FLUCTUATIONS OF THE EXTREME EIGENVALUES OF FINITE RANK DEFORMATIONS OF RANDOM MATRICES

F. BENAYCH-GEORGES*, A. GUIONNET⋆, M. MAIDA♯.

Abstract. Consider a deterministic self-adjoint matrix X_n with spectral measure converging to a compactly supported probability measure, the largest and smallest eigenvalues converging to the edges of the limiting measure. We perturb this matrix by adding a random finite rank matrix with delocalized eigenvectors and study the extreme eigenvalues of the deformed model. We give necessary conditions on the deterministic matrix X_n so that the eigenvalues converging out of the bulk exhibit Gaussian fluctuations, whereas the eigenvalues sticking to the edges are very close to the eigenvalues of the non-perturbed model and fluctuate in the same scale.
We generalize these results to the case when X_n is random and get similar behavior when we deform some classical models such as Wigner or Wishart matrices with rather general entries or the so-called matrix models.

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1. Introduction

Most of the spectrum of a large matrix is not much altered if one adds a finite rank perturbation to the matrix, simply because of Weyl's interlacement properties of the eigenvalues. But the extreme eigenvalues which, depending on the strength of the perturbation, should either stick to the extreme eigenvalues of the non-perturbed matrix or

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* UPMC Univ Paris 6, LPMA, Case courier 188, 4, Place Jussieu, 75252 Paris Cedex 05, France, florent.benaych@upmc.fr,
⋆ UMPA, ENS Lyon, 46 allée d'Italie, 69364 Lyon Cedex 07, France, aguionne@umpa.ens-lyon.fr,
♯ Université Paris-Sud, Laboratoire de Mathématiques, Bâtiment 425, Faculté des Sciences, 91405 Orsay Cedex, France, mylene.maida@math.u-psud.fr.
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deviate to some larger values. This phenomenon was made precise in [9], where a sharp phase transition, known as the *BBP transition* [34, 27, 38, 29], was exhibited for finite rank perturbations of a complex Gaussian Wishart matrix. In this case, it was shown that if the strength of the perturbation is above a threshold, the largest eigenvalue of the perturbed matrix deviates away from the bulk and has then Gaussian fluctuations, otherwise it sticks to the bulk and fluctuates according to the Tracy-Widom law. The fluctuations of the extreme eigenvalues which deviate from the bulk were studied as well when the non-perturbed matrix is a Wishart (or Wigner) matrix with non-Gaussian entries; they were shown to be Gaussian if the perturbation is chosen randomly with i.i.d. entries in [7], or with completely delocalised eigenvectors [18, 19], whereas in [12], a non-Gaussian behaviour was exhibited when the perturbation has localised eigenvectors. The influence of the localisation of the eigenvectors of the perturbation was studied more precisely in [13].

In this paper, we also focus on the behavior of the extreme eigenvalues of a finite rank perturbation of a large matrix, this time in the framework where the large matrix is deterministic whereas the perturbation has delocalised random eigenvectors. We show that the eigenvalues which deviate away from the bulk have Gaussian fluctuations whereas those which stick to the bulk are extremely close to the extreme eigenvalues of the non-perturbed matrix. In a one-dimensional perturbation situation, we can as well study the fluctuations of the next eigenvalues, for instance showing that if the first eigenvalue deviates from the bulk, the second eigenvalue will stick to the first eigenvalue of the non-perturbed matrix, whereas if the first eigenvalue sticks to the bulk, the second eigenvalue will be very close to the second eigenvalue of the non-perturbed matrix. Hence, for a one dimensional perturbation, the eigenvalues which stick to the bulk will fluctuate as the eigenvalues of the non-perturbed matrix. We can also extend these results beyond the case when the non-perturbed matrix is deterministic. In particular, if the non-perturbed matrix is a Wishart (or Wigner) matrix with rather general entries, or a matrix model, we can use the universality of the fluctuations of the extreme eigenvalues of these random matrices, to show that the pth extreme eigenvalue which sticks to the bulk fluctuates according to the pth dimensional Tracy-Widom law. This proves the universality of the BBP transition at the fluctuation level, provided the perturbation is delocalised and random.
The reader should notice however that we do not deal with the asymptotics of eigenvalues corresponding to critical deformations. This probably requires a case-by-case analysis and may depend on the model considered.

Let us now describe more precisely the models we will be dealing with. We consider a deterministic self-adjoint matrix X_n with eigenvalues $\lambda_1^n \leq \cdots \leq \lambda_n^n$ satisfying the following hypothesis.

Hypothesis 1.1. *The spectral measure $\mu_n := n^{-1} \sum_{l=1}^n \delta_{\lambda_l^n}$ of X_n converges towards a deterministic probability measure μ_X with compact support. Moreover, the smallest and largest eigenvalues of X_n converge respectively to a and b, the lower and upper bounds of the support of μ_X.*

We study the eigenvalues $\widetilde{\lambda}_1^n \leq \cdots \leq \widetilde{\lambda}_n^n$ of a perturbation $\widetilde{X_n} := X_n + R_n$ obtained from X_n by adding a finite rank matrix $R_n = \sum_{i=1}^r \theta_i u_i^n u_i^{n^*}$. We shall assume r and the

θ_i's to be deterministic and independent of n, but the column vectors $(u_i^n)_{1\leq i\leq r}$ chosen randomly as follows. Let ν be a probability measure on $\mathbb{R}$ or $\mathbb{C}$ satisfying

Assumption 1.2. *The probability measure ν satisfies a log-Sobolev inequality, is centred and has variance one. If ν is not concentrated on $\mathbb{R}$, we assume moreover that its real part and its imaginary part are independent and identically distributed (i.i.d.).*

We consider now a random vector $v^n = \frac{1}{\sqrt{n}}(x_1, \ldots, x_n)^T$ with $(x_i)_{1\leq i\leq n}$ i.i.d. real or complex random variables with law ν. Then

(1) Either the u_i^n's $(i = 1, \ldots, r)$ are independent copies of v^n
(2) Or $(u_i^n)_{1\leq i\leq r}$ are obtained by the Gram-Schmidt orthonormalisation of r independent copies of a vector v^n.

We shall refer to the model (1) as the *i.i.d. model* and to the model (2) as the *orthonormalised model.*

Before giving a rough statement of our results, let us make a few remarks.
In the orthonormalised model, if ν is the standard real (resp. complex) Gaussian law, $(u_i^n)_{1\leq i\leq r}$ follows the uniform law on the set of orthogonal random vectors on the unit sphere of $\mathbb{R}^n$ (resp. $\mathbb{C}^n$) and by invariance by conjugation, the model coincides with the one studied in [10].
For a general ν satisfying Assumption 1.2, the r i.i.d. random vectors obtained are not necessarily linearly independent almost surely so that the orthonormal vectors described in (2) are not always almost surely well defined. However, as the dimension goes to infinity, they are well defined with overwhelming probability. This means the following: we shall say that a sequence of events $(C_n)_{n\geq 1}$ occurs with *overwhelming probability*[1] if there exists two constants $C, \eta > 0$ independent of n such that for n large enough,

$$\mathbb{P}(C_n) \geq 1 - Ce^{-n^\eta}.$$

Consequently, in the sequel, we shall restrict ourselves to the event when the model (2) is well defined without mentioning it explicitly.

In this work, we study the asymptotics of the eigenvalues of $\widetilde{X_n}$ outside of the spectrum of X_n.

It has already been observed in similar situations, see [9], that these eigenvalues converge to the boundary of the support of X_n if the θ_i's are small enough, whereas for sufficiently large values of the θ_i's, they stay away from the bulk of X_n. More precisely, if we let G_{μ_X} be the Cauchy-Stieltjes transform of μ_X, defined, for $z < a$ or $z > b$, by the formula

$$G_{\mu_X}(z) = \int \frac{1}{z-x} \mathrm{d}\mu_X(x),$$

then the eigenvalues of $\widetilde{X_n}$ outside the bulk converge to the solutions of $G_{\mu_X}(z) = \theta_i^{-1}$ if they exist.

[1]Note that this is a bit different from what is called *overwhelming probability* by Tao and Vu but will be sufficient for our purpose.

Indeed, if we let

$$\overline{\theta} := \frac{1}{\lim_{z\downarrow b} G_{\mu_X}(z)} \geq 0, \qquad \underline{\theta} := \frac{1}{\lim_{z\uparrow a} G_{\mu_X}(z)} \leq 0$$

and

$$\rho_\theta := \begin{cases} G_{\mu_X}^{-1}(1/\theta) & \text{if } \theta \in (-\infty, \underline{\theta}) \cup (\overline{\theta}, +\infty), \\ a & \text{if } \theta \in [\underline{\theta}, 0), \\ b & \text{if } \theta \in (0, \overline{\theta}], \end{cases}$$

then we have the following theorem.

Theorem 1.3. *Assume that Hypothesis 1.1 and Assumption 1.2 are satisfied. Let $r_0 \in \{0, \ldots, r\}$ be such that*

$$\theta_1 \leq \cdots \leq \theta_{r_0} < 0 < \theta_{r_0+1} \leq \cdots \leq \theta_r.$$

Then for all $i \in \{1, \ldots, r_0\}$ we have

$$\widetilde{\lambda}_i^n \xrightarrow{a.s.} \rho_{\theta_i}$$

and for all $i \in \{r_0 + 1, \ldots, r\}$,

$$\widetilde{\lambda}_{n-r+i}^n \xrightarrow{a.s.} \rho_{\theta_i}.$$

Moreover, for all $i > r_0$ (resp. for all $i \geq r - r_0$),

$$\widetilde{\lambda}_i^n \xrightarrow{a.s.} a \qquad (\text{resp. } \widetilde{\lambda}_{n-i}^n \xrightarrow{a.s.} b).$$

The uniform case was proved in [10, Theorem 2.1] and we will follow a similar strategy to prove it under our assumptions in Section 2 (see Lemma 2.1).

We study the fluctuations of the extreme eigenvalues of $\widetilde{X_n}$. Precise statements will be given in Theorems 3.2, 3.4, 4.3 and 4.4 and Corollary 4.5 but the results roughly state as follows.

Theorem 1.4. *Under additional hypotheses,*

(1) *Let $\alpha_1 < \cdots < \alpha_q$ be the different values of the θ_i's such that $\rho_{\theta_i} \notin \{a, b\}$ and denote, for each j, by I_j the set of indices i so that $\theta_i = \alpha_j$. Set $k_j = |I_j|$ and q_0 the largest index so that $\alpha_{q_0} < 0$. Then, the law of the random vector*

$$\left(\sqrt{n}(\widetilde{\lambda}_i^n - \rho_{\alpha_j}), i \in I_j\right)_{1\leq j\leq q_0} \cup \left(\sqrt{n}(\widetilde{\lambda}_{n-r+i}^n - \rho_{\alpha_j}), i \in I_j\right)_{q_0+1\leq j\leq q}$$

converges to the law of the eigenvalues of $(c_j M_{k_j})_{1\leq j\leq q}$ with the M_{k_j}'s being independent matrices following the law of a $k_j \times k_j$ matrix from the GUE or the GOE, depending whether ν is supported on the complex plane or the real line. The constant c_j is explicitly defined in Equation (6).

(2) *If none of the θ_i's are critical, with overwhelming probability, the extreme eigenvalues converging to a or b are at distance at most $n^{-1+\epsilon}$ of the extreme eigenvalues of X_n for some $\epsilon > 0$.*

(3) *If $r = 1$ and $\theta_1 = \theta > 0$, we have the following more precise picture about the next eigenvalues*

- *If* $\rho_\theta > b$, $\sqrt{n}(\widetilde{\lambda}_n^n - \rho_\theta)$ *converges towards a Gaussian variable, whereas* $n^{1-\epsilon}(\widetilde{\lambda}_{n-i}^n - \lambda_{n-i+1})$ *vanishes in probability as* n *goes to infinity for any fixed* $i \geq 1$ *and some* $\epsilon > 0$.
- *If* $\rho_\theta = b$ *and* $\theta \neq \overline{\theta}$, $n^{1-\epsilon}(\widetilde{\lambda}_{n-i}^n - \lambda_{n-i})$ *vanishes in probability as* n *goes to infinity for any fixed* $i \geq 1$ *and some* $\epsilon > 0$.

The first part of this theorem will be proved in Section 3, whereas Section 4 will be devoted to the study of the eigenvalues sticking to the bulk, i.e. to the proof of the second and third parts of the theorem. Moreover, our results can be easily generalised to non-deterministic self-adjoint matrices X_n that satisfy our hypotheses with probability tending to one. This will allow us to study in Section 5 the deformations of various classical models. This will include the study of the Gaussian fluctuations away from the bulk for rather general Wigner and Wishart matrices, hence providing a new proof of the first part of [18, Theorem 1.1] and of [5, Theorem 3.1] but also a new generalisation to non-white ensembles. The study of the eigenvalues that stick to the bulk requires a finer control on the eigenvalues of X_n in the vicinity of the edges of the bulk, which we prove for random matrices such as Wigner and Wishart matrices with entries having a sub-exponential tail. This result complements [18, Theorem 1.1] where the fluctuations of the largest eigenvalue of a non-Gaussian Wishart matrix perturbed by a delocalised but deterministic rank one perturbation was studied. One should remark that our result depends very little on the law ν (only through its fourth moment in fact).

Our approach is based upon a determinant computation (see Lemma 6.1), which shows that the eigenvalues of $\widetilde{X_n}$ we are interested in are the solutions of the equation

$$f_n(z) := \det\left(\left[G_{i,j}^n(z)\right]_{i,j=1}^r - \mathrm{diag}(\theta_1^{-1}, \ldots, \theta_r^{-1})\right) = 0, \tag{3}$$

with

$$G_{i,j}^n(z) := \langle u_i^n, (z - X_n)^{-1} u_j^n \rangle,$$

where $\langle \cdot, \cdot \rangle$ denotes the usual scalar product in $\mathbb{C}^n$.
By the law of large numbers for i.i.d. vectors, by [10, Proposition 9.3] for uniformly distributed vectors or by applying Theorem 6.4 (with $A^n = (z - X_n)^{-1}$), it is easy to see that for any z outside the bulk,

$$\lim_{n\to\infty} G_{i,j}^n(z) = \mathbb{1}_{i=j} G_{\mu_X}(z)$$

and hence it is clear that one should expect the eigenvalues of $\widetilde{X_n}$ outside of the bulk to converge to the solutions of $G_{\mu_X}(z) = \theta_i^{-1}$ if they exist. Studying the fluctuations of these eigenvalues amounts to analyze the behavior of the solutions of (3) around their limit. Such an approach was already developed in several papers (see e.g [7] or [12]). However, to our knowledge, the model we consider, with a fixed deterministic matrix X_n, was not yet studied and the fluctuations of the eigenvalues which stick to the bulk of X_n was never achieved in such a generality.

For the sake of clarity, throughout the paper, we will call "hypothesis" any hypothesis we need to make on the deterministic part of the model X_n and "assumption" any hypothesis we need to make on the deformation R_n.
Moreover, because of concentration considerations that are developed in the Appendix of the paper, the proofs will be quite similar in the i.i.d. and orthonormalised models.

Therefore, we will detail each proof in the i.i.d. model, which is simpler and then check that the argument is the same in the orthonormalised model or detail the slight changes to make in the proofs.

2. Almost sure convergence of the extreme eigenvalues

For the sake of completeness, we prove in this section Theorem 1.3.

Using [10, Lemma 6.1], Theorem 1.3 will be a direct consequence of the following Lemma.

Lemma 2.1. *Assume that Hypothesis 1.1 and Assumption 1.2 are satisfied. Let $\delta > 0$ and $S_\delta = [a-\delta, b+\delta] \cup (\cup_{1\le i\le r}[\rho_{\theta_i} - \delta, \rho_{\theta_i} + \delta])$. Then, for any $\delta > 0$, the eigenvalues of $\widetilde{X_n}$ belong to S_δ with overwhelming probability.*

Proof. To prove the first statement, by (3), it is enough to prove that f_n does not vanish on S_δ^c.
The i.i.d. model. Fix some $z \in S_\delta^c$ and n large enough. By Proposition 6.2 with $A = (z - X_n)^{-1}$, whose operator norm is bounded by $2\delta^{-1}$, we find that for any $\epsilon > 0$, there exists $c > 0$ such that

$$\mathbb{P}\left(\left|G^n_{i,j}(z) - 1_{i=j}\frac{1}{n}\mathrm{Tr}((z-X_n)^{-1})\right| \ge \frac{\delta^{-1}}{n^{1/2-\epsilon}}\right) \le 4e^{-cn^{2\epsilon}}. \tag{4}$$

By convergence of the spectral measure, $\frac{1}{n}\mathrm{Tr}((z-X_n)^{-1})$ converges towards the Stieltjes transform $G_{\mu_X}(z)$ and hence $f_n(z)$ is arbitrarily close to $f(z) := \prod_{i=1}^r (G_{\mu_X}(z) - \frac{1}{\theta_i})$ with overwhelming probability.

Note now that $z \in S_\delta^c \mapsto f_n(z)$ is Lipschitz with constant of order δ^{-2} and therefore, with $z_k = kn^{-1}, k \in [-Mn, Mn]$ integer and M large enough, we have

$$\sup_{z\in[-M,M]\setminus S_\delta} |f_n(z) - f(z)| \le \max_{k\in[-Mn,Mn], z_k \in S_\delta^c} |f_n(z_k) - f(z_k)| + C\delta^{-2}n^{-1},$$

which insures with the above control that for $\delta \ge Cn^{-\frac{1}{2}+\epsilon}$, for any $\epsilon > 0$,

$$\mathbb{P}\left(\sup_{z\in[-M,M]\setminus S_\delta} |f_n(z) - f(z)| \ge \frac{2\delta^{-1}}{n^{1/2-\epsilon}}\right) \le 8Mne^{-cn^{2\epsilon}}. \tag{5}$$

Note also that the eigenvalues are bounded by $1 + \max\{|a|, |b|\} + \sum_{i=1}^r |\theta_i|$ for n large enough and take M greater than this constant. Since f does not vanish on S_δ^c, we conclude that f_n does not vanish either on S_δ^c and therefore that the extreme eigenvalues of $\widetilde{X_n}$ belong to S_δ with overwhelming probability.
The orthonormalised model can be treated similarly, by writing $U_n = W^n G_n$ with $\sqrt{n}W^n$ a matrix converging to identity with overwhelming probability by Proposition 6.3.

□

3. Fluctuations of the eigenvalues away from the bulk

Let p_+ be the number of i's such that $\rho_{\theta_i} > b$ and p_- be the number of i's such that $\rho_{\theta_i} < a$. In this section, we study the fluctuations of the eigenvalues of $\widetilde{X_n}$ with limit out of the bulk, that is $(\widetilde{\lambda}_1^n, \ldots, \widetilde{\lambda}_{p_-}^n, \widetilde{\lambda}_{n-p_++1}^n, \ldots, \widetilde{\lambda}_n^n)$. We shall assume throughout this section that the spectral measure of X_n converges to μ_X faster than $1/\sqrt{n}$. More precisely,

Hypothesis 3.1. *For all* $z \in \{\rho_{\alpha_1}, \ldots, \rho_{\alpha_q}\}$, $\sqrt{n}(G_{\mu_n}(z) - G_{\mu_X}(z))$ *converges to* 0.

Our theorem concerns the limiting joint distribution of the following random variables

$$\begin{aligned} \gamma_i^n &= \sqrt{n}(\widetilde{\lambda}_i^n - \rho_{\theta_i}) & \text{if } i \leq p_- \\ \gamma_{p_-+p_+-r+i}^n &= \sqrt{n}(\widetilde{\lambda}_{n-r+i}^n - \rho_{\theta_i}) & \text{if } r - p_+ + 1 \leq i \leq r. \end{aligned}$$

Let us recall that for $k \geq 1$, GOE(k) (resp. GUE(k)) is the distribution of a $k \times k$ symmetric (resp. Hermitian) random matrix $[g_{i,j}]_{i,j=1}^k$ such that the random variables $\{\frac{1}{\sqrt{2}} g_{i,i}\,;\, 1 \leq i \leq k\} \cup \{g_{i,j}\,;\, 1 \leq i < j \leq k\}$ (resp. $\{g_{i,i}\,;\, 1 \leq i \leq k\} \cup \{\sqrt{2}\Re(g_{i,j})\,;\, 1 \leq i < j \leq k\} \cup \{\sqrt{2}\Im(g_{i,j})\,;\, 1 \leq i < j \leq k\}$) are independent standard Gaussian random variables.

The limiting behaviour of the eigenvalues with limit outside the bulk will depend on the law ν through the following quantity, called the fourth cumulant of ν

$$\kappa_4(\nu) := \begin{cases} \int x^4 \mathrm{d}\nu(x) - 3 & \text{in the real case,} \\ \int |z|^4 \mathrm{d}\nu(z) - 2 & \text{in the complex case.} \end{cases}$$

Note that if ν is Gaussian standard, then $\kappa_4(\nu) = 0$.

We recall that the α_j's and the k_j's have been defined in Theorem 1.4.

Theorem 3.2. *Suppose that Assumption 1.2 holds with* $\kappa_4(\nu) = 0$, *as well as Hypotheses 1.1 and 3.1. Then the law of*

$$(\gamma^n_{\sum_{\ell=1}^{i-1} k_\ell + i}, 1 \leq i \leq k_j)_{1 \leq j \leq q}$$

converges to the law of the eigenvalues of $(c_j M_j)_{1 \leq j \leq q}$ *with* M_j *being independent matrices following the law of a* $k_j \times k_j$ *matrix from the GUE (resp. the GOE) if* ν *is supported on the complex plane (resp. the real line). The constant* c_j *is given by*

$$c_j^2 = \begin{cases} \dfrac{1}{\int (\rho_{\alpha_j} - x)^{-2} \mathrm{d}\mu_X(x)} & \text{in the i.i.d. model,} \\[2ex] \dfrac{\int \frac{\mathrm{d}\mu_X(x)}{(\rho_{\alpha_j} - x)^2} - \frac{1}{\alpha_j^2}}{\left(\int (\rho_{\alpha_j} - x)^{-2} \mathrm{d}\mu_X(x)\right)^2} & \text{in the orthonormalised model.} \end{cases} \tag{6}$$

When $\kappa_4(\nu) \neq 0$, we need a bit more than Hypothesis 3.1, namely

Hypothesis 3.3. *For all* $z \in \mathbb{R} \backslash [a,b]$, *there is a finite number* $l(z)$ *such that*

$$\begin{cases} \frac{1}{n} \sum_{i=1}^n ((z - X_n)^{-1})_{i,i}^2 \underset{n \to \infty}{\longrightarrow} l(z) & \text{in the i.i.d. model,} \\ \frac{1}{n} \sum_{i=1}^n (((z - X_n)^{-1})_{i,i} - \frac{1}{n} \operatorname{Tr}((z - X_n)^{-1}))^2 \underset{n \to \infty}{\longrightarrow} l(z) & \text{in the orthonormalised model.} \end{cases}$$

We then have a similar result.

Theorem 3.4. *In the case when Assumption 1.2 holds with $\kappa_4(\nu) \neq 0$, under Hypotheses 1.1, 3.1 and 3.3, Theorem 3.2 stays true, replacing the matrices $c_j M_j$ by matrices $c_j M_j + D_j$ where the D_j's are independent diagonal random matrices, independent of the M_j's, and such that for all j, the diagonal entries of D_j are independent centred real Gaussian random variables, with variance $-l(\rho_{\alpha_j})\kappa_4(\nu)/G'_{\mu_X}(\rho_{\alpha_j})$.*

Let us prove Theorems 3.2 and 3.4. For any real numbers

$$x_1(i) < y_1(i) < x_2(i) < y_2(i) < \cdots < y_{k_i}(i) \qquad (1 \leq i \leq q),$$

since, by Theorem 1.3, for all $\varepsilon > 0$, for n large enough, f_n vanishes exactly at $p_- + p_+$ points in $\mathbb{R}\backslash[a-\varepsilon, b+\varepsilon]$, we have that

$$\left[x_\ell(i) < \gamma^n_{\sum_{m=1}^{i-1} k_m + \ell} < y_\ell(i), \qquad \forall \ell = 1, \ldots, k_i, \quad \forall i = 1, \ldots q\right]$$
$$\Longleftrightarrow$$
$$\Big[\forall i = 1, \ldots, q,$$
$$f_n\left(\rho_{\alpha_i} + \tfrac{y_1(i)}{\sqrt{n}}\right) f_n\left(\rho_{\alpha_i} + \tfrac{x_1(i)}{\sqrt{n}}\right) < 0, \ \ldots, \ f_n\left(\rho_{\alpha_i} + \tfrac{y_{k_i}(i)}{\sqrt{n}}\right) f_n\left(\rho_{\alpha_i} + \tfrac{x_{k_i}(i)}{\sqrt{n}}\right) < 0\Big].$$

Therefore, to study the asymptotics of the joint law of the γ^n_i's, we have to understand those of the $f_n(\rho_{\alpha_i} + \frac{x}{\sqrt{n}})$'s. We set $\rho^i_n(x) := \rho_{\alpha_i} + \frac{x}{\sqrt{n}}$. They are given by the following

Lemma 3.5. *Under the hypotheses of Theorem 3.2, each finite dimensional marginal of the random process*

$$\left(\frac{n^{\frac{k_i}{2}}}{G'_{\mu_X}(\rho_{\alpha_i})^{k_i}} \det\left(\left[G^n_{s,t}\left(\rho^i_n(x)\right)\right]_{s,t\in I_i} - \frac{1}{\alpha_i} I\right) \prod_{\substack{1\leq s\leq r \\ s\notin I_i}} \left(G^n_{s,s}\left(\rho^i_n(x)\right) - \frac{1}{\theta_s}\right)\right)_{1\leq i\leq q,\, x\in\mathbb{R}}$$

converges weakly to the corresponding marginal of

$$\left(\det[xI - c_{\alpha_i} M_{\alpha_i}] \prod_{\substack{1\leq s\leq r \\ s\notin I_i}} \frac{\theta_s - \alpha_i}{\alpha_i \theta_s}\right)_{1\leq i\leq q,\, x\in\mathbb{R}}$$

Theorem 3.2 is then a direct consequence of the following lemma, which shows that the first order of f_n around some ρ_{α_i} is dominated by the convergence stated in Lemma 3.5, so that it changes sign at the eigenvalues of $c_{\alpha_i} M_{\alpha_i}$.

Lemma 3.6. *Let us fix $i \in \{1, \ldots, q\}$. The following convergence in probability holds uniformly as x varies in any compact subset of $\mathbb{R}$:*

$$n^{\frac{k_i}{2}} \left(f_n(\rho^i_n(x)) - \det\left([G^n_{s,t}(\rho^i_n(x))]_{s,t\in I_i} - \frac{1}{\alpha_i} I\right) \prod_{\substack{1\leq s\leq r \\ s\notin I_i}} \left(G^n_{s,s}(\rho^i_n(x)) - \frac{1}{\theta_s}\right)\right) \underset{n\to\infty}{\longrightarrow} 0.$$

Proof of Lemma 3.5. We shall only treat the i.i.d. model (the orthonormalised one can be treated in the same way).

Firstly, by (4), we have the almost sure convergence (for each i and x)

$$\prod_{\substack{1\le s\le r\\ s\notin I_i}}\left(G^n_{s,s}\left(\rho_{\alpha_i}+\frac{x}{\sqrt{n}}\right)-\frac{1}{\theta_s}\right)\underset{n\to\infty}{\longrightarrow}\prod_{\substack{1\le s\le r\\ s\notin I_i}}\frac{\theta_s-\alpha_i}{\alpha_i\theta_s}. \tag{7}$$

The rest of the proof is based on a Central Limit Theorem for quadratic forms that we detail in the Appendix. Indeed, we need to give the joint limit distribution, as n goes to infinity, of

$$M^n_{s,t}(i,x):=\sqrt{n}\left(G^n_{s,t}(\rho^i_n(x))-\frac{1}{\alpha_i}\mathbb{1}_{s=t}\right)=:M^{n,1}_{s,t}(i,x)+M^{n,2}_{s,t}(i,x)+M^{n,3}_{s,t}(i,x)$$

where

$$\begin{aligned}M^{n,1}_{s,t}(i,x)&:=\sqrt{n}\left(\langle u^n_s,(\rho^i_n(x)-X_n)^{-1}u^n_t\rangle-\mathbb{1}_{s=t}\frac{1}{n}\mathrm{Tr}((\rho^i_n(x)-X_n)^{-1})\right),\\ M^{n,2}_{s,t}(i,x)&:=\mathbb{1}_{s=t}\sqrt{n}(G_{\mu_n}(\rho^i_n(x))-G_{\mu_n}(\rho_{\alpha_i})),\\ M^{n,3}_{s,t}(i,x)&:=\mathbb{1}_{s=t}\sqrt{n}(G_{\mu_n}(\rho_{\alpha_i})-G_{\mu_X}(\rho_{\alpha_i})).\end{aligned}$$

By Remark 6.5, $((M^{n,1}_{s,t}(i,x))_{s,t\in I_i})_{1\le i\le q,x\in\mathbb{R}}$ converges to a family of Gaussian Wigner matrices $(G_i(x))_{1\le i\le q,x\in\mathbb{R}}$, where the $G_i(0)$'s are independent and for all i, the matrices $(G_i(x))_{x\in\mathbb{R}}$ are in fact all equal, with a variance given in Theorem 6.4 and which depends on

$$\lim_{n\to\infty}\frac{1}{n}\mathrm{Tr}((\rho^i_n(x)-X_n)^{-2})=-G'_{\mu_X}(\rho_{\alpha_i}). \tag{8}$$

Moreover, again because ρ_{α_i} is at distance of order one from the support of X_n, we can expand $x/\sqrt{n}$ in $M^{n,2}_{s,t}(i,x)$ to deduce that

$$\lim_{n\to\infty}M^{n,2}_{s,t}(i,x)=xG'_{\mu_X}(\rho_{\alpha_i})\mathbb{1}_{s=t}. \tag{9}$$

Finally, by Hypothesis 3.1, we have

$$\lim_{n\to\infty}M^{n,3}_{s,t}(i,x)=0. \tag{10}$$

Equations (7), (8), (9) and (10) prove the lemma (using the fact that M_{α_i} has the same law as $-M_{\alpha_i}$). □

Proof of Lemma 3.6. Firstly, note that by the convergence of $M^n_{s,t}(i,x)$ obtained in the proof of the previous lemma, we have for all $s,t\in\{1,\dots,r\}$ such that $s\ne t$ or $s\in I_i$, for all $\kappa<1/2$,

$$n^\kappa\left(G^n_{s,t}(\rho^i_n(x))-\mathbb{1}_{s=t}\frac{1}{\theta_s}\right)\underset{n\to\infty}{\longrightarrow}0\quad\text{(convergence in probability)}. \tag{11}$$

Let us prove the lemma. By the formula

$$f_n(\rho_n)=\sum_{\sigma\in S_r}\mathrm{sgn}(\sigma)\prod_{s=1}^r\left(G^n_{s,\sigma(s)}(\rho^i_n(x))-\mathbb{1}_{s=\sigma(s)}\frac{1}{\theta_s}\right),$$

it suffices to prove that for any $\sigma\in S_r$ such that for some $i_0\in\{1,\dots,r\}\backslash I_i$, $\sigma(i_0)\ne i_0$,

$$n^{\frac{k_i}{2}}\prod_{s=1}^r\left(G_{\mu^n_{s,\sigma(s)}}(\rho^i_n(x))-\mathbb{1}_{s=\sigma(s)}\frac{1}{\theta_s}\right)\underset{n\to\infty}{\longrightarrow}0\quad\text{(convergence in probability)}. \tag{12}$$

It follows immediately from (11) since for any $\kappa < 1/2$, in the above product, all the terms with index in I_i are of order at most $n^{-\kappa}$, giving a contribution $n^{-k_i\kappa}$, and i_0 is not in I_i and satisfies $\sigma(i_0) \neq i_0$, yielding another term of order at most $n^{-\kappa}$. Hence, the other terms being bounded because ρ_n stays bounded away from $[a,b]$, the above product is at most of order $n^{-\kappa(k_i+1)}$ and so taking $\kappa \in (\frac{k_i}{2(k_i+1)}, \frac{1}{2})$ proves (12). $\square$

Remark 3.7 (Gelfand-Telstin pattern). *Let us fix $\theta < \underline{\theta}$ and let the rank of the deformation increase in the following way: we define*

$$\begin{array}{lll}
\gamma_i^n(1) & := & \sqrt{n}(\lambda_i(X_n + \theta u_1^n u_1^{n^*}) - \rho_\theta) \qquad (1 \le i \le n) \\
\gamma_i^n(2) & := & \sqrt{n}(\lambda_i(X_n + \theta u_1^n u_1^{n^*} + \theta u_2^n u_2^{n^*}) - \rho_\theta) \qquad (1 \le i \le n) \\
\gamma_i^n(3) & := & \sqrt{n}(\lambda_i(X_n + \theta u_1^n u_1^{n^*} + \theta u_2^n u_2^{n^*} + \theta u_3^n u_3^{n^*}) - \rho_\theta) \qquad (1 \le i \le n) \\
\vdots & \vdots &
\end{array}$$

One can easily adapt our proofs to show that under Hypotheses 1.1 and 3.1, if $\kappa_4(\nu) = 0$, the finite dimensional marginals of the process

$$\begin{array}{ccccccc}
 & & & \gamma_1^n(1) & & & \\
 & & \gamma_1^n(2) & & \gamma_2^n(2) & & \\
 & \gamma_1^n(3) & & \gamma_2^n(3) & & \gamma_3^n(3) & \\
\cdots & & \cdots & & \cdots & & \cdots
\end{array}$$

converge to the ones of the ordered eigenvalues of the principal minors of cM, where M is an infinite GUE (resp. GOE) matrix and the constant c is defined by (6).

4. The sticking eigenvalues

4.1. **Statement of the results.** To study the fluctuations of the eigenvalues which stick to the bulk, we need a more precise information on the eigenvalues of X_n in the vicinity of their extremes. More explicitly, we shall need the following additional hypothesis, which depends on a positive integer p and a real number $\alpha \in (0,1)$. Note that this hypothesis has two versions: one adapted to the study of the smallest eigenvalues (it is the version detailed below) and one adapted to the study of the largest eigenvalues (this version is only outlined below).

Hypothesis 4.1. $[p, \alpha]$ *There exists a sequence m_n of positive integers tending to infinity such that $m_n = O(n^\alpha)$, $\eta_2 > 0$ and $\eta_4 > 0$, so that for any $\delta > 0$, for n large enough*

$$\sum_{i=m_n+1}^{n} \frac{1}{(\lambda_p^n - \lambda_i^n)^2} \le n^{2-\eta_2}, \qquad \frac{1}{n} \sum_{i=m_n+1}^{n} \frac{1}{\lambda_p^n - \lambda_i^n} \ge \frac{1}{\underline{\theta}} - \delta. \tag{13}$$

$$\text{and} \sum_{i=m_n+1}^{n} \frac{1}{(\lambda_p^n - \lambda_i^n)^4} \le n^{4-\eta_4} \tag{14}$$

(respectively we replace $\lambda_p^n - \lambda_i^n$ by $\lambda_{n-p+1}^n - \lambda_{n-i+1}^n$, and the second inequality becomes

$$\frac{1}{n} \sum_{i=m_n+1}^{n} \frac{1}{\lambda_{n-p+1}^n - \lambda_{n-i+1}^n} \le \frac{1}{\overline{\theta}} + \delta \text{).}$$

For rank one perturbation, we will only require the two first conditions (13) whereas for higher rank perturbations, we will need in addition (14) to control the off-diagonal terms of the determinant.

Moreover, we shall not study the critical case where for some i, $\theta_i \in \{\underline{\theta}, \overline{\theta}\}$.

Assumption 4.2. *For all i, $\theta_i \neq \underline{\theta}$ (respectively for all i, $\theta_i \neq \overline{\theta}$).*

The fact that the eigenvalues of the non-perturbed matrix are sufficiently spread at the edges to insure the above hypothesis allow the eigenvalues of the perturbed matrix to be very close to them, as stated in the following theorem.

Theorem 4.3. *Let $I_a = \{i \in [1, r] : \rho_{\theta_i} = a\} = [p_- + 1, r_0]$ (resp. $I_b = \{i \in [1, r] : \rho_{\theta_i} = b\} = [r_0 + 1, r - p_+]$) be the set of indices corresponding to the eigenvalues $\widetilde{\lambda}_i^n$ (resp. $\widetilde{\lambda}_{n-r+i}^n$) converging to the lower (resp. upper) bound of the support of μ_X. Let us suppose Hypothesis 1.1, Hypothesis 4.1[r, α] and Assumptions 1.2 and 4.2 to hold. Then for any $\alpha' > \alpha$, we have, for all $i \in I_a$ (resp. $i \in I_b$),*

$$\min_{1\leq k\leq i+r-r_0} |\widetilde{\lambda}_i^n - \lambda_k^n| \leq n^{-1+\alpha'},$$

$$\text{(resp. } \min_{n-r+i-r_0\leq k\leq n} |\widetilde{\lambda}_{n-r+i}^n - \lambda_n^n| \leq n^{-1+\alpha'}\text{)}$$

with overwhelming probability.

Moreover, in the case where the perturbation has rank one, we can locate exactly in the neighborhood of which eigenvalues of the non-perturbed matrix the eigenvalues of the perturbed matrix lie.

We state hereafter the result for the smallest eigenvalues, but of course a similar statement holds for the largest ones.

Theorem 4.4. *Let $(\widetilde{\lambda}_i^n)_{i\geq 1}$ be the eigenvalues of $X_n + \theta u_1 u_1^*$. Then, under Assumption 1.2 and Hypothesis 1.1, if (13) in Hypothesis 4.1 [p,α] holds for some $\alpha \in (0,1)$ and a positive integer p, then for any $\alpha' > \alpha$, we have*

- *If $\theta < \underline{\theta}$, $\widetilde{\lambda}_1^n$ converges to $\rho_\theta < a$ whereas $n^{1-\alpha'}(\widetilde{\lambda}_{i+1}^n - \lambda_i^n)_{1\leq i\leq p-1}$ vanishes in probability as n goes to infinity,*
- *If $\theta \in (\underline{\theta}, 0)$, $n^{1-\alpha'}(\widetilde{\lambda}_i^n - \lambda_i^n)_{1\leq i\leq p}$ vanishes in probability as n goes to infinity.*

Note moreover that in the rank one case, we do not need (14) to hold (indeed, it is used to neglect the off diagonal terms $(G_{ij}^n(z), 1 \leq i < j \leq r)$). At least in the i.i.d. model, this is enough to precisely localise the eigenvalues which stick to the bulk, and complement Theorem 4.3.

Corollary 4.5. *Consider the i.i.d. model and let $(\widetilde{\lambda}_i^n)_{i\geq 1}$ be the eigenvalues of $X_n + \sum_{i=1}^r \theta_i u_i u_i^*$. We assume Assumptions 1.2 and 4.2, Hypothesis 1.1, that Hypothesis 4.1 [p,α] (at both extremes) holds for some $\alpha \in (0,1)$ and a positive integer p, and that for some $\alpha' > \alpha$,*

$$\lim_{n\to\infty} n^{1-\alpha'} \max_{1\leq i\leq p} |\lambda_i^n - \lambda_{i+1}^n| = +\infty\,.$$

Then, with p_- (resp. p_+) the number of indices i so that $\rho_{\theta_i} < a$ (resp. $\rho_{\theta_i} > b$), for all finite integer $i \leq p - (p_- + p_+)$,

$$n^{1-\alpha'}(\widetilde{\lambda}^n_{p_-+i} - \lambda^n_{p_++i}) \qquad \text{and} \qquad n^{1-\alpha'}(\widetilde{\lambda}^n_{n-p_+-i} - \lambda^n_{n-p_--i})$$

both vanish in probability as n goes to infinity.

4.2. **Proofs.** Let us first prove Theorem 4.3. Let us choose $i_0 \in I_a$ and study the behaviour of $\widetilde{\lambda}^n_{i_0}$ (the case of the largest eigenvalues can be treated similarly). We assume throughout the section that Hypotheses 1.1, 4.1 [r, α] and Assumptions 1.2 and 4.2 are satisfied. We also fix $\alpha' > \alpha$.

We know, by Lemma 6.1, that the eigenvalues of $\widetilde{X_n}$ which are not eigenvalues of X_n are the z's such that

$$\text{the matrix } M_n(z) := \left[G^n_{i,j}(z)\right]_{i,j=1}^r - \operatorname{diag}(\theta_1^{-1}, \ldots, \theta_r^{-1}) \text{ is not invertible,} \tag{15}$$

where for all i, j,

$$G^n_{i,j}(z) = \langle u^n_i, (z - X_n)^{-1} u^n_j \rangle.$$

Recall that by Weyl's interlacing inequalities,

$$\widetilde{\lambda}^n_{i_0} \leq \lambda^n_{i_0+r-r_0}.$$

Let ζ be a fixed constant such that $\max_{1 \leq i \leq p_-} \rho_{\theta_i} < \zeta < a$. By Lemma 2.1, we know that

Lemma 4.6. *With overwhelming probability, $\widetilde{\lambda}^n_{i_0} > \zeta$.*

We want to show that (15) is not possible on

$$\Omega_n := \left\{ z \in [\zeta, \lambda^n_{i_0+r-r_0}] \, ; \min_{1 \leq k \leq i_0+r-r_0} |z - \lambda^n_k| > n^{-1+\alpha'} \right\}.$$

The following lemma deals with the asymptotic behaviour of the *off-diagonal terms* of the matrix $M_n(z)$ of (15).

Lemma 4.7. *For $i \neq j$ and $\kappa > 0$ small enough,*

$$\sup_{z \in \Omega_n} |G^n_{i,j}(z)| \leq n^{-\kappa}$$

with overwhelming probability.

The following lemma deals with the asymptotic behaviour of the *diagonal terms* of the matrix of (15).

Lemma 4.8. *For any $\delta > 0$,*

$$\inf_{z \in \Omega_n} \min_{1 \leq i \leq r} G^n_{i,i}(z) \geq \frac{1}{\underline{\theta}} - \delta$$

with overwhelming probability, and there exists a finite M so that

$$\sup_{z \in \Omega_n} |G^n_{i,i}(z)| \leq M \tag{16}$$

with overwhelming probability.

Let us assume these lemmas proven for a while and complete the proof of Theorem 4.3. By these two lemmas, for $z \in \Omega_n$, we find by expanding the determinant that

$$\det(M_n(z)) = \prod_{i=1}^{r} \left(G^n_{i,i}(z) - \frac{1}{\theta_i} \right) + O(n^{-\kappa}).$$

But for all $i \in I_a$, by Lemma 4.8,

$$G^n_{i,i}(z) - \frac{1}{\theta_i} \geq \frac{1}{\underline{\theta}} - \frac{1}{\theta_i} - \delta$$

is bounded from below by a positive constant if δ is chosen small enough because we have $\underline{\theta} < \theta_i < 0$.

Moreover, for $z \in \Omega_n$, $z \geq \zeta$, thus for all $i \notin I_a$, $G^n_{i,i}(z) - \frac{1}{\theta_i} \leq G^n_{i,i}(\zeta) - \frac{1}{\theta_i}$, which, with overwhelming probability, is bounded from above by a negative constant, by definition of ζ and by Proposition 6.2.

We conclude that $\det(M_n(z))$, $z \in \Omega_n$, is bounded away from zero, and hence $\widetilde{\lambda}_{i_0} \notin \Omega_n$, by (15), with overwhelming probability. It completes the proof of the theorem. □

We finally prove the two last lemmas.

Proof of Lemma 4.7. We first prove this estimate for a fixed $z \in \Omega_n$. Moreover, we treat simultaneously the orthonormalised model and the i.i.d. model (in the i.i.d. model, one just takes $W^n = I$ and replaces $\|(G^n(W^n)^T)_i\|_2$ by $\sqrt{n}$ in the proof below). Observe that if we write $X_n = O^* D_n O$ with $D_n = (\lambda^n_1, \ldots, \lambda^n_n)$ and O a unitary or orthogonal matrix,

$$\begin{aligned} G^n_{i,j}(z) &= \langle u^n_i, (z - X_n)^{-1} u^n_j \rangle \\ &= \sum_{l=1}^{n} \frac{(Ou^n_i)_l \overline{(Ou^n_j)_l}}{z - \lambda^n_l} \end{aligned}$$

The first step is to show that for any $\epsilon > 0$, with overwhelming probability,

$$\max_{l,i \in \{1,\ldots,n\}} |(Ou^n_i)_l| \leq n^{-\frac{1}{2}+\epsilon}. \tag{17}$$

Indeed, with O_l the lth row vector of O and using the notations of Section 6.2,

$$(Ou^n_i)_l = \langle O_l, u^n_i \rangle = \frac{1}{\|(G^n(W^n)^T)_i\|_2} \sum_{j=1}^{r} W^n_{i,j} \langle O_l, g^n_j \rangle.$$

But $g \mapsto \langle O_l, g^n_i \rangle$ is Lipschitz for the Euclidean norm with constant one. Hence, by concentration inequality due to the log-Sobolev hypothesis (see e.g. [1, section 4.4]), there exists $c > 0$ such that for all $\delta > 0$,

$$\mathbb{P}\left(|\langle O_l, g^n_i \rangle| > \delta \right) \leq 4e^{-c\delta^2}$$

so that

$$\mathbb{P}\left(\max_{l,i \in \{1,\ldots,n\}} |\langle O_l, g^n_i \rangle| \geq n^{\epsilon} \right) \leq 4n^4 e^{-cn^{2\epsilon}}.$$

From Proposition 6.3, we know that with overwhelming probability, $\|(G^n(W^n)^T)_i\|_2$ is bounded below by $\sqrt{n}n^{-\epsilon}$ and the entries of W^n are of order one. This gives therefore (17).

We now make the following decomposition

$$G^n_{i,j}(z) = \underbrace{\sum_{l=1}^{m_n} \frac{\overline{(Ou^n_i)_l}(Ou^n_j)_l}{z-\lambda^n_l}}_{:=A_n(z)} + \underbrace{\sum_{l=m_n+1}^{n} \frac{\overline{(Ou^n_i)_l}(Ou^n_j)_l}{z-\lambda^n_l}}_{:=B_n(z)}.$$

Note that as $|(Ou^n_i)_l|, 1 \le l \le m_n$, are smaller than $n^{-\frac{1}{2}+\epsilon'}$ by (17), for any $\epsilon' > 0$, with overwhelming probability, we have, uniformly on $z \in \Omega_n$,

$$|A_n(z)| \le m_n n^{1-\alpha'} n^{-1+2\epsilon'} = O(n^{\alpha-\alpha'+2\epsilon'})$$

We choose $0 < \epsilon' \le (\alpha' - \alpha)/4$ and now study $B_n(z)$ which can be written

$$B_n(z) = \langle u^n_i, P(z-X_n)^{-1} P u^n_j \rangle$$

with P the orthogonal projection onto the eigenvectors of X_n corresponding to the eigenvalues $(\lambda^n_{m_n+1}, \ldots, \lambda^n_n)$. By the second point in Proposition 6.2, with $z \in \Omega_n$, for all $s \neq t$,

$$\mathbb{P}\left(\left|\langle g^n_s, P(z-X_n)^{-1} P g^n_t \rangle\right| \ge \delta \sqrt{\mathrm{Tr}(P(z-X_n)^{-2}) + \kappa \sqrt{\mathrm{Tr}(P(z-X_n)^{-4})}} \right) \le 4e^{-c\delta} + 4e^{-c\min(\kappa,\kappa^2)}.$$

Moreover, by Hypothesis 4.1, for n large enough, for all $z \in \Omega_n$,

$$\mathrm{Tr}(P(z-X_n)^{-2}) \le n^{2-\eta_2} \text{ and } \mathrm{Tr}(P(z-X_n)^{-4}) \le n^{4-\eta_4}.$$

We deduce that there is $C, \eta > 0$ such that for all $z \in \Omega_n$,

$$\mathbb{P}\left(\left| \frac{1}{n} \langle g^n_s, P(z-X_n)^{-1} P g^n_t \rangle \right| > n^{-\frac{\eta_2 \wedge \eta_4}{8}} \right) \le Ce^{-n^\eta} \tag{18}$$

A similar control is verified for $s = t$ since we have, by Proposition 6.2,

$$\mathbb{P}\left(\left| \frac{1}{n} \langle g_i, P(z-X_n)^{-1} P g_i \rangle - \frac{1}{n} \mathrm{Tr}\left(P(z-X_n)^{-1}\right) \right| \ge \delta \right) \le 4e^{-c\delta^2 n^{\eta_2}} \tag{19}$$

whereas Hypothesis 4.1 insures that the term $\frac{1}{n}\mathrm{Tr}(P(z-X_n)^{-1})$ is bounded uniformly on Ω_n. Thus, up to a change of the constants C and η, there is a constant M such that for all $z \in \Omega_n$,

$$\mathbb{P}\left(\left| \frac{1}{n} \langle g_i, P(z-X_n)^{-1} P g_i \rangle \right| \ge M \right) \le Ce^{-n^\eta}.$$

Therefore, with Proposition 6.3 and developing the vectors u^n_i's as the normalised column vectors of $G^n(W^n)^T$, we conclude that, up to a change of the constants C and η, for all $z \in \Omega_n$,

$$\mathbb{P}\left(|B_n(z)| \ge n^{-\frac{\eta_2 \wedge \eta_4}{8}} \right) \le Ce^{-n^\eta}. \tag{20}$$

Hence, we have proved that there exists $\kappa > 0, C$ and $\eta > 0$ so that for all $z \in \Omega_n$,

$$\mathbb{P}\left(\left|G^n_{i,j}(z)\right| \ge n^{-\kappa} \right) \le Ce^{-n^\eta}.$$

We finally obtain this control uniformly on $z \in \Omega_n$ by noticing that $z \to G^n_{i,j}(z)$ is Lipschitz on Ω_n, with constant bounded by $(\min |z - \lambda_i|)^{-2} \le n^{-2+2\alpha'}$. Thus, if we take a grid $(z^n_k)_{0 \le k \le cn^2}$ of Ω_n with mesh $\le n^{-2+2\alpha'-\kappa}$ (there are about n^2 such z^n_k's) we have

$$\sup_{z \in \Omega_n} \left| G^n_{i,j}(z) \right| \le \max_{1 \le k \le cn^2} \left| G^n_{i,j}(z^n_k) \right| + n^{-\kappa}.$$

Since there are at most cn^2 such k and n^2 possible i, j, we conclude that

$$\mathbb{P}\left(\sup_{z \in \Omega_n} |G^n_{i,j}(z)| \ge 2n^{-\kappa} \right) \le c^2 n^4 C e^{-n^\eta}$$

which completes the proof. □

Proof of Lemma 4.8. Again, we first prove the estimate for a fixed $z \in \Omega_n$, the uniform estimate on z being obtained by a grid argument as in the previous proof (a key point being that the constants C and η of the definition of *overwhelming probability* are independent of the choice of $z \in \Omega_n$). We recall that P is the orthogonal projection on the vector space generated by the eigenvectors of X_n with eigenvalues $(\lambda^n_{m_n+1}, \dots, \lambda^n_n)$ and write

$$\begin{aligned} G^n_{i,i}(z) &= \langle u^n_i, P(z - X_n)^{-1} P u^n_i \rangle + \langle u^n_i, (1-P)(z - X_n)^{-1}(1-P) u^n_i \rangle \\ &\ge \langle u^n_i, P(\lambda^n_{i_0+r-r_0} - X_n)^{-1} P u^n_i \rangle - n^{1-\alpha'} \|(1-P)u^n_i\|_2^2, \end{aligned}$$

where we used the inequalities $z \le \lambda^n_{i_0+r-r_0}$, $P(\lambda^n_{i_0+r-r_0} - X_n)P \le 0$ and $|z - \lambda^n_k| > n^{-1+\alpha'}$ for all $1 \le k \le m_n$. But as in the previous proof, we have

$$\langle u^n_i, P(\lambda^n_{i_0+r-r_0} - X_n)^{-1} P u^n_i \rangle = \frac{n}{\|(G^n (W^n)^T)_i\|_2^2} \sum_{j,k=1}^{i} W^n_{i,k} \overline{W^n_{i,j}} \frac{1}{n} \langle g^n_j, P(\lambda^n_{i_0+r-r_0} - X_n)^{-1} P g^n_k \rangle$$

with, by (18), the off diagonal terms $j \neq k$ of order $n^{-\eta_2 \wedge \eta_4/8}$ with overwhelming probability, whereas the diagonal terms are close to $\frac{1}{n}\mathrm{Tr}(P(\lambda^n_{i_0+r-r_0} - X_n)^{-1})$ with overwhelming probability by (19). Hence, we deduce with Proposition 6.3 that for any $\delta > 0$,

$$\left| \langle u^n_i, P(\lambda^n_{i_0+r-r_0} - X_n)^{-1} P u^n_i \rangle - \frac{1}{n} \mathrm{Tr}(P((\lambda^n_{i_0+r-r_0} - X_n)^{-1})) \right| \le \delta$$

with overwhelming probability. Hence, by Hypothesis 4.1, for any $\delta > 0$ and n large enough

$$\langle u^n_i, P(\lambda^n_{i_0+r-r_0} - X_n)^{-1} P u^n_i \rangle \ge \frac{1}{\underline{\theta}} - \delta \tag{21}$$

with overwhelming probability. On the other hand

$$\|(1-P)u^n_i\|_2^2 = \frac{1}{\|(G^n (W^n)^T)_i\|_2^2} \sum_{j,k=1}^{r} W^n_{i,j} \overline{W^n_{i,k}} \langle (1-P) g^n_j, (1-P) g^n_k \rangle$$

By Proposition 6.3, the denominator is of order n with overwhelming probability, whereas by Proposition 6.2, the numerator is of order $m_n + n^\epsilon \sqrt{m_n}$ (since $\mathrm{Tr}(1-P) = m_n$) with overwhelming probability. As W^n is bounded by Proposition 6.3 we conclude that

$$\|(1-P)u^n_i\|_2^2 \le 2 \frac{m_n}{n}$$

with overwhelming probability. Putting everything together we have proved that for any $z \in \Omega_n$, any $\delta > 0$,

$$G^n_{i,i}(z) \geq \frac{1}{\underline{\theta}} - \delta$$

with overwhelming probability. Finally, we also have

$$G^n_{i,i}(z) \leq \langle u^n_i, P(\zeta - X_n)^{-1} P u^n_i \rangle + n^{1-\alpha'} \|(1-P)u^n_i\|_2^2$$

and we can bound the above right hand side by the same arguments to obtain (16) for a fixed $z \in \Omega_n$. We do not detail the grid argument which is similar to what we did in the proof of the previous lemma. □

Proof of Theorem 4.4. In the one dimensional case, the eigenvalues of $\widetilde{X}_n$ which do not belong to the spectrum of X_n are the zeroes of

$$f_n(z) = \frac{1}{n}\langle g, (z - X_n)^{-1} g\rangle - \varepsilon_n(g)\frac{1}{\theta} \tag{22}$$

with $\varepsilon_n(g) = 1$ or $\|g\|_2^2/n$ according to the model we are considering. A straightforward study of the function f_n tells us that the eigenvalues of $\widetilde{X}_n$ are distinct from those of X_n as soon as X_n has no multiple eigenvalue and

$$(\text{matrix of the eigenvectors of } X_n)^* \times g$$

has no null entry, which we can always assume up to modify X_n and g so slightly that the fluctuations of the eigenvalues are not affected. We do not detail these arguments but the reader can refer to Lemmas 9.3, 9.4 and 11.2 of [11] for a full proof in the finite rank case.

Therefore, (22) characterises all the eigenvalues of $\widetilde{X}_n$. Moreover, by Weyl's interlacing properties, for $\theta < 0$,

$$\widetilde{\lambda}^n_1 < \lambda^n_1 < \widetilde{\lambda}^n_2 < \lambda^n_2 < \cdots < \widetilde{\lambda}^n_n < \lambda^n_n .$$

Theorems 1.3 and 4.3 thus already settle the study of $\widetilde{\lambda}^n_1$. We consider $\alpha' > \alpha$ and $i \in \{2, \ldots, p\}$ and define

$$\Lambda_n := \left]\lambda^n_{i-1} + \frac{n^{-1+\alpha'}}{2}, \lambda^n_i - \frac{n^{-1+\alpha'}}{2}\right[$$

Note first that if Λ_n is empty, then the eigenvalue of $\widetilde{X_n}$ which lies between λ^n_{i-1} and λ^n_i is within $n^{-1+\alpha'}$ to both λ^n_{i-1} and λ^n_i, so we have nothing to prove. Now we want to prove that f_n does not vanish on Λ_n and that according to the sign of $\frac{1}{\theta} - \frac{1}{\underline{\theta}}$, it vanishes on one side or the other of Λ_n in $]\lambda^n_{i-1}, \lambda^n_i[$.

The proof of this fact will follow the same lines as the proof of Lemma 4.8 and we recall that P was defined above as the projection onto the eigenspace of the $(\lambda^n_{m_n+1}, \ldots, \lambda^n_n)$. Then, exactly as for (21), we can show that for all $\delta > 0$ and n large enough,

$$\sup_{z \in [\lambda^n_1, \lambda^n_p]} \left| \frac{1}{n}\langle g, P(z - X_n)^{-1} P g\rangle - \frac{1}{\underline{\theta}} \right| \leq \delta$$

with overwhelming probability. Moreover, for any $z \in \Lambda_n$, for any $j \in [1, m_n]$, we have

$$|z - \lambda^n_j| \geq \min\{z - \lambda^n_{i-1}, \lambda^n_i - z\} \geq \frac{n^{-1+\alpha'}}{2}$$

and for any $\epsilon > 0$,

$$\sup_{z\in\Lambda_n}\left|\frac{1}{n}\langle g,(1-P)(z-X_n)^{-1}(1-P)g\rangle\right| \le 2n^{-\alpha'}\langle g,(1-P)g\rangle \le n^{\epsilon}n^{-\alpha'}m_n$$

with overwhelming probability. We choose ϵ in such a way that the latter right hand side goes to zero. Therefore, we know that uniformly on Λ_n,

$$f_n(z) = \frac{1}{\underline{\theta}} - \frac{1}{\theta} + o(1)$$

with overwhelming probability. Since for all n, f_n is decreasing, going to $+\infty$ (resp. $-\infty$) as z goes to any λ_{i-1}^n on the right (resp. λ_i^n on the left), it follows that according to the sign of $\frac{1}{\underline{\theta}} - \frac{1}{\theta}$, the zero of f_n in $]\lambda_{i-1}^n, \lambda_i^n[$ is either in $]\lambda_{i-1}^n, \lambda_{i-1}^n + n^{-1+\alpha'}[$ or in $]\lambda_i^n - n^{-1+\alpha'}, \lambda_i^n[$. □

Proof of Corollary 4.5. We can finally prove Corollary 4.5 by induction. We first add the small perturbations to X_n, that is consider $\tilde{X}_n^1 = X_n + \theta uu^*$ with $\theta \in (\underline{\theta}, \overline{\theta})$. In this setting, Theorem 4.4 shows that the p largest (resp. smallest) eigenvalues are at distance smaller than $n^{-1+\alpha'}$ from the eigenvalues of X_n. Moreover, by the interlacing properties, for all $p < i$,

$$0 \le \frac{1}{\tilde{\lambda}_i^n - \tilde{\lambda}_p^n} \le \frac{1}{\lambda_{i-1}^n - \lambda_{p+1}^n}$$

so that if X_n verifies Hypothesis 4.1[p,α], $\tilde{X}_n^1$ verifies Hypothesis 4.1[p-1,α]. Thus, we can proceed with $\tilde{X}_n^1$ instead of X_n and conclude that when we have added all these small perturbations, the resulting matrix have extreme eigenvalues which are at distance smaller than $n^{-1+\alpha'}$ from the eigenvalues of X_n and it satisfies Hypothesis 4.1[$p-r+p_-+p_+, \alpha$]. We next add the big perturbation with positive coefficients, $\tilde{X}_n^{r-p_--p_++1} = \tilde{X}_n^{r-p_--p_+} + \theta_r u_r u_r^*$. We can apply Theorem 4.4 and conclude that the largest eigenvalues of $\tilde{X}_n^{r-p_--p_++1}$ which stick to the bulk are at distance smaller than $n^{-1+\alpha'}$ from the largest eigenvalues of X_n. Moreover, the same argument as before shows that the same is true for the smallest eigenvalues except the smallest eigenvalue of $\tilde{X}_n^{r-p_--p_++1}$ sticks to the second smallest eigenvalue of X_n, etc. Again, we check that Hypothesis 4.1[$p-r+p_-+p_+-1, \alpha$] is satisfied. We then can continue to add the p_+th positive perturbation, giving a matrix $\tilde{X}_n^{r-p_-}$ with p_+ eigenvalues away from the bulk, the ith (resp. $n-i-p_+$th) eigenvalue of $\tilde{X}_n^{r-p_-}$ being at distance of order $n^{-1+\alpha'}$ of the $(i+p_+)$th (resp. $n-i$th) eigenvalue of X_n. We next add the perturbation with negative coefficients. Considering the largest eigenvalues, we see that the new matrix keeps eigenvalues in the small $n^{-1+\alpha'}$ neighborhood of the large isolated non-perturbed matrix, whereas inside the bulk, the first pth eigenvalue inside $[\lambda_{n-p}^n - cn^{-1+\alpha'}, \lambda_{n-p+1}^n + cn^{-1+\alpha'}]$ is close to λ_{n-p}^n. For the smallest, one eigenvalue deviates from the bulk whereas the second one is close to $\lambda_{p_+}^n$. We can then continue by induction to finish the proof of Corollary 4.5. □

5. Application to classical models of matrices

Our goal in this section is to show that if X_n belongs to some classical ensembles of matrices, the extreme eigenvalues of perturbations of such matrices have their asymptotics obeying to Theorems 1.3, 3.2 and 4.3. For that, a crucial step will be the following statement. If (X_n) is a sequence of random matrices, we say that it satisfies an hypothesis

H in probability if the probability that X_n satisfies H converges to one as n goes to infinity (for example, if H states a convergence to a limit ℓ, "H in probability" is the convergence in probability to ℓ).

Theorem 5.1. *Let (X_n) be a sequence of random matrices independent of the u_i^n's. Under Assumption 1.2,*

(1) *If Hypothesis 1.1 holds in probability, Theorem 1.3 holds.*
(2) *If $\kappa_4(\nu) = 0$ and Hypotheses 1.1 and 3.1 hold in probability, Theorem 3.2 holds. If $\kappa_4(\nu) \neq 0$ and Hypotheses 1.1 and 3.3 hold in probability, Theorem 3.4 holds.*
(3) *Under Assumption 4.2, if Hypotheses 1.1 and 4.1 hold in probability, Theorem 4.3 holds "with probability converging to one" instead of "with overwhelming probability"; Theorems 4.4 and Corollary 4.5 hold.*

This result follows from the results with deterministic sequences of matrices X_n. Indeed, to prove that a sequence converges to a limit ℓ in a metric space, it suffices to prove that any of its subsequences has a subsequence converging to ℓ. If the convergences of the hypotheses hold in probability, then from any subsequence, one can extract a subsequence for which they hold almost surely. Then up to a conditioning by the σ-algebra generated by the X_n's, the hypotheses of the various theorems hold.

The remaining of this section is devoted to showing that such results hold if X_n, independent of $(u_i^n)_{1\le i\le r}$, is a Wigner or a Wishart matrix or a random matrix which law has density proportional to $e^{-\operatorname{Tr} V}$ for a certain potential V. In each case, we have to check that the hypotheses hold in probability.

5.1. **Wigner matrices.** Let μ_1 be a centred distribution on $\mathbb{R}$ (respectively on $\mathbb{C}$) and μ_2 be a centred distribution on $\mathbb{R}$, both having a finite fourth moment (in the case where μ_1 is not supported on the real line, we assume that the real and imaginary part are independent). We define $\sigma^2 = \int_{z\in\mathbb{C}} |z|^2 \mathrm{d}\mu_1(z)$.

Let $(x_{i,j})_{i,j\ge 1}$ be an infinite Hermitian random matrix which entries are independent up to the condition $x_{j,i} = \overline{x_{i,j}}$ such that the $x_{i,i}$'s are distributed according to μ_2 and the $x_{i,j}$'s ($i \neq j$) are distributed according to μ_1. We take $X_n = \frac{1}{\sqrt{n}}\left[x_{i,j}\right]_{i,j=1}^n$, which is said to be a *Wigner matrix*. For certain results, we will also need an additional hypothesis, which we present here:

Hypothesis 5.2. *The probability measures μ_1 and μ_2 have a sub-exponential decay, that is there exists positive constants C, C' such that if X is distributed according to μ_1 or μ_2, for all $t \ge C'$,*

$$\mathbb{P}(|X| \ge t^C) \le e^{-t}.$$

Moreover, μ_1 and μ_2 are symmetric.

The following Proposition generalizes some results of [36, 18, 12, 13] which study the effect of a finite rank perturbation on a non-Gaussian Wigner matrix. In particular, it includes the study of the eigenvalues which stick to the bulk.

Proposition 5.3. *Let X_n be a Wigner matrix. Assume that Assumption 1.2 holds. The limits of the extreme eigenvalues of $\widetilde{X_n}$ are given by Theorem 1.3 and the fluctuations of* *the ones which limits are out of* $[-2\sigma, 2\sigma]$ *are given by Theorem 3.2, where the parameters* $a, b, \rho_\theta, c_\alpha$ *are given by the following formulas :* $b = -a = 2\sigma$,

$$\rho_\theta := \begin{cases} \theta + \frac{\sigma^2}{\theta} & \text{if } |\theta| > \sigma, \\ 2\sigma & \text{if } 0 < \theta \le \sigma, \\ -2\sigma & \text{if } -\sigma \le \theta < 0, \end{cases}$$

and

$$c_\alpha = \begin{cases} \sqrt{\alpha^2 - \sigma^2} & \text{in the i.i.d. model,} \\ \\ \frac{\sigma\sqrt{\alpha^2-\sigma^2}}{\alpha} & \text{in the orthonormalized model.} \end{cases}$$

Assume moreover that, for all i, $\theta_i \notin \{-\sigma, \sigma\}$ *and Hypothesis 5.2 holds. If the perturbation has rank one, we have the following precise description of the fluctuations of the sticking eigenvalues :*

- *If* $\theta > \sigma$ *(resp.* $\theta < -\sigma$*), for all* $p \ge 2$, $n^{2/3}(\widetilde{\lambda}^n_{n-p+1} - 2\sigma)$ *(resp.* $n^{2/3}(\widetilde{\lambda}^n_p - 2\sigma)$*) converges in law to the* $p-1$*th Tracy Widom law.*
- *If* $0 \le \theta < \sigma$ *(resp.* $-\sigma < \theta \le 0$*), for all* $p \ge 1$, $n^{2/3}(\widetilde{\lambda}^n_{n-p+1} - 2\sigma)$ *(resp.* $n^{2/3}(\widetilde{\lambda}^n_p - 2\sigma)$*) converges in law to the* p*th Tracy Widom law.*

If the perturbation is rank more than one and Assumption 4.2 holds, the extreme eigenvalues of $\widetilde{X_n}$ *are at distance less than* $n^{-1+\epsilon}$ *for any* $\epsilon > 0$ *to the extreme eigenvalues of* X_n*, which have Tracy-Widom fluctuations.*

Remark 5.4. *All the Tracy-Widom laws involved in the statement of the proposition above, are the ones corresponding respectively to the GOE if* μ_1 *is supported on* $\mathbb{R}$ *and to the GUE if* μ_1 *is supported on* $\mathbb{C}$.

According to Theorem 5.1, it suffices to verify that the hypotheses hold in probability for $(X_n)_{n\ge1}$. We study separately the eigenvalues which stick to the bulk and those which deviate from the bulk.

•*Deviating eigenvalues.*

If X_n is a Wigner matrix (that is, with our terminology, with entries having a finite fourth moment), the fact that X_n satisfies Hypothesis 1.1 in probability is a well known result (see for example [4, Th. 5.2]) for μ_X the semicircle law with support $[-2\sigma, 2\sigma]$. The formulas for ρ_θ and c_α can be checked with the well known formula [1, Sect. 2.4]:

$$\forall z \in \mathbb{R}\backslash[-2\sigma, 2\sigma], \qquad G_{\mu_X}(z) = \frac{z - \operatorname{sgn}(z)\sqrt{z^2 - 4\sigma^2}}{2\sigma^2}.$$

Moreover, [5, Th. 1.1] shows that $\operatorname{Tr}(f(X_n)) - n\int f(x)d\sigma(x)$ converges in law to a Gaussian distribution for any function f which is analytic in a neighborhood of $[-2\sigma, 2\sigma]$. For any fixed $z \notin [-2\sigma, 2\sigma]$, applied for $f(t) = \frac{1}{z-t}$, we get that $n(G_{\mu_n}(z) - G_{\mu_X}(z))$ converges in law to a Gaussian distribution, hence $\sqrt{n}(G_{\mu_n}(z) - G_{\mu_X}(z))$ converges in probability to zero, so that Hypothesis 3.1 holds in probability.

•*Sticking Eigenvalues.*

We now assume moreover that the laws of the entries satisfy Hypothesis 5.2. Let us first recall that by [41, 39], the extreme eigenvalues of the non-perturbed matrix X_n, once re-centred and renormalised by $n^{2/3}$, converge to the Tracy-Widom law (which depends on whether the entries are complex or real). We need to verify that Hypothesis 4.1[p,α] for any finite p and an $\alpha < 1/3$ is fulfilled in probability. By [41], the spacing between the two smallest eigenvalues of X_n is of order greater than $n^{-\gamma}$ for $\gamma > 2/3$ with probability going to one and therefore, by the inequality

$$\sum_{i=m_n+1}^{n} \frac{1}{(\lambda_p^n - \lambda_i^n)^k} \le (\lambda_{p+1}^n - \lambda_p^n)^{1-k} \times \sum_{i=m_n+1}^{n} \frac{1}{\lambda_i^n - \lambda_p^n}, \qquad (k = 2 \text{ or } 4),$$

it is sufficient to prove the third point of Hypothesis 4.1[p,α]. We shall prove it by replacing first the smallest eigenvalue by the edge -2 thanks to a lemma that Benjamin Schlein [40] kindly communicated to us. We will then prove that the sum of the inverse of the distance of the eigenvalues to the edge indeed converges to the announced limit, thanks to both Soshnikov paper [41] (for sub-Gaussian tails) or [39] (for finite moments), and Tao and Vu article [42].

Lemma 5.5 (B. Schlein)**.** *Suppose the entries of X_n have a uniform sub-exponential tail. Then for all $\delta > 0$, for all integer number p,*

$$\lim_{n\to\infty} \mathbb{P}\left(\left|\frac{1}{n}\sum_{j=p+1}^{n} \frac{1}{\lambda_j^n - \lambda_p^n} - \frac{1}{n}\sum_{j=p+1}^{n} \frac{1}{\lambda_j^n + 2}\right| \ge \delta\right) = 0.$$

Proof. We write

$$\frac{1}{n}\sum_{j=p+1}^{n} \frac{1}{\lambda_j^n - \lambda_p^n} - \frac{1}{n}\sum_{j=p+1}^{n} \frac{1}{\lambda_j^n + 2} = \frac{\lambda_p^n + 2}{n} \sum_{j=p+1}^{n} \frac{1}{(\lambda_j^n - \lambda_p^n)(\lambda_j^n + 2)}.$$

Hence for any $K_1 > 0$,

$$\begin{aligned} \mathbb{P}&\left(\left|\frac{1}{n}\sum_{j=p+1}^{n} \frac{1}{\lambda_j^n - \lambda_p^n} - \frac{1}{n}\sum_{j=p+1}^{n} \frac{1}{\lambda_j^n + 2}\right| \ge \delta\right) \\ &\le \mathbb{P}(|\lambda_p^n + 2| \ge K_1 n^{-2/3}) \\ &\quad + \mathbb{P}\left(\frac{K_1}{n^{5/3}} \sum_{j=p+1}^{n} \frac{1}{|(\lambda_j^n - \lambda_p^n)(\lambda_j^n + 2)|} \ge \delta \text{ and } |\lambda_p^n + 2| < K_1 n^{-2/3}\right). \end{aligned} \tag{23}$$

Now, for any $K_2 > K_1$, on the event $\{|\lambda_p^n + 2| < K_1 n^{-2/3}\}$, for any $\kappa > 0$, we have

$$\begin{aligned} \frac{K_1}{n^{5/3}} \sum_{j=p+1}^{n} \frac{1}{|(\lambda_j^n - \lambda_p^n)(\lambda_j^n + 2)|} \le\ & \frac{K_1}{n^{5/3}} \sum_{\ell=0}^{+\infty} \frac{\mathcal{N}_n[2K_2 n^{-2/3} + \ell n^{-\kappa}, 2K_2 n^{-2/3} + (\ell+1) n^{-\kappa}]}{(K_2 n^{-2/3} + \ell n^{-\kappa})^2} \\ & + \frac{K_1}{n^{5/3}} \sum_{j=p+1}^{n} \frac{\mathbb{1}_{\lambda_j + 2 \le 2K_2 n^{-2/3}}}{|(\lambda_j^n - \lambda_p^n)(\lambda_j^n + 2)|}, \end{aligned} \tag{24}$$

where $\mathcal{N}_n[a, b] := \sharp\{i\,;\, -2 + a \le \lambda_i^n \le -2 + b\}$. Note that, from the upper bound on the density of eigenvalues in microscopic intervals, due to [15, Theorem 4.6], we know that for any $\kappa < 1$, there is a constant M independent of n so that for all $\ell \ge 1$

$$\mathbb{E}(\mathcal{N}_n[2K_2 n^{-2/3} + \ell n^{-\kappa}, 2K_2 n^{-2/3} + (\ell+1) n^{-\kappa}]) \le M n^{1-\kappa}. \tag{25}$$

Let us fix $\kappa \in (\frac{2}{3}, 1)$. It follows that the first term of the r.h.s. of (24) can be estimated by

$$\begin{aligned}
&\mathbb{P}\left(\frac{K_1}{n^{5/3}}\sum_{\ell=0}^{+\infty}\frac{\mathcal{N}_n[2K_2n^{-2/3}+\ell n^{-\kappa}, 2K_2n^{-2/3}+(\ell+1)n^{-\kappa}]}{(K_2n^{-2/3}+\ell n^{-\kappa})^2} \geq \frac{\delta}{2}\right) \\
&\leq \frac{2K_1}{\delta n^{5/3}}\sum_{\ell=0}^{+\infty}\frac{\mathbb{E}(\mathcal{N}_n[2K_2n^{-2/3}+\ell n^{-\kappa}, 2K_2n^{-2/3}+(\ell+1)n^{-\kappa}])}{(K_2n^{-2/3}+\ell n^{-\kappa})^2} \\
&\leq \frac{2MK_1}{\delta n^{2/3}}\frac{1}{n^{\kappa}}\sum_{\ell=0}^{+\infty}\frac{1}{(K_2n^{-2/3}+\ell n^{-\kappa})^2} \\
&\leq \frac{2MK_1}{\delta n^{2/3}}\frac{1}{n^{\kappa}(K_2n^{-2/3})^2} + \frac{2MK_1}{\delta n^{\frac{2}{3}}}\int_0^{+\infty}\frac{\mathrm{d}t}{(t+K_2n^{-\frac{2}{3}})^2} \\
&\leq \frac{2MK_1}{\delta K_2^2 n^{\kappa-2/3}} + \frac{2MK_1}{\delta K_2}.
\end{aligned} \tag{26}$$

Let us now estimate the second term of the r.h.s. of (24). For any positive integer K_3, we have

$$\begin{aligned}
&\mathbb{P}\left(\frac{K_1}{n^{5/3}}\sum_{j=p+1}^{n}\frac{\mathbb{1}_{|\lambda_j^n+2|\leq 2K_2n^{-2/3}}}{|(\lambda_j^n-\lambda_p^n)(\lambda_j^n+2)|} \geq \frac{\delta}{2}\right) \\
&\leq \mathbb{P}\left(\mathcal{N}_n(-\infty, 2K_2n^{-2/3}] \geq K_3\right) + \mathbb{P}\left(\frac{K_1K_3}{n^{5/3}}\frac{1}{\min_{p+1\leq j\leq K_3}|(\lambda_j^n-\lambda_p^n)(\lambda_j^n+2)|} \geq \frac{\delta}{2}\right) \\
&\leq \mathbb{P}\left(\lambda_{K_3}^n \leq -2+2K_2n^{-2/3}\right) + \mathbb{P}\left(\min_{p\leq j\leq K_3}|\lambda_j^n+2| \leq \frac{\sqrt{2K_1K_3}n^{-5/6}}{\sqrt{\delta}}\right) \\
&\quad +\mathbb{P}\left(|\lambda_p^n-\lambda_{p+1}^n| \leq \frac{\sqrt{2K_1K_3}n^{-5/6}}{\sqrt{\delta}}\right)
\end{aligned} \tag{27}$$

From (23), (24), (26) and (27), we conclude that

$$\begin{aligned}
&\mathbb{P}\left(\left|\frac{1}{n}\sum_{j=p+1}^{n}\frac{1}{\lambda_j^n-\lambda_1^n} - \frac{1}{n}\sum_{j=p+1}^{n}\frac{1}{\lambda_j^n+2}\right| \geq \delta\right) \\
&\leq \mathbb{P}(|\lambda_1^n+2| \geq K_1n^{-2/3}) + \frac{2MK_1}{\delta K_2} + \mathbb{P}\left(\lambda_{K_3} \leq -2+2K_2n^{-2/3}\right) \\
&\quad +\mathbb{P}\left(\min_{1\leq j\leq K_3}|\lambda_j^n+2| \leq \frac{\sqrt{2K_1K_3}n^{-5/6}}{\sqrt{\delta}}\right) + \mathbb{P}\left(|\lambda_2^n-\lambda_1^n| \leq \frac{\sqrt{2K_1K_3}n^{-5/6}}{\sqrt{\delta}}\right)
\end{aligned}$$

for arbitrary $0 < K_1 < K_3$ and $K_3 \geq 1$. Taking the limit $n \to \infty$, the last two terms disappear, because by [42, Th. 1.16], the distribution of the smallest K_3 eigenvalues lives on scales of order $n^{-2/3} \gg n^{-5/6}$. Therefore,

$$\begin{aligned}
&\lim_{n\to\infty}\mathbb{P}\left(\left|\frac{1}{n}\sum_{j=2}^{n}\frac{1}{\lambda_j^n-\lambda_1^n} - \frac{1}{n}\sum_{j=2}^{n}\frac{1}{\lambda_j^n+2}\right| \geq \delta\right) \\
&\qquad \leq \lim_{n\to\infty}\mathbb{P}(|\lambda_1^n+2| \geq K_1n^{-2/3}) + \frac{2MK_1}{\delta K_2} + \lim_{n\to\infty}\mathbb{P}\left(\lambda_{K_3} \leq -2+2K_2n^{-2/3}\right),
\end{aligned}$$

still for any $0 < K_1 < K_3$ and $K_3 \geq 1$. Now, note that for K_1 large enough, the first term can be made as small as we want. Then, keeping K_1 fixed, K_2 can be chosen in such a way to make the second term as small as we want too. At last, keeping K_2 fixed, one can choose K_3 large enough to make the third term as small as we want (as can be computed since the limit is given by the K_3 correlation function of the Airy kernel). □

To complete the proof of Hypothesis 4.1, we therefore need to show that

Lemma 5.6. *Assume that the entries of X_n satisfy Hypothesis 5.2. Then, for any $\delta > 0$, any finite integer number p,*

$$\lim_{n\to\infty} \mathbb{P}\left(\left|\frac{1}{n}\sum_{j=p+1}^{n}\frac{1}{\lambda_j^n+2}-2\right| > \delta\right) = 0$$

Proof. Notice that by [41, 39] we know that the p smallest eigenvalues of X_n converge in law towards the Tracy-Widom law, so that

$$\lim_{\epsilon\downarrow 0}\lim_{n\to\infty}\mathbb{P}\left(\min_{1\leq j\leq p}|\lambda_j^n+2| < \epsilon n^{-2/3}\right) = 0.$$

Thus, for any finite p, with large probability,

$$\frac{1}{n}\sum_{j=2}^{p}\frac{1}{|\lambda_j^n+2|} \leq p\epsilon^{-1}n^{-\frac{1}{3}}$$

and therefore it is enough to prove the lemma for any particular p. As in the previous proof, we choose p large enough so that $\lambda_p^n \geq -2+n^{-\frac{2}{3}}$ with probability greater than $1-\delta(p)$ with $\delta(p)$ going to zero as p goes to infinity. We shall prove that with high probability

$$\lim_{\gamma\downarrow 0}\lim_{n\to\infty}\frac{1}{n}\sum_{j=p}^{[\gamma n]}\frac{1}{\lambda_j^n+2} \leq 0. \tag{28}$$

This is enough to prove the statement as for any $\gamma > 0$, $2+\lambda_{[n\gamma]}^n$ converges to $\delta(\gamma) > 0$ so that $\mu_{sc}([\delta(\gamma),2]) = 1-\gamma$, see [43, Theorem 1.3],

$$\lim_{n\to\infty}\frac{1}{n}\sum_{i=[n\gamma]}^{n}\frac{1}{\lambda_i^n+2} = \int_{\delta(\gamma)}^{2}\frac{1}{2+x}d\mu_{sc}(x),$$

which converges as γ goes to zero to $\int(2+x)^{-1}d\mu_{sc}(x) = 2$. To prove (28), we choose $\rho \in (2/3, \sqrt{2/3})$ and write, on the event $\lambda_j^n+2 \geq \lambda_p^n+2 \geq n^{-\frac{2}{3}} \geq n^{-\rho}$ for $j \geq p$,

$$\frac{1}{n}\sum_{j=p}^{[\gamma n]}\frac{1}{\lambda_j^n+2} \leq \sum_{1\leq k\leq K} n^{\rho^k-1}\mathcal{N}_n[n^{-\rho^k}, n^{-\rho^{k+1}}] + \sum_{j=2}^{[\gamma n]}\frac{1_{\lambda_j^n\geq -2+n^{-\rho^{K+1}}}}{n(\lambda_j^n+2)} =: A_n + B_n.$$

For the first term, we use Sinai-Soshnikov bound, which under the weakest hypothesis are given in [39, Theorem 2.1] which implies that with probability going to one with M going to infinity, for $s_n = o(n^{2/3})$ going to infinity,

$$\sum_{i=1}^{n}\left(\frac{\lambda_i^n}{2}\right)^{s_n} \leq M\frac{n}{s_n^{\frac{3}{2}}}.$$

This implies, by Tchebychev's inequality and taking $s_n = n^{+\rho^{k+1}}$ that

$$\mathcal{N}_n[n^{-\rho^k}, n^{-\rho^{k+1}}] \leq \sharp\left\{i : \left|\frac{\lambda_i}{2}\right| \geq 1 - n^{-\rho^{k+1}}\right\} \leq (1 - n^{-\rho^{k+1}})^{-s_n} \sum_{i=1}^n \left|\frac{\lambda_i^n}{2}\right|^{s_n} \leq eMn^{1-\frac{3}{2}\rho^{k+1}}.$$

Consequently we deduce that

$$A_n \leq eM \sum_{1\leq k\leq K} n^{\rho^k} n^{-\frac{3}{2}\rho^{k+1}} \leq Cn^{-\rho^K(\frac{3}{2}\rho-1)}$$

which goes to zero as $\rho > 2/3$. For the second term B_n, note that by [42, Theorem 1.10], for any $\epsilon > 0$ small enough,

$$\left|\mathcal{N}_n[n^{-\epsilon}\ell, n^{-\epsilon}(\ell+1)] - n\mu_{sc}([-2+n^{-\epsilon}\ell, -2+n^{-\epsilon}(\ell+1)])\right| \leq n^{1-\delta(\epsilon)}$$

with $\delta(\epsilon) = \frac{2\epsilon-1}{10}$. Hence, since $\mu_{sc}([-2+n^{-\epsilon}\ell, -2+n^{-\epsilon}(\ell+1)]) \sim n^{-\frac{3\epsilon}{2}}\sqrt{\ell}$, we deduce for ϵ small enough that for all $\ell \geq 1$,

$$\mathcal{N}_n[n^{-\epsilon}\ell, n^{-\epsilon}(\ell+1)] \leq 2n^{1-\frac{3\epsilon}{2}}\sqrt{\ell}.$$

This allows to bound B_n by

$$B_n \leq 2\sum_{\ell=1}^{[\gamma n^\epsilon]} \frac{n^\epsilon}{\ell} n^{-\frac{3\epsilon}{2}}\sqrt{\ell} \leq 2\int_0^\gamma \frac{1}{\sqrt{x}}dx = 2\sqrt{\gamma}$$

which goes to zero as n goes to infinity and then γ goes to zero. □

5.2. **Coulomb Gases.** We can also consider random matrices X_n which law is invariant under the action of the unitary or the orthogonal group and with eigenvalues with law given by

$$dP_n(\lambda_1, \ldots, \lambda_n) = \frac{1}{Z_n}|\Delta(\lambda)|^\beta e^{-n\beta\sum_{i=1}^n V(\lambda_i)} \prod_{i=1}^n d\lambda_i \tag{29}$$

with a polynomial function V of even degree and positive leading coefficient and $\beta = 1, 2$ or 4. We assume moreover that V is such that the limiting spectral measure μ_V of (X_n) is connected and compact and that its smallest and largest eigenvalues converge to the boundaries of the support. This set of hypotheses is often referred to as the "one-cut assumption". It holds in particular if V is strictly convex and this includes the classical Gaussian ensembles GOE and GUE (with $V(x) = x^2/4$ and $\beta = 1, 2$).

Proposition 5.7. *Under the above hypothesis on V, the extreme eigenvalues of X_n converge to the boundary of the support. The convergence of the extreme eigenvalues of $\widetilde{X_n}$ is given by Theorem 1.3. These eigenvalues have Gaussian fluctuations as stated in Theorem 3.2 if they deviate away from the bulk.*
Suppose moreover that Assumption 4.2 holds.
If the perturbation is of rank one and is strong enough so that the largest eigenvalues deviates from the bulk, for all $k \geq 2$, the rescaled kth largest eigenvalue $n^{\frac{2}{3}}(\widetilde{\lambda}_{n-k+1}^n - b_V)$ converges weakly towards the $k-1$-th Tracy Widom law. If the perturbation is of rank one and is weak enough, for all $k \geq 1$, the rescaled kth largest eigenvalue $n^{\frac{2}{3}}(\widetilde{\lambda}_{n-k+1}^n - b_V)$ converges weakly towards the k-th Tracy Widom law.
If the perturbation is of rank more than one, the extreme eigenvalues of $\widetilde{X_n}$ sticking to the bulk are at distance less than $n^{-1+\epsilon}$ for any $\epsilon > 0$ from the eigenvalues of X_n.

Proof. As explained above, it suffices to verify that the hypotheses hold in probability for $(X_n)_{n\geq 1}$.

Note that the convergence of the spectral measure, of the edges and the fluctuations of the extreme eigenvalues were obtained in [47]. The fact that $\sqrt{n}(G_{\mu_n}(z) - G_{\rm sc}(z))$ converges in probability to zero is a consequence of [28] so that Hypothesis 3.1 holds.

We next check Hypothesis 4.1[p,α] for the matrix model P_n. We shall prove it for any $\alpha > 1/3$ and any integer p. We first show that

$$\lim_{n\to\infty} \mathbb{E}\left[\frac{1}{n}\sum_{i\neq p}\frac{1}{\lambda_i^n - \lambda_p^n}\right] = -V'(a_V)\,. \tag{30}$$

Indeed, the joint distribution of $(\lambda_1^n,\ldots,\lambda_n^n)$ is

$$\frac{1}{Z_n^\beta} e^{-n\sum_{i=1}V(\lambda_i)} \prod_{1\leq i<j\leq n}^{n} (\lambda_i-\lambda_j)^\beta \mathbb{1}_{\Delta_n} \mathrm{d}\lambda_1\cdots \mathrm{d}\lambda_n,$$

with $\beta = 1, 2$ or 4, Z_n^β is the normalising constant and $\Delta_n = \{\lambda_1 < \cdots < \lambda_n\}$. Therefore,

$$\begin{aligned}
\mathbb{E}\left[\beta\sum_{i\neq p}\frac{1}{\lambda_i^n-\lambda_p^n}\right] &= -\frac{1}{Z_n^\beta}\int_{\Delta_n} e^{-n\beta\sum_{i=1}^n V(\lambda_i)}\frac{\partial}{\partial\lambda_p}\prod_{1\leq i<j\leq n}^{n}(\lambda_i-\lambda_j)^\beta \mathrm{d}\lambda_1\cdots\mathrm{d}\lambda_n,\\
&= \frac{1}{Z_n^\beta}\int_{\Delta_n}\frac{\partial}{\partial\lambda_p}\left(e^{-n\beta\sum_{i=1}^n V(\lambda_i)}\right)\prod_{1\leq i<j\leq n}^{n}(\lambda_i-\lambda_j)^\beta \mathrm{d}\lambda_1\cdots\mathrm{d}\lambda_n,\\
&= -n\beta\mathbb{E}\left[V'(\lambda_p^n)\right],
\end{aligned}$$

by integration by parts. Equation (30) follows, since λ_p^n converges almost surely to a_V (and concentration inequalities insures $V'(\lambda_p^n)$ is uniformly integrable). But, for any $\epsilon > 0$,

$$\frac{1}{n}\sum_{i\neq p}\frac{1}{\lambda_i^n-\lambda_p^n} \geq \frac{1}{n}\sum_{i\neq p}\frac{1}{\epsilon+\lambda_i^n-\lambda_p^n}$$

with, by convergence of the spectral measure and of λ_p^n, the right hand side converging to $-G_{\mu_X}(-a_V-\epsilon)$ which converges as ϵ decreases to zero to $-G_{\mu_X}(-a_V) = -V'(a_V)$. Hence, $\frac{1}{n}\sum_{i\neq p}\frac{1}{\lambda_i^n-\lambda_p^n}$ is bounded below by $-V'(a_V)$ with large probability for large n, and converges in expectation to $-V'(a_V)$, and therefore converges in probability to $-V'(a_V)$.

Moreover, by [47] (see [45] in the Gaussian case), the joint law of

$$\left(n^{2/3}(\lambda_1^n - a_V), n^{2/3}(\lambda_2^n - a_V), \ldots, n^{2/3}(\lambda_p^n - a_V)\right)$$

converges weakly towards a probability measure which is absolutely continuous with respect to Lebesgue measure. As a consequence, we also deduce from the first point that $n^{-1}\sum_{i<m_n}(\lambda_p^n - \lambda_i^n)^{-1}$ vanishes as n goes to infinity in probability for $m_n \ll n^{1/3}$ and therefore (30) proves the lacking point of Hypothesis 4.1.

For the two other points, observe that [47] implies that for any $\epsilon > 0$, $\mathbb{P}(|\lambda_2^n - \lambda_1^n| \leq n^{-\frac{2}{3}-\epsilon}) \underset{n\to\infty}{\longrightarrow} 0$. On the event $\{|\lambda_2^n - \lambda_1^n| > n^{-\frac{2}{3}-\epsilon}\}$, we have $|\lambda_i^n - \lambda_1^n| > n^{-\frac{2}{3}-\epsilon}$ for all

$i \in [2, n-1]$, so that

$$\frac{1}{n^2}\sum_{i=2}^{n}\frac{1}{(\lambda_i^n-\lambda_1^n)^2} \le n^{-\frac{1}{3}+\epsilon}\frac{1}{n}\sum_{i=2}^{n}\frac{1}{\lambda_i^n-\lambda_1^n}$$
$$\frac{1}{n^4}\sum_{i=2}^{n}\frac{1}{(\lambda_i^n-\lambda_1^n)^4} \le n^{-1+3\epsilon}\frac{1}{n}\sum_{i=2}^{n}\frac{1}{\lambda_i^n-\lambda_1^n}$$

so that by (30) and Markov's inequality, Hypothesis 4.1 holds in probability for any $\eta < 1/3$, $\eta_4 < 1$ and $\alpha > 1/3$. □

5.3. **Wishart matrices.** Let G_n be an $n \times m$ real (or complex) matrix with i.i.d. centred entries with law μ such that $\int z \mathrm{d}\mu(z) = 0$, $\int |z|^2 \mathrm{d}\mu(z) = 1$ and $\int |z|^4 \mathrm{d}\mu(z) < \infty$. Let $X_n = G_n G_n^*/m$. The following Proposition generalises some results first appeared in [9, 19].

Proposition 5.8. *Let n, m tend to infinity in such a way that $n/m \to c \in (0,1)$. The limits of the extreme eigenvalues of $\widetilde{X}_n$ are given by Theorem 1.3 and the fluctuations of those which limits are out of $[a,b]$ are given by Theorem 3.2, where the parameters $a, b, \rho_\theta, c_\alpha$ are given by the following formulas: $a = (1-\sqrt{c})^2$, $b = (1+\sqrt{c})^2$*

$$\rho_\theta := \begin{cases} \theta + \frac{\theta}{\theta - c} & \text{if } |\theta - c| > \sqrt{c}, \\ b & \text{if } |\theta - c| \le \sqrt{c} \text{ and } \theta > 0, \\ a & \text{if } |\theta - c| \le \sqrt{c} \text{ and } \theta < 0, \end{cases}$$

and

$$c_\alpha^2 = \begin{cases} \alpha^2\left(1 - \frac{c}{(\alpha-c)^2}\right) & \text{in the i.i.d. model,} \\ \\ \frac{\alpha^2 c}{(\alpha-c)^2}\left(1 - \frac{c}{(\alpha-c)^2}\right) & \text{in the orthonormalised model.} \end{cases}$$

Assume now that the law of the entries satisfy Hypothesis 5.2. If the perturbation has rank one, we have the following precise description of the fluctuations of the extreme eigenvalues of $\widetilde{X}_n$:

- *If $\theta > c + \sqrt{c}$ (resp. $\theta < c - \sqrt{c}$), for all $p \ge 2$, $n^{2/3}(\widetilde{\lambda}_{n-p+1}^n - 2\sigma)$ (resp. $n^{2/3}(\widetilde{\lambda}_p^n - 2\sigma)$) converges in law to the $p-1$th Tracy Widom law.*
- *If $0 \le \theta < c + \sqrt{c}$ (resp. $c - \sqrt{c} < \theta \le 0$), for all $p \ge 1$, $n^{2/3}(\widetilde{\lambda}_{n-p+1}^n - 2\sigma)$ (resp. $n^{2/3}(\widetilde{\lambda}_p^n - 2\sigma)$) converges in law to the pth Tracy Widom law.*

If the perturbation has rank more than one and for all i, $\theta_i \notin \{c+\sqrt{c}, c-\sqrt{c}\}$, the extreme eigenvalues of $\widetilde{X}_n$ are at distance less than $n^{-1+\epsilon}$ for any $\epsilon > 0$ to the extreme eigenvalues of X_n, which have Tracy-Widom fluctuations.

Proof. Again, it suffices to verify that the hypotheses hold in probability for $(X_n)_{n\ge1}$.

It is known, [32], that the spectral measure of X_n converges to the so-called Marčenko-Pastur distribution

$$\mathrm{d}\mu_X(x) := \frac{1}{2\pi c x}\sqrt{(b-x)(x-a)}\mathbb{1}_{[a,b]}(x)\mathrm{d}x,$$

where $a = (1-\sqrt{c})^2$ and $b = (1+\sqrt{c})^2$. It is known, [4, Th. 5.11], that the extreme eigenvalues converge to the bounds of this support. The formula

$$G_{\mu_X}(z) = \frac{z+c-1-\operatorname{sgn}(z-a)\sqrt{(z-c-1)^2-4c}}{2cz} \qquad (z \in \mathbb{R}\backslash[a,b])$$

allows to compute ρ_θ and c_α. Moreover, by [3, Th. 1.1] or [4, Th. 9.10], we also know that a central limit theorem holds for the linear statistics of Wishart matrices, giving Hypothesis 3.1 as in the Wigner case.

For Hypothesis 4.1, the proof is similar to the Wigner case. The convergence to the Tracy-Widom law of the non-perturbed matrix is due to S. Péché [37] (see [33] and [20] for the Gaussian case). The approximation of the eigenvalues by the quantiles of the limiting law can be found in [17, Theorem 9.1] whereas the absolute continuity property needed to prove Lemma 5.5 is derived in [17, Lemma 8.1]. This allows to prove Hypothesis 4.1 in this setting as in the Wigner case, we omit the details. □

5.4. **Non-white ensembles.** In the case of non-white matrices, we can only study the fluctuations away from the bulk (since we do not have the appropriate information about the top eigenvalues to prove Hypothesis 4.1). We illustrate this generalisation in a few cases, but it is rather clear that Theorem 3.2 applies in a much wider generality.

5.4.1. *Non-white Wishart matrices.* The first statement of Proposition 5.8 can be generalised to matrices X_n of the type $X_n = \frac{1}{m}T_n^{1/2}G_nG_n^*T_n^{1/2}$ or $\frac{1}{m}G_nT_nG_n^*$, where $G_n$is an $n\times m$ real (or complex) matrix with i.i.d. centred entries with law μ such that $\int z\mathrm{d}\mu(z) = 0$, $\int |z|^2\mathrm{d}\mu(z) = 1$ and $\int |z|^4\mathrm{d}\mu(z) < \infty$ and T_n is a positive non random Hermitian $n \times n$ matrix with bounded operator norm, with a converging empirical spectral law and with no eigenvalues outside any neighborhood of the support of the limiting measure for sufficiently large n. Indeed, in this case, everything, in the proof, stays true (use [2, Th.1.1] and [4, Th. 5.11]). However, when the limiting empirical distribution of T_n is not a Dirac mass, the computation of the ρ_θ's and the c_α's is not easy.

5.4.2. *Non-white Wigner matrices.* There are less results in the literature about the central limit theorem for band matrices (with centring with respect to the limit) and the convergence of the spectrum. We therefore concentrate on a special case, namely a Hermitian matrix X_n with independent Gaussian centred entries so that $E[|X_{ij}|^2] = n^{-1}\sigma(i/n, j/n)$ with a stepwise constant function

$$\sigma(x,y) = \sum_{i,j=1}^{k} 1_{\substack{\frac{i-1}{k}\le x<\frac{i}{k} \\ \frac{i-1}{k}\le y<\frac{i}{k}}}\sigma_{i,j}.$$

In [31], matrices of the form $S_n = \sum_{j=1}^{k(k+1)} a_j \otimes X_j^{(n)}$ with some independent matrices $X_j^{(n)}$ from the GUE and self-adjoint matrices a_j were studied. Taking $a_j = (\epsilon_{p,\ell}+\epsilon_{\ell,p})\sigma_{p,\ell}$ or $i(\epsilon_{p,\ell}-\epsilon_{\ell,p})\sigma_{p,\ell}$ with $\epsilon_{p,\ell}$ the matrix with null entries except at (p,ℓ) and $1 \le p \le \ell \le k$, we find that $X_n = S_n$. Then it was proved [31, (3.8)] that there exists $\alpha, \epsilon, \gamma > 0$ so that for z with imaginary part greater than $n^{-\gamma}$ for some $\gamma > 0$,

$$\left| E\left[\frac{1}{n}\mathrm{Tr}(z-X_n)^{-1}\right] - G(z)\right| \le (\Im z)^{-\alpha} n^{-1-\epsilon} \tag{31}$$

which entails the convergence of the spectrum of X_n towards the support of the limiting measure [31, Proposition 11] with exponential speed by [31, Proof of Lemma 14]. Thus X_n satisfies Hypothesis 1.1. Hypothesis 3.1 can be checked by modifying slightly the proof of (31) which is based on an integration by parts to be able to take z on the real line but away from the limiting support. Indeed, as in [23, Section 3.3], we can add a smooth cut-off function in the expectation which vanishes outside of the event A_n that X_n has all its eigenvalues within a small neighborhood of the limiting support. This additional cut-off will only give a small error in the integration by parts due to the previous point. Then, (31), but with an expectation restricted to this event, is proved exactly in the same way, except that $\Im z$ can be replaced by the distance of z to the neighborhood of the limiting support where the eigenvalues of X_n lives. Finally, concentration inequalities, in the local version [22, Lemma 5.9 and Part II], insure that on A_n,

$$\frac{1}{n}\mathrm{Tr}(z-X_n)^{-1} - E\left[1_{A_n}\frac{1}{n}\mathrm{Tr}(z-X_n)^{-1}\right]$$

is at most of order $n^{-1+\epsilon}$ with overwhelming probability. This completes the proof of Hypothesis 3.1.

5.5. **Some models for which our hypothesis are not satisfied.**

We gather hereafter a few remarks about some models for which the hypothesis we made on X_n are not satisfied. For sake of simplicity, we present hereafter only the case of i.i.d. perturbations (1).

5.5.1. *I.i.d. eigenvalues with compact support.* We assume that X_n is diagonal with i.i.d. entries which law μ is compactly supported. As in the core of the paper, we denote by a (resp. b) the left (resp. right) edge of the support of μ. We also denote by F_μ its cumulative distribution function and assume that there is $\kappa > 0$ such that for all $c > 0$,

$$\lim_{x\to 0^+} \frac{1-F_\mu(b-cx)}{1-F_\mu(b-x)} = c^\kappa \tag{32}$$

In this situation, it is easy to check that Hypothesis 1.1 holds in probability with $\mu_X = \mu$. But Hypothesis 3.1 is not satisfied. Indeed, by classical CLT, we have, for $\rho_\alpha \notin [a,b]$,

$$W_\alpha^n = \sqrt{n}(G_{\mu_n}(\rho_\alpha) - G_\mu(\rho_\alpha))$$

converges in law, as n goes to infinity to a Gaussian variable W_α with variance $-G'_\mu(\rho_\alpha) - G_\mu(\rho_\alpha)^2$. Moreover,

$$E[W_\alpha W_{\alpha'}] = \int \frac{1}{(\rho_\alpha-\lambda)(\rho_{\alpha'}-\lambda)}d\mu(\lambda) - G_\mu(\rho_\alpha)G_\mu(\rho_{\alpha'}).$$

Nevertheless, Theorem 3.2 holds for this model. Indeed, the whole proof of this theorem goes through in this context, except the proof of Lemma 3.5, where we have to make the following decomposition $M^n_{s,t}(i,x) = M^{n,1}_{s,t}(i,x) + M^{n,2}_{s,t}(i,x) + M^{n,3}_{s,t}(i,x)$ with the difference that this time $M^{n,3}_{s,t}$ does not go to zero but converges towards W_{α_i}. Hence, the eigenvalues fluctuate according to the distribution of the eigenvalues of $(c_j M_j + W_{\alpha_j} I_{k_j})_{1\le j\le q}$, with c_j and M_j as in the statement of Theorem 3.2 and I_{k_j} denotes the $k_j \times k_j$ identity matrix.

Let us now consider the fluctuations near the bulk. We first detail the fluctuations of the extreme eigenvalues of X_n. According to [26], the fluctuations of the largest eigenvalues of X_n are determined by the parameter κ defined in (32), that is, if $v_n = F_\mu(b-1/n)$, then the law of $\frac{b-\lambda_n^n}{b-v_n}$ converges weakly to the law with density proportional to e^{-x^κ} on $\mathbb{R}^+$. Otherwise stated, the fluctuations of λ_n^n are of order $n^{-1/\kappa}$ with asymptotic distribution the Gumbel distribution of type 2. One can check that if $\kappa \le 1$, then $\overline{\theta} = 0$.
One can show that, for any fixed p, for Hypothesis 4.1[p, α] to hold, we need $\alpha > \frac{1}{\kappa} - \frac{1}{2}$ and we then obtain that the distance of the extreme eigenvalues of the deformed matrix is at distance less that $n^{-1+\alpha'}$ for any $\alpha' > \alpha$. Therefore if $\kappa > 4/3$, this theorem allows us to deduce that the fluctuations of the extreme eigenvalues of the deformed matrix are the same as those of the non-deformed matrix.

5.5.2. *Coulomb gases with non-convex potentials.* In [35], Pastur showed that for a Coulomb gas law (29) with a potential V so that the equilibrium measure has a disconnected support, the central limit theorem does not hold in the sense that the variance may have different limits according to subsequences (see [35, (3.4)]. Moreover the asymptotics of $\sqrt{n}(\mathrm{Tr}(X_n) - \mu(x))$ can be computed sometimes and do not lead to a Gaussian limit. We might expect then that also $\sqrt{n}(G_{\mu_n}(x) - G_\mu(x))$ converges to a non-Gaussian limit, which would then result with non-Gaussian fluctuations for the eigenvalues outside of the bulk.

6. Appendix

6.1. **Determinant formula.** We here state formula (3), which can be deduced from the well known formula $\det \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \det(D)\det(A - BD^{-1}C)$.

Lemma 6.1. *Let $z \in \mathbb{C}\backslash\{\lambda_1^n, \ldots, \lambda_n^n\}$ and $\theta_1, \ldots, \theta_r \neq 0$. Set $D = \mathrm{diag}(\theta_1, \ldots, \theta_r)$ and let V be any $n \times r$ matrix. Then*

$$\det(z - X_n - VDV^*) = \det(z - X_n)\det(D)\det\left(D^{-1} - V^*(z - X_n)^{-1}V\right)$$

6.2. **Concentration estimates.**

Proposition 6.2. *Under Assumption 1.2, there exists a constant $c > 0$ so that for any matrix $A := (a_{jk})_{1 \le j,k \le n}$ with complex entries, for any $\delta > 0$, for any $g = (g_1, \ldots, g_n)^T$ with i.i.d. entries $(g_i)_{1 \le i \le n}$ with law ν,*

$$\mathbb{P}\left(|\langle g, Ag\rangle - \mathbb{E}[\langle g, Ag\rangle]| > \delta\right) \le 4e^{-c\min\{\frac{\delta}{C}, \frac{\delta^2}{C^2}\}}$$

if $C^2 = \mathrm{Tr}(AA^*)$ and if $\tilde{g}$ is an independent copy of g, for any $\delta, \kappa > 0$,

$$\mathbb{P}\left(|\langle g, A\tilde{g}\rangle| > \delta\sqrt{\mathrm{Tr}(AA^*) + \kappa\sqrt{\mathrm{Tr}((AA^*)^2)}}\right) \le 4e^{-c\delta^2} + 4e^{-c\min\{\kappa,\kappa^2\}}.$$

Proof. The first point is due to Hanson-Wright Theorem [24], see also [15, Proposition 4.5]. For the second, we use concentration inequalities, see e.g. [1, Lemma 2.3.3], based on the remark that for any fixed $\tilde{g}$, $g \to \langle g, A\tilde{g}\rangle$ is Lipschitz with constant $\sqrt{\langle \tilde{g}, AA^*\tilde{g}\rangle}$ and therefore, conditionally to $\tilde{g}$, for any $\delta > 0$,

$$\mathbb{P}\left(|\langle g, A\tilde{g}\rangle| > \delta\sqrt{\langle \tilde{g}, AA^*\tilde{g}\rangle}\right) \le 4e^{-c\delta^2}$$

On the other hand, the previous estimate shows that

$$\mathbb{P}\left(|\langle \tilde{g}, AA^*\tilde{g}\rangle - \mathrm{Tr}(AA^*)| > \kappa\sqrt{\mathrm{Tr}(AA^*)^2}\right) \le 4e^{-c\min\{\kappa,\kappa^2\}}.$$

As a consequence, we deduce the second point of the proposition. $\square$

Let $G^n = \left[g_1^n \cdots g_r^n\right]$ be an $n \times r$ matrix which columns $g_1^n, \dots, g_r^n$, are independent copies of an $n \times 1$ matrix with i.i.d. entries with law ν and define

$$V_{i,j}^n = \frac{1}{n}\langle g_i^n, g_j^n\rangle, \quad 1 \le i, j \le r,$$

and, for $j \le i-1$, if $\det[V_{k,l}^n]_{k,l=1}^{i-1} \neq 0$,

$$W_{i,j}^n = \frac{\det[\gamma_{k,l}^{n,j}]_{k,l=1}^{i-1}}{\det[V_{k,l}^n]_{k,l=1}^{i-1}}, \text{ with } \gamma_{k,l}^{n,j} = \begin{cases} V_{k,l}^n, & \text{if } l \neq j, \\ -V_{k,i}^n, & \text{if } l = j. \end{cases}$$

On $\det[V_{k,l}^n]_{k,l=1}^{i-1} = 0$, we give to $W_{i,j}^n$ an arbitrary value, say one. Putting $W_{ii}^n = 1$ and $W_{ij}^n = 0$ for $j \ge i+1$, it is a standard linear algebra exercise to check that the column vectors

$$v_i^n = \sum_{j=1}^r W_{i,j}^n g_j^n = i\text{th column of } G^n(W^n)^T$$

are orthogonal in $\mathbb{C}^n$. Let us introduce, for M an $r \times r$ matrix, $\|M\|_\infty = \sup_{1\le i,j\le r}|M_{i,j}|$. We next prove

Proposition 6.3. *For any $\gamma > 0$, there exists finite positive constants c, C (depending on r) so that for $Z^n = V^n$ or W^n,*

$$\mathbb{P}\left(\|Z^n - I\|_\infty \ge n^{-\frac{1}{2}}\gamma\right) \le C\left[e^{-4^{-1}c\gamma^2} + e^{-c\sqrt{n}}\right].$$

Moreover, with $\|v\|_2^2 = \sum_{i=1}^n |v_i|^2$, for any $\gamma \in (0, \sqrt{n}(2^{-r} - \epsilon)$ for some $\epsilon > 0$,

$$\mathbb{P}\left(\max_{1\le i\le r}\left|\frac{1}{n}\|\sum_{j=1}^r Z_{ij}^n g_j^n\|_2^2 - 1\right| \ge n^{-\frac{1}{2}}\gamma\right) \le C\left[e^{-4^{-1}c2^{-r}\gamma^2} + 4e^{-c\sqrt{n}}\right].$$

Proof. We first consider the case $Z^n = V^n$. The maximum of $|V_{ij}^n - \delta_{ij}|$ is controlled by the previous proposition with $A = n^{-1}I$, and the result follows from $\mathrm{Tr}AA^* = n^{-1}$ and $\mathrm{Tr}((AA^*)^2) = n^{-3}$, and choosing $\delta = \gamma/\sqrt{2}$, $\kappa = \sqrt{n}$. The result for W^n follows as on $\|V^n - I\|_\infty \le \gamma n^{-\frac{1}{2}} \le 1$

$$|\det[V_{k,l}]_{k,l=1}^{i-1} - 1| \le 2^r\gamma n^{-\frac{1}{2}},$$

whereas

$$|\det[\gamma_{k,l}^{n,j}]_{k,l=1}^{i-1}| \le 2^r\gamma n^{-\frac{1}{2}}.$$

For the last point, we just notice that since $\frac{1}{n}\|\sum_{j=1}^r Z_{i,j}^n g_j^n\|_2^2 = (ZVZ^*)_{i,i}$, we have

$$\max_{1\le i\le r}\left|\frac{1}{n}\|\sum_{j=1}^r Z_{ij}^n g_j^n\|_2^2 - 1\right| \le C(r)\max_{Z^n=V^n \text{ or } W^n}\|Z^n\|_\infty^2 \max_{Z^n=V^n \text{ or } W^n}\|Z^n - I\|_\infty$$

for a finite constant $C(r)$ which only depends on r. Thus the result follows from the previous point. $\square$

6.3. **Central Limit Theorem for quadratic forms.**

Theorem 6.4. *Let us fix $r \geq 1$ and let, for each n, $A^n(s,t)$ $(1 \leq s,t \leq r)$ be a family of $n \times n$ real (resp. complex) matrices such that for all s,t, $A^n(t,s) = A^n(s,t)^*$ and such that for all $s,t = 1,\ldots,r$,*

- *in the i.i.d. model,*

$$\frac{1}{n}\mathrm{Tr}[A^n(s,t)A^n(s,t)^*] \underset{n\to\infty}{\longrightarrow} \sigma_{s,t}^2, \quad \frac{1}{n}\sum_{i=1}^n |A^n(s,s)_{i,i}|^2 \underset{n\to\infty}{\longrightarrow} \omega_s, \tag{33}$$

- *in the orthonormalised model,*

$$\frac{1}{n}\mathrm{Tr}[|A^n(s,t) - \frac{1}{n}\mathrm{Tr}\, A^n(s,t)|^2] \underset{n\to\infty}{\longrightarrow} \sigma_{s,t}^2, \quad \frac{1}{n}\sum_{i=1}^n \left|A^n(s,s)_{i,i} - \frac{1}{n}\mathrm{Tr}\, A^n(s,t)\right|^2 \underset{n\to\infty}{\longrightarrow} \omega_s. \tag{34}$$

for some finite numbers $\sigma_{s,t}, \omega_s$ (in the case where $\kappa_4(\nu) = 0$, the part of the hypothesis related to ω_s can be removed). For each n, let us define the $r \times r$ random matrix

$$G_n := \left[\sqrt{n}\left(\langle u_s^n, A^n(s,t)u_t^n\rangle - \mathbb{1}_{s=t}\frac{1}{n}\mathrm{Tr}(A^n(s,s))\right)\right]_{s,t=1}^r.$$

Then the distribution of G_n converges weakly to the distribution of a real symmetric (resp. Hermitian) random matrix $G = [g_{s,t}]_{s,t=1}^r$ such that the random variables

$$\{g_{s,t}\,;\, 1 \leq s \leq t \leq r\}$$

$$\text{(resp. } \{g_{s,s}\,;\, 1 \leq s \leq r\} \cup \{\Re(g_{s,t})\,;\, 1 \leq s < t \leq r\} \cup \{\Im(g_{s,t})\,;\, 1 \leq s < t \leq r\})$$

are independent and for all s, $g_{s,s} \sim \mathcal{N}(0, 2\sigma_{s,s}^2 + \kappa_4(\nu)\omega_s)$ (resp. $g_{s,s} \sim \mathcal{N}(0, \sigma_{s,s}^2 + \kappa_4(\nu)\omega_s)$) and for all $s \neq t$, $g_{s,t} \sim \mathcal{N}(0, \sigma_{s,t}^2)$ (resp. $\Re(g_{s,t}), \Im(g_{s,t}) \sim \mathcal{N}(0, \sigma_{s,t}^2/2)$).

Remark 6.5. *Note that if the matrices $A^n(s,t)$ depend on a real parameter x in such a way that for all s,t, for all $x,x' \in \mathbb{R}$,*

$$\frac{1}{n}\mathrm{Tr}(A^n(s,t)(x) - A^n(s,t)(x'))^2 \underset{n\to\infty}{\longrightarrow} 0,$$

then it follows directly from Theorem 6.4 and from a second moment computation that each finite dimensional marginal of the process

$$\left[\sqrt{n}\left(\langle u_s^n, A^n(s,t)(x_{s,t})u_t^n\rangle - \mathbb{1}_{s=t}\frac{1}{n}\mathrm{Tr}(A^n(s,s)(x_{s,s}))\right)\right]_{1\leq s,t\leq r\,,\, x_{s,t}\in\mathbb{R}\,,\, x_{s,t}=x_{t,s}}$$

converges weakly to the law of $[g_{s,t}]_{1\leq s,t\leq r\,,\, x_{s,t}\in\mathbb{R}\,,\, x_{s,t}=x_{t,s}}$.

Proof. • Let us first consider the model where the $(\sqrt{n}u_s^n)_{1\leq s\leq r}$ are i.i.d. vectors with i.i.d. entries with law ν satisfying Assumption 1.2. Note that for all $s,t = 1,\ldots,r$, by (33), the sequence $\frac{1}{n}\sum_{i,j=1}^n A^n(s,t)_{i,j}^2$ is bounded. Hence up to the extraction of a subsequence, one can suppose that it converges to a limit $\tau_{s,t} \in \mathbb{C}$. Since the conclusion of the theorem does not depend on the numbers $\tau_{s,t}$ and the weak convergence is metrisable, one can ignore the fact that these convergences are only along a subsequence. In the case where $\kappa_4(\nu) = 0$, we can in the same way add the part of the hypothesis related to ω_s.

We have to prove that for any real symmetric (resp. Hermitian) matrix $B := [b_{s,t}]_{s,t=1}^r$, the distribution of $\mathrm{Tr}(BG_n)$ converges weakly to the distribution of $\mathrm{Tr}(BG)$. Note that

$$\mathrm{Tr}(BG_n) \;=\; \frac{1}{\sqrt{n}}(U_n^* C^n U_n - \mathrm{Tr}\, C^n),$$

where C^n is the $rn \times rn$ matrix and U_n is the $rn \times 1$ random vector defined by

$$C^n = \begin{bmatrix} b_{1,1}A^n(1,1) & \cdots & b_{1,r}A^n(1,r) \\ \vdots & & \vdots \\ b_{r,1}A^n(r,1) & \cdots & b_{r,r}A^n(r,r) \end{bmatrix}, \quad U_n = \sqrt{n}\begin{bmatrix} u_1^n \\ \vdots \\ u_r^n \end{bmatrix}.$$

In the real (resp. complex) case, let us now apply Theorem 7.1 of [7] in the case $K = 1$. It follows that the distribution of

$$\mathrm{Tr}(BG_n) = \sum_{s=1}^r b_{s,s} G_{n,s,s} + \sum_{1\le s<t\le r} 2\Re(b_{s,t})\Re(G_{n,s,t}) + 2\Im(b_{s,t})\Im(G_{n,s,t})$$

converges weakly to a centred real Gaussian law with variance

$$\begin{cases} \sum_{s=1}^r b_{s,s}^2(2\sigma_{s,s}^2 + \kappa_4(\nu)\omega_s) + \sum_{1\le s<t\le r}(2b_{s,t})^2\sigma_{s,t}^2 & \text{in the real case,} \\ \sum_{s=1}^r b_{s,s}^2(\sigma_{s,s}^2 + \kappa_4(\nu)\omega_s) + \sum_{1\le s<t\le r}(2\Re(b_{s,t}))^2\frac{\sigma_{s,t}^2}{2} + (2\Im(b_{s,t}))^2\frac{\sigma_{s,t}^2}{2} & \text{in the complex case.} \end{cases}$$

It completes the proof in the i.i.d. model.

• In the orthonormalised model, we can write $u_s^n = \frac{1}{\|\sum_{i=1}^s W_{si}^n g_i\|_2}\sum_{j=1}^s W_{sj}^n g_j$, where the matrix W^n is the one introduced in this section. It follows that, with

$$B^n(s,t) = A^n(s,t) - \frac{1}{n}\mathrm{Tr}(A^n(s,t)),$$

by orthonormalization of the u_s^n's

$$\begin{aligned} &\sqrt{n}\left(\langle u_s^n, A^n(s,t)u_t^n\rangle - \frac{\mathbb{1}_{s=t}}{n}\mathrm{Tr}(A^n(s,t))\right) \\ =\;& \sqrt{n}\langle u_s^n, B^n(s,t)u_t^n\rangle \\ =\;& \frac{n}{\|\sum_{i=1}^s W_{si}^n g_i\|_2 \|\sum_{i=1}^t W_{ti}^n g_i\|_2}\sum_{j,i=1}^r W_{si}^n \bar{W}_{tj}^n \frac{1}{\sqrt{n}}\langle g_i, B^n(s,t)g_j\rangle. \end{aligned}$$

But, by the previous result, if $i \neq j$,

$$\frac{1}{\sqrt{n}}\langle g_i, B(s,t)g_j\rangle$$

converges in distribution to a Gaussian law, whereas if $i = j$,

$$\frac{1}{\sqrt{n}}\langle g_i, B(s,t)g_i\rangle$$

$$= \frac{1}{\sqrt{n}}\left(\langle g_i, A(s,t)g_i\rangle - \mathbb{E}[\langle g_i, A(s,t)g_i\rangle]\right) + \frac{\mathrm{Tr}(A(s,t))}{\sqrt{n}}\left(\langle g_i, g_i\rangle - \mathbb{E}[\langle g_i, g_i\rangle]\right)$$

where both terms converge to a Gaussian. Thus this term is also bounded as n goes to infinity.

Hence, by Proposition 6.3, we may and shall replace W^n by the identity (since the error term would be of order at most $n^{-\frac{1}{2}+\epsilon}$), which yields

$$\sqrt{n}\langle u_s^n, B^n(s,t)u_t^n\rangle \approx \sqrt{n}^{-1}\langle g_s, B(s,t)g_t\rangle$$

so that we are back to the previous setting with B instead of A. $\square$

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