\left(L_{N,\epsilon_.}\in B(P_\epsilon*\mu_.,\frac{\delta}{3})\right)-\mathbb{P}(\mathcal{K}_\epsilon^N(K_L)^c)\\
&\quad-\mathbb{P}\left(D(L_{N,\epsilon}(.),L_N(.))\ge\frac{\delta}{3},\mathcal{K}_\epsilon^N(K_L)\right)=I-II-III.
\end{aligned}$$

(13.33) implies, up to terms of smaller order, that

$$II\le e^{-N^2(S_{\mu_D}(\mu)+1)}.$$

Lemma 13.17 shows that $III=0$ for ϵ small enough and N large, while Corollary 13.16 implies that for any $\eta>0$, N large and $\epsilon>0$

$$I\ge e^{-N^2 S_{P_\epsilon*\mu_D}(P_\epsilon*\mu)-N^2\eta}\ge e^{-N^2 S_{\mu_D}(\mu)-N^2\eta}.$$

Theorem 13.18 is proved.

$\square$

Proof of Lemma 13.13. Following [55], we take $f(x,t) := e^{i\lambda x}$ for some $\lambda \in \mathbb{R}$ in (13.20) and denote by $\mathcal{L}_t(\lambda) = \int e^{i\lambda x} d\nu_t(x)$ the Fourier transform of ν_t. $\nu \in \mathcal{MC}([0,1],\mathcal{P}(\mathbb{R}))$ implies that if k is the field associated with ν,

$$k_t(x) = C_t + \int e^{i\lambda x}\hat{k}_t(\lambda)d\lambda$$

with $\int_0^1 \max_\lambda e^{-2\epsilon|\lambda|}|\hat{k}_t(\lambda)|^2 dt \le C$ for a given $\epsilon > 0$. Then, we find that for $t \in [0,1]$,

$$\begin{aligned}\mathcal{L}_t(\lambda) =& \mathcal{L}_0(\lambda) - \frac{\lambda^2}{2}\int_0^t\int_0^1 \mathcal{L}_s(\alpha\lambda)\mathcal{L}_s((1-\alpha)\lambda)d\alpha ds \\ &+ i\lambda\int_0^t\int \mathcal{L}_s(\lambda+\lambda')\hat{k}(\lambda',s)d\lambda' ds + i\lambda\int_0^t \mathcal{L}_s(\lambda)C_s ds.\end{aligned} \tag{13.34}$$

Multiplying both sides of this equality by $e^{-\frac{\epsilon}{4}|\lambda|}$ gives, with $\mathcal{L}_t^\epsilon(\lambda) = e^{-\frac{\epsilon}{4}|\lambda|}\mathcal{L}_t(\lambda)$,

$$\begin{aligned}\mathcal{L}_t^\epsilon(\lambda) =& \mathcal{L}_0^\epsilon(\lambda) - \frac{\lambda^2}{2}\int_0^t\int_0^1 \mathcal{L}_s^\epsilon(\alpha\lambda)\mathcal{L}_s^\epsilon((1-\alpha)\lambda)d\alpha ds \\ &+ i\lambda\int_0^t\int \mathcal{L}_s^\epsilon(\lambda+\lambda')e^{\frac{\epsilon}{4}|\lambda+\lambda'|-\frac{\epsilon}{4}|\lambda|}\hat{k}(\lambda',s)d\lambda' ds + i\lambda\int_0^t \mathcal{L}_s^\epsilon(\lambda)C_s ds.\end{aligned} \tag{13.35}$$

Therefore, if $\nu, \tilde{\nu}'$ are two solutions with Fourier transforms $\mathcal{L}$ and $\tilde{\mathcal{L}}$ respectively and if we set $\Delta_t^\epsilon(\lambda) = |\mathcal{L}_t^\epsilon(\lambda) - \tilde{\mathcal{L}}_t^\epsilon(\lambda)|$, we deduce from (13.35) that if we denote $D_s = \sup_{\lambda\in\mathbb{R}} e^{2\epsilon|\lambda|}|\hat{k}(\lambda,s)|$,

$$\begin{aligned}\Delta_t^\epsilon(\lambda) &\le \lambda^2\int_0^t\int_0^1 \Delta_s^\epsilon(\alpha\lambda)e^{-\frac{1}{4}(1-\alpha)\epsilon\lambda}d\alpha ds + |\lambda|\int_0^t \Delta_s^\epsilon(\lambda)C_s ds \\ &\quad + |\lambda|\int_0^t\int \Delta_s^\epsilon(\lambda+\lambda')D_s e^{\frac{\epsilon}{4}|\lambda+\lambda'|-\frac{\epsilon}{4}|\lambda|-\epsilon|\lambda'|}d\lambda' ds \\ &\le \frac{4|\lambda|}{\epsilon}\int_0^t \sup_{|\lambda'|\le|\lambda|}\Delta_s^\epsilon(\lambda')ds + |\lambda|\int_0^t \Delta_s^\epsilon(\lambda)C_s ds \\ &\quad + |\lambda|\int_0^t D_s[\sup_{|\lambda'|\le R}\Delta_s^\epsilon(\lambda') + 2e^{-\frac{\epsilon}{4}R}]\int e^{\frac{\epsilon}{4}|\lambda+\lambda'|-\frac{\epsilon}{4}|\lambda|-\epsilon|\lambda'|}d\lambda' ds\end{aligned}$$

where R is any positive constant and we used that $\Delta_t^\epsilon(\lambda) \le 2e^{-\frac{\epsilon}{4}|\lambda|}$. Considering $\bar{\Delta}_t^\epsilon(R) = \sup_{|\lambda'|\le R}\Delta_s^\epsilon(\lambda')$, we therefore obtain, since $|\lambda|+|\lambda'| \ge |\lambda+\lambda'|$,

$$\bar{\Delta}_t^\epsilon(R) \le \frac{R}{\epsilon}\int_0^t (D_s + C_s + 4)\bar{\Delta}_s^\epsilon(R)ds + 2\frac{R}{\epsilon}e^{-\frac{\epsilon}{4}R}\int_0^t D_s ds$$

By Gronwall's lemma, we deduce that

$$\bar{\Delta}_t^\epsilon(R) \le 2\frac{R}{\epsilon}e^{-\frac{\epsilon}{4}R}\int_0^t D_s e^{\frac{R}{\epsilon}\int_s^t (D_u+C_u+4)du} ds.$$

Now, since we assumed $D^2 := \int_0^1 D_s^2 ds < \infty$ and $C^2 := \int_0^1 C_s^2 ds < \infty$, $\int_0^t (D_s + C_s)ds \le (C+D)\sqrt{t}$, we have

$$\bar{\Delta}_t^\epsilon(R) \le 2\frac{R}{\epsilon}De^{-\frac{\epsilon}{4}R}e^{\frac{R}{\epsilon}[(C+D)\sqrt{t}+4t]}.$$

Thus $\bar{\Delta}_t^\epsilon(\infty) = 0$ for $t < \tau \equiv (\frac{\epsilon^2}{4(C+D+4)})^2$. By induction over the time, we conclude that $\bar{\Delta}_t^\epsilon(\infty) = 0$ for any time $t \le 1$, and therefore that $\nu = \tilde{\nu}$. $\square$

Bibliographical notes. The previous large deviation estimates were proved in [109]. Further analysis of the rate function was performed in [101], and of related PDE questions in [5, 134, 142]. On less rigorous ground, we refer to [146]. In particular, the rate function $S_{\mu_D}(\mu)$ is given as an infimum achieved at the solution of a complex Burgers equation [101]. By completely different techniques, Kenyon, Okounkov, Sheffield [128] obtained large deviation principles for the law of the random surfaces given by dimers. Interestingly, the limiting shapes are also described by the complex Burgers equation [127]. This point may indicate that there is a link between random matrices and random partitions at the level of large deviations, generalizing the well-known local connection [17, 161].

14

Asymptotics of Harish–Chandra–Itzykson–Zuber integrals and of Schur polynomials

Let $Y^{N,\beta}$ be the random matrix $\mathbf{D}^N + X^{N,\beta}$ with a deterministic diagonal matrix $\mathbf{D}^N$ and $X^{N,\beta}$ a Gaussian Wigner matrix. We now show how the deviations of the law of the spectral measure of $Y^{N,\beta}$ are related to the asymptotics of the Harish–Chandra–Itzykson–Zuber (or spherical) integrals

$$I_N(A, B) = \int e^{N \mathrm{Tr}(AUBU^*)} dm_N^\beta(U)$$

where $m_N^\beta(U)$ is the Haar measure on $U(N)$ when $\beta = 2$ and $O(N)$ when $\beta = 1.m_N$ will stand for m_N^2 to simplify the notations. Here, $I_N(A, B)$ makes sense for any $A, B \in \mathcal{M}_n(\mathbb{C})$, but we shall consider asymptotics only when $A, B \in \mathcal{H}_N^{(\beta)}(\mathbb{C})$ (the extension of our results to non-self-adjoint matrices is still open). To this end, we shall make the following hypothesis:

Assumption 1. 1. There exists $d_{max} \in \mathbb{R}^+$ such that for any integer number N, $L_{D_N}(\{|x| \geq d_{max}\}) = 0$ and that L_{D_N} converges weakly to $\mu_D \in \mathcal{P}(\mathbb{R})$.
2. L_{E_N} converges to $\mu_E \in \mathcal{P}(\mathbb{R})$ while $L_{E_N}(x^2)$ stays uniformly bounded.

Theorem 14.1. *Under Assumption 1:*

1) There exists a function $g : [0,1] \times \mathbb{R}^+ \mapsto \mathbb{R}^+$, depending on μ_E only, such that $g(\delta, L) \to_{\delta \to 0} 0$ for any $L \in \mathbb{R}^+$, and, for $\hat{E}_N, \bar{E}_N$ such that

$$d(L_{\hat{E}_N}, \mu_E) + d(L_{\bar{E}_N}, \mu_E) \leq \delta/2 \tag{14.1}$$

and

$$\int x^2 dL_{\bar{E}_N}(x) + \int x^2 dL_{\hat{E}_N}(x) \leq L, \tag{14.2}$$

and it holds that

$$\limsup_{N \to \infty} \left| \frac{1}{N^2} \log \frac{I_N^{(\beta)}(D_N, \hat{E}_N)}{I_N^{(\beta)}(D_N, \bar{E}_N)} \right| \leq g(\delta, L)\,.$$

We define

$$\bar{I}^{(\beta)}(\mu_D, \mu_E) = \limsup_{N\uparrow\infty} \frac{1}{N^2} \log I_N^{(\beta)}(D_N, E_N)$$
$$\underline{I}^{(\beta)}(\mu_D, \mu_E) = \liminf_{N\uparrow\infty} \frac{1}{N^2} \log I_N^{(\beta)}(D_N, E_N)\,,$$

$\bar{I}^{(\beta)}(\mu_D, \mu_E)$ and $\underline{I}^{(\beta)}(\mu_D, \mu_E)$ are continuous functions on $\{(\mu_E, \mu_D) \in \mathcal{P}(\mathbb{R})^2 : \int x^2 d\mu_E(x) + \int x^2 d\mu_D(x) \le L\}$ for any $L < \infty$.

2) For any probability measure $\mu \in \mathcal{P}(\mathbb{R})$,

$$\begin{aligned}\inf_{\delta\to 0} \liminf_{N\to\infty} \frac{1}{N^2} &\log \mathbb{P}\left(d(L_{Y^{N,\beta}}, \mu) < \delta\right) \\ &= \inf_{\delta\to 0} \limsup_{N\to\infty} \frac{1}{N^2} \log \mathbb{P}\left(d(L_{Y^N,\beta}, \mu) < \delta\right) \\ &:= -J_\beta(\mu_D, \mu).\end{aligned}$$

3) We let, for any $\mu \in \mathcal{P}(\mathbb{R})$,

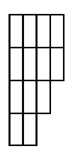
$$I_\beta(\mu) = \frac{\beta}{4} \int x^2 d\mu(x) - \frac{\beta}{2} \int \int \log|x-y| d\mu(x) d\mu(y).$$

If L_{E_N} converges to $\mu_E \in \mathcal{P}(\mathbb{R})$ with $I_\beta(\mu_E) < \infty$, we have

$$\begin{aligned}I^{(\beta)}(\mu_D, \mu_E) &:= \bar{I}^{(\beta)}(\mu_D, \mu_E) = \underline{I}^{(\beta)}(\mu_D, \mu_E) \\ &= -J_\beta(\mu_D, \mu_E) + I_\beta(\mu_E) - \inf_{\mu\in\mathcal{P}(\mathbb{R})} I_\beta(\mu) + \frac{\beta}{4} \int x^2 d\mu_D(x).\end{aligned}$$

Before going any further, let us point out that these results give interesting asymptotics for Schur polynomials that are defined as follows.

- A Young shape λ is a finite sequence of non-negative integers $(\lambda_1, \lambda_2, \ldots, \lambda_l)$ written in non-increasing order. One should think of it as a diagram whose ith line is made of λ_i empty boxes: for example,

  corresponds to $\lambda_1 = 4, \lambda_2 = 4, \lambda_3 = 3, \lambda_4 = 2$.

  We denote by $|\lambda| = \sum_i \lambda_i$ the total number of boxes of the shape λ.
  In the sequel, when we have a shape $\lambda = (\lambda_1, \lambda_2, \ldots)$ and an integer N greater than the number of lines of λ having a strictly positive length, we will define a sequence l associated to λ and N, that is an N-tuple of integers $l_i = \lambda_i + N - i$. In particular we have that $l_1 > l_2 > \ldots > l_N \ge 0$ and $l_i - l_{i+1} \ge 1$.

- For some fixed $N \in \mathbb{N}$, a Young tableau will be any filling of the Young shape with integers from 1 to N that is non-decreasing on each line and (strictly) increasing on each column. For each such filling, we define the content of a Young tableau as the N-tuple $(\mu_1, \dots, \mu_N)$ where μ_i is the number of i's written in the tableau.

  For example, $\begin{array}{|c|c|c|}\hline 1&1&2\\ \hline 2&3\\ \cline{1-2} 3\\ \cline{1-1}\end{array}$ is allowed (and has content $(2, 2, 2)$),

  whereas $\begin{array}{|c|c|c|}\hline 1&1&2\\ \hline 1&3\\ \cline{1-2} 3\\ \cline{1-1}\end{array}$ is not.

  Notice that, for $N \in \mathbb{N}$, a Young shape can be filled with integers from 1 to N if and only if $\lambda_i = 0$ for $i > N$.
- For a Young shape λ and an integer N, the Schur polynomial s_λ is an element of $\mathbb{C}[x_1, \dots, x_N]$ defined by
$$s_\lambda(x_1, \dots, x_N) = \sum_T x_1^{\mu_1} \dots x_N^{\mu_N}, \tag{14.3}$$
  where the sum is taken over all Young tableaux T of fixed shape λ and $(\mu_1, \dots, \mu_N)$ is the content of T. On a statistical point of view, one can think of the filling as the heights of a surface sitting on the tableau λ, μ_i being the height of the surface at i. s_λ is then a generating function for these heights when one considers the surfaces uniformly distributed under the constraints prescribed for the filling. Note that s_λ is positive whenever the x_i's are and, although it is not obvious from this definition (cf. for example [175] for a proof), s_λ is a symmetric function of the x_i's and actually (s_λ, λ) form a basis of symmetric functions and hence play a key role in representation theory of the symmetric group. If A is a matrix in $\mathcal{M}_N(\mathbb{C})$, then define $s_\lambda(A) \equiv s_\lambda(A_1, \dots, A_N)$, where the A_i's are the eigenvalues of A. Then, by Weyl's formula (cf. [204, Theorem 7.5.B]), for any matrices V, W,
$$\int s_\lambda(UVU^*W) dm_N(U) = \frac{1}{d_\lambda} s_\lambda(V) s_\lambda(W), \tag{14.4}$$
  with $d_\lambda = s_\lambda(I) = \prod_{i<j}(l_i - l_j) / \prod_{i=1}^{N-1} i!$ with $l_i = \lambda_i - i + N$. s_λ can also be seen as a generating function for the number of surfaces constructed on the Young shape with prescribed level areas.

The Schur function s_λ has a determinantal formula (cf. [175] and [98]);
$$s_\lambda(x) = \frac{\det(x_i^{l_j})_{1 \le i,j \le N}}{\Delta(x)}$$
with $\Delta(x)$ the Vandermonde determinant $\Delta(x) = \prod_{i<j}(x_i - x_j)$. Since also the spherical integral $I_N^{(2)}$ has a determinantal expression for $A = \operatorname{diag}(a_1, \dots, a_N)$ and $B = \operatorname{diag}(b_1, \dots, b_N)$,

$$I_N^{(2)}(A,B) = \frac{\det(e^{a_i b_j})_{1\le i,j\le N}}{\Delta(a)\Delta(B)},$$

with $\Delta(A) = \Delta(a), \Delta(B) = \Delta(a)$, we deduce

$$s_\lambda(M) = I_N^{(2)}\left(\log M, \frac{l}{N}\right) \Delta\left(\frac{l}{N}\right) \frac{\Delta(\log M)}{\Delta(M)}, \tag{14.5}$$

where $\frac{l}{N}$ denotes the diagonal matrix with entries $N^{-1}(\lambda_i - i + N)$. Therefore, we have the following immediate corollary to Theorem 14.1:

Corollary 14.2. *Let λ^N be a sequence of Young shapes and set $D_N = (N^{-1}(\lambda_i^N - i + N))_{1\le i\le N}$. We pick a sequence of Hermitian matrices $(E_N)_{N\ge 0}$ and assume that $(D_N, E_N)_{N\in\mathbb{N}}$ satisfy hypothesis 1 and that $\Sigma(\mu_D) > -\infty$. Then,*

$$\begin{aligned}&\lim_{N\to\infty} \frac{1}{N^2} \log s_{\lambda^N}(e^{E_N})\\ &= I^{(2)}(\mu_E, \mu_D) - \frac{1}{2}\int \log\left[\int_0^1 e^{\alpha x + (1-\alpha)y} d\alpha\right] d\mu_E(x) d\mu_E(y) + \frac{1}{2}\Sigma(\mu_D).\end{aligned}$$

Proof of Theorem 14.1:. To simplify, let us assume that E_N and $\hat{E}_N$ are uniformly bounded by a constant M. Let $\delta' > 0$ and $\{A_j\}_{j\in\mathcal{J}}$ be a partition of $[-M, M]$ such that $|A_j| \in [\delta', 2\delta']$ and the end points of A_j are continuity points of μ_E. Define

$$\hat{I}_j = \{i : \hat{E}_N(ii) \in A_j\}, \quad \bar{I}_j = \{i : \bar{E}_N(ii) \in A_j\}.$$

By (14.1),

$$|\mu_E(A_j) - |\hat{I}_j|/N| + |\mu_E(A_j) - |\bar{I}_j|/N| \le \delta.$$

We construct a permutation σ_N so that $|\hat{E}(ii) - \bar{E}(\sigma_N(i), \sigma_N(i))| < 2\delta$ except possibly for very few i's as follows. First, if $|\bar{I}_j| \le |\hat{I}_j|$ then $\tilde{I}_j := \bar{I}_j$, whether if $|\bar{I}_j| > |\hat{I}_j|$ then $|\tilde{I}_j| = |\hat{I}_j|$ while $\tilde{I}_j \subset \bar{I}_j$. Then, choose and fix a permutation σ_N such that $\sigma_N(\tilde{I}_j) \subset \hat{I}_j$. Then, one can check that if $\mathcal{J}_0 = \{i : |\hat{E}(ii) - \bar{E}(\sigma_N(i), \sigma_N(i))| < 2\delta\}$,

$$\begin{aligned}|\mathcal{J}_0| \ge |\cup_j \sigma_N(\tilde{I}_j)| &= \sum_j |\sigma_N(\tilde{I}_j)| \ge N - \sum_j |\bar{I}_j \backslash \tilde{I}_j|\\ &\ge N - \max_j(|\bar{I}_j| - |\tilde{I}_j|)|\mathcal{J}| \ge N - 2\delta N \frac{M}{\delta'}.\end{aligned}$$

Next, note the invariance of $I_N^{(\beta)}(D_N, E_N)$ to permutations of the matrix elements of D_N. That is,

$$
\begin{aligned}
I_N^{(\beta)}(D_N, \bar{E}_N) &= \int \exp\left\{N\frac{\beta}{2}\mathrm{Tr}(UD_NU^*\bar{E}_N)\right\} dm_N^\beta(U) \\
&= \int \exp\left\{N\frac{\beta}{2}\sum_{i,k} u_{ik}^2 D_N(kk)\bar{E}_N(ii)\right\} dm_N^\beta(U) \\
&= \int \exp\left\{N\sum_{i,k} u_{ik}^2 D_N(kk)\bar{E}_N(\sigma_N(i)\sigma_N(i))\right\} dm_N^\beta(U)\,.
\end{aligned}
$$

But, with $d_{\max} = \max_k |D_N(kk)|$ bounded uniformly in N,

$$
\begin{aligned}
& N^{-1}\sum_{i,k} u_{ik}^2 D_N(kk)\bar{E}_N(\sigma_N(i)\sigma_N(i)) \\
&= N^{-1}\sum_{i\in\mathcal{J}_0}\sum_k u_{ik}^2 D_N(kk)\bar{E}_N(\sigma_N(i)\sigma_N(i)) \\
&\quad + N^{-1}\sum_{i\notin\mathcal{J}_0}\sum_k u_{ik}^2 D_N(kk)\bar{E}_N(\sigma_N(i)\sigma_N(i)) \\
&\le N^{-1}\sum_{i,k} u_{ik}^2 D_N(kk)(\hat{E}_N(ii)+2\delta) + N^{-1}d_{\max}M|\mathcal{J}_0^c| \\
&\le N^{-1}\sum_{i,k} u_{ik}^2 D_N(kk)\hat{E}_N(ii) + d_{\max}\frac{M^2\delta}{\delta'}.
\end{aligned}
$$

Hence, we obtain, taking $d_{\max}\frac{M^2\delta}{\delta'} = \sqrt{\delta}$,

$$
I_N^{(\beta)}(D_N, \bar{E}_N) \le e^{N\sqrt{\delta}} I_N^{(\beta)}(D_N, \hat{E}_N)
$$

and the reverse inequality by symmetry. This proves the first point of the theorem when $(\bar{E}_N, \hat{E}_N)$ are uniformly bounded. The general case (which is not much more complicated) is proved in [109] and follows from first approximating $\bar{E}_N$ and $\hat{E}_N$ by bounded operators using (14.2).

The second and the third points are proved simultaneously: in fact, writing

$$
\begin{aligned}
& \mathbb{P}\left(d(L_Y,\mu)<\delta\right) \\
&= \frac{1}{Z_N^\beta}\int_{d(L_Y,\mu)<\delta} e^{-\frac{N\beta}{4}\mathrm{Tr}((Y^{N,\beta}-D_N)^2)} dY^{N,\beta} \\
&= \frac{e^{-\frac{N\beta}{4}\mathrm{Tr}(D_N^2)}}{Z_N^\beta}\int_{d(\frac{1}{N}\sum_{i=1}^N \delta_{\lambda_i},\mu)<\delta} I_N^{(\beta)}(D(\lambda), D_N) e^{-\frac{N\beta}{4}\sum_{i=1}^N \lambda_i^2}\Delta(\lambda)^\beta \prod_{i=1}^N d\lambda_i
\end{aligned}
$$

with Z_N^β the normalizing constant

$$
Z_N^\beta = \int e^{-\frac{N}{2}\mathrm{Tr}((Y^{N,\beta}-D_N)^2)} dY^{N,\beta} = \int e^{-\frac{N}{2}\mathrm{Tr}((Y^{N,\beta})^2)} dY^{N,\beta},
$$

we see that the first point gives, since $I_N^{(\beta)}(D(\lambda), D_N)$ is approximately constant on $\{d(N^{-1}\sum \delta_{\lambda_i}, \mu) < \delta\} \cap \{d(L_{D_N}, \mu_D) < \delta\}$,

$$\begin{aligned}
&\mathbb{P}\left(d(L_Y, \mu) < \delta\right) \\
&\approx \frac{e^{N^2(I^{(\beta)}(\mu_D,\mu) - \frac{\beta}{4} L_{D_N}(x^2))}}{Z_N^\beta} \int_{d(\frac{1}{N}\sum_{i=1}^N \delta_{\lambda_i}, \mu)<\delta} e^{-\frac{N\beta}{4}\sum_{i=1}^N \lambda_i^2} \Delta(\lambda)^\beta \prod_{i=1}^N d\lambda_i \\
&= e^{-\frac{N^2\beta}{4} L_{D_N}(x^2) + N^2 I^{(\beta)}(\mu_D,\mu)} \mathbb{P}\left(d(L_X, \mu) < \delta\right)
\end{aligned}$$

where $A_{N,\delta} \approx B_{N,\delta}$ means that $N^{-2}\log A_{N,\delta} B_{N,\delta}^{-1}$ goes to zero as N goes to infinity first and then δ goes to zero.

The large deviation principle proved in the Chapter 10 shows 2) and 3). $\square$

Note for 3) that if $I_\beta(\mu_E) = +\infty$, $J(\mu_D, \mu_E) = +\infty$ so that in this case the result is empty since it leads to an indetermination. Still, if $I_\beta(\mu_D) < \infty$, by symmetry of $I^{(\beta)}$, we obtain a formula by exchanging μ_D and μ_E. If both $I_\beta(\mu_D)$ and $I_\beta(\mu_E)$ are infinite, we can only argue, by continuity of $I^{(\beta)}$, that for any sequence $(\mu_E^\epsilon)_{\epsilon>0}$ of probability measures with uniformly bounded variance and finite entropy I_β converging to μ_E,

$$I^{(\beta)}(\mu_D, \mu_E) = \lim_{\epsilon\to\infty}\{-J_\beta(\mu_D, \mu_E^\epsilon) + I_\beta(\mu_E^\epsilon)\} - \inf I_\beta + \frac{\beta}{4}\int x^2 d\mu_D(x).$$

A more explicit formula is not yet available.

Bibliographical notes. Note that the convergence of the spherical integral is in fact not obvious and is given by the large deviation principle for the law of the spectral measure of non-centered Wigner matrices. Such types of convergence were shown to hold for more general integrals, but in a small-parameters region, in [66].

Harish–Chandra–Itzykson–Zuber integral was studied intensively [49, 89, 210, 65]. Our approach is based on its relation with the large deviation principle for the law of the Hermitian Brownian motion. A parallel approach uses the heat kernel [146]. An important tool, when the integral holds over the unitary group, is the use of the Harish–Chandra formula that expresses this integral as a determinant [115, 116, 49]. The asymptotics of the Harish–Chandra–Itzykson–Zuber integral when one of the matrices has a rank that is negligible with respect to N were studied in [103, 67]; they are given by the so-called R-transform.

15

Asymptotics of some matrix integrals

We would like to consider integrals of more than one matrix. The simplest interaction that one can think of is the quadratic one. Such an interaction describes already several classical models in random-matrix theory; We refer here to the works of M. Mehta, A. Matytsin, A. Migdal, V. Kazakov, P. Zinn Justin and B. Eynard for instance. We list below a few models that were studied.

- The random Ising model on random graphs is described by the Gibbs measure

$$\mu^N_{Ising}(dA, dB) = \frac{1}{Z^N_{Ising}} e^{N\mathrm{Tr}(AB) - N\mathrm{Tr}(P_1(A)) - N\mathrm{Tr}(P_2(B))} dAdB$$

  with Z^N_{Ising} the partition function

$$Z^N_{Ising} = \int e^{N\mathrm{Tr}(AB) - N\mathrm{Tr}(P_1(A)) - N\mathrm{Tr}(P_2(B))} dAdB$$

  and two polynomial functions P_1, P_2. The limiting free energy for this model was calculated by M. Mehta [153] in the case $P_1(x) = P_2(x) = x^2 + gx^4$ and integration holds over $\mathcal{H}_N$. The limit was studied in [43]. However, the limiting spectral measures of A and B under μ^N_{Ising} were not considered in these papers. A discussion about this problem can be found in P. Zinn Justin [209].
- One can also define the $q-1$ Potts model on random graphs described by the Gibbs measure

$$\mu^N_{Potts}(d\mathbf{A}_1, ..., d\mathbf{A}_q)$$
$$= \frac{1}{Z^N_{Potts}} \prod_{i=2}^{q} e^{N\mathrm{Tr}(\mathbf{A}_1\mathbf{A}_i) - N\mathrm{Tr}(P_i(\mathbf{A}_i))} d\mathbf{A}_i e^{-N\mathrm{Tr}(P_1(\mathbf{A}_1))} d\mathbf{A}_1.$$

  The limiting spectral measures of $(\mathbf{A}_1, \ldots, \mathbf{A}_q)$ were first discussed in [80].

- As a straightforward generalization, one can consider matrices coupled in chain following S. Chadha, G. Mahoux and M. Mehta [60] given by

$$\mu^N_{chain}(d\mathbf{A}_1, ..., d\mathbf{A}_q)$$
$$= \frac{1}{Z^N_{chain}} \prod_{i=2}^{q} e^{N\mathrm{Tr}(\mathbf{A}_{i-1}\mathbf{A}_i) - N\mathrm{Tr}(P_i(\mathbf{A}_i))} d\mathbf{A}_i e^{-N\mathrm{Tr}(P_1(\mathbf{A}_1))} d\mathbf{A}_1.$$

  q can eventually go to infinity as in [147].

The first-order asymptotics of these models can be studied thanks to the control of spherical integrals obtained in the last chapter.

Theorem 15.1. *Assume that $P_i(x) \geq c_i x^4 + d_i$ with $c_i > 0$ and some finite constants d_i. Hereafter, $\beta = 1$ (resp. $\beta = 2$, resp. $\beta = 4$) when dA denotes the Lebesgue measure on $\mathcal{S}_N$ (resp. $\mathcal{H}_N$, resp. $\mathcal{H}_N$ with N even). Then, with $c = \inf_{\nu \in \mathcal{P}(\mathbb{R})} I_\beta(\nu)$,*

$$\begin{aligned} F_{Ising} &= \lim_{N\to\infty} \frac{1}{N^2} \log Z^N_{Ising} \\ &= -\inf\left\{\mu(P) + \nu(Q) - I^{(\beta)}(\mu,\nu) - \frac{\beta}{2}\Sigma(\mu) - \frac{\beta}{2}\Sigma(\nu)\right\} - 2c \end{aligned} \tag{15.1}$$

$$\begin{aligned} F_{Potts} &= \lim_{N\to\infty} \frac{1}{N^2} \log Z^N_{Potts} \\ &= -\inf\left\{\sum_{i=1}^q \mu_i(P_i) - \sum_{i=2}^q I^{(\beta)}(\mu_1,\mu_i) - \frac{\beta}{2}\sum_{i=1}^q \Sigma(\mu_i)\right\} - qc \end{aligned} \tag{15.2}$$

$$\begin{aligned} F_{chain} &= \lim_{N\to\infty} \frac{1}{N^2} \log Z^N_{chain} \\ &= -\inf\left\{\sum_{i=1}^q \mu_i(P_i) - \sum_{i=2}^q I^{(\beta)}(\mu_{i-1},\mu_i) - \frac{\beta}{2}\sum_{i=1}^q \Sigma(\mu_i)\right\} - qc \end{aligned} \tag{15.3}$$

Remark 6. (1) The above theorem actually extends to polynomial functions going to infinity like x^2. However, the case of quadratic polynomials is trivial since it boils down to the Gaussian case and therefore the next interesting case is the quartic polynomial as above. Moreover, Theorem 15.2 fails in the case where P, Q go to infinity only like x^2. However, all our proofs would extend easily for any continuous functions $P_i's$ such that $P_i(x) \geq a|x|^{2+\epsilon} + b$ with some $a > 0$ and $\epsilon > 0$.

(2) Note that we did not assume here that potentials are small perturbations of the Gaussian potential as in Part III.

(3)The above free energies are not very explicit and not easy to analyze. To give a taste of the kind of information we have been able to establish so far, we state below a result about the Ising model.

Proof of Theorem 15.1. It is enough to notice that, when diagonalizing the matrices $\mathbf{A}_i$'s, the interaction in the models under consideration is expressed in terms of spherical integrals since, under dA, $A = UD_AU^*$, with D_A diagonal, U independent from D_A following the Haar measure on $U(N)$ when $\beta = 2$ and $O(N)$ when $\beta = 1$, so that

$$\mathbb{E}[e^{N\mathrm{Tr}(AB)}|\lambda_N^i(A), \cdots, \lambda_N^i(B), 1 \le i \le N] = I_N^{(\beta)}(A_N, B_N).$$

Laplace's (or saddle point) method then gives the result (up to the boundedness of the matrices A_i's in the spherical integrals,that can be obtained by approximation). We shall not detail it here and refer the reader to [101]. $\square$

We shall then study the variational problems for the above energies; indeed, by standard large deviation considerations, it is clear that the spectral measures of the matrices $(\mathbf{A}_i)_{1\le i\le d}$ will concentrate on the set of the minimizers defining the free energies, and in particular converge to these minimizers when they are unique. We prove in [101] the following for the Ising model.

Theorem 15.2. *Assume $P_1(x) \ge ax^4 + b, P_2(x) \ge ax^4 + b$ for some positive constant a. Then:*

(0) The infimum in F_{Ising} is achieved at a unique couple (μ_A, μ_B) of probability measures.

(1) (L_A, L_B) converges almost surely to (μ_A, μ_B).

(2) (μ_A, μ_B) are compactly supported with finite non-commutative entropy

$$\Sigma(\mu) = \int\int \log|x-y|d\mu(x)d\mu(y).$$

(3) There exists a couple $(\rho^{A\to B}, u^{A\to B})$ of measurable functions on $\mathbb{R} \times (0,1)$ such that $\rho_t^{A\to B}(x)dx$ is a probability measure on $\mathbb{R}$ for all $t \in (0,1)$ and $(\mu_A, \mu_B, \rho^{A\to B}, u^{A\to B})$ are characterized uniquely as the minimizer of a strictly convex function under a linear constraint.

In particular, $(\rho^{A\to B}, u^{A\to B})$ are solution of the Euler equation for isentropic flow with negative pressure $p(\rho) = -\frac{\pi^2}{3}\rho^3$ such that, for all (x,t) in the interior of $\Omega = \{(x,t) \in \mathbb{R} \times [0,1]; \rho_t^{A\to B}(x) \ne 0\}$,

$$\begin{cases} \partial_t \rho_t^{A\to B} + \partial_x(\rho_t^{A\to B} u_t^{A\to B}) = 0 \\ \partial_t(\rho_t^{A\to B} u_t^{A\to B}) + \partial_x(\rho_t^{A\to B}(u_t^{A\to B})^2 - \frac{\pi^2}{3}(\rho_t^{A\to B})^3) = 0 \end{cases} \tag{15.4}$$

with the probability measure $\rho_t^{A\to B}(x)dx$ weakly converging to $\mu_A(dx)$ (resp. $\mu_B(dx)$) as t goes to zero (resp. one). Moreover, we have

$$P'(x) - x - \frac{\beta}{2}u_0^{A\to B}(x) - \frac{\beta}{2}H\mu_A(x) = 0 \quad \mu_A - a.s$$

$$\text{and} \quad Q'(x) - x + \frac{\beta}{2}u_1^{A\to B}(x) - \frac{\beta}{2}H\mu_B(x) = 0 \quad \mu_B - a.s.$$

For the other models, uniqueness of the minimizers is not always clear. For instance, we obtain uniqueness of the minimizers for the q-Potts models only for $q \leq 2$, whereas it is also expected for $q = 3$ (when the potential is convex, uniqueness is, however, always true, see [106]). For the description of these minimizers, I refer the reader to [101].

15.1 Enumeration of maps from matrix models

As we have seen in Part III, the enumeration of maps with one color is related with matrix integrals of the form

$$Z_{\mathbf{t}}^N = \int e^{-N\mathrm{Tr}(V_{\mathbf{t}}(X))} d\mu^N(X)$$

with $V_{\mathbf{t}} = \sum_{i=1}^n t_i x^i$ a polynomial function depending on parameters $\mathbf{t} = (t_1, \cdots, t_n)$. When n is even and $t_n > 0$, the above matrix integral is finite and we can apply the results of Theorem 10.1 to see that

$$\lim_{N\to\infty} \frac{1}{N} \log Z_{\mathbf{t}}^N = \sup_{\mu\in\mathcal{P}(\mathbb{R})} \left\{ \int\int \log|x-y| d\mu(x)d\mu(y) - \int V_{\mathbf{t}}(x)d\mu(x) \right\} - c.$$

By Lemma 10.2, there is a unique optimizer $\mu_{\mathbf{t}}$ to the above supremum and it is characterized by the fact that there exists a constant ℓ such that

$$\ell = -2\int \log|x-y| d\mu_{\mathbf{t}}(y) + V_{\mathbf{t}}(x) + \frac{1}{2}x^2 \quad \mu_{\mathbf{t}} \text{ a.s.} \tag{15.5}$$

$$\ell \leq -2\int \log|x-y| d\mu_{\mathbf{t}}(y) + V_{\mathbf{t}}(x) + \frac{1}{2}x^2 \quad \mu_{\mathbf{t}} \text{ everywhere.} \tag{15.6}$$

The large deviation principle of Theorem 10.1 as well as the uniqueness of the minimizers assert that under the Gibbs measure

$$d\mu_{\mathbf{t}}^N(X) = (Z_{\mathbf{t}}^N)^{-1} e^{-N\mathrm{Tr}(V_{\mathbf{t}}(X))} d\mu^N(X)$$

$\hat{\mathbf{L}}^N$ converges almost surely towards $\mu_{\mathbf{t}}$. Assume now that there exists $c > 0$ such that $V_{\mathbf{t}}$ is c-convex. Then, if the parameters $(t_i)_{1\leq i\leq n}$ are small enough, Theorem 8.4 and Corollary 8.6 assert that the limit $\mu_{\mathbf{t}}$ is also a generating function for planar maps;

$$\mu_{\mathbf{t}}(x^p) = \sum_{k\in\mathbb{N}^n} \prod_{i=1}^n \frac{(-t_i)^{k_i}}{k_i!} \mathcal{M}_{\mathbf{k}}(x^p)$$

with $\mathcal{M}_{\mathbf{k}}(x^p)$ the number of planar maps with k_i stars of type x^i and one star of type x^p.

Let us show how to deduce formulae for $\mathcal{M}_{\mathbf{k}}(x^p)$ when $V_{\mathbf{t}}(x) = tx^4$ from the above large deviation result, i.e., count quadrangulations and recover the result of Tutte [194] from the matrix-model approach. The analysis below is inspired from [32]. As can be guessed, formulae become more complicated as $V_{\mathbf{t}}$ becomes more complex (see [72] for a more general treatment).

To find an explicit formula for $\mu_{\mathbf{t}}$ from (15.5) and (15.6), let us observe first that differentiating (15.5) (or using directly the Schwinger–Dyson's equation) and integrating with respect to $(z-x)^{-1}d\mu_{\mathbf{t}}(x)$ gives

$$G\mu_{\mathbf{t}}(z)^2 = -4t(\alpha_t + z^2) - 1 + 4tz^3 G\mu_{\mathbf{t}}(z) + zG\mu_{\mathbf{t}}(z)$$

with $G\mu_{\mathbf{t}}(z) = \int (z-x)^{-1} d\mu_{\mathbf{t}}(x)$, $z \in \mathbb{C}\backslash\mathbb{R}$ and $\alpha_t = \int x^2 d\mu_t(x)$. Solving this equation yields

$$G\mu_{\mathbf{t}}(z) = \frac{1}{2}\left(4tz^3 + z - \sqrt{(4tz^3+z)^2 - 4(4t(\alpha_t + z^2) + 1)}\right)$$

where we have chosen the solution so that $G\mu_{\mathbf{t}}(z) \approx z^{-1}$ as $|z| \to \infty$. The square root is chosen as the analytic continuation in $\mathbb{C}\backslash\mathbb{R}^-$ of the square root on $\mathbb{R}^+$. Recall that if p_ϵ is the Cauchy law with parameter $\epsilon > 0$, for $x \in \mathbb{R}$,

$$-\Im(G\mu_{\mathbf{t}}(x+i\epsilon)) = \int \frac{\epsilon}{(x-y)^2+\epsilon^2} d\mu_{\mathbf{t}}(y) = \pi p_\epsilon * \mu_{\mathbf{t}}(x).$$

Hence, if $-\Im(G\mu_{\mathbf{t}}(x+i\epsilon))$ converges as ϵ decreases towards zero, its limit is the density of $\mu_{\mathbf{t}}$. Thus, we in fact have

$$\frac{d\mu_{\mathbf{t}}}{dx} = -\frac{1}{\pi}\lim_{\epsilon\downarrow 0}\Im\left(\sqrt{(4t(x+i\epsilon)^3 + (x+i\epsilon))^2 - 4(4t(\alpha_t + (x+i\epsilon)^2) + 1)}\right).$$

To analyze this limit, we write

$$(4tz^3+z)^2 - 4(4t(\alpha_t+z^2)+1) = (4t)^2(z^2-a_1)(z^2-a_2)(z^2-a_3)$$

for some $a_1, a_2, a_3 \in \mathbb{C}$. Note that since $G\mu_{\mathbf{t}}$ is analytic on $\mathbb{C}\backslash\mathbb{R}$, either we have a double root and a real non-negative root, or three real non-negative roots. We now argue that when $V_{\mathbf{t}}$ is convex, $a_1 = a_2$ and $a_3 \in \mathbb{R}^+$. In fact, the function

$$f(x) := -2\int \log|x-y| d\mu_{\mathbf{t}}(y) + V_{\mathbf{t}}(x) + \frac{1}{2}x^2$$

is strictly convex on $\mathbb{R}\backslash\mathrm{support}(\mu_{\mathbf{t}})$ and it is continuous at the boundaries of the support of $\mu_{\mathbf{t}}$ since $\mu_{\mathbf{t}}$ as a bounded density. Since f equals ℓ on the support of $\mu_{\mathbf{t}}$ and is greater or equal to ℓ outside, we deduce that if there is a hole in the support of $\mu_{\mathbf{t}}$, f must also be constant equal to ℓ on this hole. This contradicts the strict convexity of f outside the support. Hence, the support S of $\mu_{\mathbf{t}}$ must be an interval and $G\mu_{\mathbf{t}}$ must be analytic outside S. Thus, we must have $a_1 = a_2 := b \in \mathbb{R}$ and $a_3 := a \in \mathbb{R}^+$ and $S = [-\sqrt{a}, +\sqrt{a}]$. Plugging back this equality gives the system of equations

$$a + 2b = -\frac{1}{2t}, \quad -2ba + b^2 = \frac{1}{4t^2} - \frac{1}{t}, \quad 4t^2ab^2 = -(4t\alpha_t + 1),$$

which has a unique solution $(a, b) \in \mathbb{R}^+ \times \mathbb{R}$, which in turn prescribes α_t uniquely. Thus, we now have

$$\frac{d\mu_{\mathbf{t}}}{dx}(x) = c_t 1_{[-\sqrt{a},\sqrt{a}]}(x^2-b)\sqrt{a-x^2}$$
$$c_t^{-1} = \int_{[-\sqrt{a},\sqrt{a}]}(x^2-b)\sqrt{a-x^2}dx = \frac{\pi}{2}a\left[\frac{a}{4} - b\right].$$

In particular, this expression allows us to write all moments of $\mu_{\mathbf{t}}$ in terms of Catalan numbers since

$$\begin{aligned}\int x^{2p} d\mu_{\mathbf{t}}(x) &= c_t \int_{-\sqrt{a}}^{\sqrt{a}} x^{2p}(x^2 - b)\sqrt{a - x^2}dx \\ &= \frac{4}{a[\frac{a}{4} - b]}(a/4)^p \int [ax^2 - b]x^{2p} d\sigma(x) \\ &= \frac{4}{a[\frac{a}{4} - b]}(a/4)^p [aC_{p+1} - bC_p]\end{aligned}$$

where we finally used Property 1.11. Thus, we have found exact formulae for $\mathcal{M}((x^4, k), (x^p, 1))$.

Remark. Note that the connectivity argument for the support of the optimizing measure is valid for any c-convex potential, $c > 0$. It was shown in [72] that the optimal measure has always the form, in the small-parameters region,

$$d\mu_{\mathbf{t}}(x) = ch(x)\sqrt{(x - a_1)(a_2 - x)}dx$$

with h a polynomial. However, as the degree of $V_{\mathbf{t}}$ grows, the equations for the parameters of h become more and more complicated.

15.2 Enumeration of colored maps from matrix models

In the context of colored maps, exact computations are much more scarce. However, for the Ising model, some results can again be obtained (it corresponds to maps with colored vertices of a given degree, say p, corresponding to monomials $V_1(A) = A^p$ and $V_2(B) = B^p$) that can be glued together by a bi-colored straight line (corresponding to the monomial AB). For the Ising model with quartic polynomial (the case $p = 4$), M. Mehta [152] obtained an explicit expression for the free energy corresponding to the potential $V(A, B) = V_{Ising}(A, B) = -cAB + V_1(A) + V_2(B)$ when $V_1 = V_2 = (g/4)x^4$. The corresponding results for the enumeration of colored quadrangulation was recently recovered by M. Bousquet-Melou and G. Scheaffer. Let us emphasize that there is a general approach also based on Schwinger–Dyson equations that should allow us to understand these results, see B. Eynard [87]. The remark is that by the Schwinger-Dyson equation we know that the limiting state $\mu_{\mathbf{t}}$ satisfies

$$\mu_{\mathbf{t}}((W_1'(A) - B)P) = \mu_{\mathbf{t}} \otimes \mu_{\mathbf{t}}(\partial_A P),$$

$$\mu_{\mathbf{t}}((W_2'(B) - A)P) = \mu_{\mathbf{t}} \otimes \mu_{\mathbf{t}}(\partial_B P).$$

Taking $P = P(A) = (x - A)^{-1}$ in the second equation and $P(A, B) = \frac{1}{(x-A)} \frac{(W_2'(y) - W_2'(B))}{(y-B)}$ we find that if

$$E(x,y) = (y - W_1'(x))(x - W_2'(y)) + 1 - Q(x,y)$$

with

$$Q(x,y) = \mu_{\mathbf{t}}\left(\frac{W_1'(x) - W_1'(A)}{(x-A)}\frac{W_2'(y) - W_2'(B)}{(y-B)}\right),$$

$$E(x, W_1'(x) - G\mu_A(x)) = 0$$

where $G\mu_A(x) = \mu_{\mathbf{t}}((x-A)^{-1})$. Hence, as for one-matrix models, $G\mu_A$ is solution to an algebraic equation, with some unknown coefficients $\mu_{\mathbf{t}}(A^i B^j)$ with i, j smaller are equal to the degree of W_1' (resp. W_2') minus one. The large deviations theorem 15.1 should now show us (but we have not yet been able to prove it) that for small enough parameters, the support of μ_A and μ_B are connected. Connectivity of the support was in fact proved by using dynamics in a general convex potential setting, including the Ising model, in [106]. This information should also prescribe uniquely the solution.

Bibliographical notes. The question of enumerating maps was first tackled by Tutte [194, 193, 192] who enumerated rooted planar triangulations and quadrangulations (see, e.g., E. Bender and E. Canfield [29] for generalizations). In general, the equations obtained by Tutte's approach are not exactly solvable; their analysis was the subject of subsequent developments (see [97]). Because this last problem is in general difficult, a bijective approach was developed after the work of Cori and Vauquelin [69] and Schaeffer's thesis (see e.g [176]). It was shown that planar triangulations and quadrangulations can be encoded by labeled trees, which are much easier to count. This idea proved to be very fruitful in many respects and was generalized in many ways [24, 46]. It allows us not only to study the number of maps but also part of their geometry; P. Chassaing and G. Schaeffer [62] could prove that the diameter of uniformly distributed quadrangulations with n vertices behaves like $n^{\frac{1}{4}}$. This in particular allowed a limiting object for random planar maps to be defined [135, 144]. Such results seem out of reach of random-matrix techniques. The case of planar bi-colored maps related to the so-called Ising model on random planar graphs could also be studied [44]. However, there are still many several-colors problems that could be solved by using random matrices but not on a direct combinatorial approach, see e.g the Potts model [80], the dually weighted graph model [124, 102].

Part VI

Free probability

Free probability is a probability theory for non-commutative variables. In this field, random variables are usually bounded operators on a Hilbert space. The law of a self-adjoint operator T is given as the evaluation $(\langle \zeta, T^n \zeta \rangle)_{n\geq 0}$ of its moments in the direction of a fixed vector ζ of this Hilbert space. Large $N \times N$ matrices can be seen to fit in this framework as bounded operators on the Hilbert space $\mathbb{C}^N$ equipped for instance with the Euclidean scalar product. We will see in fact that free probability is the right framework to consider random matrices as their size goes to infinity.

For the sake of completeness, but actually not needed for our purpose, we shall recall some notions of operator algebras. We shall then describe free probability as a probability theory on non-commutative functionals (a point of view that forgets the space of realizations of the laws) equipped with the notion of freeness that generalizes the idea of independence to this non-commutative setting. We will then focus on large random matrices and show that their asymptotics are related with freeness. In particular, independent Wigner's matrices converge to free semi-circular operators and the Hermitian Brownian motion converges to the free Brownian motion. Conversely, large random matrices can be seen as an approximation to a large class of (and maybe all) operators. In particular, ideas from classical probability, once applied to large random matrices, can be imported to operator algebra theory via such an approximating scheme. In this part, we shall emphasize the uses of stochastic dynamics, as applied to the Hermitian and the free Brownian motions, to obtain large deviations estimates and study free entropies.

16

Free probability setting

16.1 A few notions about algebras and tracial states

Definition 16.1. *A C^*-algebra $(\mathcal{A}, *)$ is a complex algebra equipped with an involution $*$ and a norm $||.||_{\mathcal{A}}$ such that $\mathcal{A}$ is complete for the norm $\|\cdot\|_{\mathcal{A}}$ and, for any $X, Y \in \mathcal{A}$,*

$$\|XY\|_{\mathcal{A}} \leq \|X\|_{\mathcal{A}} \, \|Y\|_{\mathcal{A}}, \quad \|X^*\|_{\mathcal{A}} = \|X\|_{\mathcal{A}}, \quad \|XX^*\|_{\mathcal{A}} = \|X\|_{\mathcal{A}}^2.$$

$X \in \mathcal{A}$ is self-adjoint iff $X^* = X$. $\mathcal{A}_{sa}$ denote the set of self-adjoint elements of $\mathcal{A}$. A C^*-algebra $(\mathcal{A}, *)$ is unital if it contains a neutral element I.

$\mathcal{A}$ can always be realized as a sub-C^*-algebra of the space $B(H)$ of bounded linear operators on a Hilbert space H. For instance, if $\mathcal{A}$ is a unital C^*-algebra furnished with a positive linear form τ, one can always construct such a Hilbert space H by completing and separating $L^2(\tau)$ (this is the Gelfand–Neumark–Segal construction, see [186, Theorem 2.2.1]). We shall restrict ourselves to this case in the sequel and denote by H a Hilbert space equipped with a scalar product $\langle .,. \rangle_H$ such that $\mathcal{A} \subset B(H)$.

Definition 16.2. *If $\mathcal{A}$ is a sub-C^*-algebra of $B(H)$, $\mathcal{A}$ is a von Neumann algebra iff it is closed for the weak topology, generated by the semi-norms $\{p_{\xi,\eta}(X) = \langle X\xi, \eta \rangle_H, \xi, \eta \in H\}$.*

Let us notice that by definition, a von Neumann algebra contains only bounded operators. The theory nevertheless allows us to consider unbounded operators thanks to the notion of affiliated operators. A densely defined self-adjoint operator X on H is said to be affiliated to $\mathcal{A}$ iff for any Borel function f on the spectrum of X, $f(X) \in \mathcal{A}$ (see [167, p.164]). Here, $f(X)$ is well defined for any operator X as the operator with the same eigenvectors as X and eigenvalues given by the image of those of X by the map f. Murray and von Neumann have proved that if X and Y are affiliated with $\mathcal{A}$, $aX + bY$ is also affiliated with $\mathcal{A}$ for any $a, b \in \mathbb{C}$.

A state τ on a unital von Neumann algebra $(\mathcal{A}, *)$ is a linear form on $\mathcal{A}$ such that $\tau(\mathcal{A}_{sa}) \subset \mathbb{R}$ and:
1. **Positivity** $\tau(\mathbf{A}\mathbf{A}^*) \geq 0$, for any $\mathbf{A} \in \mathcal{A}$.
2. **Total mass** $\tau(I) = 1$.

A tracial state satisfies the additional hypothesis:

3. **Traciality** $\tau(\mathbf{AB}) = \tau(\mathbf{BA})$ for any $\mathbf{A}, \mathbf{B} \in \mathcal{A}$.

The couple $(\mathcal{A}, \tau)$ of a von Neumann algebra equipped with a state τ is called a W^*- probability space.

Exercise 16.3. *1. Let $n \in \mathbb{N}$, and consider $\mathcal{A} = M_n(\mathbb{C})$ as the set of bounded linear operators on $\mathbb{C}^n$. For any $v \in \mathbb{C}^n$, $\langle v, v \rangle_{\mathbb{C}^n} = \sum_{i=1}^n |v_i|^2 = \|v\|_{\mathbb{C}^n}^2 = 1$,*

$$\tau_v(M) = \langle v, Mv \rangle_{\mathbb{C}^n}$$

is a state. There is a unique tracial state on $M_n(\mathbb{C})$ that is the normalized trace

$$\frac{1}{n}\mathrm{Tr}(M) = \frac{1}{n}\sum_{i=1}^n M_{ii}.$$

2. Let $(X, \Sigma, d\mu)$ be a classical probability space. Then $\mathcal{A} = L^\infty(X, \Sigma, d\mu)$ equipped with the expectation $\tau(f) = \int f d\mu$ is a (non-)commutative probability space. Here, $L^\infty(X, \Sigma, d\mu)$ is identified with the set of bounded linear operators on the Hilbert space H obtained by separating $L^2(X, \Sigma, d\mu)$ (by the equivalence relation $f \simeq g$ iff $\mu((f-g)^2) = 0$). The identification follows from the multiplication operator $M(f)g = fg$. Observe that $\mathcal{A}$ is weakly closed for the semi-norms $(\langle f, .g \rangle_H, f, g \in L^2(\mu))$ as $L^\infty(X, \Sigma, d\mu)$ is the dual of $L^1(X, \Sigma, d\mu)$.

3. Let G be a discrete group, and $(e_h)_{h \in G}$ be a basis of $\ell^2(G)$. Let $\lambda(h)e_g = e_{hg}$. Then, we take $\mathcal{A}$ to be the von Neumann algebra generated by the linear span of $\lambda(G)$. The (tracial) state is the linear form such that $\tau(\lambda(g)) = 1_{g=e}$($e$ = neutral element).
We refer to [200] for further examples and details.

The notion of law $\tau_{X_1,\dots,X_m}$ of m operators $(X_1, \dots, X_m)$ in a W^*-probability space $(\mathcal{A}, \tau)$ is simply given by the restriction of the trace τ to the algebra generated by $(X_1, \dots, X_m)$, that is by the values

$$\tau_{X_1,\dots,X_m}(P) := \tau(P(X_1, \dots, X_m)), \quad P \in \mathbb{C}\langle X_1, \dots X_m \rangle$$

where $\mathbb{C}\langle X_1, \dots X_m \rangle$ denotes the set of polynomial functions of m noncommutative variables.

16.2 Space of laws of m non-commutative self-adjoint variables

Following the above description, laws of m non-commutative self-adjoint variables can be seen as elements of the set $\mathcal{M}^{(m)}$ of linear forms τ on the set of polynomial functions of m non-commutative variables $\mathbb{C}\langle X_1, \dots X_m\rangle$ furnished with the involution

$$(X_{i_1} X_{i_2} \cdots X_{i_n})^* = X_{i_n} X_{i_{n-1}} \cdots X_{i_1}$$

and such that:

1. **Positivity** $\tau(PP^*) \geq 0$, for any $P \in \mathbb{C}\langle X_1, \dots X_m\rangle$,
2. **Traciality** $\tau(PQ) = \tau(QP)$ for any $P, Q \in \mathbb{C}\langle X_1, \dots X_m\rangle$,
3. **Total mass** $\tau(I) = 1$.

This point of view is identical to the previous one. Indeed, by the Gelfand–Neumark–Segal construction, being given $\mu \in \mathcal{M}^{(m)}$, we can construct a W^*-probability space $(\mathcal{A}, \tau)$ and operators $(X_1, \dots, X_m)$ such that

$$\mu = \tau_{X_1,\dots,X_m}. \tag{16.1}$$

This construction can be summarized as follows. Consider the bilinear form on $\mathbb{C}\langle X_1, \dots X_m\rangle^2$ given by

$$\langle P, Q\rangle_\tau = \tau(PQ^*).$$

We let H be the Hilbert space obtained as follows. We set

$$L^2(\tau) = \overline{\mathbb{C}\langle X_1, \dots X_m\rangle}^{||\cdot||_\tau}$$

to be the completion of $\mathbb{C}\langle X_1, \dots X_m\rangle$ for the norm $||.||_\tau = \langle ., .\rangle_\tau^{\frac{1}{2}}$. We then separate $L^2(\tau)$ by taking the quotient by the left ideal

$$L_\mu = \{F \in L^2(\tau) : ||F||_\tau = 0\}.$$

Then $H = L^2(\tau)/L_\mu$ is a Hilbert space with scalar product $\langle ., .\rangle_\tau$. The non-commutative polynomials $\mathbb{C}\langle X_1, \dots X_m\rangle$ act by left multiplication on $L^2(\tau)$ and we can consider the completion of these multiplication operators for the semi-norms $\{\langle P, .Q\rangle_H; P, Q \in L^2(\tau)\}$, which forms a von Neumann algebra $\mathcal{A}$ equipped with a tracial state τ satisfying (16.1). In this sense, we can think about $\mathcal{A}$ as the set of bounded measurable functions $L^\infty(\tau)$. The topology under consideration is usually in free probability the $\mathbb{C}\langle X_1, \dots X_m\rangle$-* topology that is $\{\tau_{X_1^n,\dots,X_m^n}\}_{n\in\mathbb{N}}$ converges to $\tau_{X_1,\dots,X_m}$ iff for every $P \in \mathbb{C}\langle X_1, \dots X_m\rangle$,

$$\lim_{n\to\infty} \tau_{X_1^n,\dots,X_m^n}(P) = \tau_{X_1,\dots,X_m}(P).$$

If $(X_1^n, \dots, X_m^n)_{n\in\mathbb{N}}$ are non-commutative variables whose law $\tau_{X_1^n,\dots,X_m^n}$ converges to $\tau_{X_1,\dots,X_m}$, then we shall also say that $(X_1^n, \dots, X_m^n)_{n\in\mathbb{N}}$ converges in law (or in distribution) to $(X_1, \dots, X_m)$.

Such a topology is reasonable when one deals with uniformly bounded non-commutative variables. In fact, if we consider for $R \in \mathbb{R}^+$,

$$\mathcal{M}_R^{(m)} := \{\mu \in \mathcal{M}^{(m)} : \mu(X_i^{2p}) \leq R^p,\ \forall p \in \mathbb{N},\ 1 \leq i \leq m\},$$

then $\mathcal{M}_R^{(m)}$, equipped with this $\mathbb{C}\langle X_1, \dots X_m\rangle$-* topology, is a Polish space (i.e a complete metric space). In fact, $\mathcal{M}_R^{(m)}$ is compact by the Banach–Alaoglu theorem. A distance is for instance given by

$$d(\mu, \nu) = \sum_{n\geq 0} \frac{1}{2^n} |\mu(P_n) - \nu(P_n)|$$

where $\{P_n\}_{n\in\mathbb{N}}$ is a dense sequence of polynomials with operator norm bounded by one when evaluated at any set of self-adjoint operators with operator norms bounded by R.

This notion is the generalization of laws of m real-valued variables bounded by a given finite constant R, in which case the $\mathbb{C}\langle X_1, \dots X_m\rangle$-* topology driven by polynomial functions is the same as the standard weak topology. Actually, it is not hard to check that $\mathcal{M}_R^{(1)} = \mathcal{P}([-R, R])$. However, it may be useful to consider more general topologies, compatible with the existence of unbounded operators, as might be encountered for instance when considering the deviations of large random matrices. One way to do that is to change the set of test functions (as one does in the case $m = 1$ where bounded continuous test functions are often chosen to define the standard weak topology). In [55], the set of test functions was chosen to be the complex vector space $\mathcal{CC}_{st}^m(\mathbb{C})$ generated by the Stieltjes functionals

$$ST^m(\mathbb{C}) = \left\{ \prod_{1\leq i\leq n}^{\rightarrow} \left(z_i - \sum_{k=1}^m \alpha_i^k X_k \right)^{-1} ; \quad z_i \in \mathbb{C}\backslash\mathbb{R}, \alpha_i^k \in \mathbb{Q}, n \in \mathbb{N} \right\} \tag{16.2}$$

where $\prod^{\rightarrow}$ denotes the non-commutative product. It can be checked easily that, with such type of test functions, $\mathcal{M}^{(m)}$ is again a Polish space.

A particular important example of non-commutative laws is given by the empirical distribution of matrices.

Definition 16.4. *Let $N \in \mathbb{N}$ and consider m Hermitian matrices $A_1^N, \cdots, A_m^N \in \mathcal{H}_N^m$ with spectral radius $||A_i^N||_\infty \leq R$, $1 \leq i \leq m$. Then, the empirical distribution of the matrices $(A_1^N, \dots, A_m^N)$ is given by*

$$\mathbf{L}_{A_1^N,\dots,A_m^N}(P) := \frac{1}{N} \mathrm{Tr}\left(P(A_1^N, \dots, A_m^N)\right), \quad \forall P \in \mathbb{C}\langle X_1, \cdots X_m\rangle.$$

Exercise 16.5. *Show that* $\mathbf{L}_{A_1^N,\ldots,A_m^N}$ *is an element of the set* $\mathcal{M}_R^{(m)}$ *of non-commutative laws. Moreover, if* $(A_1^N,\ldots,A_m^N)_{N\in\mathbb{N}}$ *is a sequence such that*

$$\lim_{N\to\infty} \mathbf{L}_{A_1^N,\ldots,A_m^N}(P) = \tau(P), \quad \forall P \in \mathbb{C}\langle X_1,\cdots X_m\rangle,$$

show that $\tau \in \mathcal{M}_R^{(m)}$.

It is actually a long-standing question posed by A. Connes to know whether all $\tau \in \mathcal{M}^{(m)}$ *can be approximated in such a way. In the case* $m = 1$*, the question amounts to asking if for all* $\mu \in \mathcal{P}([-R,R])$*, there exists a sequence* $(\lambda_1^N,\ldots,\lambda_N^N)_{N\in\mathbb{N}}$ *such that*

$$\lim_{N\to\infty} \frac{1}{N}\sum_{i=1}^{N} \delta_{\lambda_i^N} = \mu.$$

This is well known to be true by the Birkhoff's theorem (which is based on the Krein–Milman theorem), but still an open question when $m \geq 2$.

Bibliographical Notes. Introductory notes to free probability can be found in [200, 203, 118, 202]. Basics on operator algebra theory are taken from [167, 83].

17

Freeness

In this chapter we first define freeness as a non-commutative analog of independence. We then show how independent matrices, as their size goes to infinity, converge to free variables.

17.1 Definition of freeness

Free probability is not only a theory of probability for non-commutative variables; it contains also the central notion of freeness, that is, the analog of independence in standard probability.

Definition 17.1. *The variables $(X_1, \ldots, X_m)$ and $(Y_1, \ldots, Y_n)$ are said to be free iff for any $(P_i, Q_i)_{1 \le i \le p} \in (\mathbb{C}\langle X_1, \ldots, X_m\rangle \times \mathbb{C}\langle X_1, \ldots, X_n\rangle)^p$,*

$$\tau\left(\prod_{1 \le i \le p}^{\rightarrow} P_i(X_1, \ldots, X_m) Q_i(Y_1, \ldots, Y_n)\right) = 0 \tag{17.1}$$

as soon as

$$\tau\left(P_i(X_1, \ldots, X_m)\right) = 0, \qquad \tau\left(Q_i(Y_1, \ldots, Y_n)\right) = 0, \qquad \forall i \in \{1, \ldots, p\}.$$

More generally, let $(\mathcal{A}, \phi)$ be a non-commutative probability space and consider unital subalgebras $\mathcal{A}_1, \ldots, \mathcal{A}_m \subset \mathcal{A}$. Then, $\mathcal{A}_1, \ldots, \mathcal{A}_m$ are free if and only if for any $(a_1, \ldots, a_n) \in \mathcal{A}$, $a_j \in \mathcal{A}_{i_j}$

$$\phi(a_1 \cdots a_n) = 0$$

as soon as $i_j \neq i_{j+1}$ for $1 \le j \le n-1$ and $\phi(a_i) = 0$ for all $i \in \{1, \ldots, n-1\}$.

Observe that the assumption that $\tau(Q_p(Y_1, \ldots, Y_n)) = 0$ can be removed by linearity.

Remark 17.2. *(1) The notion of freeness defines uniquely the law of*

$$\{X_1, \dots, X_m, Y_1, \dots, Y_n\}$$

once the laws of $(X_1, \dots, X_m)$ and $(Y_1, \dots, Y_n)$ are given (in fact, check that every expectation of any polynomial is given uniquely by induction over the degree of this polynomial).

(2) If X and Y are free variables with joint law τ, and $P, Q \in \mathbb{C}\langle X\rangle$ such that $\tau(P(X)) = 0$ and $\tau(Q(Y)) = 0$, it is clear that $\tau(P(X)Q(Y)) = 0$ as it should for independent variables, but also $\tau(P(X)Q(Y)P(X)Q(Y)) = 0$ which is very different from what happens with usual independent commutative variables where $\mu(P(X)Q(Y)P(X)Q(Y)) = \mu(P(X)^2Q(Y)^2) > 0$.

(3)The above notion of freeness is related with the usual notion of freeness in groups as follows. Let $(x_1, \dots, x_m, y_1, \dots, y_n)$ be elements of a group. Then, $(x_1, \dots, x_m)$ is said to be free from $(y_1, \dots, y_n)$ if any non-trivial words in these elements is not the neutral element of the group, i.e., that for every monomials $P_1, \dots, P_k \in \mathbb{C}\langle X_1, \dots, X_m\rangle$ and $Q_1, \dots, Q_k \in \mathbb{C}\langle X_1, \dots, X_n\rangle$, $P_1(x)Q_1(y)P_2(x)\cdots Q_k(y)$ is not the neutral element as soon as the $Q_k(y)$ and the $P_i(x)$ are not the neutral element. If we consider, following example 16.3.3), the map that is one on trivial words and zero otherwise and extend it by linearity to polynomials, we see that this defines a tracial state on the operators of left multiplication by the elements of the group and that the two notions of freeness coincide.

(4) We shall see below that examples of free variables naturally show up when considering random matrices with size going to infinity.

17.2 Asymptotic freeness

Definition 17.3. *Let $(X_1^N, \dots, X_m^N)_{N\geq 0}$ and $(Y_1^N, \dots, Y_n^N)_{N\geq 0}$ be two families of $N \times N$ random Hermitian matrices on a probability space (Ω, P). $(X_1^N, \dots, X_m^N)_{N\geq 0}$ and $(Y_1^N, \dots, Y_n^N)_{N\geq 0}$ are asymptotically free almost surely (respectively in expectation) iff $\mathbf{L}_{X_1^N,\dots,X_m^N,Y_1^N,\dots,Y_n^N}(Q)$ (respectively $\mathbb{E}[\mathbf{L}_{X_1^N,\cdots X_m^N,Y_1^N,\cdots Y_n^N}(Q)]$) converges for all polynomial Q to $\tau(Q)$ and τ satisfies* (17.1).

We claim that:

Lemma 17.4. *Let $(X_1^N, \dots, X_m^N)_{N\in\mathbb{N}}$ be m independent matrices taken from the GUE. Then, $(X_1^N, \dots, X_{m-1}^N)_{N\in\mathbb{N}}$ and $(X_m^N)_{N\in\mathbb{N}}$ are asymptotically free almost surely (and in expectation).*

Proof. By Theorem 8.4 with $V = 0$, $\mathbf{L}_{X_1^N,\cdots X_m^N}$ converges almost surely and in expectation as N goes to infinity. Moreover, the limit satisfies the Schwinger–Dyson equation $\mathbf{SD}[\mathbf{0}]$

$$\tau(X_iP) = \tau \otimes \tau(\partial_i P).$$

We check by induction over the degree of P that $\mathbf{SD}[\mathbf{0}]$ implies (17.1). Indeed, (17.1) is true if the total degree of $R = \prod^{\rightarrow} P_i Q_i$ is one by definition. Let us assume this relation is true for any monomial $R = \prod^{\rightarrow} P_j Q_j$ of degree less than k with $\tau(P_j(X_1, \ldots, X_{m-1})) = 0$ and $\tau(Q_j(X_m)) = 0$ except possibly for the first P_j. If now $R' = X_i R$ for some $i \leq m-1$, the Schwinger–Dyson equation $\mathbf{SD}[\mathbf{0}]$ gives

$$\tau(R') = \sum_{R = R_1 X_i R_2} \tau(R_1)\tau(R_2)$$

where R_1 and R_2 are polynomials of degree less than k so that all monomials in R_2 are centered, except possibly the first one. Thus, $\tau(R_2) = 0$ by induction unless $R_2 = 1$ in which case $\tau(R_1)$ vanishes since by traciality it can be written as $\tau(P_1' R_1')$ with R_1' an alternated product of centered monomials. Hence, $\tau(R') = 0$. We can prove similarly that $\tau(X_m \prod^{\rightarrow} Q_j P_j)$ vanishes. This proves the induction. □

We next show that Lemma 17.4 can be generalized to laws that are invariant by multiplication by unitary matrices. Assume that the algebra generated by $(A_1^N, \cdots A_m^N)_{N \geq 0}$ is closed under the involution $*$ and that the operator norm of the matrix A_N^i is bounded independently of N. Finally, suppose that

$$\lim_{N \to \infty} \mathbf{L}_{A_1^N, \ldots, A_m^N} = \mu.$$

Let $U_1^N, \ldots, U_m^N$ be m independent unitary matrices, independent of the A_N^i's, following the Haar measure on $\mathcal{U}(N)$. Then:

Theorem 17.5. $\{A_i^N\}_{1 \leq i \leq m}$ *and* $\{U_i^N, (U_i^N)^{-1}\}_{1 \leq i \leq m}$ *are asymptotically free almost surely and in expectation. Moreover, the variables* $\{U_i^N, (U_i^N)^{-1}\}_{1 \leq i \leq m}$ *are asymptotically free almost surely with limit law* τ *such that* $\tau(U_i^n) = 1_{n=0}$ *for all* $n \in \mathbb{Z}$. *In particular,* $(U_i A_i U_i^{-1})_{1 \leq i \leq m}$ *are free variables under* τ.

Exercise 17.6. *Extend the theorem to the case where the Haar measure on* $\mathcal{U}(N)$ *is replaced by the Haar measure on* $\mathcal{O}(N)$ *Hint: extend the proof below.*

The proof we shall provide here follows the change of variable trick that we used to analyze matrix models. It can be found in [199] (and is taken, in the form below, from a work with B. Collins and E. Maurel Segala [66]).

Proof. 1. *Corresponding Schwinger–Dyson equation.* We denote by m_N the Haar measure on $\mathcal{U}(N)$. By definition, m_N is invariant under left multiplication by a unitary matrix. In particular, if $P \in \mathbb{C}\langle A_i, U_i, U_i^{-1}, 1 \leq i \leq m\rangle$, we have for all $kl \in \{1, \ldots, N\}^2$,

$$\partial_t \int \left(P(A_i, e^{tB_i} U_i, U_i^* e^{-tB_i})\right)_{kl} dm_N(U_1) \cdots dm_N(U_m) = 0$$

for any antihermitian matrices B_i ($B_i^* = -B_i$). Taking B_i null except for $i = i_0$ and B_{i_0} null except at the entries qr and rq, we find that

$$\int [\partial_{i_0} P]_{kr,ql}(A_i, U_i, U_i^*) dm_N(U_1) \cdots dm_N(U_m) = 0$$

with ∂_i the derivative that obeys Leibniz's rule

$$\partial_i(PQ) = \partial_i P \times 1 \otimes Q + P \otimes 1 \times \partial_i Q \tag{17.2}$$

so that

$$\partial_i U_j = 1_{j=i} U_j \otimes 1, \partial_i U_j^* = -1_{j=i} 1 \otimes U_j^* \tag{17.3}$$

and $[A \otimes B]_{kr,ql} := A_{kr} B_{ql}$. Taking $k = r$ and $q = l$ and summing over r, q gives

$$\mathbb{E}\left[\mathbf{L}_{A_i^N, U_i^N, (U_i^N)^*, 1 \le i \le m} \otimes \mathbf{L}_{A_i^N, U_i^N, (U_i^N)^*, 1 \le i \le m}(\partial_i P)\right] = 0.$$

Observe that $\mathbf{L}_{A_i^N, U_i^N, (U_i^N)^*, 1 \le i \le m}(P)$ is a Lipschitz function of the unitary matrices $\{U_i^N, (U_i^N)^*\}_{1 \le i \le p}$ with uniformly bounded constant since we assumed that the A_i^N's are uniformly bounded for the operator norm. Thus, we can use the concentration result of Theorem 6.16 to deduce that for all $P \in \mathbb{C}\langle A_i, U_i, U_i^{-1}, 1 \le i \le m\rangle$

$$\lim_{N \to \infty} \mathbb{E}[\mathbf{L}_{A_i^N, U_i^N, (U_i^N)^*, 1 \le i \le m}] \otimes \mathbb{E}[\mathbf{L}_{A_i^N, U_i^N, (U_i^N)^*, 1 \le i \le m}](\partial_i P) = 0. \tag{17.4}$$

Observe that $\mathbb{E}[\mathbf{L}_{A_i^N, U_i^N, (U_i^N)^*, 1 \le i \le m}]$ can be identified with the noncommutative law of self-adjoint variables $\mathbb{E}[\mathbf{L}_{A_i^N, V_i^N, W_i^N, 1 \le i \le m}]$ where $V_i^N = U_i^N + (U_i^N)^*$ and $W_i^N = (U_i^N - (U_i^N)^*)/\sqrt{-1}$ up to an obvious change of variables. If A denotes a uniform bound on the spectral radius of the $\{A_i^N\}_{1 \le i \le m}$, $\mathbb{E}[\mathbf{L}_{A_i^N, V_i^N, W_i^N, 1 \le i \le m}]$ belongs to the compact set $\mathcal{M}_{A \vee 2}^{(3m)}$. Thus, we can consider limit points of $\mathbb{E}[\mathbf{L}_{A_i^N, V_i^N, W_i^N, 1 \le i \le m}]$, and therefore of $\mathbb{E}[\mathbf{L}_{A_i^N, U_i^N, (U_i^N)^*,, 1 \le i \le m}]$. If τ is such a limit point, we deduce from (17.4) that it must satisfy the Schwinger–Dyson equation

$$\tau \otimes \tau(\partial_i P) = 0 \tag{17.5}$$

for all $i \in \{1, \dots, m\}$ and $P \in \mathbb{C}\langle A_i, U_i, U_i^{-1}, 1 \le i \le m\rangle$. Note here that we can bypass the concentration argument by using the change of variable trick that shows that the above equation is satisfied almost surely asymptotically (see [66]).

2. *Uniqueness of the solution to* (17.5).
Let τ be a tracial solution to (17.5) and P be a monomial. Note that if P depends only on the A_i's, $\tau(P) = \mu(P)$ is uniquely determined. If P belongs to $\mathbb{C}\langle A_i, U_i, U_i^{-1}, 1 \le i \le m\rangle \backslash \mathbb{C}\langle A_i, 1 \le i \le m\rangle$, we can always write $\tau(P) = \tau(QU_i)$ or $\tau(P) = \tau(U_i^{-1}Q)$ for some monomial Q. Let us consider the first case (the second can then be deduced from the fact that $\tau(U_i^{-1}Q) = \tau(Q^* U_i)$). Then, we have

$$\partial_i(QU_i) = \partial_i Q \times 1 \otimes U_i + (QU_i) \otimes 1$$

and so (17.5) gives

$$\tau(QU_i) = -\tau \otimes \tau(\partial_i Q \times 1 \otimes U_i)$$
$$= -\sum_{Q=Q_1 U_i Q_2} \tau(Q_1 U_i)\tau(Q_2 U_i) + \sum_{Q=Q_1 U_i^* Q_2} \tau(Q_1)\tau(Q_2)$$

where we used $\tau(U_i^{-1} Q_2 U_i) = \tau(Q_2)$ by traciality. Each term in the above right-hand side is the expectation under τ of a polynomial of degree strictly smaller in U_i and U_i^{-1} than QU_i. Hence, this relation defines uniquely τ by induction.

3. *The solution is the law of free variables.* It is enough to show by the previous point that if the algebra generated by $\{U_i, U_i^{-1}, 1 \leq i \leq m\}$ is free from the $A_i's$, then the corresponding tracial state on $\mathbb{C}\langle A_i, U_i, U_i^{-1}, 1 \leq i \leq m\rangle$ satisfies (17.5). So take $P = U_{i_1}^{n_1} B_1 \cdots U_{i_p}^{n_p} B_p$ with some B_k's in the algebra generated by $(A_i, 1 \leq i \leq m)$. We wish to show that for all $i \in \{1, \ldots, m\}$,
$$\mu \otimes \mu(\partial_i P) = 0.$$
By linearity, it is enough to prove this equality when $\mu(B_j) = 0$ for all j. Using repeatedly (17.2) and (17.3), we find that
$$\partial_i P = \sum_{k: i_k = i, n_k > 0} \sum_{l=1}^{n_k} U_{i_1}^{n_1} B_1 \cdots B_{k-1} U_i^l \otimes U_i^{n_k - l} B_k \cdots U_{i_p}^{n_p} B_p$$
$$- \sum_{k: i_k = i, n_k < 0} \sum_{l=0}^{n_k - 1} U_{i_1}^{n_1} B_1 \cdots B_{k-1} U_i^{-l} \otimes U_i^{n_k + l} B_k \cdots U_{i_p}^{n_p} B_p$$
Taking the expectation on both sides, since $\mu(U_j^i) = 0$ and $\mu(B_j) = 0$ for all $i \neq 0$ and j, we see that freeness implies that the right-hand side vanishes (recall here that in the definition of freeness, two consecutive elements have to be in free algebras but the first and the last element can be in the same algebra). Thus, $\mu \otimes \mu(\partial_i P)$ vanishes, which proves the claim.

The last point of the theorem is a direct consequence of the asymptotic freeness of the algebra which implies that for all $B_i \in \mathbb{C}\langle A_i, 1 \leq i \leq m\rangle$ such that $\mu(B_i) = \tau(U_j B_i U_j^{-1}) = 0$
$$\tau(B_1 U_{i_1} B_2 U_{i_1}^{-1} B_3 \cdots U_{ip}^{-1}) = 0$$
and therefore $(U_i^N A_i^N (U_i^N)^{-1})_{1 \leq i \leq m}$ are asymptotically free.

□

17.3 The combinatorics of freeness

It is a natural question to wonder, if a, b are two free bounded variables in a non-commutative probability space, what is the law of their sum $a+b$ or of their product ab. The study of this question is the object of this section. We restrict ourselves to bounded variables.

17.3.1 Free cumulants

In practice, the notion of cumulants often appears to be easier to work with to compute laws than the direct use of (17.1). We recall below the definition of free cumulants that is based on non-crossing partitions.

Definition 17.7.
- *A partition of a the set $S := \{1, \ldots, n\}$ is a decomposition*

  $$\pi = \{V_1, \ldots, V_r\}$$

  of S into disjoint and non empty sets V_i.
- *The set of all partitions of S is denoted by $\mathcal{P}(S)$, and for short by $\mathcal{P}(n)$ if $S := \{1, \ldots, n\}$.*
- *The $V_i, 1 \leq i \leq r$, are called the blocks of the partition and we say that $p \sim_\pi q$ if p, q belong to the same block of the partition π.*

The central result of this section is that freeness is related with *non-crossing* partitions:

Definition 17.8.
- *A partition π of $\{1, \ldots, n\}$ is said to be crossing if there exists $1 \leq p_1 < q_1 < p_2 < q_2 \leq n$ with*

  $$p_1 \sim_\pi p_2 \not\sim_\pi q_1 \sim_\pi q_2.$$

  It is non- crossing otherwise.
- *The set of non-crossing partitions of $\{1, \ldots, n\}$ is denoted by $NC(n)$.*
- *We let $\leq$ be the partial order on $NC(n)$; if $\pi, \pi' \in NC(n)$, $\pi \leq \pi'$ iff every block of π is included into a block of π'. For this partial order, $\mathbf{0_n} := \{(1), \ldots, (n)\}$ (resp. $\mathbf{1_n} := \{(1, \ldots, n)\}$) is the "smallest" (resp. "largest") element of $NC(n)$.*
- *We write $NC(S_1) \equiv NC(S_2)$ iff S_1 and S_2 have the same number of elements.*

A pictural description of non-crossing versus crossing partitions was given in Figure 1.4. In practice, the following recursive definition of non-crossing partition shall be used.

Property 17.9. *A partition π of $S = \{1, \ldots, n\}$ is non-crossing iff there is at least one block V of π that is an interval (i.e., of the form $\{p, p+1, \ldots, p+q\}$ for some $q \geq 0$ and $1 \leq p \leq p+q \leq n$) and the restriction of π to $S \backslash V$ is non-crossing.*

Proof. If π is non-crossing, we consider the block V that contains 1. If it is an interval, we are done, and otherwise we consider a block S' in $S\backslash V$ that is contained in between the first and the last elements of V (here, elements of a block are ordered) and whose first element is the smaller between the possible choices of S'. Since the restriction of π to this block is non-crossing, we can reiterate this procedure. Since the cardinality of elements of S' is strictly less than the cardinality of S, the iteration of this decomposition will lead us to a block W of the partition that is of the form $\{p, p+1, \ldots, p+q\}$ with $q \geq 0$. Moreover, the restriction of π to $S\backslash W$ is non-crossing. Reciprocally, since the restriction of π to an interval is non-crossing and we assumed π non-crossing once restricted to $S\backslash V$, π is non-crossing. $\square$

Definition 17.10. *Let $(\mathcal{A}, \phi)$ be a non-commutative probability space. The free cumulants are defined as a collection of multi-linear functionals*

$$k_n : \mathcal{A}^n \to \mathbb{C} \quad (n \in \mathbb{N})$$

by the following system of equations:

$$\phi(a_1 \cdots a_n) = \sum_{\pi \in NC(n)} k_\pi(a_1, \ldots, a_n)$$

with, if $\pi = (V_1, \ldots, V_r)$ with $V_i = \{v_1^i, \cdots, \ldots, v_{l_i}^i\}$ for $1 \leq i \leq r$,

$$k_\pi(a_1, \ldots, a_n) = k_{l_1}(a_{v_1^1}, \ldots, a_{v_{l_1}^1}) k_{l_2}(a_{v_1^2}, \ldots, a_{v_{l_2}^2}) \cdots k_{l_r}(a_{v_1^r}, \ldots, a_{v_{l_r}^r}).$$

Observe that the above system of equations is implicit but defines uniquely the cumulants k_n since

$$k_n(a_1, \ldots, a_n) = \phi(a_1 \cdots a_n) - \sum_{\substack{\pi \in NC(n) \\ \pi \neq \mathbf{1}_n}} k_\pi(a_1, \ldots, a_n)$$

where the last term in the right-hand side only depends on $(k_l, l \leq n-1)$. This last definition of k_n also shows its existence by induction over n (remark that k_π is multilinear).

Example 17.11.
- *$n = 1$, $k_1(a_1) = \phi(a_1)$.*
- *$n = 2$, $\phi(a_1 a_2) = k_2(a_1, a_2) + k_1(a_1) k_1(a_2)$ and so*

$$k_2(a_1, a_2) = \phi(a_1 a_2) - \phi(a_1)\phi(a_2).$$

- *$n = 3$,*

$$\begin{aligned} k_3(a_1, a_2, a_3) = \phi(a_1 a_2 a_3) &- \phi(a_1)\phi(a_2 a_3) - \phi(a_1 a_3)\phi(a_2) \\ &- \phi(a_1 a_2)\phi(a_3) + 2\phi(a_1)\phi(a_2)\phi(a_3). \end{aligned}$$

We now turn to the description of freeness in terms of cumulants.

Theorem 17.12. *Let $(\mathcal{A}, \phi)$ be a non-commutative probability space and consider unital subalgebras $\mathcal{A}_1, \ldots, \mathcal{A}_m \subset \mathcal{A}$. Then, $\mathcal{A}_1, \ldots, \mathcal{A}_m$ are free if and only if for all $n \geq 2$ and for all $a_i \in \mathcal{A}_{j(i)}$ with $1 \leq j(1), \ldots, j(n) \leq m$,*

$$k_n(a_1, \ldots, a_n) = 0 \quad \textit{if there exists } 1 \leq l, k \leq n \textit{ with } j(l) \neq j(k). \tag{17.6}$$

Observe here that the description of freeness by cumulants does not require any centering of the variables; all the questions of centering concern only the cumulant k_1. In fact, we have:

Property 17.13. *Let $(\mathcal{A}, \phi)$ be a non-commutative probability space and $a_1, \ldots, a_n$ be elements of $\mathcal{A}$. Assume $n \geq 2$. If there is $i \in \{1, \ldots, n\}$ so that $a_i = 1$, then*

$$k_n(a_1, \ldots, a_n) = 0.$$

As a consequence, for $n \geq 2$, any $a_1, \ldots, a_n \in \mathcal{A}$,

$$k_n(a_1, \ldots, a_n) = k_n(a_1 - \phi(a_1), a_2 - \phi(a_2), \ldots, a_n - \phi(a_n)).$$

Proof. We prove this result by induction over $n \geq 2$. First, for $n = 2$ we have, since $k_1(a) = \phi(a)$

$$\phi(a_1 a_2) = k_2(a_1, a_2) + \phi(a_1)\phi(a_2)$$

and so if $a_1 = 1$, we deduce, since $\phi(1) = 1$, that

$$\phi(a_2) = \phi(1 a_2) = k_2(a_1, a_2) + \phi(a_2) \Rightarrow k_2(a_1, a_2) = 0.$$

The same argument holds when $a_2 = 1$. Let us assume that for $p \leq n - 1$, $k_p(a_1, \ldots, a_p) = 0$ if one of the a_p is the neutral element. Consider the step n with $a_i = 1$. Then

$$\phi(a_1 \cdots a_n) = k_n(a_1, \ldots, a_n) + \sum_{\substack{\pi \in NC(n) \\ \pi \neq \mathbf{1}_n}} k_\pi(a_1, \ldots, a_n) \tag{17.7}$$

where by our induction hypothesis all the partitions π in the above sum where the element i is not a block of the partition do not contribute. But then

$$\begin{aligned} \sum_{\substack{\pi \in NC(n) \\ \pi \neq \mathbf{1}_n}} k_\pi(a_1, \ldots, a_n) &= \sum_{\pi \in NC(n-1)} k_\pi(a_1, \ldots, a_{i-1}, a_{i+1}, \ldots, a_n) \\ &= \phi(a_1 \cdots a_{i-1} a_{i+1} \cdots a_n) = \phi(a_1 \cdots a_n) \end{aligned}$$

which proves $k_n(a_1, \ldots, a_n) = 0$ with (17.7). □

Proof of Theorem 17.12. Assume that the cumulants vanish when evaluated at elements of different algebras $\mathcal{A}_1, \ldots, \mathcal{A}_m$ and consider, for $a_i \in \mathcal{A}_{j(i)}$,

$$\phi((a_1 - \phi(a_1)) \cdots (a_n - \phi(a_n))) = \sum_{\pi \in NC(n)} k_\pi(a_1, \ldots, a_n).$$

By our hypothesis, k_π vanishes as soon as a block of π contains $p, q \in \{1, \ldots, n\}$ so that $j(p) \neq j(q)$. Therefore, if we assume $j(p) \neq j(p+1)$, we see that the contribution in the above sum comes from partitions π whose blocks cannot contain two nearest neighbors $\{p, p+1\}$. By Property 17.9, this implies that π contains an interval of the form $V = \{p\}$. But then k_π also vanishes since $k_1 = 0$ by centering of the variables. Therefore, if for $1 \leq p \leq n-1$ $j(p) \neq j(p+1)$, we get

$$\phi((a_1 - \phi(a_1)) \cdots (a_n - \phi(a_n))) = 0,$$

that is ϕ satisfies (17.1).

Reciprocally, let us assume that ϕ satisfies (17.1). We prove that (17.6) is satisfied by induction over n. It is clear for $n = 2$ since then we saw that $k_2(a_1, a_2) = \phi(a_1 a_2) - \phi(a_1)\phi(a_2)$. Let us assume it is true for $p \leq n-1$, $n \geq 3$.

We first prove that $k_n(a_1, \ldots, a_n) = 0$ when $a_i \in \mathcal{A}_{j(i)}, 1 \leq i \leq n$ with $j(i) \neq j(i+1)$ for all $1 \leq i \leq n-1$. Indeed, for any $\pi \in NC(n)$, $\pi \neq 1_n$, k_π will vanish as soon as it contains two nearest neighbors by our induction hypothesis. But again by property 17.9, this implies that π contains a singleton. Thus $k_\pi(a_1 - \phi(a_1), \ldots, a_n - \phi(a_n)) = 0$ since $k_1(a_i - \phi(a_i)) = 0$ and since also $\phi((a_1 - \phi(a_1)) \cdots (a_n - \phi(a_n))$ vanish, we deduce that $k_n(a_1 - \phi(a_1), \ldots, a_n - \phi(a_n))$ vanishes. Since k_n does not depend on the centering by the previous property, we have shown that $k_n(a_1, \ldots, a_n) = 0$ when $j(i) \neq j(i+1)$ for all $1 \leq i \leq n-1$, $j(n) \neq j(1)$.

To prove that $k_n(a_1, \ldots, a_n)$ vanishes as soon as a couple of a_i's belong to different subalgebras, we shall show how to come back to the situation of alternating moments by the next lemma.

Lemma 17.14. *Consider $n \geq 2$, $a_1, \ldots, a_n \in \mathcal{A}$ and $1 \leq p \leq n-1$. Then,*

$$\begin{aligned} &k_{n-1}(a_1, \ldots, a_{p-1}, a_p a_{p+1}, a_{p+2}, \ldots, a_n) \\ &= k_n(a_1, \cdots a_p, a_{p+1}, \ldots, a_n) \\ &\quad + \sum_{\substack{\pi \in NC(n) \\ \sharp\pi = 2, p \not\sim_\pi p+1}} k_\pi(a_1, \ldots, a_p, a_{p+1}, \ldots, a_n) \end{aligned}$$

where $\sharp\pi = 2$ means that π has exactly two blocks.

We complete the proof of the theorem before proving the lemma. We have $a_i \in \mathcal{A}_{j(i)}$ with some $j(l) \neq j(p)$. If $j(p) \neq j(p+1)$ for all $1 \leq p \leq n-1$ then we are done by the previous consideration. Otherwise, there is a p so that $j(p) = j(p+1)$. We can then use Lemma 17.14 to reduce the number of variables. By our induction hypothesis $k_{n-1}(a_1, \cdots, a_{p-1}, a_p a_{p+1}, a_{p+2}, \ldots, a_n)$ vanishes, but also $k_\pi(a_1, \ldots, a_p, a_{p+1}, \ldots, a_n)$ since π decomposes into two blocks of size strictly smaller than n, one of which containing an element of a subalgebra free with $\mathcal{A}_{j(p)}$. Therefore, $k_n(a_1, \cdots a_p, a_{p+1}, \ldots, a_n)$ vanishes and the theorem is proved. $\square$

Proof of Lemma 17.14. Again, we prove it by induction over n. For $n = 2$, the equality reads

$$k_1(a_1 a_2) = k_2(a_1, a_2) + k_1(a_1)k_1(a_2) \Leftrightarrow k_2(a_1, a_2) = \phi(a_1 a_2) - \phi(a_1)\phi(a_2)$$

that we have already seen. We thus assume the equality true for $p \leq n-1$ for some $n \geq 3$. For $\pi \in NC(n)$, let us denote by $\pi|_{p=p+1}$ the partition obtained by identifying p and $p+1$, namely if $\pi = \{V_1, \ldots, V_r\}$ with $p \in V_j, p+1 \in V_l$,

$$\pi|_{p=p+1} = \{V_1, \ldots, V_j \cup V_l \backslash \{p+1\}, \ldots, V_r\}.$$

Note that $\pi|_{p=p+1} \in NC(n-1)$. In terms of such a partition, the equality of the lemma can be restated as

$$k_{\mathbf{1}_{n-1}}(a_1, \ldots, a_p a_{p+1}, \ldots, a_n) = \sum_{\substack{\pi \in NC(n) \\ \pi|_{p=p+1} = \mathbf{1}_{n-1}}} k_\pi(a_1, \ldots, a_p, a_{p+1}, \ldots, a_n)$$

Since we assumed that this equality is true for $l < n$, we deduce that for any $\sigma \in NC(n-1)$, $\sigma \neq \mathbf{1}_{n-1}$ so that the block containing $a_p a_{p+1}$ has length strictly smaller than $n-1$,

$$k_\sigma(a_1, \ldots, a_p a_{p+1}, \cdots, a_n) = \sum_{\substack{\pi \in NC(n) \\ \pi|_{p=p+1} = \sigma}} k_\pi(a_1, \ldots, a_p, a_{p+1}, \ldots, a_n).$$

Therefore, we have

$$\begin{aligned}
&k_{\mathbf{1}_{n-1}}(a_1, \ldots, a_{p-1}, a_p a_{p+1}, a_{p+2}, \ldots, a_n) \\
&= \phi(a_1 \cdots a_n) - \sum_{\substack{\sigma \in NC(n-1) \\ \sigma \neq \mathbf{1}_n}} k_\sigma(a_1, \ldots, a_p a_{p+1}, \ldots, a_n) \\
&= \phi(a_1 \cdots a_n) - \sum_{\substack{\sigma \in NC(n-1) \\ \sigma \neq \mathbf{1}_{n-1}}} \sum_{\substack{\pi \in NC(n) \\ \pi|_{p=p+1} = \sigma}} k_\pi(a_1, \ldots, a_p, a_{p+1}, \ldots, a_n) \\
&= \sum_{\sigma \in NC(n)} k_\sigma(a_1, \ldots, a_n) - \sum_{\substack{\pi \in NC(n) \\ \pi|_{p=p+1} \neq \mathbf{1}_{n-1}}} k_\pi(a_1, \ldots, a_n) \\
&= \sum_{\substack{\sigma \in NC(n) \\ \sigma|_{p=p+1} = \mathbf{1}_{n-1}}} k_\sigma(a_1, \ldots, a_n)
\end{aligned}$$

which proves the claim. □

Bibliographical notes. This section followed quite closely R. Speicher [185]. Note that in classical probability, cumulants play also a similar role but then partition can be crossing (e.g., Shiryaev [178, p. 290]).

We next exhibit a consequence of free independence, namely free harmonic analysis. The problem of interest is to determine the law of $a+b$ when a, b are free. Since the law of (a, b) with a, b free is uniquely determined by the laws μ_a of a and μ_b of b, the law of their sum is a function of μ_a and μ_b denoted by $\mu_a \boxed{+} \mu_b$. There are several approaches to the problem; we shall present the combinatorial approach based on free cumulants and refer the interested reader to [159] for more details.

17.3.2 Free additive convolution

Definition 17.15. *Let a, b be two operators in a non-commutative probability space $(\mathcal{A}, \phi)$ with law μ_a, μ_b respectively. If a, b are free, the law of $a+b$ is denoted by $\mu_a \boxed{+} \mu_b$.*

We write for short $k_n(a) = k_n(a, \ldots, a)$ as the nth cumulant of the variable a.

Lemma 17.16. *Let a, b be two bounded operators in a non-commutative probability space $(\mathcal{A}, \phi)$. If a and b are free, for all $n \geq 1$*

$$k_n(a+b) = k_n(a) + k_n(b).$$

Proof. The result is obvious for $n = 1$ by linearity of k_1. Moreover, for all $n \geq 2$, by multilinearity of the cumulants,

$$k_n(a+b) = \sum_{\epsilon_i = 0,1} k_n(\epsilon_1 a + (1-\epsilon_1)b, \ldots, \epsilon_n a + (1-\epsilon_n)b) = k_n(a) + k_n(b)$$

where the second line is a direct consequence of Theorem 17.12. □

As a consequence, let us define the following generating function of cumulants:

Definition 17.17. *For a bounded operator a the formal power series*

$$R_a(z) = \sum_{n \geq 0} k_{n+1}(a) z^n$$

is called the R-transform of the law μ_a of the operator a. We also write $R_{\mu_a} := R_a$ since R only depends on the law μ_a.

Then, the R-transform is to free probability what the log-Fourier transform is to classical probability in the sense that it is linear for free convolution.

Corollary 17.18. *Let a, b be two bounded operators in a non-commutative probability space $(\mathcal{A}, \phi)$. If a and b are free,*

$$R_{a+b} = R_a + R_b \Leftrightarrow R_{\mu_a \boxed{+} \mu_b} = R_{\mu_a} + R_{\mu_b}.$$

We next provide a more tractable definition of the R-transform in terms of the Cauchy transform. Suppose that $\mu : \mathbb{C}[X] \to \mathbb{C}$ is a distribution with all moments. Then we may define G_μ as the formal series

$$G_\mu(z) = \sum_{n \geq 0} \mu(X^n) z^{-(n+1)}. \tag{17.8}$$

Let $K_\mu(z)$ be the formal inverse of G_μ, i.e., $G_\mu(K_\mu(z)) = z$. The *formal* power series expansion of K_μ is

$$K_\mu(z) = \frac{1}{z} + \sum_{n=1}^{\infty} C_n z^{n-1}.$$

Then, we shall prove the following:

Lemma 17.19. *Let μ be a compactly supported probability measure. For all $n \in \mathbb{N}$, $C_n = k_{n+1}$. Therefore, $R_\mu(z) = K_\mu(z) - 1/z$.*

Proof. To prove this lemma, we compare the generating function of the cumulants as the formal power series

$$C_a(z) = 1 + \sum_{n=1}^{\infty} k_n(a) z^n$$

and the generating function of the moments as the formal power series

$$M_a(z) = 1 + \sum_{n=1}^{\infty} m_n(a) z^n$$

with $m_n(a) := \mu(a^n)$. Then, we shall prove that

$$C_a(zM_a(z)) = M_a(z). \tag{17.9}$$

The rest of the proof is pure algebra since

$$G_a(z) := G_{\mu_a}(z) = z^{-1} M_a(z^{-1}), R_a(z) := z^{-1}(C_a(z) - 1)$$

then gives $C_a(G_a(z)) = zG_a(z)$ and so by composition by K_a

$$zR_a(z) + 1 = C_a(z) = zK_a(z).$$

This equality is formal and only proves $k_{n+1} = C_n$. We thus need to derive (17.9). To do so, we show that

$$m_n(a) = \sum_{s=1}^{n} \sum_{\substack{i_1,\ldots,i_s \in \{0,1\cdots,n-s\} \\ i_1+\cdots+i_s=n-s}} k_s(a) m_{i_1}(a) \cdots m_{i_s}(a). \tag{17.10}$$

Once (17.10) holds, (17.9) follows readily since

$$\begin{aligned} M_a(z) &= 1 + \sum_{n=1}^{\infty} z^n m_n(a) \\ &= 1 + \sum_{n=1}^{\infty} \sum_{s=1}^{n} \sum_{\substack{i_1,\ldots,i_s \in \{0,1\cdots,n-s\} \\ i_1+\cdots+i_s=n-s}} k_s(a) z^s m_{i_1}(a) z^{i_1} \cdots m_{i_s}(a) z^{i_s} \\ &= 1 + \sum_{s=1}^{\infty} k_s(z) z^s \left(\sum_{i=0}^{\infty} z^i m_i(a) \right)^s = C_a(zM_a(z)). \end{aligned}$$

To prove (17.10), recall that by definition of the cumulants,

$$m_n(a) = \sum_{\pi \in NC(n)} k_\pi(a).$$

Let $\pi = \{V_1, \ldots, V_r\} \in NC(n)$ be given, and let us fix its first block $V_1 = (1, v_2, \ldots, v_s)$ with $s = |V_1| \in \{1, \ldots, n\}$. Being given V_1, since π is non-crossing, we see that for any $l \in \{2, \ldots, r\}$, there exists $k \in \{1, \ldots, s\}$ so that the elements of V_l lies between v_k and v_{k+1}. Here $v_{s+1} = n + 1$ by convention. This means that π decomposes into V_1 and at most s other partitions $\tilde{\pi}_1, \ldots, \tilde{\pi}_s$. Therefore

$$k_\pi = k_s k_{\tilde{\pi}_1} \cdots k_{\tilde{\pi}_s}.$$

If we denote by i_k the number of elements in $\tilde{\pi}_k$, we thus have proved that

$$\begin{aligned} m_n(a) &= \sum_{s=1}^{n} k_s(a) \sum_{\substack{\tilde{\pi}_k \in NC(i_k), \\ i_1+\cdots+i_s=n-s}} k_{\tilde{\pi}_1}(a) \cdots k_{\tilde{\pi}_s}(a) \\ &= \sum_{s=1}^{n} k_s(a) \sum_{\substack{i_1+\cdots+i_s=n-s \\ i_k \geq 0}} m_{i_1}(a) \cdots m_{i_s}(a) \end{aligned}$$

where we used again the relation between cumulants and moments. The proof is thus complete. $\square$

Example 17.20. *Let $\nu_a = \sigma(x)dx$ be the standard semicircle, and note that $G_{\nu_a}(z) = -S_{\nu_a}(z)$. We saw in Corollary 1.12 that $G_{\nu_a}(1/\sqrt{z}) = (1 - \sqrt{1-4z})/2\sqrt{z}$. Thus, $G_{\nu_a}(z) = (z \pm \sqrt{z^2 - 4})/2$, and the correct choice of the branch of the square-root, dictated by the fact that $\Im z > 0$ implies $\Im G_{\nu_a}(z) < 0$, leads to the formula*

$$G_{\nu_a}(z) = \frac{z - \sqrt{z^2 - 4}}{2}.$$

Thus,

$$z = \frac{K_{\nu_a}(z) - \sqrt{K_{\nu_a}^2(z) - 4}}{2}$$

with solution $K_{\nu_a}(z) = z^{-1} + z$. *In particular, the R-transform of the semicircle is the linear function* z, *and summing two (freely independent) semicircles yields again a semicircle with a different variance. Indeed, repeating the computation above, the R-transform of a semicircle with support* $[-\alpha, \alpha]$ *(or equivalently with covariance* $\alpha^2/2$*) is* $\alpha^2 z/2$. *Note here that the linearity of the R-transform is equivalent to* $k_n(a) = 0$ *except if* $n = 2$, *and* $k_2(a) = \alpha^2/2 = \phi(a^2)$.

Exercise 17.21. *Let* $\mu = \frac{1}{2}(\delta_{+1} + \delta_{-1})$. *Then,* $G_\mu(z) = (z^2 - 1)^{-1} z$ *and*

$$R_\mu(z) = \frac{\sqrt{1 + 4z^2} - 1}{2z}.$$

Show that $\mu \boxed{+} \mu$ *is absolutely continuous with respect to the Lebesgue measure and with density const.*$/\sqrt{4 - x^2}$.

Bibliographical notes. In these notes, we only considered the R-transform as formal series. It is, however, possible to see it as an analytic function once restricted to an appropriate subset of the complex plane. The study of multiplicative convolution can be performed similarly, see, e.g., [159]. This section closely follows the lecture notes of Roland Speicher [185]. Lots of refinements of the relation between free cumulants and freeness can be found for instance in the book by Nica and Speicher [159]. Here, we only considered free convolution of bounded operators; the generalization holds for unbounded operators and can be found in [30]. Our last example is a particularly simple example of infinite divisibility; the theory of free infinite divisibility parallels the classical one (in particular, a Levy–Khitchine formula does exist to characterize them) (see cf. [30],[20]). Related with free convolution come natural questions such as the regularizing effect of free convolution. A detailed study of free convolution by a semi-circular variable was done by P. Biane [34].

Free entropy

Free entropy was defined by Voiculescu as a generalization of classical entropy to the non-commutative context. There are several definitions of free entropy; we shall concentrate on two of them. The first is the so-called microstates entropy that measures a volume of matrices with empirical distribution approximating a given law. The second, called the microstates-free entropy, is defined via a non-commutative version of Fisher information. The classical analog of these definitions is, on one hand, the definition of the entropy of a measure μ as the volume of points whose empirical distribution approximates μ, and, on the other hand, the well-known entropy $\int \frac{d\mu}{dx} \log \frac{d\mu}{dx} dx$. In this classical setting, Sanov's theorem shows that these two entropies are equal. The free analog statement is still open but we shall give in this section bounds to compare the microstates and the microstates-free entropies. The ideas come from [55, 56, 37] but we shall try to simplify the proof to hopefully make it more accessible to non-probabilists (the original proof uses Malliavin calculus but we shall here give an elementary version of the few properties of Malliavin calculus we need). In the following, we consider only laws of self-adjoint variables (i.e., $A_i^* = A_i$ for $1 \le i \le m$). We do not lose generality since any operator can be decomposed as the sum of two self-adjoint operators.

Definition 18.1. *Let $\tau \in \mathcal{M}^m$. Let $R \in \mathbb{R}^+$, $\epsilon > 0$ and $k, N \in \mathbb{N}$. We then define the microstate*

$$\begin{aligned} \Gamma_N(\tau; \epsilon, k, R) = \{A_1, \ldots, A_m \in \mathcal{H}_N^m : \|A_i\|_\infty \le R, \\ |\mathbf{L}_{A_1,\ldots,A_m}(X_{i_1} \cdots X_{i_p}) - \tau(X_{i_1} \cdots X_{i_p})| \le \epsilon \\ \textit{for all } i_j \in \{1, \ldots, m\}, p \le k\}. \end{aligned}$$

We then define the microstates entropy of τ) by

$$\chi(\tau) = \limsup_{\substack{\epsilon \to 0, L \to \infty \\ k \to \infty}} \limsup_{N \to \infty} \frac{1}{N^2} \log \mu_N^{\otimes m} \left(\Gamma_N(\tau; \epsilon, k, L) \right).$$

Remark 18.2. • *Voiculescu's original definition $\chi_{original}$ of the entropy consists in taking the Lebesgue measure over $\mathcal{H}_N^m$ rather than the Gaussian measure $\mu_N^{\otimes m}$. However, both definitions are equivalent up to a quadratic weight since as soon as $k \geq 2$, the Gaussian weight is almost constant on a small microstate. Hence, we have (see [56])*

$$\chi(\tau) = \chi_{original}(\tau) - \frac{1}{2}\sum_{i=1}^{m} \tau(X_i^2) - mc$$

with

$$c = \lim_{N\to\infty} \frac{1}{N^2} \log \int_{\mathcal{H}_N} e^{-\frac{N}{2}\mathrm{Tr}(A^2)} dA = \sup_{\mu \in \mathcal{P}(\mathbb{R})} \left\{ \Sigma(\mu) - \frac{1}{2}\mu(x^2) \right\}.$$

- *It was proved by Belinschi and Bercovici [23] that in the definition of χ, one does not need to take L going to infinity but rather any L fixed greater than R if $\tau \in \mathcal{M}_R^m$. For the same reason, χ can be defined as the asymptotic volume of $\Gamma_N(\tau; \epsilon, k, \infty)$.*
- *The classical analog is, if γ is the standard Gaussian law, to take a probability measure μ on $\mathbb{R}$ and define, if d is a distance on $\mathcal{P}(\mathbb{R})$ compatible with the weak topology,*

$$S(\mu) = \limsup_{\epsilon\to\infty} \limsup_{N\to\infty} \frac{1}{N^2} \log \gamma^{\otimes N} \left(d(\frac{1}{N}\sum_{i=1}^{N} \delta_{x_i}, \mu) < \epsilon \right).$$

  the main difference is here that we take bounded continuous test functions, instead of polynomials, and so do not need the cutoff $\cap_i\{\|A_i\|_\infty \leq L\}$. We shall later on also adopt this point of view in the proofs, to avoid dealing with the cutoff.
- *It is an open problem whether one can replace the limsup by a liminf in the definition of χ without changing its value. This question can be seen to be equivalent to the convergence of $N^{-2} \log \int_{\|A_i\|_\infty \leq R} e^{\mathrm{Tr}\otimes\mathrm{Tr}(V)} d\mu_N^{\otimes m}$ as N goes to infinity for any polynomial V in $\mathbb{C}\langle X_1, \ldots, X_m\rangle^{\otimes 2}$ and all $R \in \mathbb{R}^+$ large enough (see Property 18.3).*

Hereafter, when no confusion is possible, $\mathbf{L}^\mathbf{N}$ will write for short $\mathbf{L}_{A_1,\ldots,A_m}$ with $A_1, \ldots, A_m$ generic Hermitian $N \times N$ matrices.

There is a dual definition to the microstates entropy χ, namely:

Property 18.3. *Let $F \in \mathbb{C}\langle X_1, \ldots, X_m\rangle \otimes \mathbb{C}\langle X_1, \ldots, X_m\rangle$ and define, for $L \in \mathbb{R}^+$, its Legendre transform by*

$$\Lambda_L(F) = \limsup_{N\to\infty} \frac{1}{N^2} \log \int_{\|A_i\|_\infty \leq L} e^{N^2 \mathbf{L}^\mathbf{N} \otimes \mathbf{L}^\mathbf{N}(F)} d\mu_N^{\otimes m}.$$

Then, if $\tau \in \mathcal{M}_R^m$ is such that $\chi(\tau) > -\infty$, for any $L > R$,

$$\chi(\tau) = \inf_F \{-\tau \otimes \tau(F) + \Lambda_L(F)\}.$$

Proof. Clearly, for all $F \in \mathbb{C}\langle X_1, \ldots, X_m\rangle \otimes \mathbb{C}\langle X_1, \ldots, X_m\rangle$,

$$
\begin{aligned}
&\mu_N^{\otimes m}\left(\Gamma_N(\tau; \epsilon, k, L)\right) \\
&= \mu_N^{\otimes m}\left(1_{\Gamma_N(\tau;\epsilon,k,L)} e^{N^2 \mathbf{L}^{\mathbf{N}} \otimes \mathbf{L}^{\mathbf{N}}(F) - N^2 \mathbf{L}^{\mathbf{N}} \otimes \mathbf{L}^{\mathbf{N}}(F)}\right) \\
&\leq e^{-N^2(\tau\otimes\tau(F)+\delta(F,\epsilon))} \mu_N^{\otimes m}\left(1_{\Gamma_N(\tau;\epsilon,k,L)} e^{N^2 \mathbf{L}^{\mathbf{N}} \otimes \mathbf{L}^{\mathbf{N}}(F)}\right) \qquad (18.1) \\
&\leq e^{-N^2(\tau\otimes\tau(F)+\delta(F,\epsilon))} \mu_N^{\otimes m}\left(1_{\max_i \|A_i\|_\infty \leq L} e^{N^2 \mathbf{L}^{\mathbf{N}} \otimes \mathbf{L}^{\mathbf{N}}(F)}\right) \qquad (18.2)
\end{aligned}
$$

where we assumed in (18.1) that k is larger than the degree of all monomials in F so that $\delta(F, \epsilon)$ goes to zero with ϵ. Taking the logarithm and the large N limit and then the small ϵ limit (together with the remark of Belinschi and Bercovici) we conclude that for L sufficiently large,

$$\chi(\tau) \leq -\tau \otimes \tau(F) + \Lambda_L(F),$$

which gives the upper bound by optimizing over F. For the lower bound, remark that we basically need to show that the inequalities in (18.1) and (18.2) are almost equalities on the large deviation scale for some F. The candidate for F will be given as a multiple of

$$
\begin{aligned}
F := \sum_{\ell=1}^{k} (m+1)^{-\ell} \sum_{i_1,\ldots,i_\ell=1}^{m} & \left(X_{i_1} \cdots X_{i_\ell} - \tau(X_{i_1} \cdots X_{i_\ell})\right) \\
& \otimes \left(X_{i_1} \cdots X_{i_\ell} - \tau(X_{i_1} \cdots X_{i_\ell})\right)
\end{aligned}
$$

so that

$$\mu \otimes \mu(F) = \sum_{\ell=1}^{k} (m+1)^{-\ell} \sum_{i_1,\ldots,i_\ell=1}^{m} \left(\mu(X_{i_1} \cdots X_{i_\ell}) - \tau(X_{i_1} \cdots X_{i_\ell})\right)^2.$$

Note that for matrices bounded by L, $A_1, \ldots, A_m \in \Gamma_N(\tau, \epsilon, k, L)$ if

$$0 \leq \mathbf{L}_{A_1,\ldots,A_m} \otimes \mathbf{L}_{A_1,\ldots,A_m}(F) \leq (m+1)^{-k} \epsilon^2$$

so that for all $L \geq 0$, if we set $F_N := \mathbf{L}^N \otimes \mathbf{L}^N(F)$,

$$
\begin{aligned}
&\lim_{\epsilon \downarrow 0} \limsup_{N\to\infty} \frac{1}{N^2} \log \mu_N^{\otimes m}\left(\Gamma_N(\tau; \epsilon, k, L)\right) \\
&\quad \geq \lim_{\epsilon \downarrow 0} \limsup_{N\to\infty} \frac{1}{N^2} \log \mu_N^{\otimes m}\left(\max_i \|A_i\|_\infty \leq L, F_N \leq \epsilon\right).
\end{aligned} \qquad (18.3)
$$

But, since $F_N \geq 0$, for any $\gamma > 0$,

$$\begin{aligned}&\mu_N^{\otimes m}\left(\max_i\|A_i\|_\infty \le L, F_N \le \epsilon\right) \ge \mu_N^{\otimes m}\left(1_{\max_i\|A_i\|_\infty \le L, F_N\le\epsilon} e^{-\gamma N^2 F_N}\right)\\ &= \mu_N^{\otimes m}\left(1_{\max_i\|A_i\|_\infty \le L} e^{-\gamma N^2 F_N}\right) - \mu_N^{\otimes m}\left(1_{\max_i\|A_i\|_\infty \le L, F_N>\epsilon} e^{-\gamma N^2 F_N}\right)\\ &=: I_N^1 - I_N^2\end{aligned}$$

For the first term, for any ϵ', k' and L we have

$$I_N^1 \ge \mu_N^{\otimes m}\left(1_{\Gamma_N(\tau;\epsilon',k',L)} e^{-\gamma N^2 F_N}\right) \ge e^{-\gamma N^2(m+1)(\epsilon')^2}\mu_N^{\otimes m}\left(\Gamma_N(\tau;\epsilon',k',L)\right).$$

Therefore, for L large enough (but finite according to [23])

$$\limsup_{N\to\infty}\frac{1}{N^2}\log I_N^1 \ge \chi(\tau) > -\infty.$$

On the other hand, $I_N^2 \le e^{-\gamma N^2\epsilon}$ is negligible with respect to I_N^1 if $\gamma\epsilon > -\chi(\tau)$. Thus, we conclude by (18.3) that

$$\lim_{\epsilon\downarrow 0}\limsup_{N\to\infty}\frac{1}{N^2}\log\mu_N^{\otimes m}\left(\Gamma_N(\tau;\epsilon,k,L)\right) \ge \limsup_{N\to\infty}\frac{1}{N^2}\log I_N^1 = \Lambda_L(-\gamma F) \tag{18.4}$$

and therefore that, with $G = -\gamma F$,

$$\begin{aligned}\lim_{\epsilon\downarrow 0}\limsup_{N\to\infty}\frac{1}{N^2}\log\mu_N^{\otimes m}\left(\Gamma_N(\tau;\epsilon,k,L)\right) &\ge -\tau\otimes\tau(G) + \Lambda_L(G)\\ &\ge \inf_F\{-\tau\otimes\tau(F) + \Lambda_L(F)\}.\end{aligned}$$

We finally take the limit k, L going to infinity to conclude. □

Remark 18.4. *In the classical case, it is enough to take a linear function of* $\mathbf{L^N}$*. This in particular implies that the rate function (corresponding to* $-\chi$*) is convex. This cannot be the case in the non-commutative case since Voiculescu (see [198]) proved that if* $\chi(\tau_1) > -\infty$ *and* $\chi(\tau_2) > -\infty$, $\chi(\alpha\tau_1 + (1-\alpha)\tau_2) = -\infty$ *for all* $\alpha \in]0,1[$.

Let us now introduce the microstates-free entropy. Its definition is based on the notion of free Fisher information which is given, for a tracial state τ, by

$$\Phi^*(\tau) = 2\sum_{i=1}^m \sup_{P\in\mathbb{C}\langle X_1,\dots,X_m\rangle}\left\{\tau\otimes\tau(\partial_i P) - \frac{1}{2}\tau(P^2)\right\}$$

with ∂_i the non-commutative derivative defined in section 7.2.2. Then, we define the microstates-free entropy χ^* by

$$\chi^*(\tau) = -\frac{1}{2}\int_0^1 \Phi^*(\tau_{tX+\sqrt{t(1-t)}S})dt$$

with S an m-dimensional semicircular law, free from X.

Theorem 18.5. *There exists $\chi^{**} \le 0$ so that for all $\tau \in \mathcal{M}_R^m$,*

$$\chi^{**}(\tau) \le \chi(\tau) \le \chi^*(\tau).$$

Remark 18.6. *Many questions around microstates-free entropy remain open and important. It would be very interesting to show that the limsup in its definition can be replaced by liminf. The two bounds above still hold in fact if we perform this change. It would be great to prove that $\chi = \chi^*$ at least on $\chi < \infty$. It is not clear at all that $\chi^{**} = \chi^*$ in general (since this amounts to saying that any law can be achieved (at least approximatively) as the marginal at time one of a very smooth process) but should be expected for instance for laws such as those encountered in Part III).*

The idea is to try to compute the Legendre transforms $\Lambda_\infty(F)$ for F in $\mathbb{C}\langle X_1, \ldots, X_m\rangle^{\otimes 2}$. However, this is not easy directly. We shall follow a standard path of thoughts in large deviation theory and consider the problem in a bigger space; namely instead of random Wigner matrices, we shall consider Hermitian Brownian motions (which, at time one, have the same law than Gaussian Wigner matrices) and generalize the ideas of Part V to the multi-matrix setting. We then study the deviations of the empirical distributions of m independent Hermitian Brownian motions. It is simply defined as a linear form on the set of polynomials of the indeterminates $(X_{t_j}^{i_j}, 1 \le j \le n)$ for any choices of times $(t_j, 1 \le j \le n)$. It might be possible to argue that we can use the polynomial topology, but it is in fact easier to use a topology of bounded functions at this point, since it is easier for us to estimate Laplace transforms without cutoff, a cutoff that is necessary in general to insure that the polynomial topology is good. The space of test functions we shall use is the set of Stieltjes functionals as introduced in (16.2). To go from processes with continuous-time parameters to finite-time marginals, a standard way is to consider *continuous processes* and to make sure that large deviations hold in this space of continuous processes. A final point is that changing the topology (from polynomials to Stieltjes functionals) does not change the entropy of laws in $\mathcal{M}_R^m$, as was proved in Lemma 7.1 of [37]. We next describe the setting of continuous processes and show how Laplace transforms of time marginals can be computed.

1. *Space of laws of continuous processes.* We let $\mathcal{F}$ be the space of functions on processes indexed by $t \in [0,1]$ such that $F \in \mathcal{F}$ iff there exists $n \in \mathbb{N}$, $t_1, \ldots, t_n \in [0,1]^n$, $i_1, \ldots, i_n \in \{1, \ldots, m\}^n$ and $S \in ST^n(\mathbb{C})$ such that

$$F(X_.^1, \ldots, X_.^m) = T(X_{t_1}^{i_1}, \ldots, X_{t_n}^{i_n}).$$

$ST^n(\mathbb{C})$ contains the multiplicative neutral element 1 for all $n \in \mathbb{N}$ and is equipped with the involution

$$\left(\prod_{1\le i\le p}^{\rightarrow} (z_i - \sum_{k=1}^m \alpha_i^k X_k)^{-1} \right)^* = \prod_{p\le i\le 1}^{\rightarrow} (\bar{z}_i - \sum_{k=1}^m \alpha_i^k X_k)^{-1}$$

and so $\mathcal{F}$ is equipped with its natural extension.
We let $\mathcal{M}^P$ be the subset of linear forms τ on $\mathcal{F}$ such that

$$\tau(FG) = \tau(GF) \quad \tau(FF^*) \geq 0 \quad \tau(1) = 1.$$

We endow $\mathcal{M}^P$ with its weak topology. $\mathcal{M}^P$ is a metric space; a distance can for instance be given in the spirit of Levy's distance

$$d(\tau, \nu) = \sum_{k \geq 0} \frac{1}{2^k} |\tau(P_k) - \nu(P_k)|$$

with P_k a countable family of uniformly bounded functions dense in $\mathcal{F}$ (for instance by restricting the parameters α_i^j to take rational values, as well as the complex parameters).
Note that if we restrict $\tau \in \mathcal{M}^P$ to $F = (z - \sum_{i=1}^n \alpha_i X_{t_i}^{j_i})^{-1}$ with some fixed α_i but varying z, then this retriction is a linear form on the set of functionals $(z - .)^{-1}, z \in \mathbb{C} \backslash \mathbb{R}$. By the GNS construction, since τ is a tracial state, $Y = \sum_{i=1}^n \alpha_i X_{t_i}^{j_i}$ can be seen as the law of a self-adjoint operator on a C^*-algebra. Therefore, $\tau|_Y$ can be seen as a positive measure on the real line (by Riesz's theorem). Since we also have $\tau(1) = 1$, $\tau|_Y$ is a probability measure on the real line, it is the spectral measure of Y. Note in particular that for every $\tau \in \mathcal{M}^P$, any $z \in \mathbb{C} \backslash \mathbb{R}$ and $\alpha_i \in \mathbb{R}$

$$\begin{aligned}
&\left\| \left(z - \sum_{i=1}^n \alpha_i X_{t_i}^{j_i} \right)^{-1} \right\|_\infty^\tau \\
&= \lim_{n \to \infty} \left(\tau(\left(\left(z - \sum_{i=1}^n \alpha_i X_{t_i}^{j_i} \right)^{-1} \left(\bar{z} - \sum_{i=1}^n \alpha_i X_{t_i}^{j_i} \right)^{-1} \right)^n) \right)^{\frac{1}{2n}} \\
&\leq \frac{1}{|\Im(z)|}.
\end{aligned}$$

Hence, by non-commutative Hölder inequality Theorem 19.5, for all $\tau \in \mathcal{M}^P$, all $z_i \in \mathbb{C} \backslash \mathbb{R}$, all $\alpha_i^k \in \mathbb{R}$,

$$\left| \tau \left(\prod_{1 \leq i \leq p}^{\rightarrow} (z_i - \sum_{k=1}^m \alpha_i^k X_k)^{-1}) \right) \right| \leq \prod_{1 \leq i \leq p} \frac{1}{|\Im(z_i)|}.$$

We let $\mathcal{M}_c^P$ be the set of laws of *continuous processes*, i.e., the set of $\tau \in \mathcal{M}^P$ such that for all, $n \in \mathbb{N}$, $i_1, \ldots, i_n \in \{1, \ldots, m\}^n$ and $T \in ST^n(\mathbb{C})$ such that

$$t_1, \ldots, t_n \in [0,1]^n \to \tau\left(T(X_{t_1}^{i_1}, \ldots, X_{t_n}^{i_n})\right)$$

is continuous. Note that for all $n \in \mathbb{N}$, $ST^n(\mathbb{C})$ is countable, and therefore by the Arzela–Ascoli theorem, compact subsets of $\mathcal{M}_c^P$ are for example

$$\cap_{n\geq 0}\cap_{p\geq 0}\left\{\tau\in\mathcal{M}_c^P:\sup_{|t_i-s_i|\leq\eta_p^n}\max_{i_1,\ldots,i_n}|f_{t_1,\ldots,t_n}^{\tau,T_n}-f_{s_1,\ldots,s_n}^{\tau,T_n}|\leq\epsilon_n^p\right\}$$

with $f_{t_1,\ldots,t_n}^{\tau,T}=\tau\left(T_n(X_{t_1}^{i_1},\ldots,X_{t_n}^{i_n})\right)$, $\{T_n\}_{n\in\mathbb{N}}$ is some sequence that is dense in the countable space $\cup_{n\geq 0}ST^n(\mathbb{C})$, and ϵ_n^p goes to zero as p goes to infinity for all n while $\eta_p^n>0$.
We let $\mathbf{L_N^P}$ be the element of $\mathcal{M}^P$ given, for $F\in\mathcal{F}$, by

$$\mathbf{L_N^P}(F)=\frac{1}{N}\mathrm{Tr}(F(H_N^1,\ldots,H_N^m))$$

with $H_N^1,\ldots,H_N^m$ m independent Hermitian Brownian motions (and denote $\mathbb{P}_N$ the law of one Hermitian Brownian motion).

2. *Exponential tightness.* We next prove:

Lemma 18.7. *For all $L\in R^+$, there exists a compact set $\mathcal{K}(L)$ of $\mathcal{M}_c^P$ such that*

$$\limsup_{N\to\infty}\frac{1}{N^2}\log\mathbb{P}_N^{\otimes m}\left(\mathbf{L_N^P}\in\mathcal{K}(L)^c\right)\leq -L.$$

Proof. Note that for all $T\in ST^n(\mathbb{C})$, all $\tau\in\mathcal{M}^P$,

$$\begin{aligned}
&\tau\left(T(X_{t_1}^{i_1},\ldots,X_{t_n}^{i_n})\right)-\tau\left(T(X_{s_1}^{i_1},\ldots,X_{s_n}^{i_n})\right)\\
&=\sum_{j=1}^n\int_0^1 d\alpha\,\tau\Big(\partial_i T(\alpha X_{t_1}^{i_1}+(1-\alpha)X_{s_1}^{i_1},\ldots,\\
&\qquad\qquad \alpha X_{t_n}^{i_n}+(1-\alpha)X_{s_n}^{i_n})\sharp(X_{t_j}^{i_j}-X_{s_j}^{i_j})\Big)d\alpha\\
&=\sum_{j=1}^n\int_0^1 \tau\Big(\partial_j T(\alpha X_{t_1}^{i_1}+(1-\alpha)X_{s_1}^{i_1},\ldots,\\
&\qquad\qquad \alpha X_{t_n}^{i_n}+(1-\alpha)X_{s_n}^{i_n})\sharp(X_{t_j}^{i_j}-X_{s_j}^{i_j})\Big)d\alpha\\
&=\sum_{j=1}^n\int_0^1 \tau\Big(D_j T(\alpha X_{t_1}^{i_1}+(1-\alpha)X_{s_1}^{i_1},\ldots,\\
&\qquad\qquad \alpha X_{t_n}^{i_n}+(1-\alpha)X_{s_n}^{i_n})\times(X_{t_j}^{i_j}-X_{s_j}^{i_j})\Big)d\alpha
\end{aligned}$$

with ∂_j (resp. D_j) the non-commutative derivative (resp. cyclic derivative) with respect to the jth variable (see Section 7.2.2). Now, $D_jT(\alpha X_{t_1}^{i_1}+(1-\alpha)X_{s_1}^{i_1},\ldots,\alpha X_{t_n}^{i_n}+(1-\alpha)X_{s_n}^{i_n}$ belongs to $\mathcal{F}$ for any $\alpha\in[0,1]$ and is therefore uniformly bounded (independently of α and τ). Thus, there exists a finite constant c, that depends only on T such that

$$|\tau\left(T(X_{t_1}^{i_1},\ldots,X_{t_n}^{i_n})\right)-\tau\left(T(X_{s_1}^{i_1},\ldots,X_{s_n}^{i_n})\right)|\leq c\sum_{i=1}^n\tau(|X_{t_j}^{i_j}-X_{s_j}^{i_j}|^2)^{\frac{1}{2}}.$$

By the characterization of the compact sets of $\mathcal{M}_c^P$, it is therefore enough to show that for all $L > 0$ and $\epsilon > 0$, there exists $\eta > 0$ such that

$$\mathbb{P}_N\left(\sup_{\substack{|t-s|\le\eta\\0\le s\le t\le 1}}\frac{1}{N}\mathrm{Tr}((H_N(t)-H_N(s))^2)\ge\epsilon\right)\le e^{-N^2L}.$$

But

$$\begin{aligned}
&\mathbb{P}_N\left(\sup_{\substack{|t-s|\le\eta\\0\le s\le t\le 1}}\frac{1}{N}\mathrm{Tr}((H_N(t)-H_N(s))^2)\ge\epsilon\right)\\
&\le\mathcal{W}^{\otimes N^2}\left(\sup_{\substack{|t-s|\le\eta\\0\le s\le t\le 1}}\frac{1}{N^2}\sum_{1\le i,j\le N}((B_{ij}(t)-B_{ij}(s))^2)\ge\epsilon\right)\\
&\le e^{-N^2\epsilon\lambda}\mathcal{W}(e^{\lambda\sup_{\substack{|t-s|\le\eta\\0\le s\le t\le 1}}(B_{ij}(t)-B_{ij}(s))^2})^{N^2}
\end{aligned}$$

where $\mathcal{W}$ is the Wiener law. Using the fact (see [137, theorem 4.1]) that there exists $\alpha > 0$ such that

$$\mathcal{W}(e^{\alpha\delta^{-\frac{1}{4}}\sup_{|t-s|\le\delta}|B(t)-B(s)|})<\infty$$

we see that we can take above $\lambda = \alpha\eta^{-\frac{1}{4}}$ to conclude that if $\epsilon\eta^{-\frac{1}{4}} \ge L$,

$$\mathbb{P}_N\left(\sup_{\substack{|t-s|\le\eta\\0\le s\le t\le 1}}\frac{1}{N}\mathrm{Tr}((H_N(t)-H_N(s))^2)\ge\epsilon\right)\le e^{-N^2(L+C)}$$

for some finite constant C. $\square$

3. *Statement of the large deviation result for processes.* According to Lemma 18.7, the rate function for a large deviation principle for the law of $\mathbf{L_N^P}$ under $P_N^{\otimes m}$ has to be infinite outside $\mathcal{M}_c^P$. For $\tau \in \mathcal{M}_c^P$, we let τ^t be the law of $(X^i_{s\wedge t} + S^i_{s-t\vee 0}, i \in \{1,\dots,m\}, 0 \le s \le t)$ with X with law τ, and S an m-dimensional free Brownian motion, free with X. For $F(X^1,\dots,X^m) = T(X_{t_1} - X_{t_0}, \cdots X_{t_p} - X_{t_{p-1}}) \in \mathcal{F}$ with $T \in ST^{pm}(\mathbb{C})$, $0 \le t_0 \le t_1 \cdots \le t_m \le 1$ and $s \in [0,1]$, we let

$$D_sF(X^1,\dots,X^m)=\sum_{i=1}^{p}1_{s\in[t_{i-1},t_i]}D_iT$$

where D_i is the cyclic gradient with respect to the ith variable (it is m dimensional). Finally, for $\tau \in \mathcal{M}^P$, we let $\tau(.|\mathcal{B}_s)$ be the $L^2(\tau)$ projection over the algebra $\mathcal{B}_s$ generated by the set $\mathcal{F}_s$ of functions of $\mathcal{F}$ that only depend on $X_u, u \le s$:

$$\tau(\tau(P|\mathcal{B}_s)Q) = \tau(PQ) \quad \forall P \in \mathcal{F}, \forall Q \in \mathcal{F}_s.$$

Then we have

Theorem 18.8. *a) The law of* $\mathbf{L_N^P}$ *under* $P_N^{\otimes m}$ *satisfies a large deviation upper bound for the weak* $\mathcal{F}$*-topology in the scale* N^2 *with good rate function*

$$I(\tau) = \sup_{t\in[0,1]} \sup_{F\in\mathcal{F}} \left\{ \tau^t(F) - \tau^0(F) - \frac{1}{2}\int_0^t \tau[|\tau^s(D_s F|\mathcal{B}_s)|^2] ds \right\}.$$

b) If $I(\tau) < \infty$*, there exists a map* $s \to K_s \in L^2(\mathcal{F}_s, \tau)^m$ *such that*

i. $\inf_{F\in\mathcal{F}} \int_0^1 \tau[\|\tau^s(D_s F|\mathcal{B}_s) - K_s\|^2] ds = 0.$

ii. For any $P \in \mathcal{F}$ *any* $t \in [0,1]$

$$\tau^t(F) = \tau^0(F) + \int_0^t \tau(\tau^s(D_s F|\mathcal{B}_s).K_s) ds. \tag{18.5}$$

Moreover, we then have

$$I(\tau) = \frac{1}{2}\int_0^1 \tau(\|K_s\|^2) ds.$$

c) When the infimum in b. i) above is achieved (i.e., there exists $F \in \mathcal{F}$ *such that* $K_s = \tau^s(D_s F|\mathcal{B}_s)$*),* τ *is uniquely determined by* (18.5) *and is the strong solution of the free stochastic differential equation*

$$dX_t = dS_t + K_t(X) dt.$$

d) If τ *is such that the infimum in b.i) above is achieved, the large deviation lower bound is also given by* $I(\tau)$*, i.e*

$$\liminf_{\epsilon\to 0} \liminf_{N\to\infty} \frac{1}{N^2} \log \mathbb{P}_N^{\otimes m}\left(d(\mathbf{L_N^P}, \tau) \le \epsilon\right) \ge -I(\tau)$$

where d *is a Levy distance compatible with the weak* $\mathcal{F}$*-topology.*

If we now use the contraction principle we deduce:

Corollary 18.9. *The law of* $\mathbf{L}^N = \mathbf{L}_{H_1^N(1),\cdots H_m^N(1)}$ *under* $\mathbb{P}_N^{\otimes m}$ *satisfies a large deviation upper bound with rate function*

$$\chi^*(\mu) = -\inf\{I(\tau); \tau|_{X_1^1,\ldots,X_1^m} = \mu\}$$

and a large deviation lower bound with rate function

$$\chi(\mu) = -\inf\{I(\tau); \tau|_{X_1^1,\ldots,X_1^m} = \mu,$$
$$\tau \text{ is such that the infimum in b. i) is achieved } \}$$

To complete the proof and identify χ^* with Voiculescu's original definition, one can use an abstract argument (cf. [56]) to see that the infimum has to be taken at the law of the free Brownian bridge:

$$dX_t = dS_t + \frac{X_t - X}{t-1} dt.$$

Taking the previsible representation of the above process, we get that $K_t = \tau(\frac{X_t - X}{t-1} | X_t) = t^{-1} X_t - \mathcal{J}^{\tau_t}$ with $\mathcal{J}^{\tau_t}$ the conjuguate variable of the law of $X_t = tX + \sqrt{t(1-t)}S$. Plugging this result into the definition of I shows that χ^* is indeed the integral of the free Fisher information along free Brownian motion paths.

4. *Large deviation upper bound.* In the classical case where one considers large deviations for the empirical measure $N^{-1} \sum_{i=1}^{N} \delta_{B_t^i, t \in [0,1]}$, that are solved by Sanov's theorem, it can be seen that deviations (i.e., deviations with finite rate function) occur only along laws that are absolutely continuous with respect to Wiener law. By Girsanov's theorem, one knows that such laws are obtained as weak solutions of the SDE

$$dX_t = dB_t + b(t, (X_s)_{s \le t}) dt$$

for some drift b. These heuristics will extend, as we shall see, to the non-commutative case (once one defines the right notion of weak solution to the SDE). We now compute a few Laplace transforms of quantities depending on a finite number of time marginals under $\mathbf{L}_{\mathbf{N}}^{\mathbf{P}}$. We shall give a pedestrian way to understand the Clark–Ocone formula used in [37].

• First, let $0 = t_0 \le t_1 \cdots \le t_{n-1} \le t_n = 1$ and let us consider $T_k^i \in ST^k(\mathbb{C})^{sa}$ for $k \in \{0, \ldots, n-1\}$, $i \in \{1, \ldots, m\}$. Then, if we denote $\Delta_k t = t_{k+1} - t_k$, $\Delta_k H_N^i = H_N^i(t_{k+1}) - H_N^i(t_k)$ and $\Delta_k H_N = (\Delta_k H_N^i)_{1 \le i \le m}$,

$$\begin{aligned}
\Lambda_N &:= \mathbb{E}\Big[\exp\big\{N \sum_{k=0}^{n-1} \sum_{i=1}^{m} \\
&\quad \mathrm{Tr}\left(T_k^i(\Delta_l H_N, l < k) \Delta_k H_N^i - \frac{1}{2} T_k^i(\Delta_l H_N, l < k)^2 \Delta_k t \right) \big\}\Big] \\
&= \int \prod \frac{d\Delta_k H_N^i}{\sqrt{2\pi \Delta_k t}^{N^2}} \\
&\quad \prod_{i,k} \exp\left\{ -\frac{N}{2\Delta_k} \mathrm{Tr}\left((\Delta_k H_N^i - T_k^i(\Delta_l H_N, l < k)\Delta_k)^2 \right) \right\} \\
&= 1
\end{aligned} \tag{18.6}$$

where we have used that since the above centering of the Gaussian variables only depends on the past, it can be considered as constant and therefore does not change the integral. Putting $\Delta_k X^i = X^i(t_{k+1}) - X^i(t_k)$ and

$$f(\tau) = \sum_{k=0}^{n-1} \sum_{i=1}^{m} \left(\tau \left(T_k^i(\Delta_l X, l < k) \Delta_k X^i - \frac{1}{2} T_k^i(\Delta_l X, l < k)^2) \Delta_k t \right) \right)$$

we deduce that

$$\mathbb{E}[e^{N^2 f(\mathbf{L}_\mathbf{N}^\mathbf{P})}] = \Lambda_N = 1.$$

Hence, this simple computation already shows that considering processes allows us to compute the Laplace transforms $\mathbb{E}[e^{N^2 f(\mathbf{L}_\mathbf{N}^\mathbf{P})}]$ for all functions f as above.

However, the above computation is not sufficient to get a good upper bound since for instance it does not allow to compute the Laplace transform of $\mathbf{L}_\mathbf{N}^\mathbf{P}((\Delta_k X^i)^2)$. To do such a computation, we shall need infinitesimal calculus. Note that the upper bound that one would obtain by using only the previous functionals would allow to show already that the laws with finite entropy are such that there exists a drift K so that $dX_t - K(X_t)dt$ are the increments of a martingale. We could not, however, deduce that it has to be a free Brownian motion without using differential calculus (instead of finite variations).

• The idea to compute more general Laplace transforms is to generalize (18.6) by constructing more martingales. Indeed, the fact that $\Lambda_N(T,t)$ equals one can be seen as a consequence of the fact that

$$t \to \exp\{N^2 \sum_{k=0}^{n-1} \sum_{i=1}^{m} \Big(\mathbf{L}_\mathbf{N}^\mathbf{P} \Big(T_k^i(\Delta_l X, l < k) \Delta_k^t X^i - \frac{1}{2} T_k^i(\Delta_l X, l < k)^2 \Delta_k^t t \Big) \Big) \}$$

is a martingale for the canonical filtration of the underlying Brownian motions if $\Delta_k^t X^i := X^i_{t_{k+1} \wedge t} - X^i_{t_k \wedge t}$ and $\Delta_k^t t = t_{k+1} \wedge t - t_k \wedge t$. A simple way to construct martingales is simply to consider

$$M_t^F := \mathbb{E}[\mathrm{Tr}(F(H_1^N, \ldots, H_m^N)) | \mathcal{F}_t]$$

with $F \in \mathcal{F}$ (and $\mathcal{F}_t$ the canonical filtration of Brownian motion) and the associated exponential martingale

$$E_t^F = \exp\left\{ N M_t^F - \frac{N^2}{2} \langle M^F \rangle_t \right\}.$$

Then, since $F \in \mathcal{F}$ is uniformly bounded, E_t^F is a martingale implying that $\mathbb{E}[E_t^F] = \mathbb{E}[E_0^F] = 1$. Now, we can always write, with $H^N = (H_1^N, \ldots, H_m^N)$,

$$\mathbb{E}[\mathrm{Tr}(F(H^N)) | \mathcal{F}_t] = \mathbb{E}_{\tilde{H}}[\mathrm{Tr}(F(H^N(s \wedge t) + \tilde{H}^N(s-t))_{0 \le s \le 1})]$$

with $\tilde{H}^N$ a Hermitian Brownian motion, independent from H^N. Taking $F(X) = T(X_{t_1}^{i_1}, \ldots, X_{t_n}^{i_n})$ and applying Itô's calculus, one finds that

$$M_t^F = M_0^F + \int_0^t \mathrm{Tr}\left(\mathbb{E}_{\tilde{H}}[D_s F(H^N(s\wedge t) + \tilde{H}^N(t-s))_{0\le s\le 1}].dH_s^N\right).$$

Indeed, when one performs Itô's calculus on $H^N(s) + \tilde{H}^N(t-s)$ the infinitesimal generator appears twice; once from $H^N(s)$ and once from $\tilde{H}^N(t-s)$ and the two contributions cancel. This shows that

$$\langle M^F\rangle_t = \frac{1}{N}\int_0^t \mathrm{Tr}\left(\|\mathbb{E}_{\tilde{H}}[D_s F(H^N(s\wedge t) + \tilde{H}^N(s-t))_{0\le s\le 1}]\|^2\right) ds$$

It was proved in [37] that for all $t \in [0,1]$,

$$\begin{aligned} G_F^t(\mathbf{L}_\mathbf{N}^\mathbf{P}) &:= \mathbb{E}[N^{-1}\mathrm{Tr}(F(H^N))|\mathcal{F}_t] - \mathbb{E}[N^{-1}\mathrm{Tr}(F(H^N))] \\ &- \frac{1}{2}\int_0^t \mathrm{Tr}\left(\|\mathbb{E}_{\tilde{H}}[D_s F(H^N(s\wedge t) + \tilde{H}^N(s-t))_{0\le s\le 1}]\|^2\right) ds \end{aligned}$$

is a continuous function of $\mathbf{L}_\mathbf{N}^\mathbf{P}$(actually of its resctriction to $\mathcal{B}_t$ measurable functions). Furthermore, as $\mathbf{L}_\mathbf{N}^\mathbf{P}$ goes to τ, $G_F^t(\mathbf{L}_\mathbf{N}^\mathbf{P})$ goes to

$$G_F^t(\tau) = \tau^t(F) - \tau^0(F) - \frac{1}{2}\int_0^t \tau[|\tau^s(D_s F|\mathcal{B}_s)|^2]ds.$$

Therefore, since $E[e^{N^2 G_F^t(\mathbf{L}_\mathbf{N}^\mathbf{P})}] = 1$, we readily get the large deviation upper bound by taking a distance d compatible with our topology and picking an $\epsilon > 0$ to get

$$\begin{aligned} &\mathbb{P}_N^{\otimes m}\left(d(\mathbf{L}_\mathbf{N}^\mathbf{P}, \tau) < \epsilon\right) \\ &= \mathbb{P}_N^{\otimes m}\left(1_{d(\mathbf{L}_\mathbf{N}^\mathbf{P},\tau)<\epsilon} e^{N^2(G_F^t(\mathbf{L}_\mathbf{N}^\mathbf{P}) - G_F^t(\mathbf{L}_\mathbf{N}^\mathbf{P}))}\right) \\ &\le e^{-N^2 G_F^t(\tau) + N^2\kappa(\epsilon)} \mathbb{P}_N^{\otimes m}\left(1_{d(\mathbf{L}_\mathbf{N}^\mathbf{P},\tau)<\epsilon} e^{N^2(G_F^t(\mathbf{L}^\mathbf{N}))}\right) \\ &\le e^{-N^2 G_F^t(\tau) + N^2\kappa(\epsilon)} \mathbb{P}_N^{\otimes m}\left(e^{N^2(G_F^t(\mathbf{L}_\mathbf{N}^\mathbf{P}))}\right) \\ &= e^{-N^2 G_F^t(\tau) + N^2\kappa(\epsilon)} \end{aligned}$$

where $\kappa(\epsilon)$ goes to zero with ϵ. We finally can take the logarithm, divide by N^2, let N going to infinity, ϵ to zero and finally optimize over F to conclude.

• *Uniqueness of the solutions with smooth drift.* The second point of Theorem 18.8 is a consequence of Riesz's theorem. Moreover, taking $F = (X_t - X_s)G(X)$ with G $\mathcal{B}_s$ measurable, we deduce that

$$\tau\left(\left(X_t - X_s - \int_s^t K_u du\right) G(X)\right) = 0.$$

Hence

$$H_t = X_t - X_0 - \int_0^t K_u du$$

is a free martingale. To show that it is a free Brownian motion, we use a free version of Paul Lévy's well-known theorem on the characterization of Brownian motion as the unique martingale with continuous paths and square bracket equal to t, and that may be of independent interest (see [37, theorem 6.2])

Lemma 18.10. *Let $(\mathcal{B}_s; s \in [0,1])$ be an increasing family of von Neumann subalgebras, in a non-commutative probability space $(\mathcal{A}, \tau)$, and let*

$$(Z_s = (Z_s^1, \ldots, Z_s^m); s \in [0,1])$$

be an m-tuple of self-adjoint processes adapted to $(\mathcal{B}_s; s \in [0,1])$, such that Z is bounded, $Z_0 = 0$, and for all $s < t$ one has:
(a) $\tau(Z_t|\mathcal{B}_s) = Z_s$.
(b) $\tau(|Z_t - Z_s|^4) \leq K(t-s)^2$ for some constant K.
(c) For any $l, p \in \{1, \ldots, m\}$, and all $A, B \in \mathcal{B}_s$, one has

$$\tau(AZ_t^l BZ_t^p) = \tau(AZ_s^l BZ_s^p) + 1_{p=l}(t-s)\tau(A)\tau(B) + o(t-s),$$

then Z is a free Brownian motion, i.e., for all $s < t$ the elements $Z_t^l - Z_s^l; l \in \{1, \ldots, m\}$ are free with $\mathcal{B}_s$, and have a semi-circular distribution of covariance $(t-s)I_m$.

Proof. Because of the invariance of the conditions under time translation, it is enough to prove that $Z_t - Z_0$ is free with $\mathcal{B}_0$, and of semi-circular distribution with covariance tI_m. We can assume that $Z_0 = 0$, and one has for any $i_1, \ldots i_n \in \{1, \ldots, m\}$,

$$\tau(Z_t^{i_1} \ldots Z_t^{i_n}) = \tau((Z_s^{i_1} + (Z_t^{i_1} - Z_s^{i_1})) \ldots (Z_s^{i_n} + (Z_t^{i_n} - Z_s^{i_n}))).$$

From condition (a) we get $\tau(Z_t^l - Z_s^l|\mathcal{B}_s) = 0$, and expanding the above product using (b) and (c) gives

$$\begin{aligned}
&\tau(Z_t^{i_1} \ldots Z_t^{i_n}) - \tau(Z_s^{i_1} \ldots Z_s^{i_n}) \\
&= (t-s) \sum_{0 \leq k+p \leq n-2} \sum_{i_k = i_p} \tau(Z_s^{i_1} \ldots Z_s^{i_{k-1}} Z_s^{i_{k+p+1}} \ldots Z_s^{i_n}) \\
&\qquad \tau(Z_s^{i_{k+1}} \ldots Z_s^{i_{k+p-1}}) + o(t-s)
\end{aligned}$$

where we have used non-commutative Hölder's inequality in order to bound the terms containing at least three $(Z_t^l - Z_s^l)$ factors. It follows that the quantities $\tau(Z_t^{i_1} \ldots Z_t^{i_n})$ satisfy a system of differential equations

whose initial conditions are known. It is easy to see that this system has a unique solution, resulting in the observation that there exists at most one process (in distribution) satisfying (a), (b) and (c).
Since the free m-dimensional Brownian motion also satisfies (a), (b) and (c), we conclude that $Z_t - Z_0$ is a free Brownian motion. For the freeness property with respect to $\mathcal{B}_0$, we consider a quantity of the form

$$\tau(A_1 Z_t^{i_1} A_2 Z_t^{i_2} \dots A_n Z_t^{i_n})$$

that again satisfies the same differential equation as when Z_t is a free Brownian motion free with $\mathcal{B}_0$. $\square$

In order to apply Theorem 18.10 to the process Y we have to check the three conditions. First we apply (18.5) to $P = (X_t^l - X_s^l)Q_s$, where $Q_s \in \mathcal{B}_s \cap \mathcal{F}$. Although P does not belong to $\mathcal{F}$ one can again check that it is a limit of a sequence of P_n in $\mathcal{F}$, such that $\nabla_s P_n$ converges to $\nabla_s P$, so there is no problem in applying formula (18.5). One has $\nabla_u^k P = \delta_{kl} 1_{u\in[s,t]} Q_s + W$ where $\tilde{\tau}^u(W|\mathcal{B}_s) = 0$. We thus find that for all $Q_s \in \mathcal{B}_s$, one has

$$\begin{aligned}
&\tau((X_t^l - X_s^l)Q_s) \\
&= \tilde{\tau}^0((X_t^l - X_s^l)Q_s) + \int_0^1 \tau(\tilde{\tau}^u(\nabla_u[(X_t^l - X_s^l)Q_s]|\mathcal{B}_u).K_u)du \\
&= \tau\left(\int_s^t Q_s K_u^l du\right)
\end{aligned}$$

from which we get that condition (a) is satisfied by $X_t - \int_0^t K_s ds$.
We now apply (18.5) to $P = (X_t^l - X_s^l)^4$ (the same remark as above applies). Since $\tilde{\tau}^0((X_t^l - X_s^l)^4) = 2(t-s)^2$, $\nabla_u^k[(X_t^l - X_s^l)^4] = 0$ for $u \notin [s,t]$ and $\nabla_u^k[(X_t^l - X_s^l)^4] = 4\delta_{kl}(X_t^l - X_s^l)^3$ for $u \in [s,t]$, one has

$$\tau((X_t^l - X_s^l)^4) = 2(t-s)^2 + \int_s^t \tau(\tilde{\tau}^u(4(X_t^l - X_s^l)^3|\mathcal{B}_u)K_u^{\tau,l})du$$

Since $K_u^{\tau,l}$ is uniformly bounded in norm, using Hölder's inequality and Gronwall lemma, we get the bound (b).
Condition (c) can be checked in a similar way as condition (a).
We conclude that X is solution to the stochastic differential equation

$$X_t = S_t + \int_0^t K_s ds$$

with $K_s = \tau(\nabla_s K|\mathcal{B}_s)$. Observing that for $K \in \mathcal{F}_{[0,1]}^m$, $X \to K_s(X)$ is uniformly Lipschitz, e.g., there exists a finite constant C such that for all $s \in [0,1]$,

$$||K_s(X) - K_s(Y)||_\infty \le C \sup_{u\le s} ||X_u - Y_u||_\infty,$$

we can use the usual Gronwall argument to prove the uniqueness of the solution to this equation, establishing the uniqueness of τ.

•*Large deviation lower bound.* If τ is the law of the solution of

$$dX_t = dS_t + \tau^t(D_t K|\mathcal{B}_t)dt$$

for some $K \in \mathcal{F}$, we know that the unique strong solution of

$$dX_t^N = dH_t^N + \tau^t(D_t K|\mathcal{F}_t)(X^N)dt$$

will converge weakly to τ. Moreover, this law is absolutely continuous with respect to the law of the m-dimensional Hermitian Brownian motion H^N with density (see, e.g., (18.6))

$$d_N = \exp\left\{N^2\left(\mathbf{L}_\mathbf{N}^\mathbf{P}(K) - \sigma(K) - \frac{1}{2}\int_0^1 \mathbf{L}_\mathbf{N}^\mathbf{P}(D_t K|\mathcal{F}_t)^2 dt\right)\right\}.$$

Since $G_K^1(\mathbf{L}_\mathbf{N}^\mathbf{P}) = N^{-2}\log d_N$ is a continuous function of $\mathbf{L}_\mathbf{N}^\mathbf{P}$, we get the desired lower bound by the following chain of inequalities:

$$\begin{aligned}
&\mathbb{P}_N^{\otimes m}\left(d(\mathbf{L}_\mathbf{N}^\mathbf{P}, \tau) < \epsilon\right) \\
&= \mathbb{P}_N^{\otimes m}\left(1_{d(\mathbf{L}_\mathbf{N}^\mathbf{P}, \tau)<\epsilon} e^{N^2(G_K^1(\mathbf{L}_\mathbf{N}^\mathbf{P}) - G_K^1(\mathbf{L}_\mathbf{N}^\mathbf{P}))}\right) \\
&\geq e^{-N^2 G_K^1(\tau) - N^2\kappa(\epsilon)} \mathbb{P}_N^{\otimes m}\left(1_{d(\mathbf{L}_\mathbf{N}^\mathbf{P}, \tau)<\epsilon} e^{N^2(G_K^1(\mathbf{L}_\mathbf{N}^\mathbf{P}))}\right) \\
&= e^{-N^2 G_K^1(\tau) - N^2\kappa(\epsilon)} \mathbb{P}_N^{\otimes m}\left(d(\mathbf{L}_\mathbf{N}^\mathbf{P}, \tau) < \epsilon\right) \\
&= e^{-N^2 G_K^1(\tau) - N^2\kappa(\epsilon)} \geq e^{-N^2 I(\tau) - N^2\kappa(\epsilon)}
\end{aligned}$$

with $\kappa(\epsilon)$ going to zero as ϵ goes to zero.

Bibliographical notes. A very nice introductory review on free entropy was written by Voiculescu [201]. The results of this section were proved in [37]. They were used in relation with the entropy dimension in [68, 156]. The problem of proving that in the definition of entropy one can replace the lim sup by a lim inf is still open, as well as the equality with the microstates-free free entropy.

Part VII

Appendix

19

Basics of matrices

19.1 Weyl's and Lidskii's inequalities

Theorem 19.1 (Weyl). *Denote $\lambda_1(C) \le \lambda_2(C) \le \cdots \le \lambda_N(C)$ the (real) eigenvalues of an $N \times N$ Hermitian matrix C. Let A, B be $N \times N$ Hermitian matrices. Then, for any $j \in \{1, \ldots, N\}$,*

$$\lambda_j(A) + \lambda_1(B) \le \lambda_j(A+B) \le \lambda_j(A) + \lambda_N(B).$$

In particular,

$$|\lambda_j(A+B) - \lambda_j(A)| \le \left(\mathrm{Tr}(B^2)\right)^{\frac{1}{2}} . \tag{19.1}$$

Theorem 19.2 (Courant–Fischer).

Let $A \in \mathcal{H}_N^{(2)}$ with ordered eigenvalues $\lambda_1(A) \le \cdots \le \lambda_N(A)$. For $k \in \{1, \ldots, N\}$,

$$\lambda_k(A) := \min_{w_1,\ldots,w_{N-k} \in \mathbb{C}^N} \max_{\substack{x \neq 0, x \in \mathbb{C}^N \\ x \perp w_1,\ldots,w_{N-k}}} \frac{x^* A x}{x^* x}.$$

Proof. We can without loss of generality assume that A is diagonal up to rotate the vectors $w_1, \ldots, w_{N-k}$ Then

$$\begin{aligned}
\max_{\substack{x \neq 0, x \in \mathbb{C}^N \\ x \perp w_1,\ldots,w_{N-k}}} \frac{x^* A x}{x^* x} &= \max_{\substack{\|x\|_2 = 1, x \in \mathbb{C}^N \\ x \perp w_1,\ldots,w_{N-k}}} \sum_{i=1}^N \lambda_i(A) |x_i|^2 \\
&\ge \max_{\substack{\|x\|_2 = 1, x \in \mathbb{C}^N, x_j = 0, j \le k \\ x \perp w_1,\ldots,w_{N-k}}} \sum_{i=1}^N \lambda_i(A) |x_i|^2 \\
&\ge \lambda_k(A)
\end{aligned}$$

and equality holds when $w_i = u_{N-i+1}$ is the eigenvector corresponding to the eigenvalue $\lambda_{N-i+1}(A)$. Taking the minimum over the vectors w_i thus completes the proof. $\square$

Proof of Weyl's inequalities Theorem 19.1. Let $(u_1, \ldots, u_{N-j})$ be the eigenvectors of the $N-j$ largest eigenvalues of A. Then, by Theorem 19.2,

$$\begin{aligned}
\lambda_j(A+B) &= \min_{w_1,\ldots,w_{N-j}\in\mathbb{C}^N} \max_{\substack{x\neq 0, x\in\mathbb{C}^N\\ x\perp u_1,\ldots,u_{N-j}}} \frac{x^*(A+B)x}{x^*x} \\
&\leq \max_{\substack{x\neq 0, x\in\mathbb{C}^N\\ x\perp u_1,\ldots,u_{N-j}}} \frac{x^*(A+B)x}{x^*x} \\
&\leq \max_{\substack{x\neq 0, x\in\mathbb{C}^N\\ x\perp u_1,\ldots,u_{N-j}}} \frac{x^*Ax}{x^*x} + \max_{x\neq 0} \frac{x^*Bx}{x^*x} \\
&= \lambda_j(A) + \lambda_N(B)
\end{aligned}$$

Replacing A, B by $-A, -B$ we obtain the second inequality.

□

Theorem 19.3 (Lidskii). *Let $A \in \mathcal{H}_N^{(2)}$, $\eta \in \{+1, -1\}$ and $z \in \mathbb{C}^N$. We order the eigenvalues of $A + \eta zz^*$ in increasing order. Then*

$$\lambda_k(A+\eta zz^*) \leq \lambda_{k+1}(A) \leq \lambda_{k+2}(A+\eta zz^*).$$

Proof. Using the Courant–Fischer theorem one gets for $k \geq 2$,

$$\begin{aligned}
\lambda_k(A+\eta zz^*) &:= \min_{w_1,\ldots,w_{N-k}\in\mathbb{C}^N} \max_{\substack{x\neq 0, x\in\mathbb{C}^N\\ x\perp w_1,\ldots,w_{N-k}}} \frac{x^*(A+\eta zz^*)x}{x^*x} \\
&\geq \min_{w_1,\ldots,w_{N-k}\in\mathbb{C}^N} \max_{\substack{x\neq 0, x\in\mathbb{C}^N\\ x\perp z, w_1,\ldots,w_{N-k}}} \frac{x^*Ax}{x^*x} \\
&\geq \min_{w_1,\ldots,w_{N-k+1}\in\mathbb{C}^N} \max_{\substack{x\neq 0, x\in\mathbb{C}^N\\ x\perp w_1,\ldots,w_{N-k+1}}} \frac{x^*Ax}{x^*x} \\
&= \lambda_{k-1}(A).
\end{aligned}$$

Replacing $A' = A + \eta zz^*$, and η by $-\eta$ we also have proved $\lambda_k(A' - \eta zz^*) \geq \lambda_k(A')$, i.e., $\lambda_k(A) \geq \lambda_{k-1}(A + \eta zz^*)$. □

One also has [33, Proposiion 28.2]:

Theorem 19.4 (Löwner). *Let $A, E \in \mathcal{H}_N^{(2)}$.*

$$\sum_{k=1}^{N} |\lambda_k(A+E) - \lambda_k(A)|^2 \leq \sum_{k=1}^{N} \lambda_k(E)^2 \,. \tag{19.2}$$

19.2 Non-commutative Hölder inequality

The following can be found in [158].

Theorem 19.5 (Nelson). *For any* $P_1, \ldots, P_q \in \mathbb{C}\langle X_1, \ldots, X_m \rangle$*, any matrices* $\mathbf{A} = (A_1, \ldots, A_m)$ *in* $\mathcal{M}_N$ *any* $p_1, \ldots, p_q \in [0,1]^q$ *so that* $\sum p_i^{-1} = 1$*,*

$$|\mathrm{Tr}(P_1(\mathbf{A}) \cdots P_q(\mathbf{A}))| \leq \prod_{i=1}^{q} [\mathrm{Tr}(|P_i(\mathbf{A})|^{q_i})]^{\frac{1}{q_i}}$$

with $|P| = \sqrt{PP^*}$*. This non-commutative Hölder inequality extends when* Tr *is replaced by any tracial states.*

20

Basics of probability theory

20.1 Basic notions of large deviations

This appendix recalls basic definitions and main results of large deviations theory. We refer the reader to [75] and [74] for a full treatment.

In what follows, X will be assumed to be a Polish space (that is a complete separable metric space). We recall that a function $f : X \to \mathbb{R}$ is *lower semicontinuous* if the level sets $\{x : f(x) \leq C\}$ are closed for any constant C.

Definition 20.1. *A sequence $(\mu_N)_{N\in\mathbb{N}}$ of probability measures on X satisfies a large deviation principle with speed a_N (going to infinity with N) and rate function I iff*

$$I : X \to [0,\infty] \text{ is lower semicontinuous.} \tag{20.1}$$

$$\text{For any open set } O \subset X,\ \liminf_{N\to\infty} \frac{1}{a_N} \log \mu_N(O) \geq -\inf_O I. \tag{20.2}$$

$$\text{For any closed set } F \subset X,\ \limsup_{N\to\infty} \frac{1}{a_N} \log \mu_N(F) \leq -\inf_F I. \tag{20.3}$$

When it is clear from the context, we omit the reference to the speed or rate function and simply say that the sequence $\{\mu_N\}$ satisfies the LDP. Also, if x_N are X-valued random variables distributed according to μ_N, we say that the sequence $\{x_N\}$ satisfies the LDP if the sequence $\{\mu_N\}$ satisfies the LDP.

Definition 20.2. *A sequence $(\mu_N)_{N\in\mathbb{N}}$ of probability measures on X satisfies a weak large deviation principle if (20.1) and (20.2) hold, and in addition (20.3) holds for all compact sets $F \subset X$.*

The proof of a large deviation principle often proceeds first by the proof of a weak large deviation principle, in conjuction with the so-called exponential tightness property.

Definition 20.3. *a. A sequence $(\mu_N)_{N\in\mathbb{N}}$ of probability measures on X is exponentially tight iff there exists a sequence $(K_L)_{L\in\mathbb{N}}$ of compact sets such that*

$$\limsup_{L\to\infty} \limsup_{N\to\infty} \frac{1}{a_N} \log \mu_N(K_L^c) = -\infty.$$

b. A rate function I is good if the level sets $\{x \in X : I(x) \le M\}$ are compact for all $M \ge 0$.

The interest in these concepts lies in the following:

Theorem 20.4. *a. ([74, Lemma 1.2.18]) If $\{\mu_N\}$ satisfies the weak LDP and it is exponentially tight, then it satisfies the full LDP, and the rate function I is good.*
b. ([74, Exercise 4.1.10]) If $\{\mu_N\}$ satisfies the upper bound (20.3) with a good rate function I, then it is exponentially tight.

A weak large deviation principle is itself equivalent to the estimation of the probability of deviations towards small balls:

Theorem 20.5. *[74, Theorem 4.1.11] Let $\mathcal{A}$ be a base of the topology of X. For every $A \in \mathcal{A}$, define*

$$\mathcal{L}_A = -\liminf_{N\to\infty} \frac{1}{a_N} \log \mu_N(A)$$

and

$$I(x) = \sup_{A\in\mathcal{A}:x\in A} \mathcal{L}_A.$$

Suppose that for all $x \in X$,

$$I(x) = \sup_{A\in\mathcal{A}:x\in A} \left\{ -\limsup_{N\to\infty} \frac{1}{a_N} \log \mu_N(A) \right\}.$$

Then, μ_N satisfies a weak large deviation principle with rate function I.

Let d be the metric in X, and set $B(x,\delta) = \{y \in X : d(y,x) < \delta\}$.

Corollary 20.6. *Assume that for all $x \in X$*

$$\begin{aligned} -I(x) &:= \limsup_{\delta\to 0} \limsup_{N\to\infty} \frac{1}{a_N} \log \mu_N(B(x,\delta)) \\ &= \liminf_{\delta\to 0} \liminf_{N\to\infty} \frac{1}{a_N} \log \mu_N(B(x,\delta)). \end{aligned}$$

Then, μ_N satisfies a weak large deviation principles with rate function I.

From a given large deviation principle one can deduce large deviation principle for other sequences of probability measures by using either the so-called contraction principle or Laplace's method.

Theorem 20.7 (Contraction principle). *[74, Theorem 4.2.1] Assume that the sequence of probability measures $(\mu_N)_{N\in\mathbb{N}}$ on X satisfies a large deviation principle with good rate function I. Then, for any function $F : X \to Y$ with values in a Polish space Y which is continuous, the image $(F\sharp\mu_N)_{N\in\mathbb{N}} \in M_1(Y)^{\mathbb{N}}$ defined as $F\sharp\mu_N(A) = \mu_N\left(F^{-1}(A)\right)$ also satisfies a large deviation principle with the same speed and rate function given for any $y \in Y$ by*

$$J(y) = \inf\{I(x) : F(x) = y\}.$$

Theorem 20.8 (Varadhan's lemma). *[74, Theorem 4.3.1] Assume that $(\mu_N)_{N\in\mathbb{N}}$ satisfies a large deviation principle with good rate function I. Let $F : X \to \mathbb{R}$ be a bounded continuous function. Then,*

$$\lim_{N\to\infty} \frac{1}{a_N} \log \int e^{a_N F(x)} d\mu_N(x) = \sup_{x\in X}\{F(x) - I(x)\}.$$

Moreover, the sequence

$$\nu_N(dx) = \frac{1}{\int e^{a_N F(y)} d\mu_N(y)} e^{a_N F(x)} d\mu_N(x) \in M_1(X)$$

satisfies a large deviation principle with good rate function

$$J(x) = I(x) - F(x) - \sup_{y\in X}\{F(y) - I(y)\}.$$

Large deviation principles are quite robust to exponential equivalence that we now define.

Definition 20.9. *Let (X, d) be a metric space. Let $(\mu_N)_{N\in\mathbb{N}}$ and $(\tilde{\mu}_N)_{N\in\mathbb{N}}$ be two sequences of probability measures on X. $(\mu_N)_{N\in\mathbb{N}}$ and $(\tilde{\mu}_N)_{N\in\mathbb{N}}$ are said to be exponentially equivalent if there exists probability spaces $(\Omega, \mathcal{B}_N, P_N)$ and two families of random variables $Z_N, \tilde{Z}_N$ on Ω with values in X with joint distribution P_N and marginals μ_N and $\tilde{\mu}_N$ respectively so that for each $\delta > 0$*

$$\limsup_{N\to\infty} P_N\left(d(Z_N, \tilde{Z}_N) > \delta\right) = -\infty.$$

We then have:

Lemma 20.10. *[74, Theorem 4.2.13] If a large deviation principle for μ_N holds with good rate function I and $\tilde{\mu}_N$ is exponentially equivalent to μ_N, then a $\tilde{\mu}_N$ satisfies a large deviation principle with the same rate function I.*

$\mathcal{P}(\Sigma)$ possesses a useful criterion for compactness.

Theorem 20.11 (Prohorov). *Let Σ be Polish, and let $\Gamma \subset \mathcal{P}(\Sigma)$. Then $\overline{\Gamma}$ is compact iff Γ is tight.*

Since $\mathcal{P}(\Sigma)$ is Polish, convergence may be decided by sequences.

20.2 Basics of stochastic calculus

Definition 20.12 ([122], [168]). *Let $(\Omega, \mathcal{F})$ be a measurable space.*

- *A filtration $\mathcal{F}_t, t \geq 0$ is a non-decreasing family of sub-σ-fields of $\mathcal{F}$.*
- *A random time T is a stopping time of the filtration $\mathcal{F}_t, t \geq 0$ if the event $\{T \leq t\}$ belongs to the σ-field $\mathcal{F}_t$ for all $t \geq 0$.*
- *A process $X_t, t \geq 0$ is adapted to the filtration $\mathcal{F}_t, t \geq 0$ if for all $t \geq 0$ X_t is an $\mathcal{F}_t$-measurable random variable.*
- *Let $\{X_t, \mathcal{F}_t, t \geq 0\}$ be an adapted process so that $\mathbb{E}[|X_t|] < \infty$ for all $t \geq 0$. The process $\{X_t, \mathcal{F}_t, t \geq 0\}$ is said to be a martingale if for every $0 \leq s < t < \infty$,*
$$\mathbb{E}[X_t | \mathcal{F}_s] = X_s.$$
- *Let $\{X_t, \mathcal{F}_t, t \geq 0\}$ be a martingale so that $E[X_t^2] < \infty$ for all $t \geq 0$. The martingale bracket (or the quadratic variation) $\langle X \rangle$ of X is the unique adapted increasing process so that $X^2 - \langle X \rangle$ is a martingale for the filtration $\mathcal{F}$.*

Let $\{X_t, \mathcal{F}_t, t \geq 0\}$ be a real-valued adapted process and let B be a Brownian motion. Assume that $E \int_0^T X_t^2 dt] < \infty$. Then,

$$\int_0^T X_t dB_t := \lim_{n \to \infty} \sum_{k=0}^{n-1} X_{\frac{Tk}{n}} (B_{\frac{T(k+1)}{n}} - B_{\frac{Tk}{n}})$$

exists, the convergence hold in L^2 and the limit does not depend on the above choice of the discretization of $[0, T]$ (see [122, section 3]). The limit is called a stochastic integral.

One can therefore consider the problem of finding solutions to the integral equation

$$X_t = X_0 + \int_0^t \sigma(X_s) dB_s + \int_0^t b(X_s) ds \tag{20.4}$$

with a given X_0, σ and b some functions on $\mathbb{R}^n$, and B a n-dimensional Brownian motion. This can be written under the differential form

$$dX_s = \sigma(X_s) dB_s + b(X_s) ds. \tag{20.5}$$

There are at least two notions of solutions; the strong solutions and the weak solutions.

Definition 20.13. *[122, Definition 2.1] A strong solution of the stochastic differential equation* (20.5) *on the given probability space $(\Omega, \mathcal{F})$ and with respect to the fixed Brownian motion B and initial condition ξ is a process $\{X_t, t \geq 0\}$ with continuous sample paths so that*

1. X is adapted to the filtration $\mathcal{F}$ given by

$$\mathcal{G}_t = \sigma(B_s, s \le t; X_0), \mathcal{N} = \{N \subset \Omega, \exists G \in \mathcal{G}_\infty \text{ with } N \subset G, P(G) = 0\},$$
$$\mathcal{F}_t = \sigma(\mathcal{G}_t \cup \mathcal{N}).$$

2. $P(X_0 = \xi) = 1$.
3. $P(\int_0^t (|b_i(X_s)| + |\sigma_{ij}(X_s)|^2) ds < \infty) = 1$ for all $i, j \le n$.
4. (20.4) holds almost surely.

Definition 20.14. *[122, Definition 3.1] A weak solution of the stochastic differential equation (20.5) is a triple (X, B) and $(\Omega, \mathcal{F}, P)$ so that $(\Omega, \mathcal{F}, P)$ is a probability space equipped with a filtration $\mathcal{F}$, X is a continuous adapted process and B an n-dimensional Brownian motion. X satisfies (3) and (4) in Definition 20.13.*

There are also two notions of uniqueness:

Definition 20.15. *[122, Definition 3.4]*

- *We say that strong uniqueness holds if two solutions with common probability space, common Brownian motion B and common initial condition are almost surely equal at all times.*
- *We say that weak uniqueness, or uniqueness in the sense of probability, holds if any two weak solutions have the same law.*

Theorem 20.16. *[122, Theorems 2.5 and 2.9]*
Suppose that b and σ satisfy

$$\|b(t,x) - b(t,y)\| + \|\sigma(t,x) - \sigma(t,y)\| \le K\|x - y\|,$$
$$\|b(t,x)\|^2 + \|\sigma(t,x)\|^2 \le K^2(1 + \|x\|^2),$$

for some finite constant K independent of t and $\|.\|$ the Euclidean norm on $\mathbb{R}^n$, then there exists a unique strong solution to (20.5). Moreover, it satisfies

$$\mathbb{E}[\int_0^T \|b(t, X_t)\|^2 dt] < \infty$$

for all $T \ge 0$.

Theorem 20.17. *[122, Proposition 3.10]*
Any two weak solutions $(X^i, B^i, \Omega^i, \mathcal{F}^i, P^i)_{i=1,2}$ of (20.5) so that

$$\mathbb{E}[\int_0^T \|b(t, X_t^i)\|^2 dt] < \infty$$

for all $T < \infty$ and $i = 1, 2$ have the same law.

Theorem 20.18 (Itô (1944), Kunita–Watanabe (1967)). *[122, p. 149]* *Let $f : \mathbb{R} \to \mathbb{R}$ be a function of class $\mathcal{C}^2$ and let $X = \{X_t, \mathcal{F}_t; 0 \le t < \infty\}$ be a continuous semi-martingale with decomposition*

$$X_t = X_0 + M_t + A_t$$

where M is a local martingale and A the difference of continuous, adapted, non-decreasing processes. Then, almost surely,

$$f(X_t) = f(X_0) + \int_0^t f'(X_s) dM_s + \int_0^t f'(X_s) dA_s$$

$$+\frac{1}{2} \int_0^2 f''(X_s) d < M >_s, \quad 0 \le t < \infty.$$

We shall use the following well known results on martingales.

Theorem 20.19 (Burkholder–Davis–Gundy's inequality). *[122, p. 166] Let $(M_t, t \ge 0)$ be a continuous local martingale with bracket $(A_t, t \ge 0)$. There exists universal constants λ_m, Λ_m so that for all $m \in \mathbb{N}$*

$$\lambda_m E(A_T^m) \le E(\sup_{t \le T} M_t^{2m}) \le \Lambda_m E(A_T^m).$$

Theorem 20.20 (Novikov (1972)). *[122, p. 199] Let $\{X_t, \mathcal{F}_t, t \ge 0\}$ be an adapted process with values in $\mathbb{R}^d$ such that*

$$E[e^{\frac{1}{2}\int_0^T \sum_{i=1}^d (X_t^i)^2 dt}] < \infty$$

for all $T \in \mathbb{R}^+$. Then, if $\{W_t, \mathcal{F}_t, t \ge 0\}$ is a d dimensional Brownian motion,

$$M_t = \exp\left\{ \int_0^t X_u . dW_u - \frac{1}{2} \int_0^t \sum_{i=1}^d (X_u^i)^2 du \right\}$$

is a $\mathcal{F}_t$-martingale.

Theorem 20.21 (Girsanov (1960)). *[122, p. 191] Let $\{X_t, \mathcal{F}_t, t \ge 0\}$ be an adapted process with values in $\mathbb{R}^d$ such that*

$$E[e^{\frac{1}{2}\int_0^T \sum_{i=1}^d (X_t^i)^2 dt}] < \infty.$$

Then, if $\{W_t, \mathcal{F}_t, P, 0 \le t \le T\}$ is a d dimensional Brownian motion,

$$\bar{W}_t^i = W_t^i - \int_0^t X_s^i ds, 0 \le t \le T$$

is a d-dimensional Brownian under the probability measure

$$\bar{P} = \exp\{ \int_0^T X_u . dW_u - \frac{1}{2} \int_0^T \sum_{i=1}^d (X_u^i)^2 du \} P.$$

Theorem 20.22. *[122, p. 14] Let* $\{X_t, \mathcal{F}_t, 0 \le t < \infty\}$ *be a submartingale whose every path is right-continuous. Then for any* $\tau > 0$*, for any* $\lambda > 0$

$$\lambda P(\sup_{0\le t\le \tau} X_t \ge \lambda) \le E[X_\tau^+].$$

We shall use the following consequence:

Corollary 20.23. *Let* $\{X_t, \mathcal{F}_t, t \ge 0\}$ *be an adapted process with values in* $\mathbb{R}^d$ *such that*

$$\int_0^T \|X_t\|^2 dt = \int_0^T \sum_{i=1}^d (X_t^i)^2 dt$$

is uniformly bounded by A_T*. Let* $\{W_t, \mathcal{F}_t, t \ge 0\}$ *be a* d*-dimensional Brownian motion. Then for any* $L > 0$*,*

$$P\left(\sup_{0\le t\le T}\left|\int_0^t X_u.dW_u\right| \ge L\right) \le 2e^{-\frac{L^2}{2A_T}}.$$

Proof. We denote in short $Y_t = \int_0^t X_u.dW_u$ and write for $\lambda > 0$,

$$\begin{aligned} P\left(\sup_{0\le t\le T}|Y_t| \ge A\right) &\le P\left(\sup_{0\le t\le T} e^{\lambda Y_t} \ge e^{\lambda A}\right) + P\left(\sup_{0\le t\le T} e^{-\lambda Y_t} \ge e^{\lambda A}\right) \\ &\le P\left(\sup_{0\le t\le T} e^{\lambda Y_t - \frac{\lambda^2}{2}\int_0^t \|X_u\|^2 du} \ge e^{\lambda A - \frac{\lambda^2 A_T}{2}}\right) \\ &\quad + P\left(\sup_{0\le t\le T} e^{-\lambda Y_t - \frac{\lambda^2}{2}\int_0^t \|X_u\|^2 du} \ge e^{\lambda A - \frac{\lambda^2 A_T}{2}}\right). \end{aligned}$$

By Theorem 20.20, $M_t = e^{-\lambda Y_t - \frac{\lambda^2}{2}\int_0^t \|X_u\|^2 du}$ is a non-negative martingale. Thus, By Chebychev's inequality and Doob's inequality

$$\begin{aligned} P\left(\sup_{0\le t\le T} M_t \ge e^{\lambda A - \frac{\lambda^2 A_T}{2}}\right) &\le e^{-\lambda A + \frac{\lambda^2 A_T}{2}} \mathbb{E}[M_T] \\ &= e^{-\lambda A + \frac{\lambda^2 A_T}{2}} \end{aligned}$$

Optimizing with respect to λ completes the proof. □

Theorem 20.24 (Rebolledo's Theorem). *Let* $n \in \mathbb{N}$*, and let* M_N *be a sequence of continuous centered martingales with values in* $\mathbb{R}^n$ *with bracket* $\langle M_N \rangle$ *converging pointwise (i.e., for all* $t \ge 0$*) in* L^1 *towards a continuous deterministic function* $\phi(t)$*. Then, for any* $T > 0$*,* $(M_N(t), t \in [0,T])$ *converges in law as a continuous process from* $[0,T]$ *into* $\mathbb{R}^n$ *towards a Gaussian process* G *with covariance*

$$E[G(s)G(t)] = \phi(t \wedge s).$$

20.3 Proof of (2.3)

Put

$$V(\mathbf{i}^1,\dots,\mathbf{i}^l) = [[i_n^j]_{n=1}^k]_{j=1}^l\,,\ I = \bigcup_{j=1}^l \{j\}\times\{1,\dots,k\}\,,\ A = [\{i_n^j, i_{n+1}^j\}]_{(i,n)\in I}.$$

We visualize A as a left-justified table of l rows. Let $G' = (V', E')$ be any spanning forest in $G(\mathbf{i}^1,\dots,\mathbf{i}^l)$, with c connected components. Since every connected component of G' is a tree, we have

$$|V| = |V'| = c + |E'|. \tag{20.6}$$

Now let $X = \{X_{in}\}_{(i,n)\in I}$ be a table of the same "shape" as A, but with all entries equal either to 0 or 1. We call X an *edge-bounding table* under the following conditions:

- For all $(i,n)\in I$, if $X_{in} = 1$, then $A_{in}\in E'$.
- For each $e\in E'$ there exists distinct $(i_1,n_1),(i_2,n_2)\in I$ such that $X_{i_1n_1} = X_{i_2n_2} = 1$ and $A_{i_1n_1} = A_{i_2n_2} = e$.
- For each $e\in E'$ and index $i\in\{1,\dots,j\}$, if e appears in the ith row of A then there exists $(i,n)\in I$ such that $A_{in} = e$ and $X_{in} = 1$.

For any edge-bounding table X the corresponding quantity $\frac{1}{2}\sum_{(i,n)\in I} X_{in}$ bounds $|E'|$ by the second required property. At least one edge-bounding table exists, namely the table with a 1 in position (i,n) for each $(i,n)\in I$ such that $A_{in}\in E'$ and 0's elsewhere. Now let X be an edge-bounding table such that for some index i_0 all the entries of X in the i_0th row are equal to 1. Then the graph $G(\mathbf{i}_0)$ is a tree (since all edges of $G(\mathbf{i}_0)$ could be kept in G'), and hence every entry in the i_0th row of A appears there an even number of times and *a fortiori* at least twice. Now choose $(i_0,n_0)\in I$ such that $A_{i_0n_0}\in E'$ appears in another row than i_0. Let Y be the table obtained by replacing the entry 1 of X in position (i_0,n_0) by the entry 0. Then Y is again an edge-bounding table. Proceeding in this way we can find an edge-bounding table with 0 appearing at least once in every row, and hence we have $|E'|\le [\frac{|I|-l}{2}] = \frac{kl-l}{2}$. Together with (20.6) and the definition of I, this completes the proof.

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List of participants of the summer school

Lecturers

Maury BRAMSON	University of Minnesota, Minneapolis, USA
Alice GUIONNET	École Normale Supérieure de Lyon, France
Steffen LAURITZEN	University of Oxford, UK

Participants

Marie ALBENQUE	Université Denis Diderot, Paris, France
Louis-Pierre ARGUIN	Princeton University, USA
Sylvain ARLOT	Université Paris-Sud, Orsay, France
Claudio ASCI	Università La Sapienza, Roma, Italy
Jean-Yves AUDIBERT	École Nationale des Ponts et Chaussées, Marne-la-Vallée, France
Wlodzimierz BRYC	University of Cincinnati, USA
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Alain CAMANES	Université de Nantes, France
Mireille CAPITAINE	Université Paul Sabatier, Toulouse, France
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Pauline SCULLI	London School of Economics, UK
Sylvie SEVESTRE-GHALILA	Université Paris 5, France
Frederic UTZET	Universitat Autonoma de Barcelona, Spain
Nicolas VERZELEN	Université Paris-Sud, Orsay, France
Yvon VIGNAUD	Centre de Physique Théorique, Marseille, France
Pompiliu Manuel ZAMFIR	Stanford University, USA

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