ON CLASSICAL ANALOGUES OF FREE ENTROPY DIMENSION.

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Abstract. We define a classical probability analog of Voiculescu's free entropy dimension that we shall call the classical probability entropy dimension. We show that the classical probability entropy dimension is related with diverse other notions of dimension. First, it equals the fractal dimension. Second, if one extends Bochner's inequalities to a measure by requiring that microstates around this measure asymptotically satisfy the classical Bochner's inequalities, then we show that the classical probability entropy dimension controls the rate of increase of optimal constants in Bochner's inequality for a measure regularized by convolution with the Gaussian law as the regularization is removed. We introduce a free analogue of the Bochner inequality and study the related free entropy dimension quantity. We show that it is greater or equal to the non-microstates free entropy dimension.

1. Introduction.

In [11], using his notion of free entropy χ, Voiculescu introduced the free entropy dimension of a non-commutative law. If $X_1, \dots, X_n \in (M, \tau)$ are self-adjoint non-commutative random variables in a tracial W^*-probability space, then

$$\delta(X_1, \dots, X_n) = n + \limsup_{t \to 0} \frac{\chi(X_1^t, \dots, X_n^t)}{|\log t|},$$

where $X_j^t = X_j + tS_j$ and $S_1, \dots, S_n$ form a free semicircular family, free from $X_1, \dots, X_n$. Voiculescu's motivation was to introduce a kind of asymptotic Minkowski content of matricial microstate spaces associated to the joint law of $X_1, \dots, X_n$. Indeed, for a variation of the definition of free entropy dimension, K. Jung has proved a formula that involves asymptotic packing numbers [7]. Moreover, he proved (again, for a version of the definition above), that one obtains the same number whether one uses semicircular perturbations or some other perturbation $X_j^t = X_j + tY_j$, where $Y_1, \dots, Y_n$ are some n-tuple, free from $X_1, \dots, X_n$ and having finite free entropy.

The free entropy dimension is a remarkable quantity, with unexpected connections to other branches of mathematics. For example, if $X_1, \dots, X_n$ generate the group algebra of a discrete group Γ, $\delta(X_1, \dots, X_n)$ is related by an inequality to the L^2-Betti numbers of the group Γ (this is based on a number of results, see [5, 8]). Unfortunately, the exact values of free entropy dimension are known in only a few cases. For example, in the case of a single variable X with law given by a probability measure μ on $\mathbb{R}$, $\delta(\mu) = 1 - \sum_{t \in \mathbb{R}} \mu(\{t\})^2$.

One of the most important questions surrounding δ is the question of its invariance under various functional calculi. It is hoped that $\delta(X_1, \dots, X_n) = \delta(Y_1, \dots, Y_m)$ if $X_1, \dots, X_n$ and $Y_1, \dots, Y_m$ generate the same von Neumann algebra (i.e., are "non-commutative measurable

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functions of each other"). However, the question is open even if it is asked for continuous functions (that is, assuming that the C^*-algebras generated by $X_1, \ldots, X_n$ and $Y_1, \ldots, Y_m$ are the same). What is known, for a version of the definition of free entropy dimension, is that its value is preserved under algebraic changes of generators. Solving these problems would be of great interest to von Neumann algebra theory.

In the first part of the present paper, we turn to look at the classical analog of free entropy dimension. Given a probability measure μ on $\mathbb{R}^n$ (which can be though of as the law of n real random variables $X_1, \ldots, X_n$), we consider the measure $\mu_t = \mu * \nu_t$, where ν_t is the Gaussian law

$$\nu_t(\prod dx_j) = \frac{1}{(2\pi t^2)^{n/2}} \exp(-\frac{1}{2t^2} \sum x_j^2) \prod dx_j.$$

Thus μ_t is the law of $X_1^t, \ldots, X_n^t$ with $X_j^t = X_j + tG_j$, and $G_1, \ldots, G_n$ independent Gaussian random variables, independent from $X_1, \ldots, X_n$. We then set

$$\delta_c(\mu) = n - \liminf_{t \to 0} \frac{H(\mu_t)}{|\log t|},$$

where for a non negative Lebesgue absolutely-continuous measure $p(x)dx$,

$$H(p(x)dx) = \int p(x) \log p(x) dx.$$

(The change of sign here is due to the fact that $H(\mu_t) \in (-\infty, +\infty]$ behaves as the analog of $-\chi$, since the free entropy χ is valued in $[-\infty, +\infty)$).

The main result of this paper relates $\delta_c(\mu)$ with a kind of average fractal dimension of the measure μ. In particular, we prove that $\delta_c(\mu)$ remains the same if μ is replaced by a push-forward by a Lipschitz function. However, the value of $\delta_c(\mu)$ may change if we push forward μ by a continuous or measurable function.

We also prove a number of technical properties of δ_c. Among the ones of independent interest is the fact that (in the case that $\limsup$ in its definition is a limit) δ_c is affine: $\delta_c(\sum \alpha_j \mu_j) = \sum \alpha_j \delta_c(\mu_j)$ in the case that μ_j are probability measures and $\alpha_j \geq 0$, $\sum \alpha_j = 1$.

The second part of the paper relates the rate of increase of optimal constants in Bochner's inequality with entropy dimension. We say that a probability measure μ satisfies Bochner's inequality with constant $(n, K(n)) \in (\mathbb{R}^+)^2$ if for all smooth f,

$$\mu(\Gamma_2(f,f)) \geq \frac{1}{n} \mu((\Delta f)^2) - K(n) \mu(\Gamma(f,f)), \tag{1}$$

where $\Gamma(f,f)$ and $\Gamma_2(f,f)$ are the carré du champ and carré du champ itéré, respectively. Intuitively, one should think of n as the dimension of the support of μ and K as an estimate for the smallest eigenvalue of the Ricci curvature of the support in the sense that if $\mu = \delta_x$, we recover the classical Bochner inequality at the point x, with n the dimension of the manifold where x lives and $-K(n)$ a lower bound on the Ricci curvature (cf. e.g. [1, 2]). The definition is actually obtained by considering the microstates $\Gamma_N(\mu, \epsilon) := \{x_1, \cdots, x_N \in \mathbb{R}^N : d(N^{-1} \sum_{i=1}^N \delta_{x_i}, \mu) < \epsilon\}$, viewing it as a submanifold of $\mathbb{R}^N$ with some dimension $[nN]$ and Ricci curvature bounded below by $-K(n)$. Letting then N going to infinity gives (1). We now replace μ with $\mu_\varepsilon = \mu * \nu_\varepsilon$ and study the functions $\varepsilon \mapsto K(n, \varepsilon) \geq 0$ for which the inequality

$$\mu_\varepsilon(\Gamma_2(f,f)) \geq \frac{1}{n} \mu_\varepsilon((\Delta f)^2) - K(n, \varepsilon) \mu_\varepsilon(\Gamma(f,f)) \tag{2}$$

is satisfied for all smooth f and a given $n \geq 0$. We then set

$$\delta^{\square} = 1 - \inf(\liminf_{\varepsilon \to 0} \frac{\int_{\varepsilon}^{1} K(y,n)dy}{|\log \varepsilon|} + 1)n,$$

where the inf is taken over all $n \geq 0$ and functions $K(n,\varepsilon)$ for which (2) holds. We prove that with this definition, $\delta^{\square} = \delta_c$.

In the third and final part of the paper, we study the free non-commutative analog of the inequality (2) and the related free entropy dimension quantity, which we show to be less than or equal to the non-microstates free entropy dimension.

2. Equivalent definitions of δ_c.

The main result of this section is that one can replace in the definition of $\delta_c(\mu)$ the convolution with the Gaussian measure by convolution with dilations of any other probability measure ν that has finite entropy. We first consider some properties of δ_c, which are of independent interest. Throughout this section, it will be convenient to assume that ν is a finite positive measure, but to drop the assumption that its total mass is 1. We will also denote by $D_t : \mathbb{R} \to \mathbb{R}$ the dilation map $x \mapsto tx$. For simplicity of notation, we give all statements and proofs for a measure on $\mathbb{R}$. However, these go through unaltered for measures on $\mathbb{R}^n$. Also, all lim inf could be replaced by lim sup if one would prefer to define δ_c with a lim sup.

Lemma 2.1. *(a) Let ν be a Lebesgue absolutely continuous finite measure on $\mathbb{R}$, $\nu_t = D_t^*(\nu)$ (where D_t is the map $x \mapsto tx$ is a dilation). Then for any probability measure μ and any constant $\alpha > 0$ we have*

$$\liminf_{t \to 0} \frac{H(\alpha\mu_t)}{\log t} = \alpha \liminf_{t \to 0} \frac{H(\mu_t)}{|\log t|}$$

(b) Let ν be a non negative Lebesgue absolutely continuous measure for which $\nu(\mathbb{R}) = \delta < \infty$. Let μ be a probability measure on $\mathbb{R}$ and denote $\nu_t = D_t^*(\nu)$ and $\mu_t = \mu * \nu_t$. Let $p_t(x)$ be the density of μ_t.

If

$$\int \log(1 + |x|)d\nu(x) < \infty, \quad \textit{and} \quad \int \log(1 + |x|)d\mu(x) < \infty, \tag{3}$$

then

$$0 \leq \liminf_{t \to 0} \frac{H(\mu_t)}{|\log t|}.$$

On the other hand, if $H(\nu) < \infty$, then

$$\liminf_{t \to 0} \frac{H(\mu_t)}{|\log t|} \leq \limsup_{t \to 0} \frac{H(\mu_t)}{|\log t|} \leq \delta.$$

(Here and below $H(q(x)dx) = \int q(x) \log q(x)dx$ for any non-negative measurable function q, even if $q(x)dx$ is not a probability measure).

Proof. (a) follows from the formula $H(\alpha\mu) = \alpha H(\mu) + \mu(\mathbb{R}) \log \alpha$ and the fact that $\mu_t(\mathbb{R}) = \nu(\mathbb{R})$ is independent of $t \in \mathbb{R}$.

(b)We may assume without loss of generality that $\delta = 1$ by a rescaling up to using (a).

For the first inequality, recall that for any probability measure ν, any non negative function f, Jensen's inequality implies that

$$\int f(x)\log f(x)d\nu(x) \geq \int f(x)d\nu(x)\log\left(\int f(x)d\nu(x)\right).$$

Therefore, if we let $\nu(dx) = p(x)dx$ be a probability measure absolutely continuous with respect to the Lebesgue measure, we can write

$$H(f(x)dx) = \int \frac{f(x)}{p(x)}\log\frac{f(x)}{p(x)}p(x)dx + \int \log p(x)f(x)dx \geq \int \log p(x)f(x)dx$$

if $\int f(x)dx = 1$. We can for instance take $p(x) = \frac{1}{2(1+|x|)^2}$ to obtain the lower bound

$$H(f(x)dx) \geq -\log 2 - 2\int \log(1+|x|)f(x)dx$$

for all $f \geq 0$ so that $\int f(x)dx = 1$.

Now, since ν is absolutely continuous with respect to Lebesgue measure, so is the measure $\mu_t(dx) = f_t(x)dx$. Applying the above to f_t, we deduce

$$\begin{aligned} H(\mu_t) &\geq -\log 2 - 2\int \log(1+|x|)d\mu_t(x) \\ &\geq -\log 2 - 2\int \log(1+|x|)(1+t|y|)d\mu(x)d\nu(y) \\ &\geq -\log 2 - 2\int \log(1+|x|)d\mu(x) - 2\int \log(1+|y|)d\nu(y) \end{aligned}$$

where the last bound holds for $t \leq 1$. Hence, when (3) is satisfied, $H(\mu_t)$ is bounded below independently of $t \leq 1$, which gives the desired lower bound.

We next prove the upper bound. By the entropy power inequality (see e.g. [10]), we have that

$$\begin{aligned} \exp(-2H(\mu_t)) &\geq \exp(-2H(\mu)) + \exp(-2H(\nu_t)) \\ &\geq \exp(-2H(\nu_t)) \\ &= \exp(-2H(\nu) + 2\log t). \end{aligned}$$

Thus

$$H(\mu_t) \leq H(\nu) - \log t$$

so that

$$\limsup_{t\to 0}\frac{H(\mu_t)}{|\log t|} \leq \limsup_{t\to 0}\frac{H(\nu)-\log t}{|\log t|} = 1$$

as claimed. □

Lemma 2.2. *Let $n \in \mathbb{N}$ and $\mu = \sum_{i=1}^n \mu_i$ for some non negative measures $(\mu_i, 1 \leq i \leq n)$ so that $\mu_i(\mathbb{R}) = a_i > 0$, $\sum_{i=1}^n a_i = 1$. Let ν be a probability measure on $\mathbb{R}$ so that $H(\nu) < \infty$. Then*

$$\liminf_{t\to 0}\frac{H(\mu * \nu_t)}{|\log t|} = \liminf_{t\to 0}\frac{1}{|\log t|}\sum a_i H(a_i^{-1}\mu_i * \nu_t). \tag{4}$$

Note that since $H(\nu)$ is assumed finite, $H(a_i^{-1}\mu_i * \nu_t) \leq |\log t|$ by the previous Lemma and so the sum in the right hand side of (4) is well defined.

Proof. Since ν is absolutely continuous with respect to Lebesgue measure with density p, so is $\mu * \nu_t$ and

$$p_\mu(x) = \frac{d\mu * \nu_t}{dx}(x) = \frac{1}{t}\int p\left(\frac{x-y}{t}\right) d\mu(y).$$

We assume first that $n = 2$ and denote in short $p_i(x) = a_i^{-1} p_{\mu_i}(x)$ for $i = 1, 2$, so that $\int p_i(x)dx = 1$. Then the density of $\mu * \nu_t$ is given by $\sum a_i p_i(x)$ and hence

$$H(\mu * \nu_t) \;=\; \int \sum_i a_i p_i(x) \log \sum_j a_j p_j(x)\; dx = \sum_i a_i \int p_i(x) \log(\sum_j a_j p_j(x))dx.$$

As a consequence,

$$H(\mu * \nu_t) - \sum a_i H(a_i^{-1}\mu_i * \nu_t) = \sum_i a_i \int p_i(x) \log\left(\frac{\sum_j a_j p_j(x)}{a_i p_i(x)}\right) dx + \sum_{i=1}^{2} a_i \log a_i.$$

Then for each $i = 1, 2$

$$\sum_j a_j p_j(x)/a_i p_i(x) = 1 + \frac{a_j p_j(x)}{a_i p_i(x)}$$

where in the last term $i, j \in \{1, 2\}$ and $i \neq j$.

Since for $y \geq 0$, $0 \leq \log(1 + y) \leq y$ and since $p_j(x)$, $p_i(x) \geq 0$, we conclude that

$$0 \leq \log\left(1 + \frac{a_j p_j(x)}{a_i p_i(x)}\right) \leq \frac{a_j p_j(x)}{a_i p_i(x)}.$$

Hence

$$0 \leq H(\mu * \nu_t) - \sum_{i=1}^{2} a_i H(a_i^{-1}\mu_i * \nu_t) \leq \sum_j \int a_j p_j(x)dx + \sum a_i \log a_i \leq 1 + \sum a_i \log a_i.$$

If $\mu = \sum_{i=1}^n \mu_i$ for $n > 2$, we first apply the above bound with $\mu_1' = \mu_1, \mu_2' = \sum_{i=2}^n \mu_i$ and $a_1' = a_1$, $a_2' = \sum_{i=2}^n a_i$, and then proceed by induction, replacing μ by $(\sum_{i=2}^n a_i)^{-1} \sum_{i=2}^n \mu_i$. We get in this way

$$0 \leq H(\mu * \nu_t) - \sum_{i=1}^{n} a_i H(a_i^{-1}\mu_i * \nu_t) \leq n - 1 + \sum_{i=1}^{n} a_i \log a_i.$$

Thus

$$\lim_{t\to 0} \frac{H(\mu * \nu_t) - \sum_{i=1}^n a_i H(a_i^{-1}\mu_i * \nu_t)}{|\log t|} = 0,$$

which implies the claim. ☐

We have as an immediate corollary a somewhat surprising property of δ_c:

Corollary 2.3. *Assume that μ_j are probability measures for which* $\limsup$ *in the definition of δ_c is a limit. Then the map $\mu \mapsto \delta_c(\mu)$ is affine: if $\alpha_j \geq 0$, $\sum \alpha_j = 1$, then $\delta_c(\sum \alpha_j \mu_j) = \sum \alpha_j \delta_c(\mu_j)$.*

Note that this property is very particular to the commutative case. Indeed, recall that the formula for the free entropy dimension of a single variable with law μ can be equivalently written as

$$\delta(\mu) = 1 - \sum_{t\in\mathbb{R}} \mu \times \mu(\{(t, t)\})$$

so that $\delta(\mu)$ is *quadratic* in μ. By the Cauchy-Schwartz inequality, $\delta(\sum_{i=1}^n a_i\mu_i) \geq \sum_{i=1}^n a_i\delta(\mu_i)$ but equality can hold only if for all $t \in \mathbb{R}$, $\mu_i(\{t\})$ does not depend on $i \in \{1, \cdots, n\}$.

Lemma 2.4. *Let for $n \in \mathbb{N}$, $\nu = \sum_{i=1}^n \nu^{(i)}$ so that $\nu^{(i)}(\mathbb{R}) = a_i$. Assume that $H(a_i^{-1}\nu^{(i)})$ is finite for all i. Then*

$$\liminf_{t\to 0} \frac{H(\mu * \nu_t)}{|\log t|} = \liminf_{t\to 0} \frac{1}{|\log t|} \sum a_i H(a_i^{-1}\mu * \nu_t^{(i)}).$$

Proof. The proof is very similar to that of Lemma 2.2 and we first assume $n = 2$. We let $\nu_t^{(i)} = D_t^*\nu^{(i)}$ where $D_t : \mathbb{R} \to \mathbb{R}$ is the map $D_t(x) = tx$. We have:

$$\mu * \nu_t = \sum_i \mu * \nu_t^{(i)} = \sum_i a_i(a_i^{-1}\mu * \nu_t^{(i)}).$$

Thus if we set

$$p_i(x) = d(a_i^{-1}\mu * \nu_t^{(i)})/dx$$

then the density of $\mu * \nu_t$ is given by $\sum a_i p_i(x)$ and hence

$$H(\mu * \nu_t) \quad = \quad \int \sum_i a_i p_i(x) \log \sum_j a_j p_j(x)\ dx = \sum_i a_i \int p_i(x) \log(\sum_j a_j p_j(x))dx.$$

Hence, we deduce as in the proof of Lemma 2.2 that

$$0 \leq H(\mu * \nu_t) - \sum a_i H(p_i(x)dx) \leq \sum_j \int a_j p_j(x)dx + \sum a_j \log a_j \leq 1 + \sum a_j \log a_j.$$

Thus

$$\lim_{t\to 0} \frac{H(\mu * \nu_t) - \sum a_i H(p_i(x)dx)}{|\log t|} = 0,$$

which implies the claim. □

Corollary 2.5. *Given $\nu(dx) = f(x)dx$, with $\nu(\mathbb{R}) = 1$ and $H(\nu) < \infty$, set $\nu_t = D_t^*(\nu)$ where $D_t : \mathbb{R} \to \mathbb{R}$, given by $D_t(x) = tx$. Let μ be a probability measure on $\mathbb{R}$. Then given $\varepsilon > 0$ there exists M sufficiently large so that if we denote by ν^M the measure $\nu([-M, M])^{-1}\nu|_{[-M,M]}$, $\nu_t^M = D_t(\nu^M)$ and by μ_M the measure $\mu_M = \mu[-M, M]^{-1}\mu|_{[-M,M]}$, then*

$$\left|\liminf_{t\to 0} \frac{H(\mu * \nu_t)}{|\log t|} - \liminf_{t\to 0} \frac{H(\mu_M * \nu_t^M)}{|\log t|}\right| < \varepsilon.$$

Proof. This follows from first decomposing μ as $\mu|_{[-M,M]} + \mu|_{[-M,M]^c}$, so that Lemma 2.2 shows that

$$\left|\liminf_{t\to 0} \frac{H(\mu * \nu_t)}{|\log t|} - \mu([-M, M]) \liminf_{t\to 0} \frac{H(\mu_M * \nu_t)}{|\log t|}\right|$$
$$\leq \mu([-M, M]^c) \limsup_{t\to 0} \frac{H(\mu([-M, M]^c)^{-1}\mu|_{[-M,M]^c} * \nu_t)}{|\log t|} \leq \mu([-M, M]^c)$$

where the last inequality is due to Lemma 2.1.(b) since $\nu(\mathbb{R}) = 1$.

We next decompose ν as $\nu|_{[-M,M]} + \nu|_{[-M,M]^c}$ and apply Lemma 2.4. Since $H(\nu)$ is finite, also $H(\nu|_{[-M,M]})$ and $H(\nu|_{[-M,M]^c})$ are finite and so

$$\left|\liminf_{t\to 0} \frac{H(\mu_M * \nu_t)}{|\log t|} - \nu([-M,M]) \liminf_{t\to 0} \frac{H(\mu_M * \nu_t^M)}{|\log t|}\right|$$
$$\leq \nu([-M,M]^c) \limsup_{t\to 0} \frac{H(\mu|_{[-M,M]} * D_t^*(\nu_M)}{|\log t|}$$
$$\leq \nu([-M,M]^c)$$

again by Lemma 2.1(b). Since

$$\left|(\mu([-M,M])\nu([-M,M]) - 1) \liminf_{t\to 0} \frac{H(\mu_M * \nu_t^M)}{|\log t|}\right| \leq \mu([-M,M]^c) + \nu([-M,M]^c)$$

the proof is complete if we take M big enough so that $2(\mu([-M,M]^c)+\nu([-M,M]^c)) \leq \varepsilon$. □

Lemma 2.6. *Assume that $\nu(dx) = f(x)dx$ with supp$f = E$ a bounded subset of $\mathbb{R}$, and that for some constant $C > \varepsilon > 0$, $|f - C| < \varepsilon$ on E. Let $\nu'(dx) = C\chi_E dx$ and set $\nu_t = D_t^*(\nu)$, $\nu_t' = D_t^*(\nu')$. Assume furthermore that the support of μ is a bounded subset of $\mathbb{R}$. Then*

$$\left|\liminf_{t\to 0} \frac{H(\mu * \nu_t')}{|\log t|} - \liminf_{t\to 0} \frac{H(\mu * \nu_t)}{|\log t|}\right| \leq \varepsilon\lambda(E).$$

Proof. Recall that

$$p_t(x) := \frac{d\mu * \nu_t}{dx}(x) = \int f(t^{-1}(x-y))\frac{1}{t}d\mu(y)$$
$$p_t'(x) := \frac{d\mu * \nu_t'}{dx}(x) = C\int \chi_E(t^{-1}(x-y))\frac{1}{t}d\mu(y).$$

Using the fact that μ is a probability measure, we have:

$$|p_t(x) - p_t'(x)| \leq \int \varepsilon\chi_E(t^{-1}(x-y))\frac{1}{t}d\mu(y) = \frac{\varepsilon}{C}p_t'(x).$$

In particular, we have that

$$\left|\frac{p_t(x)}{p_t'(x)} - 1\right| \leq C^{-1}\varepsilon.$$

Thus

$$\int p_t(x)\log p_t(x)dx = \int p_t(x)\log p_t'(x)dx - \int p_t(x)\log\frac{p_t(x)}{p_t'(x)}dx,$$

implies

$$|\int p_t(x)\log p_t(x)dx - \int p_t(x)\log p_t'(x)dx| \leq \max|\log(1 \pm C^{-1}\varepsilon)| = f(C^{-1}\varepsilon),$$

with $f(C^{-1}\varepsilon) \to 0$ as $C^{-1}\varepsilon \to 0$. Hence

$$\begin{aligned} |H(\mu * \nu_t') - H(\mu * \nu_t)| &\leq |\int (p_t(x) - p_t'(x))\log p_t'(x)dx| + f(C^{-1}\varepsilon) \\ &\leq \frac{\varepsilon}{C}\int p_t'(x)|\log p_t'(x)|dx + f(C^{-1}\varepsilon). \end{aligned}$$

It follows that

$$\left|\liminf_{t\to 0}\frac{H(\mu*\nu_t')}{|\log t|}-\liminf_{t\to 0}\frac{H(\mu*\nu_t)}{|\log t|}\right|\le\frac{\varepsilon}{C}\limsup_{t\to 0}\frac{\int p_t'(x)|\log p_t'(x)|dx}{|\log t|}. \tag{5}$$

Now, let $A_t=\{x: 0<p_t'(x)\le 1\}\subset tE+\mathrm{supp}\mu$. Then $\log p_t'(x)>0$ for $x\notin A_t$ and $\log p_t'(x)\le 0$ for $x\in A_t$. Therefore,

$$\int p_t'(x)|\log p_t'(x)|dx=\int p_t'(x)\log p_t'(x)dx-2\int_{A_t}p_t'(x)\log p_t'(x)dx.$$

Since for $y\in[0,1]$, the function $y\log y$ is bounded from below by $-e^{-1}$ and from above by 0, we get that for $x\in A_t$, $0\le -p_t'(x)\log p_t'(x)\le e^{-1}$. Since $A_t\subset E+\mathrm{supp}\mu$ for $t\le 1$, the Lebesgue measure $\lambda(A_t)$ is bounded uniformly in t. Thus, we find that

$$\liminf_{t\to 0}\frac{\int p_t'(x)|\log p_t'(x)|dx}{|\log t|}=\liminf_{t\to 0}\frac{\int p_t'(x)\log p_t'(x)}{|\log t|}=\liminf_{t\to 0}\frac{H(\mu*\nu_t')}{|\log t|}.$$

But since $H(\nu')=C\lambda(E)\log C$ is finite, we can use Lemma 2.1 to conclude that the right hand side above is bounded by $C\lambda(E)$, the mass of ν. Hence, we have proved with (5) that

$$\left|\liminf_{t\to 0}\frac{H(\mu*\nu_t')}{|\log t|}-\liminf_{t\to 0}\frac{H(\mu*\nu_t)}{|\log t|}\right|\le\varepsilon\lambda(E).$$

□

Theorem 2.7. *Let ν be an arbitrary probability measure with $H(\nu)$ finite. Assume that μ is a probability measure, and assume that μ and ν satisfy* (3)*.Then if we denote by D_t^* the push-forward of a measure by the dilation $x\mapsto tx$, we have that*

$$\liminf_{t\to 0}\frac{H(\mu*D_t^*(\nu))}{|\log t|}=\liminf_{t\to 0}\frac{H(\mu*D_t^*(\chi_{[0,1]}))}{|\log t|}.$$

In particular, the limit is independent of the measure ν.

Proof. Fix $\varepsilon>0$. By Corollary 2.5, we may assume, without changing $\liminf_{t\to 0}\frac{H(\mu*D_t^*(\nu))}{|\log t|}$ by more than $\varepsilon/2$, that μ and ν are supported on bounded sets. In particular, ν is Lebesgue absolutely continuous with density $q(x)\in L^1(\mathbb{R})$ with $E=\mathrm{supp}q$ a subset of finite Lebesgue measure. Given $\varepsilon>0$ we may find a subset $E_0\subset\mathbb{R}$ and a constant M so that $q(x)<M$ on E_0 and $\nu(E_0)^{-1}\le 1-\varepsilon/8$. By Corollary 2.5 we may replace ν by $\nu(E_0)^{-1}\nu|_{E_0}$ without affecting the value of $\liminf_{t\to 0}\frac{H(\mu*D_t^*(\nu))}{|\log t|}$ by more than $\varepsilon/4$. Next, since the density $p(x)$ of ν is now a bounded function on the support of ν, we may find a finite collection of disjoint subsets $E_j\subset E_0$ and constants C_j with the property that on each E_j, $|p_j-C_j|<\varepsilon/\lambda(E)8$ and that C_j is the average value of f on E_j (in particular, $\sum C_j\lambda(E_j)=\int f(x)dx=1$). According to Lemma 2.6 we may replace on each E_j $\nu|_{E_j}$ with χ_{E_j} at a penalty of at most $\varepsilon\lambda(E_j)/8$. Hence we may replace ν with the probability measure $\sum C_j\chi E_j$ at a penalty of at most $(\varepsilon\lambda(E)/8)\cdot\sum\lambda(E_j)\le\varepsilon/8$. By Lemma 2.4 it follows that

$$\liminf_{t\to 0}\frac{H(\mu*\nu_t)}{|\log t|}=\liminf_{t\to 0}\sum\frac{H(\mu*D_t^*(C_j\chi_{E_j}))}{|\log t|}.$$

Finally, by Lebesgue almost everywhere differentiability theorem, we may find, for each E_j disjoint intervals $I_1^{(j)},\dots,I_{k_j}^{(j)}$ of rational length with the property that E_j and $\cup_k I_k^{(j)}$ differ by at most $\lambda(E_j)\cdot\varepsilon/8$. Applying once again Lemma 2.2 and Lemma 2.4, we conclude that we may assume at a further penalty of $\varepsilon/8$ that $\nu = \sum K_r \chi_{E_r}$ where E_r are a finite collection of intervals. Up to subdivision, we may assume that all the E_r have the same Lebesgue measure (or length). We conclude that

$$\liminf_{t\to 0} \frac{H(\mu * \nu_t)}{|\log t|} = \liminf_{t\to 0} \sum K_r \frac{H(\mu * D_t^*(\chi_{E_r}))}{|\log t|} + o(\varepsilon),$$

where K_r is a family of non negative real numbers so that $\sum K_r \lambda(E_r) = 1$ and E_r are intervals.

Since $H(q(x)dx) = H(q(x-y)dx)$, we may replace any interval E_r in the previous formula by a shifted interval $E_j + k_j$ for any constant k_j. Hence, since all the E_r have the same length, $H(\mu * D_t^*(\chi_{E_r}))$ does not depend on r and so we have

$$\liminf_{t\to 0} \frac{H(\mu * \nu_t)}{|\log t|} = \liminf_{t\to 0} \frac{1}{\lambda(E_1)} \frac{H(\mu * D_t^*(\chi_{E_1}))}{|\log t|} + o(\varepsilon),$$

here E_1 is an interval with right hand point at the origin. Note that E_1 could be chosen as small as wished and so letting ε going to zero we have

$$\liminf_{t\to 0} \frac{H(\mu * \nu_t)}{|\log t|} = \lim_{a\downarrow 0} \liminf_{t\to 0} \frac{H(\mu * D_t^* \chi_{[0,a]})}{a|\log t|}.$$

This shows in particular that $\liminf_{t\to 0} \frac{H(\mu*\nu_t)}{|\log t|}$ does not depend on the probability measure ν with finite entropy and so we also have

$$\liminf_{t\to 0} \frac{H(\mu * \nu_t)}{|\log t|} = \liminf_{t\to 0} \frac{H(\mu * D_t^* \chi_{[0,1]})}{|\log t|}.$$

$\square$

3. δ_c AND FRACTAL DIMENSION.

If μ is a probability measure on $\mathbb{R}$, one can consider the (lower) point wise dimension of μ:

$$f^\mu(x) = \liminf_{t\to 0} \frac{\mu[x-t, x+t]}{\log t}.$$

This function quantifies the logarithmic rate of growth of the measures of t-balls around x and hence is a kind of local fractal dimension of μ. For example, certain Cantor-Lebesgue measures

$$\mu = \frac{1}{2}(\delta_{-1} + \delta_1) * \frac{1}{2}(\delta_\lambda + \delta_{-\lambda}) * \frac{1}{2}(\delta_{\lambda^2} + \delta_{-\lambda^2}) * \cdots, \qquad 0 < \lambda < 1/2,$$

satisfy $f^\mu = \alpha = -\log_2 \lambda$ on the Cantor set supporting μ and $f^\mu = 0$ outside of it. We show that δ_c is very close to the average value (computed with respect to μ) of the function f^μ, apart from the question of exchanging integration against μ and the limit $\liminf_{t\to 0}$.

Theorem 3.1. *Let μ be a probability measure on $\mathbb{R}$, and let*

$$d_t(x) = \frac{-\log \mu[x-t/2, x+t/2]}{|\log t|}.$$

Then

$$\delta_c(\mu) = \limsup_{t\to 0} \int d_t(y) d\mu(y).$$

Proof. By Theorem 2.7 we may write

$$\delta_c(\mu) = 1 - \liminf_{t\to 0} \frac{H(\mu * \nu_t)}{|\log t|},$$

where $\nu_t = D_t^* \chi_{[-1/2,1/2]} = \frac{1}{t}\chi_{[-t/2,t/2]}$. Let $p_t(x)$ be the density of μ_t:

$$p_t(x) = (\mu * D_t^* \chi_{[-1/2,1/2]})(x) = \frac{1}{t}\mu([x - t/2, x + t/2]).$$

Now,

$$\begin{aligned} H(\mu_t) &= \int p_t(x) \log p_t(x) dx \\ &= \iint \frac{1}{t}\chi_{[-t/2,t/2]}(x-y) d\mu(y) \log p_t(x) dx \\ &= \iint \frac{1}{t}\chi_{[-t/2,t/2]}(x) \log p_t(x+y) dx d\mu(y). \end{aligned}$$

Since $p_t(x+y) = \frac{1}{t}\mu[x+y-t/2, x+y+t/2]$ and $[y+x-t/2, y+x+t/2] \subset [y-t, y+t]$ as long as $-t/2 \le x \le t/2$, we find that for $|x| \le t/2$, $p_t(x+y) \le \frac{1}{t}\mu[y-t, y+t]$. Thus

$$\begin{aligned} H(\mu_t) &\le \iint \frac{1}{t}\chi_{[-t/2,t/2]}(x) \log \frac{1}{t}\mu[y-t, y+t]\ dx d\mu(y) \\ &= \int \frac{1}{t}\chi_{[-t/2,t/2]}(x) dx \int \log \frac{1}{t}\mu[y-t, y+t] d\mu(y) \\ &= \int \log \frac{1}{2t}\mu[y-t, y+t] d\mu(y) + \int \log 2 d\mu(y) = \int \log \frac{1}{2t}\mu[y-t, y+t] + \log 2 \end{aligned}$$

(since μ is a probability measure). It follows that

$$\liminf_{t\to 0} \frac{H(\mu_t)}{|\log t|} \le \liminf_{t\to 0} \frac{\int \log \frac{1}{t}\mu[y-t/2, y+t/2] d\mu(y)}{|\log t|} = \liminf_{t\to 0} \frac{\int \log p_t(y) d\mu(y)}{|\log t|}.$$

Let now $\delta > 0$ and set $C = 1 + \delta$. Let $\nu' = \chi_{[-C/2,C/2]}$, $\nu'' = \nu' - \chi_{[-1/2,1/2]}$. Let $\mu_t' = \mu * D_t^*(\nu')$, $\mu_t'' = \mu * D_t^*(\nu'')$. Thus $\mu_t' = \mu_t + \mu_t''$. Let $p_t'(x)$, $p_t''(x)$ be the densities of μ_t' and μ_t'', respectively. Then we have:

$$\begin{aligned} \int p_t(x) \log p_t'(x) dx - \int p_t(x) \log p_t(x) dx &= \int p_t(x) \log \frac{p_t'(x)}{p_t(x)} dx \\ &= \int p_t(x) \log \frac{p_t(x) + p_t''(x)}{p_t(x)} dx \\ &= \int p_t(x) \log(1 + p_t''(x)/p_t(x)) dx. \end{aligned}$$

Since $0 \le \log(1+z) \le z$ for $z \ge 0$, we conclude that

$$\begin{aligned} 0 &\le \int p_t(x) \log(1 + p_t''(x)/p_t(x)) dx \\ &\le \int p_t(x) p_t''(x)/p_t(x)\ dx \\ &= \int p_t''(x) dx = \mu_t''(\mathbb{R}) = \delta. \end{aligned}$$

It follows that

$$\left|\int p_t(x)\log p_t'(x)dx - \int p_t(x)\log p_t(x)dx\right| \le \delta. \tag{6}$$

Now, $p_t'(x) = \frac{1}{t}\mu[x - Ct/2, x + Ct/2]$. If $|x| < t/2$, then $[y - \delta t/2, y + \delta t/2] \subset [y + x - Ct/2, y + x + Ct/2]$. Thus $p_t'(x + y) \ge \frac{1}{t}\mu[y - \delta t/2, y + \delta t/2]$ as long as $|x| < t/2$. It follows that

$$\begin{aligned}
\int p_t(x)\log p_t'(x)dx &= \iint \frac{1}{t}\chi_{[-t/2,t/2]}(x)\log p_t'(x+y)\ d\mu(y)dx \\
&\ge \iint \frac{1}{t}\chi_{[-t/2,t/2]}(x)\log \frac{1}{t}\mu[y-\delta t/2, y+\delta t/2]\ d\mu(y)dx \\
&= \int \log \frac{1}{t}\mu[y-\delta t/2, y+\delta t/2]\ d\mu(y) \\
&= \int \log \frac{1}{\delta t}\mu[y-\delta t/2, y+\delta t/2]\ d\mu(y) + \log\delta. \qquad (7)
\end{aligned}$$

Thus, first by (6) and then (7) we obtain

$$\begin{aligned}
\liminf_{t\to 0}\frac{H(\mu_t)}{|\log t|} &= \liminf_{t\to 0}\frac{\int p_t(x)\log p_t'(x)dx}{|\log t|} \\
&\ge \liminf_{t\to 0}\frac{\int \log\frac{1}{\delta t}\mu[y-\delta t/2, y+\delta t/2]\ d\mu(y)}{|\log t|} = \liminf_{t\to 0}\frac{\int \log p_t(y)d\mu(y)}{|\log t|}
\end{aligned}$$

where we finally made the change of variable $t' = \delta t$. Combining this with the previous estimate proves that

$$\begin{aligned}
\delta_c(\mu) &= 1 - \liminf_{t\to 0}\frac{\int \log t^{-1}\mu[x-t/2, x+t/2]\ d\mu(x)}{|\log t|} \\
&= 1 - \liminf_{t\to 0}\int\left[\frac{\log\mu[x-t/2,x+t/2]}{|\log t|} + \frac{-\log t}{|\log t|}\right]\ d\mu(x) \\
&= \limsup_{t\to 0}\int d_t(x)d\mu(x).
\end{aligned}$$

□

Corollary 3.2. *Assume that μ is a probability measure, which is dimension regular; i.e., there exists some μ-measurable function $\alpha(x)$ and strictly positive constants C, c, and t_0 so that for any x in the support of μ and all $0 < t < t_0$ one has*

$$ct^{\alpha(x)} \le \mu[x-t/2, x+t/2] \le Ct^{\alpha(x)}. \tag{8}$$

Then $\delta_c(\mu) = \int \alpha(x)d\mu(x)$.

Note that in all the previous results, we could have change the lim inf into a lim sup and vice versa. Under the hypotheses of the Corollary we would thus obtain

$$\delta_c(\mu) = 1 - \liminf_{t\to 0}\frac{H(\mu_t)}{|\log t|} = 1 - \limsup_{t\to 0}\frac{H(\mu_t)}{|\log t|} = \int \alpha(x)d\mu(x).$$

Proof. We find that

$$d_t(x) = -\frac{\log \mu[x - t/2, x + t/2]}{|\log t|}$$

satisfies the inequalities

$$\frac{\alpha(x)\log t + \log c}{|\log t|} \le -d_t(x) \le \frac{\alpha(x)\log t + \log C}{|\log t|},$$

so that for $t < 1$,

$$\alpha(x) - \frac{\log c}{|\log t|} \ge d_t(x) \ge \alpha(x) - \frac{\log C}{|\log t|}.$$

Integrating these inequalities against $d\mu(x)$, passing to the limit as $t \to 0$ and using Theorem 3.1, we obtain that $\delta_c(\mu) = \int \alpha(x)d\mu(x)$. □

Example 3.3. (i) Let $0 < \alpha < 1$ and let μ_α be the Cantor-Lebesgue measure given by

$$\mu_\alpha = \frac{1}{2}(\delta_{-1} + \delta_1) * \frac{1}{2}(\delta_\lambda + \delta_{-\lambda}) * \frac{1}{2}(\delta_{\lambda^2} + \delta_{-\lambda^2}) * \cdots \qquad \lambda = 2^{-\alpha}$$

Then μ_α satisfies (8) with $\alpha(x) = \alpha$ for all x in the support of μ_α. Thus $\delta_c(\mu_\alpha) = \alpha$.
(ii) Let $\mu = \delta_0$ be a delta measure at 0. Then (8) is satisfied with $\alpha = 0$ on the support of μ. Hence $\delta_c(\mu) = 0$.
(iii) Let μ be Lebesgue absolutely continuous with density $p(x)$. Then $\mu = \mu_M + \mu_M^\perp$ where $\mu_M = \mu|_{\{x:p(x)\le M\}}$. Furthermore, $\mu_M^\perp(\mathbb{R}) \to 0$ as $M \to \infty$. Thus by Lemma (2.1), $\lim_{M\to\infty} \delta_c(\mu_M^\perp) = 0$ and hence by Lemma $\delta_c(\mu) = \lim_{M\to\infty} \delta_c(\mu_M) + \delta_c(\mu_M^\perp) = \lim_{M\to\infty} \delta_c(\mu_M)$. Since $H(\mu_M) < \infty$, we have that for all $t > 0$, $H(\mu_M * \nu) \le H(\mu_M)$ for any ν and so $\delta_c(\mu_M) = 1$. Thus $\delta_c(\mu) = 1$.

It is curious to note that one has a classical analog of the connection between free entropy dimension and group cohomology. In the classical case, the L^2 Betti numbers are replaced with ordinary Betti numbers and the statement greatly trivializes:

Theorem 3.4. *Let Γ be a finitely generated discrete abelian group with generators $\gamma_1, \ldots, \gamma_n$. Identify $\mathbb{C}\Gamma \subset L^\infty(\hat{\Gamma}, \mu)$, where μ is a Haar measure of $\hat{\Gamma}$, normalized to have measure 1 at each connected component of $\hat{\Gamma}$. Let ν be the law of the $2n$-tuple $X_1, \ldots, X_{2n}$, $X_{2k} = \gamma_k + \gamma_k^{-1}$, $X_{2k-1} = -i(\gamma_k - \gamma_k^{-1})$. Then*

$$\delta_c(\nu) = \dim_{\mathbb{C}} H^1(\Gamma; \mathbb{C})$$

Proof. Let $\Gamma = \Gamma_1 \oplus \Gamma_2$, where Γ_1 is a finite group of order l and Γ_2 is a free abelian group on p generators. Then $\hat{\Gamma} = \Gamma_1 \times \mathbb{T}^p$, where $\mathbb{T}$ denotes the unit circle in the complex plane. Since μ is the Haar measure on $\hat{\Gamma}$, it is dimension regular of dimension p. Hence $\delta_c(\nu) = lp$.

On the other hand, $H^1(\Gamma; \mathbb{C}) = H^1(\hat{\Gamma}; \mathbb{C}^p) = \mathbb{C}^p$ and thus also has dimension lp. □

4. δ_c via Fisher information and a notion of Ricci curvature.

In this section, we relate δ_c with quantities related with differential calculus. Let us remark, in the spirit of Voiculescu [12], that we can express δ_c via the asymptotics of the associated Fisher information. To that end, recall that for a probability measure $\mu(dx) = p(x)dx$ absolutely continuous with respect to Lebesgue measure, the Fisher information is given by

$$F(\mu) = \int (\partial_x \log p(x))^2 p(x)dx.$$

Note that if $\mu_s = \mu * \nu_s$ with ν_s the centered Gaussian law with covariance s, since $\partial_s \frac{d\nu_s}{dx} = \frac{1}{2}(\frac{d\nu_s}{dx})''$ $\partial_s H(\mu_s) = -\frac{1}{2}F(\mu_s)$ from which one sees that the entropy H and the Fisher information F are related by

$$H(\mu) - H(\mu_1) = \int_0^1 F(\mu_s)ds.$$

Taking $\mu = \mu_t$ gives, since $H(\mu_1)$ which is always bounded, that

$$\delta_c(\mu) = 1 - \liminf_{t\to 0} \frac{\int_t^1 F(\mu_s)ds}{|\log t|}. \tag{9}$$

Observe that if p_s is the density of μ_s

$$\partial_x \log p_s(x) = \frac{1}{\sqrt{s}} E[g|X + \sqrt{s}g]$$

when g is a standard Gaussian variable independent from X with law μ. This shows by Cauchy-Schwartz inequality that

$$0 \le F(\mu_s) \le \frac{1}{s} \tag{10}$$

and so proves again that $0 \le \delta_c(\mu) \le 1$. Moreover, (9) already reveals that $\delta_c(\mu)$ is related with the behaviour of the Fisher information of μ_t for small t and in fact, with the way that μ_t approaches μ as t goes to zero. Let us give some heuristics by assuming that we have the stronger statement that

$$F(\mu_t) \approx_{t\to 0} \frac{1-\delta_c(\mu)}{t}(1+o(1))$$

and show that this entails that the convergence of μ_t towards μ is at least of order $\sqrt{(1-\delta_c(\mu))t}$. In fact, Fisher's information can be equivalently defined by

$$F(\mu_t) := 2\sup_f\{\mu_t(\Delta f) - \frac{1}{2}\mu_t((f')^2)\} = \sup_f \frac{(\mu_t(\Delta f))^2}{\mu_t((f')^2)}$$

where the supremum is taken over all twice differentiable functions f (and is achieved here at $\log p_t$). Consequently, we find that for all twice differentiable function f,

$$(\mu_t(\Delta f))^2 \le F(\mu_t)\|f'\|_\infty^2.$$

As a consequence,

$$\begin{aligned}
|\mu_t(f) - \mu(f)| &\le \int_0^t |\partial_s \mu_s(f)|ds \\
&= \frac{1}{2}\int_0^t |\mu_s(\Delta f)|ds \\
&\le \frac{1}{2}\|f'\|_\infty \int_0^t \sqrt{\frac{1-\delta_c(\mu)}{s}}(1+o(1))ds \\
&\le \|f'\|_\infty \sqrt{(1-\delta_c(\mu))t}(1+o(1)).
\end{aligned}$$

Extending this inequality to all Lispchitz functions gives a bound on the Duddley distance between μ_t and μ;

$$d(\mu_t,\mu) := \sup_{f \text{ Lipschitz with norm}\le 1} |\mu_t(f) - \mu(f)| \le \sqrt{(1-\delta_c(\mu))t}(1+o(1)).$$

We believe that the relation between the short time asymptotics of μ_t and δ_c should be deeper that this result even though we could not prove it here. However, we shall prove here another definition for δ_c which is closely related with Bochner's inequality, a classical tool to estimate the short time asymptotics of the heat kernel in a compact Riemaniann manifold. We shall restrict ourselves here to measures on $\mathbb{R}$ but could as well consider measures on a compact Riemaniann manifold with Ricci curvature bounded below (eventually by a negative real number). To make this generalization more transparent, we denote Δ the Laplace Baltrami operator on $\mathbb{R}$ (i.e the second spatial derivative). We let Γ be the carré du champ given by

$$\Gamma(f,g) = \frac{1}{2}\left(\Delta(fg) - f\Delta g - g\Delta f\right),$$

and Γ_2 be the carré du champ itéré

$$\Gamma_2(f,f) = \frac{1}{2}\left(\Delta\Gamma(f,f) - 2\Gamma(f,\Delta f)\right).$$

In the case where $M = \mathbb{R}$, we simply have

$$\Gamma(f,f) = (f')^2, \quad \Gamma_2(f,f) = (f'')^2.$$

Note that in the case of a connected Riemanian manifold with metric g, Laplace Baltrami operator Δ and gradient ∇, the same definitions hold and give

$$\Gamma(f,f) = g(\nabla f, \nabla f), \Gamma_2(f,f) = (\mathrm{Hess} f, \mathrm{Hess} f)_g + \mathrm{Ric}(\nabla f, \nabla f)$$

with *Ric* the Ricci tensor. Bochner's (or curvature-dimension) inequality $CD(n,K)$ states that

$$\Gamma_2(f,f)(x) \geq \frac{1}{n}(\Delta f)^2(x) - K\Gamma(f,f)(x)$$

for all smooth function f and at all points x of the manifold. n corresponds to the dimension of the manifold whereas the best constant $-K$ corresponds to the smallest eigenvalue of the Ricci tensor. It is well known (see Bakry and Ledoux [2], Bakry and Qian [1] etc) that the coefficient n governs the short time scaling of the heat kernel (as $t^{-\frac{n}{2}}$). Here $n \geq 0$ and K is a real number which we will assume finite for a while. In the real one dimensional case, we clearly have $K = 0$ and $n = 1$, but the constant n of course is universal and does not depend on any measure. We next define the measure-dependent Bochner inequality as follows.

Notation 4.1. We write $\mu_\epsilon = P_\epsilon * \mu$, where P_ϵ is the Gaussian measure of variance ϵ.

Definition 4.2. We say that a probability measure μ on $\mathbb{R}$ satisfies Bochner's inequality with constants $\mathrm{CD}_{\mathrm{m}}(K,n)$ if there exists $\delta > 0$ so that for all $0 \leq \epsilon' \leq \delta$, all smooth functions f,

$$P_{\epsilon'} * \mu(\Gamma_2(f,f)) \geq \frac{1}{n}[P_{\epsilon'} * \mu(\Delta f)]^2 - K(\epsilon', n) P_{\epsilon'} * \mu(\Gamma(f,f)).$$

In the sequel, it will appear that interesting cases appear when the constant $K(n,\epsilon')$ may blow up with ϵ', reason why K will be later some *non negative* arbitrary function. n is some positive real number.

Remark. Note here that assuming that Bochner's inequality is true in expectation would lead to the stronger definition

$$P_{\epsilon'} * \mu(\Gamma_2(f,f)) \geq \frac{1}{n} P_{\epsilon'} * \mu[(\Delta f)^2] - K(\epsilon', n) P_{\epsilon'} * \mu(\Gamma(f,f)).$$

However, the idea is that what we want is that the points belonging to the microstates

$$\Gamma_{\delta,\mu} := \{x_1, \cdots, x_N : d(\frac{1}{N}\sum \delta_{x_i}, \mu) < \delta\}$$

approximately satisfy Bochner's inequality when N goes to infinity and ϵ goes to zero. Applying the classical Bochner's inequality to functions of the form $F(x_1, \cdots, x_N) = N^{-1}\sum f(x_i+\epsilon g_i)$ for independent standard Gaussian variables $(g_1, \cdots, g_N)$, $\epsilon > 0$ and letting N go to infinity gives our actual definition of measure-dependent Bochner's inequality. Hence, roughly speaking, $(n, -K(\epsilon, n))$ represent the dimension and the smallest eigenvalue of the Ricci tensor of a manifold where the entries $(x_1+\sqrt{\epsilon}g_1, \cdots, x_N+\sqrt{\epsilon}g_N)$ live when the $(x_1, \cdots, x_N)$ belong to $\Gamma_{\delta,\mu}$, for δ arbitrarily small.

Based on measure-dependent Bochner's inequalities we shall now define a new entropy dimension

Definition 4.3. Let μ be a probability measure on $\mathbb{R}$. We define the CD- dimension as

$$\delta^{\Box}(\mu) := 1 - \inf_{\mu \text{ satisfies } \mathrm{CD_m}(n,K)} (\liminf_{\epsilon\to 0} \frac{\int_\epsilon^1 K(y,n)dy}{\log\epsilon^{-1}} + 1)n.$$

Above, the infimum is taken over all couple $(n, K(.,n))$ such that μ satisfies $\mathrm{CD_m}(n, K)$.

We now prove that $\delta^{\Box}$ equals δ_c. We first prove that

Lemma 4.4. *For any probability measure μ on $\mathbb{R}^d$,*

$$\delta^{\Box}(\mu) \le \delta_c(\mu).$$

Proof. Note that for $d = 1$, $(\Delta f)^2 = \Gamma_2(f, f)$ but that the following argument will generalize to dimension d by Cauchy-Schwartz inequality which gives $d\Gamma_2(f, f) \ge (\Delta f)^2$. Integrating with respect to μ implies that for all $\epsilon \ge 0$

$$[\mu_\epsilon(\Delta f)]^2 \le \mu_\epsilon[(\Delta f)^2] \le \mu_\epsilon[\Gamma_2(f, f)].$$

On the other hand, with p_ϵ the density of μ_ϵ with respect to Lebesgue measure,

$$\begin{aligned} [\mu_\epsilon(\Delta f)]^2 &= (\mu_\epsilon[f'(\log p_\epsilon)'])^2 \\ &\le \mu_\epsilon[(f')^2]\mu_\epsilon[((\log p_\epsilon)')^2] \\ &= \mu_\epsilon[\Gamma_1(f, f)]F(\mu_\epsilon) \end{aligned}$$

Therefore, for all $\alpha \in [0, 1]$, we have

$$[\mu_\epsilon(\Delta f)]^2 \le \alpha\mu_\epsilon[\Gamma_2(f, f)] + (1-\alpha)F(\mu_\epsilon)\mu_\epsilon[\Gamma_1(f, f)]$$

and so μ satisfies $CD_m(n, K)$ with $n = \alpha$ and

$$K(\epsilon, n) = n^{-1}(1-n)\ F(\mu_\epsilon)$$

for all $\alpha \in [0, 1]$. Then,

$$\liminf_{\epsilon\to 0}(\log\epsilon^{-1})^{-1}\int_\epsilon^1 K(y, n)dy \le (1-n)n^{-1}\liminf_{\epsilon\to 0}(\log\epsilon^{-1})^{-1}\int_\epsilon^1 F(\mu_y)dy,$$

and so

$$\delta^{\Box}(\mu) \ge 1 - \inf_{n\le d}[n + (1-n)\liminf_{\epsilon\to 0}(\log\epsilon^{-1})^{-1}\int_\epsilon^1 F(\mu_y)dy] = \delta_c(\mu)$$

where we used $(\log \epsilon^{-1})^{-1} \int_\epsilon^1 F(\mu_y)dy \leq d = 1$ by (10) to say that the infimum is taken at $n = 0$. $\square$

Proposition 4.5. *If a probability measure μ on $\mathbb{R}$ satisfies* $\mathrm{CD_m}(K, n)$*, then*

$$\liminf_{\epsilon \to 0} (\log \epsilon^{-1})^{-1} \int_\epsilon^1 F(\mu_y)dy \leq \liminf_{\epsilon \to 0} [(\log \epsilon^{-1})^{-1} \int_\epsilon^1 K(y, n)dy + 1]n.$$

As an immediate corollary of Proposition 4.5 we have

Theorem 4.6. *For any probability measure μ on $\mathbb{R}$,*

$$\delta^\square(\mu) = \delta_c(\mu)$$

Whereas it can be easily seen that the characteristic $(n, -K)$ of a manifold are invariant by Lipschitz map (simply by taking local quadratic functions), invariance is not so transparent for measure-dependent Bochner's inequality and we could not prove interesting invariance property of $\delta^\square$. However, the above theorem and section 5 show that $\delta^\square$ is invariant under Lipschitz maps.

Proof. Let us first put $\mu_\epsilon = P_\epsilon * \mu$ with $\epsilon > 0$ and write

$$F(\mu_\epsilon) = 2 \sup_f \{P_\delta * \mu_\epsilon(\Delta f) - \frac{1}{2} P_\delta * \mu_\epsilon(\Gamma(f, f))\}$$

Now, let for $x \in [0, \delta]$, $\phi(x) = P_x * \mu_\epsilon(\Gamma(P_{\delta-x}f, P_{\delta-x}f))$. Differentiating with respect to x, we find that

$$\begin{aligned}
\phi'(x) &= P_x * \mu_\epsilon(\Gamma_2(P_{\delta-x}f, P_{\delta-x}f)) \\
&\geq \frac{1}{n}[P_x * \mu_\epsilon(\Delta P_{\delta-x}f)]^2 - K(x + \epsilon, n)P_x * \mu_\epsilon(\Gamma(P_{\delta-x}f, P_{\delta-x}f)) \\
&= \frac{1}{n}((\mu_\epsilon \Delta P_\delta f)^2) - K(x + \epsilon, n)\phi(x)
\end{aligned}$$

where we used the fact that P_x is a semigroup which commutes with the Laplacian. Also, we have used our measure-dependent Bochner's inequality with $f \to P_{\delta-x}f$ and $\epsilon' = x + \epsilon$. We set $L(x) = e^{\int_x^1 K(y,n)dy}$. Integrating $x \in [0, \delta]$, we deduce that

$$P_\delta * \mu_\epsilon(\Gamma(f, f)) \geq \mu_\epsilon(\Gamma(P_\delta f, P_\delta f)) \frac{L(\epsilon + \delta)}{L(\epsilon)} + \frac{1}{n} \mu_\epsilon((\Delta P_\delta f))^2 \int_0^\delta \frac{L(\epsilon + \delta)}{L(\epsilon + x)} dx \tag{11}$$

We thus obtain that for all $a \in [0, 1]$,

$$\begin{aligned}
F(\mu_{\epsilon+\delta}) &\leq 2 \sup_f \{a\mu_\epsilon(\Delta P_\delta f) - \frac{1}{2} \mu_\epsilon(\Gamma(P_\delta f, P_\delta f)) \frac{L(\epsilon + \delta)}{L(\epsilon)} \\
&\quad + (1 - a)\mu_\epsilon(\Delta P_\delta * f) - \frac{1}{2n} \int_0^\delta \frac{L(\epsilon + \delta)}{L(\epsilon + x)} dx (\mu_\epsilon(\Delta P_\delta f))^2\} \\
&\leq a^2 \frac{L(\epsilon)}{L(\epsilon + \delta)} F(\mu_\epsilon) + (1 - a)^2 \frac{n}{\int_0^\delta \frac{L(\epsilon+\delta)}{L(\epsilon+x)} dx}. \qquad (12)
\end{aligned}$$

The optimum with respect to a is taken at

$$a = \frac{n}{\frac{L(\epsilon)}{L(\epsilon+\delta)} \int_0^\delta \frac{L(\epsilon+\delta)}{L(\epsilon+x)} dx F(\mu_\epsilon) + n}.$$

We conclude

$$F(\mu_{\epsilon+\delta}) \leq \frac{n\frac{L(\epsilon)}{L(\epsilon+\delta)}F(\mu_\epsilon)}{\int_0^\delta \frac{L(\epsilon)}{L(\epsilon+x)}dxF(\mu_\epsilon)+n} = n\partial_\delta[\log(\int_0^\delta \frac{L(\epsilon)}{L(\epsilon+x)}dxF(\mu_\epsilon)+n)]. \tag{13}$$

Integrating with respect to $\delta \in [0, 1-\epsilon]$ thus gives

$$\begin{aligned} n^{-1}\int_\epsilon^1 F(\mu_x)dx &\leq \log(n^{-1}\int_\epsilon^1 \frac{L(\epsilon)}{L(x)}dxF(\mu_\epsilon)+1) \\ &\leq \log(n^{-1}\epsilon^{-1}\int_\epsilon^1 \frac{L(\epsilon)}{L(x)}dx+1) \end{aligned}$$

where we used again $\epsilon F(\mu_\epsilon) \leq 1$ by (10). Consequently

$$\begin{aligned} n^{-1}\liminf_{\epsilon\to 0}(\log \epsilon^{-1})^{-1}\int_\epsilon^1 F(\mu_x)dx &\leq \liminf_{\epsilon\to 0}(\log \epsilon^{-1})^{-1}\log(n^{-1}\epsilon^{-1}d\int_\epsilon^1 \frac{L(\epsilon)}{L(x)}dx+1) \\ &= \liminf_{\epsilon\to 0}(\log \epsilon^{-1})^{-1}\log(\epsilon^{-1}\int_\epsilon^1 \frac{L(\epsilon)}{L(x)}dx) \end{aligned}$$

Now,

$$\int_\epsilon^1 \frac{L(\epsilon)}{L(x)}dx \leq e^{\int_\epsilon^1 K(y,n)dy}$$

and so we arrive at

$$n^{-1}\liminf_{\epsilon\to 0}(\log \epsilon^{-1})^{-1}\int_\epsilon^1 F(\mu_x)dx \leq 1+\liminf_{\epsilon\to 0}(\log \epsilon^{-1})^{-1}\int_\epsilon^1 K(y,n)dy \tag{14}$$

which is the desired inequality. □

We finally give a lower bound of $\delta^\square$ in the spirit of [9]. To do this, let us defined, for a $\mathcal{C}_1^b(\mathbb{R},\mathbb{R})$ function g,

$$F_g(\mu) = 2\sup_f\{\mu(g\Delta f) - \frac{1}{2}\mu(\Gamma_1(f,f))\}$$

Proposition 4.7. *For any probability measure μ on $\mathbb{R}^d$,*

$$\delta_c(\mu) = \delta^\square(\mu) \geq 1 - \inf_{h\in\bar{\mathcal{F}}_\mu}\mu[(1-h)^2]$$

with $\bar{\mathcal{F}}_\mu$ the set of continuous functions so that

$$\liminf_{\delta\to\infty}(\log \delta^{-1})^{-1}\int_\delta^1 F_h(\mu_x)dx = 0.$$

This lower bound has the advantage to give a more intuitive picture of the dimension; for instance, if μ has a smooth density such that the gradient of its logarithm is uniformly bounded, on a subset A of M, we take $h = 1$ in some interior set A^s of A, $|h| \leq 1$ and $h = 0$ outside A. It is easy to see that $F_h(\mu) < \infty$ and so $h \in \mathcal{F}_\mu$. Thus, we get

$$\delta^*(\mu) = \delta^\square(\mu) \geq \mu(A).$$

Note however that such a lower bound is already contained in Theorem 3.1.

Proof. (of Proposition 4.7). We take $h \in \mathcal{F}_\mu$. We can assume without loss of generality that $\mu[(1-h)^2] \neq 0$ since otherwise the bound is trivial (h being equal to one almost surely, and hence $F_h = F$ implying that $\delta_c = \delta^\square = d$). We now write

$$\begin{aligned}\mu_\epsilon(\Delta f) &= \mu_\epsilon(h\Delta f) + \mu_\epsilon((1-h)\Delta f)\\ &= \mu_\epsilon(J_h f') + \mu_\epsilon((1-h)\Delta f)\end{aligned}$$

Now,

$$[\mu_\epsilon(J_h f')]^2 \leq F_h(\mu_\epsilon)\mu_\epsilon(\Gamma(f,f))$$

whereas

$$\begin{aligned}[\mu_\epsilon((1-h)\Delta f)]^2 &\leq \mu_\epsilon((1-h)^2)\mu_\epsilon((\Delta f)^2)\\ &\leq \mu_\epsilon((1-h)^2)[\mu_\epsilon(\Gamma_2(f,f)) + Kd\mu_\epsilon(\Gamma_1(f,f))]\end{aligned}$$

Using that for all $\alpha > 0$, for all $x, y \in \mathbb{R}$, $(x+y)^2 \leq (1+\alpha)x^2 + (1+\alpha^{-1})y^2$ we thus derive the inequality

$$[\mu_\epsilon(\Delta f)]^2 \leq (1+\alpha)F_h(\mu_\epsilon)\mu_\epsilon(\Gamma(f,f)) + (1+\alpha^{-1})\mu_\epsilon((1-h)^2)[\mu_\epsilon(\Gamma_2(f,f)) + K\mu_\epsilon(\Gamma_1(f,f))]$$

that is the $CD_m(n,K)$ inequality with

$$n = n(\epsilon) = (1+\alpha^{-1})\mu_\epsilon((1-h)^2), K(\epsilon,n) = n^{-1}[(1+\alpha)F_h(\mu_\epsilon) + (1+\alpha^{-1})K]$$

Since h is continuous, $\mu_\epsilon((1-h)^2)$ converges towards $\mu((1-h)^2) \neq 0$ and since $\liminf(\log\epsilon^{-1})^{-1}\int_\epsilon^1 F_h(\mu_x)dx$ goes to zero ,

$$\liminf_{\epsilon\to 0}(\log\epsilon^{-1})^{-1}\int_\epsilon^1 K(x,n)dx = 0.$$

Thus, $\delta^\square(\mu) \geq 1 - \inf_\alpha(1+\alpha^{-1})\mu((1-h)^2) = 1 - \mu((1-h)^2)$ and optimizing over $h \in \bar{\mathcal{F}}_\mu$ yields the desired estimate. □

5. Lipschitz invariance.

Our main result is that δ_c is invariant under push-forwards by bi-Lipschitz maps:

Theorem 5.1. *Let $f : \mathbb{R} \to \mathbb{R}$ be bi-Lipschitz, i.e., we assume that for some $m, M > 0$ and all $x, y \in \mathbb{R}$,*

$$m|x-y| \leq |f(x) - f(y)| \leq M|x-y|.$$

Let $\eta = f^*\mu$ be the push-forward of μ. Then $\delta_c(\mu) = \delta_c(\eta)$.

Proof. For any $y = f(x)$,

$$\eta[y - t/2, y + t/2] = \mu(f^{-1}[y-t/2, y+t/2]) \geq \mu[x - t/(2M), x + t/(2M)].$$

It follows that

$$\begin{aligned}\int \log\frac{1}{t}\eta[y-t/2, y+t/2]d\eta(y) &\geq \int \log\frac{1}{t}\mu[f^{-1}(y) - t/(2M), f^{-1}(y) + t/(2M)]d\eta(y)\\ &= \int \log\frac{1}{t}\mu[x - t/(2M), x + t/(2M)]d\mu(x)\\ &= \int \log\frac{1}{t/M}\mu[x - t/(2M), x + t/(2M)]d\mu(x) - \log M.\end{aligned}$$

Using Theorem 2.7 we conclude that

$$\delta_c(\eta) \leq \delta_c(\mu).$$

Replacing f by its inverse yields the reverse inequality. $\square$

It should be noted that one cannot expect much more invariance for δ_c than is given by Theorem 5.1. Indeed, Cantor sets in $\mathbb{R}$ can be made homeomorphic in a way that distorts their fractal dimensions.

6. Non-commutative Bochner's inequality

In this last section, we generalize the notion of measure-dependent Bochner's inequality of section 4. To this end, we first define the appropriate notions of carré du champ and carré du champ itéré.

6.1. **Carré du champ.** We recall first that the carré du champ and the carré du champ itéré in $\mathbb{R}^n$ are given, for $f : \mathbb{R}^n \to \mathbb{C}$ by

$$\Gamma(f, f) = \sum_{i=1}^{n} |\partial_i f|^2, \quad \Gamma_2(f, f) = \sum_{i,j=1}^{n} |\partial_{x_i}\partial_{x_j} f|^2$$

In the case of m Hermitian matrices X_N with complex entries x_{ij}^k, $1 \le i \le j \le N$, $1 \le k \le m$,

$$\Delta = 2\sum_{k=1}^{m} \sum_{1\le i<j\le N} \partial_{x_{ij}^k}\partial_{\bar{x}_{ij}^k} + \sum_{k=1}^{m} \sum_{1\le i\le N} \partial_{x_{ii}^k}\partial_{x_{ii}^k}$$

and so, if $f, g : \mathbb{R}^{2mN^2} \to \mathbb{C}$, we set

$$\Gamma_1(f, g) = 2\sum_{k=1}^{m}\sum_{i<j} \partial_{x_{ij}^k} f \partial_{\bar{x}_{ij}^k} \bar{g} + \sum_{k=1}^{m}\sum_{i<j} \partial_{x_{ii}^k} f \partial_{x_{ii}^k} \bar{g}$$

and

$$\Gamma_2(f, g) = \sum_{k,l=1}^{m}\sum_{ij}\sum_{ml} (\partial_{x_{ij}^l}\partial_{x_{ml}^k} f \partial_{\bar{x}_{ij}^l}\partial_{\bar{x}_{ml}^k}\bar{g}).$$

Again, to define the notion of carré du champ and carré du champ itéré for tracial states, the idea is that if we consider $f((x_{ml}^k)_{1\le m\le l\le N}^{1\le k\le m}) := F(X) = \mathrm{tr}(P(X_1, \cdots, X_m))$ when $\hat{\mu}^N(Q) := N^{-1}\mathrm{tr}(Q(X_1, \cdots, X_m))$ goes to $\tau(Q)$ for all polynomial Q and some non-commutative law τ. We denote $*$ the involution

$$(zX_{i_1}\cdots X_{i_k})^* = \bar{z}X_{i_k}\cdots X_{i_1}$$

for any $i_l \in \{1, \cdots, m\}$. Since $\overline{\mathrm{tr}(P)} = \mathrm{tr}(P^*)$, applying the above recipe we find,

$$\begin{aligned}
\Gamma_1^{\hat{\mu}^N}(P, Q) &=: \sum_k \sum_{i,j} \partial_{x_{ij}^k}(\mathrm{tr}(P(X_1, \cdots, X_m)))\partial_{\bar{x}_{ij}^k}(\mathrm{tr}(Q^*(X_1, \cdots, X_m))) \\
&= N^{-1}\sum_k \sum_{i,j} [D_k P(X)]_{ij}[D_k Q(X)^*]_{ji} \\
&= \sum_k N^{-1}\mathrm{tr}(D_k P(X)(D_k Q(X))^*) \\
&\approx \sum_k \tau(D_k P(X)(D_k Q(X))^*) := \Gamma_1^\tau(P, Q)
\end{aligned}$$

where we have denoted D_k the cyclic derivative on polynomial, given by $D_kP = \sum_{P=P_1X_kP_2} P_2P_1$ if P is a monomial (and extending by linearity to all polynomial then) and noticed, as can be readily checked on monomials, that $(D_kP)^* = D_kP^*$. Similarly,

$$\begin{aligned}
\Gamma_2^{\hat{\mu}^N}(P,Q) &=: \sum_{k,l=1}^m \sum_{i,j} \sum_{pq} \partial_{X^l_{ij}} \partial_{X^k_{pq}} (\mathrm{tr}(P(X_1,\cdots,X_m))) \partial_{\bar{X}^l_{ij}} \partial_{\bar{X}^k_{pq}} (\mathrm{tr}(Q^*(X_1,\cdots,X_m))) \\
&= N^{-2} \sum_{k,l=1}^m [\partial_l \circ D_k P \sharp 1_{pq}]_{ij} [\partial_l \circ D_k Q^* \sharp 1_{qp}]_{ji} \\
&\approx \sum_{k,l=1}^m \tau \otimes \tau((\partial_l \circ D_k Q)^* \star \partial_l \circ D_k P) := \Gamma_2^\tau(P,Q)
\end{aligned}$$

where ∂_k denotes the non-commutative derivative with respect to the variable X_k ($\partial_k P = \sum_{P=P_1X_kP_2} P_1 \otimes P_2$ for a monomial P), $A \otimes B \sharp C = ACB$, $(A\otimes B)^* = B^* \otimes A^*$ and $A \otimes B \star A' \otimes B' = BA' \otimes AB'$. 1_{kl} is the matrix with zeroes except in kl. Hence, we define

Definition 6.1. For any non-commutative law τ of m self-adjoint variables, we define its *non-commutative carré du champ* to be the bilinear function on $\mathbb{C}\langle X_1,\cdots,X_m\rangle$ so that for any $P,Q \in \mathbb{C}\langle X_1,\cdots,X_m\rangle$,

$$\Gamma_1^\tau(P,Q) = \sum_{i=1}^m \tau(D_iP(D_iQ)^*)$$

and its *non-commutative carré du champ itéré* to be the bilinear function on $\mathbb{C}\langle X_1,\cdots,X_m\rangle$ so that for any $P,Q \in \mathbb{C}\langle X_1,\cdots,X_m\rangle$,

$$\Gamma_2^\tau(P,Q) = \sum_{k,l=1}^m \tau\otimes\tau((\partial_l \circ D_kQ)^* \star \partial_l \circ D_kP).$$

We also denote in short

$$\Gamma_i^\tau(P,Q) =< P,Q >_{\tau,i}\ .$$

Observe that the above notation makes sense since Γ_i^τ are positive bilinear forms. This is obvious for Γ_1^τ. For Γ_2^τ, one needs to observe that if τ is a tracial state, $P,Q \to \tau\otimes\tau(P \star Q^*)$ is non negative. But if $P = \sum \alpha_i A_i \otimes B_i$,

$$\tau\otimes\tau(P\star P^*) = \sum \alpha_i \bar{\alpha}_j \tau(A_iA_j^*)\tau(B_iB_j*) \geq 0$$

since the matrices $(\tau(A_iA_j^*))_{i,j}, (\tau(B_iB_j^*))_{i,j}$ are non-negative.

Further, when the Laplacian $\Delta = \sum \partial_{x^k_{ij}} \partial_{\bar{x}^k_{ij}}$ on the entries acts on $F(X^l_{ij}) = f(X_1,\cdots,X_m)$, we get if

$$D_k^2 \equiv \frac{1}{2}(\partial_k \otimes 1 + 1 \otimes \partial_k) \circ \partial_k$$

and

$$M(A\otimes B\otimes C) \equiv B \otimes AC$$

and

$$\mathbb{L}_\tau := \sum_k \tau \otimes IM \circ \partial_k^2$$

that

$$\Delta F = \mathbb{L}_\tau f$$

when the law of X approximate τ. If $F = N^{-1}\mathrm{tr}(P)$, we get

$$\Delta F \approx \tau(\mathbb{L}_\tau P).$$

Note here that

$$\tau(\mathbb{L}_\tau P) = \sum_{i=1}^m \tau \otimes \tau(\partial_i \circ D_i P)$$

as can be readily checked by taking P to be a monomial. Let $S = (S^1, \cdots, S^m)$ be a free Brownian motion, free with $X = (X^1, \cdots, X^m)$ with law τ, and ϕ a tracial state on a von Neumann algebra containing S and X. We then have

$$P(X + S_t) = P(X) + \int_0^t \mathbb{L}_{\phi_{X+S_s}}(P)(X + S_s)ds + \int_0^t \sum_{i=1}^m \partial_i P(X + S_s)\sharp dS_s^i$$

where the last term is a martingale. We denote τ_t the distribution of $(X^1+S_t^1, \cdots, X^m+S_t^m)$.

6.2. **Non-commutative Bochner's inequality.** We recall that Bochner's inequality reads in the classical context as

$$\Gamma_2(f, f) \geq \frac{1}{n}(\Delta f)^2 - K\Gamma_1(f, f)$$

for some fixed constants $n \geq 0$, $K \in \mathbb{R}$. Remark that n is of the order of the dimension, so of order N^2 in the context of matrices, so we let $\mathcal{N} = n/N^2$ and apply this inequality to $F = \mathrm{tr}(P)$ we get if $\hat{\mu}_X^N \approx \tau$, as N goes to infinity,

$$< P, P >_{\tau,2} \geq \frac{1}{\mathcal{N}}[\tau(\mathbb{L}_\tau P)]^2 - K < P, P >_{\tau,1} .$$

Therefore,

Definition 6.2. We shall say that a non-commutative law τ satisfies a $\mathrm{CD_m}(\mathcal{K}, \mathcal{N})$ inequality iff for all ϵ small enough,

$$< P, P >_{\tau_\epsilon,2} \geq \frac{1}{\mathcal{N}}[\tau_\epsilon(\mathbb{L}_{\tau_\epsilon} P)]^2 - \mathcal{K}(\mathcal{N}, \epsilon) < P, P >_{\tau_\epsilon,1}$$

for any polynomial function P.

We can therefore define

Definition 6.3.

$$\delta^\square(\tau) = m - \inf_{\tau \text{ satisfies } \mathrm{CD_m}(\mathcal{K},\mathcal{N})} (\bar{\mathcal{K}}(\mathcal{N}) + 1)\mathcal{N}$$

where

$$\bar{\mathcal{K}}(\mathcal{N}) = \liminf_{\epsilon \to 0} (\log \epsilon^{-1})^{-1} \int_\epsilon^1 \mathcal{K}(\mathcal{N}, y)dy.$$

We next want to compare this definition of a non-commutative dimension with already existing entropy dimension. We recall that in the non-commutative setting, Voiculescu [13] defined the following notion of Fisher entropy and related entropy dimension. For a tracial state τ, we define its Fisher information by

$$\begin{aligned}\Phi^*(\tau) &= \sum_{i=1}^m \sup_{P\in\mathbb{C}\langle X_1,\cdots,X_m\rangle} \{\tau\otimes\tau(\partial_i(P+P^*)) - \tau(PP^*)\} \\ &= \sup_{P\in\mathbb{C}\langle X_1,\cdots,X_m\rangle^m} \{\sum_{i=1}^m \tau\otimes\tau(\partial_i(P_i+P_i^*)) - \sum_{i=1}^m \tau(P_iP_i^*)\}\end{aligned}$$

Then, as in (9), the microstates-free free entropy dimension is given by

$$\delta^*(\mu) = m - \liminf_{t\to 0} \frac{\int_t^1 \Phi^*(\tau_s)ds}{|\log t|}. \tag{15}$$

Here, we shall consider a variant of δ^* based on the following definition of Fisher information as found in [3]:

$$\bar{\Phi}^*(\tau) = \sup_{P\in\mathbb{C}\langle X_1,\cdots,X_m\rangle} \{\sum_{i=1}^m \tau\otimes\tau(\partial_i(D_iP + D_iP^*)) - \sum_{i=1}^m \tau(D_iPD_iP^*)\}$$

and

$$\bar{\delta}^*(\tau) = m - \liminf_{t\to 0} \frac{\int_t^1 \bar{\Phi}^*(\tau_s)ds}{|\log t|}.$$

Observe that $\bar{\Phi}^* \le \Phi^*$ and so $\bar{\delta}^*(\tau) \ge \delta^*(\tau)$. Equality is achieved if the conjugate variables belong to the cyclic gradient space, which appears to be often (if not always) the case (see Voiculescu [13] and Cabanal Duvillard-Guionnet [4]). This is the case, in particular, if we are dealing with the law τ of a single variable (i.e., $m = 1$).

In the sequel, we shall as well denote $(\mathcal{J}_\tau^i)_{1\le i\le m}$ for the projection of the conjugate variable on the cyclic gradient space, i.e

$$\tau\otimes\tau(\partial_i \circ D_iP) = \tau(\mathcal{J}_\tau^i D_iP)$$

for all polynomials P.

We next prove

Proposition 6.4.

$$\delta^{\square}(\tau) = \bar{\delta}^*(\tau).$$

In particular,

$$\delta^{\square}(\tau) \ge \bar{\delta}^*(\tau) \ge \delta(\tau)$$

where $\delta(\tau)$ denotes the microstates entropy dimension.

Proof. Let us first remark that by definition

$$\tau(\mathbb{L}_\tau P) = \sum_{i=1}^m \tau\otimes\tau(\partial_i\circ D_iP) = \sum_{i=1}^m \tau(\mathcal{J}_\tau^i D_iP)$$

and therefore

$$|\tau(\mathbb{L}_\tau P)|^2 \le \bar{\Phi}(\mu)\Gamma_1^\tau(P,P).$$

On the other hand

$$|\tau(\mathbb{L}_\tau P)|^2 \le m \sum_{i=1}^m |\tau\otimes\tau(\partial_i \circ D_i P)|^2$$

with

$$|\tau\otimes\tau(\partial_i\circ D_iP)|^2 \le \tau\otimes\tau(\partial_i\circ D_iP \star (\partial_i\circ D_iP)^*)$$

by Cauchy-Schwartz inequality, which holds because of the positivity of the positive bilinear form $P,Q\to\tau\otimes\tau(\partial_i\circ D_iP\star(\partial_i\circ D_iP)^*)$. Hence, for any $\alpha\in[0,1]$

$$|\tau(\mathbb{L}_\tau P)|^2 \quad\le\quad m\alpha\Gamma_2^\tau(P,P)+(1-\alpha)\bar\Phi(\tau)\Gamma_1^\tau(P,P).$$

This proves that Bochner's inequality is satisfied with $\mathcal{N}=m\alpha$ and $\mathcal{K}(\mathcal{N},\epsilon)=(1-\mathcal{N}/m)\bar\Phi(\tau_\epsilon)\mathcal{N}^{-1}$ from which we get

$$m-\delta^\square(\tau)=\inf\{\mathcal{N}(1+\bar{\mathcal{K}}(\mathcal{N}))\}\le \inf_{\mathcal{N}\in[0,m]}\{\mathcal{N}+(1-\mathcal{N}/m)\liminf\frac{\int_\epsilon^1\bar\Phi^*(\tau_s)ds}{|\log\epsilon|}\}=m-\bar\delta^*(\tau)$$

where we used that $\frac{\int_\epsilon^1\bar\Phi^*(\tau_s)ds}{|\log\epsilon|}\in[0,m]$ which holds since $\bar\Phi^*(\tau_s)\le s^{-1}$.

For the other inequality, let X be an m-tuple of random variables having the law $\tau_{x+\epsilon}$ obtained as free convolution of the law τ with the semicircular law of variance ϵ. Let $0<x<\delta$ and let $S_{\delta-x}$ be an m-tuple of semicircular variables of variance $\delta-x$, free from X. Denote by $\tau(\cdot|X)$ the conditional expectation onto the algebra generated by X. We then introduce, in the spirit of the proof in the classical case, the function

$$\phi(x)=\sum_{i=1}^m\tau_{x+\epsilon}\left(|D_i\tau(P(X+S_{\delta-x})|X)|^2\right)$$

(note that $\tau(P(X+S_{\delta-x})|X)$ is a polynomial in X and hence is in the domain of D_i).

We have

$$\begin{aligned}\phi'(x) &= \sum_{i=1}^m\tau_{x+\epsilon}\left(\mathbb{L}_{\tau_{x+\epsilon}}|D_i\tau(P(X+S_{\delta-x})|X)|^2\right)\\ &\quad -2\Re\tau_{x+\epsilon}\left(D_i\tau(\mathbb{L}_{\tau_{\delta+\epsilon}}P(X+S_{\delta-x})|X)(D_i\tau(P(X+S_{\delta-x})|X)^*\right) \qquad (16)\end{aligned}$$

where we used the fact that the law of $X+S_{\delta-x}$ under $\tau_{x+\epsilon}$ is the law of $X+S_{\delta-x}+\bar S_{x+\epsilon}$, with $\bar S$ a free Brownian motion independent from S,X, which has the same law $\tau_{\delta+\epsilon}$ of $X+S_{\delta+\epsilon}$. Now, let us compute $\mathbb{L}_{\tau_{x+\epsilon}}(PQ)$ for polynomials P,Q. $\mathbb{L}_{\tau_{x+\epsilon}}$ is a second order differential operator; it will either act on P, or Q, or both;

$$\mathbb{L}_{\tau_{x+\epsilon}}(PQ)=\mathbb{L}_{\tau_{x+\epsilon}}(P)Q+P\mathbb{L}_{\tau_{x+\epsilon}}(Q)+R(P,Q).$$

To compute $R(P,Q)$ note that this contribution comes from

$$\Delta_k^2(PQ)-\Delta_k^2(P)\times 1\otimes 1\otimes Q-P\otimes 1\otimes 1\times\Delta_k^2(Q)=\partial_kP\bar\star\partial_kQ$$

with $A\otimes B\bar\star A'\otimes B'=A\otimes BA'\otimes B'$. Note that

$$M(A\otimes B\bar\star A'\otimes B')=BA'\otimes AB'=A\otimes B\star A'\otimes B'.$$

Therefore

$$\begin{aligned}&\sum_{i=1}^m\tau_{x+\epsilon}\left(R(D_i\tau(P(X+S_{\delta-x})|X),D_i\tau(P(X+S_{\delta-x})|X))\right)\\ &\qquad=\Gamma_2^{\tau_{x+\epsilon}}(\tau(P(X+S_{\delta-x})|X),\tau(P(X+S_{\delta-x})|X))\end{aligned}$$

Finally, it is easy to see that

$$\mathbb{L}_{\tau_{x+\epsilon}}(D_i\tau(P(X+S_{\delta-x})|X)) = D_i\tau(\mathbb{L}_{\tau_{\delta+\epsilon}}P(X+S_{\delta-x})|X)$$

so that we have proved according to (16) that

$$\begin{aligned} \phi'(x) &= \Gamma_2^{\tau_{x+\epsilon}}(\tau(P(X+S_{\delta-x})|X)) \\ &\geq \frac{1}{\mathcal{N}}[\tau_{x+\epsilon}[\mathbb{L}_{\tau_{x+\epsilon}}(\tau(P(X+S_{\delta-x})|X)]^2 - K\Gamma_1^{\tau_{x+\epsilon}}(\tau(P(X+S_{\delta-x})|X)) \end{aligned} \tag{17}$$

We can now proceed exactly in the lines of the proof of Proposition 6.4 to conclude that $\bar{\Phi}^*(\tau_\epsilon)$ satisfies the bound

$$\bar{\Phi}^*(\tau_\epsilon) \leq \frac{\mathcal{N}\frac{L(\epsilon)}{L(\epsilon+\delta)}\bar{\Phi}^*(\tau_\epsilon)}{\int_0^\delta \frac{L(\epsilon)}{L(\epsilon+x)}dx\bar{\Phi}^*(\tau_\epsilon)+\mathcal{N}} \tag{18}$$

with $L(y) = e^{\int_y^1 \mathcal{K}(x,\mathcal{N})dx}$ as before. The rest of the proof is exactly as in the classical case. $\square$

Corollary 6.5. *If τ is the law of a single variable (i.e., $m = 1$) then*

$$\delta^\square(\tau) = \bar{\delta}^*(\tau) = \delta(\tau) = 1 - \tau\otimes\tau(\chi_\Delta)$$

where χ_Δ is the characteristic function of the diagonal $\Delta \subset \mathbb{R}^2$ and we identify τ with a measure on $\mathbb{R}$.

Proposition 6.6. *Let $X = (X_1, \ldots, X_m)$ have the given law τ , $M = W^*(X_1, \ldots, X_m)$ and let $G = (G_{ij}) \in M_{m\times m}(L^2(M\bar{\otimes}M^o))$ be a fixed matrix. Let $\bar{\Phi}_G$ be the Fisher information defined by*

$$\bar{\Phi}_G = \sup_{P\in\mathbb{C}\langle X_1,\cdots,X_m\rangle}\{\sum_{i=1}^m \tau\otimes\tau(\partial_i^G(D_iP+D_iP^*)) - \sum_{i=1}^m \tau(D_iPD_iP^*)\}$$

where $\partial_i^G(X_j) = G_{ij}$. Then

$$\bar{\delta}^*(\tau) = \delta^\square(\tau) \geq m(1 - \inf_{G\in\mathcal{F}_\tau}\tau(1-G)^2)$$

with $\mathcal{F}_\tau$ the set of $G \in M_{m\times m}(L^2(M\bar{\otimes}M^o))$ so that $(\log\epsilon^{-1})^{-1}\int_\epsilon^1 dt\bar{\Phi}_G^*(\tau_t)$ goes to zero.

The proof is exactly the same as the previous one except that the use of Bochner inequality is simply replaced by the fact that any measure satisfies $\mathrm{CD_m}(m, 0)$ as we have seen in the proof of the previous theorem.

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References

[1] D. Bakry and Z. Qian, Some new results on eigenvectors via dimension, diameter, and Ricci curvature, *Adv. Math.* **155** (2000) 98–153

[2] D. Bakry and M. Ledoux, A logarithmic Sobolev form of the Li-Yau parabolic inequality, *preprint* (2005) *http://www.lsp.ups-tlse.fr/Ledoux/*

[3] T. Cabanal Duvillard and A. Guionnet, Large deviations upper bounds for the laws of matrix-valued processes and non-communicative entropies *Ann. Probab.* **29** (2001) pp 1205–1261

[4] T. Cabanal Duvillard and A. Guionnet, Discussions around Voiculescu's free entropies, *Adv. Math.* **174** (2003) pp 167–226

[5] A. Connes, D. Shlyakhtenko, L^2-homology for von Neumann algebras, *J. reine angew. Math.* **586** (2005) 125–168

[6] A. Dembo and O. Zeitouni *Large deviations techniques and applications*, Applications of Mathematics (New York), **38**, Second Edition,Springer-Verlag, New York, (1998)

[7] K. Jung, A free entropy dimension lemma, *Pacific J. Math.* **211** (2003) 265–271.

[8] I. Mineyev, D. Shlyakhtenko, Non-microstates free entropy dimension for groups *Geom. Funct. Anal.* **15** (2005) 476–490

[9] D. Shlyakhtenko, Some estimates for non-microstates free entropy dimension, with applications to q-semicircular families, *IMRN* **51** (2004), 2757–2772

[10] A.J. Stam, Some inequalities satisfied by the quantities of Information of Fisher and Shannon' *Inform. and Contr.* **2** (1959) pp. 102–112

[11] D. Voiculescu, The analogues of entropy and of Fisher's information measure in free probability theory II, *Invent. Math.* **118** (1994) 411-440

[12] D. Voiculescu,The analogues of entropy and of Fisher's information measure in free probability theory. V. Noncommutative Hilbert transforms *Invent. Math.* **132**(1998)pp. 189–227

[13] D. Voiculescu, The analogues of entropy and of Fisher's information measure in free probability theory. VI. Liberation and mutual free information, *Adv. Math.* **146** (1999) pp. 101–166