

# Character expansion method for the first order asymptotics of a matrix integral

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## Abstract

The estimation of various matrix integrals as the size of the matrices goes to infinity is motivated by theoretical physics, geometry and free probability questions. On a rigorous ground, only integrals of one matrix or of several matrices with simple quadratic interaction (called  $AB$  interaction) could be evaluated so far (see e.g. [19], [17] or [9]). In this article, we follow an idea widely developed in the physics literature, which is based on character expansion, to study more complex interaction. In this context, we derive a large deviation principle for the empirical measure of Young tableaux. We then use it to study a matrix model defined in the spirit of the 'dually weighted graph model' introduced in [13], but with a cutoff function such that the matrix integral and its character expansion converge. We prove that the free energy of this model converges as the size of the matrices go to infinity and study the saddle points of the limit.

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## 1 Introduction

The evaluation of matrix integrals was first motivated by theoretical physics and geometry since they can be related, via Feynman diagrams expansion (see [27] for a nice introduction), to the enumeration of maps. Thanks to this relation, matrix integrals can also be used to describe some models appearing in statistical mechanics, such as the Ising model or the  $q$ -Potts model, on random graphs (instead of the usual two-dimensional lattice). Using similar ideas, string theory models can be described via matrix integrals around criticality (see the course [7] for various applications to physics). Another motivation is the study of non-commutative entropies introduced by D. Voiculescu [22] in the context of free probability. Let us roughly say that the understanding of the asymptotic behavior of all possible matrix integrals would be equivalent to the understanding of the so-called microstates entropy.

So, what is a matrix integral? If we let, for  $n \in \mathbb{N}$ ,  $\mathbb{C}\langle X_1, \dots, X_n \rangle$  be the set of polynomial functions of  $n$  non-commutative variables and if we choose, for some  $m, p \in \mathbb{N}$ ,  $P \in \mathbb{C}\langle X_1, \dots, X_{n+p} \rangle^{\otimes m}$  and  $\phi := (\phi_i)_{1 \leq i \leq n+p} \in \mathcal{C}^o(\mathbb{R})^{n+p}$ , then a matrix integral can be defined by

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$$Z_N(P, \phi) = \int e^{N^2(N^{-1}\text{tr})^{\otimes m}(P(\phi_1(A_1), \dots, \phi_{n+p}(A_{n+p})))} dA_1 \cdots dA_n.$$

where  $dA$  denotes the Lebesgue measure on the chosen state space of the matrices, included into  $\mathcal{M}_N(\mathbb{C})$ , the space of square matrices of dimension  $N$  with complex entries. In the following, the matrices will take their values in the set  $\mathcal{H}_N(\mathbb{C})$  of Hermitian matrices of dimension  $N$ . The first order asymptotics of  $Z_N(P, \phi)$  can easily be studied in the case where  $n = 1$  since then the joint law of the eigenvalues of the matrix  $A$  is known and described by the Coulomb gas law (see [1] for instance). All the correction terms have been recently studied rigorously by N. Ercolani and K. McLaughlin in [6]. To this end, they use Riemann-Hilbert techniques together with a good understanding of the asymptotic behaviour of the spectral measure of the matrix with law given by the corresponding Gibbs measure

$$\mu_N^{P, \phi}(dA_1 \cdots dA_n) = Z_N(P, \phi)^{-1} e^{N^2(N^{-1}\text{tr})^{\otimes m}(P(\phi_1(A_1), \dots, \phi_{n+p}(A_{n+p})))} dA_1 \cdots dA_n.$$

There are much less complete results in the case where  $n \geq 2$ . On a rigorous ground, let us however mention the work of M. Mehta and al. (see e.g. [19] and [17]) who considered symmetric models with  $AB$  interaction including the so-called Ising model or matrices coupled in chain model, i.e  $m = 1, p = 0$  and

$$P(A_1, \dots, A_n) = \sum_{i=1}^n P(A_i) + \sum_{i=1}^{n-1} A_i A_{i+1}.$$

By orthogonal polynomial techniques, they could obtain the asymptotic behaviour of the associated free energy when integration holds over Hermitian matrices. By using completely different techniques based on large deviations, similar asymptotics could be derived in [10] and [9] for  $AB$  interaction models where the symmetry between the matrices can be broken (i.e. we can choose  $P(A_1, \dots, A_n) = \sum_{i=1}^n P_i(A_i) + \sum_{i=1}^{n-1} A_i A_{i+1}$ , possibly with different  $P_i$ 's) and integration can also hold over the orthogonal ensemble. These techniques have moreover the advantage to allow the description of the asymptotic behaviour of the spectral measures of the matrices  $(A_1, \dots, A_n)$  with law  $\mu_N^P$ , key step to try to obtain the full expansion of  $Z_N(P)$ .

On a less rigorous ground, a few other models have been studied. The main idea to study most of them is based on character expansion, a technique which was introduced by A. Migdal in [20] and by C. Itzykson and J.-B. Zuber in their famous article on planar approximation [12], and then widely developed in the 90's by various physicists (see for example [5], [15] for the so-called  $ABAB$  model or refer to [13] for a review). This technique allows to express the involved matrix integrals in terms basically of a sum over characters which are simpler to deal with because the interaction is reduced to spherical integrals, whose asymptotics are described in [10]. However, this sum is in general an infinite signed series (which actually might diverge), point which is not addressed for instance in [13]. A formal expansion was also obtained by B. Collins in [3] in a very general setting. He could obtain a formula for the free energy of matrix integrals as a formal series and study the convergence of each terms of this series. However, he could not prove that the series in fact converges.

In the present article, we show how the idea of character expansion can be used to estimate rigorously the specific matrix integral in which,  $A_N$  and  $B_N$  being two  $N \times N$  given Hermitian

matrices, the partition function is

$$\begin{aligned} Z_N(\Phi) &\equiv \int dM e^{-\frac{N}{2} \text{tr} M^2 - \text{tr} \otimes \text{tr} \log(I \otimes I - B_N \otimes \Phi(M)_{A_N})}, \\ &= \int dM e^{-\frac{N}{2} \text{tr} M^2 + \sum_{k \geq 1} k^{-1} \text{tr}(B_N^k) \text{tr}((\Phi(M)_{A_N})^k)} \end{aligned} \quad (1)$$

with the following notations :

- $dM$  is the Lebesgue measure over the set  $\mathcal{H}_N(\mathbb{C})$  of Hermitian matrices of size  $N$ ,
- $\text{tr}$  is the usual trace on  $\mathcal{M}_N(\mathbb{C})$  and  $I$  is the identity in  $\mathcal{M}_N(\mathbb{C})$ ,
- $\Phi$  is a continuous function from  $\mathbb{R}$  into  $\mathbb{R}$ .  $\Phi(M)$  is then uniquely defined by

$$\Phi(M) = U \text{diag}(\Phi(\lambda_1), \dots, \Phi(\lambda_N)) U^* \text{ when } M = U \text{diag}(\lambda_1, \dots, \lambda_N) U^* \text{ for some } U \in \mathcal{U}_N(\mathbb{C}).$$

This model was studied in the case where  $\Phi(x) = x$  in [14] where it was called the “dually weighted graphs model”, because it describes, in the large  $N$  limit, planar graphs having arbitrary coordination dependent weights for both vertices and faces. Note that in fact, in the case where  $\Phi(x) = x$ , the expansion is diverging (see [14], (2.7)). In this work, we shall restrict ourselves to functions  $\Phi$  satisfying appropriate boundness conditions to insure that the partition function  $Z_N(\Phi)$  and its character expansion are well defined. We discuss in section 6 the relation between our result, [14] and the enumeration of maps. Our main results can be sketched as follows

**Theorem 1.1**    1. *Under appropriate assumptions (see hypotheses 2.1, 4.2),*

$$F_N(\Phi) = \frac{1}{N^2} \log Z_N(\Phi)$$

*converges as  $N$  goes to infinity and a formula is derived (see Theorem 4.3 for details).*

2. *Under appropriate additional assumptions, we can give a weak characterization of the limit points of the spectral measure of  $M$  under the Gibbs measure associated to  $Z_N(\Phi)$  (see Proposition 5.1)*

The main advantage of this model is that its character expansion is not signed (i.e is a sum of non negative terms), allowing standard Laplace method techniques. But let us explain what we mean by “character expansion”, i.e. expansion in terms of Schur polynomials. For that, we recall the following notions (see for example section 4.4. of the book [21] for more details):

- a Young shape  $\lambda$  is a finite sequence of non-negative integers  $(\lambda_1, \lambda_2, \dots, \lambda_l)$  written in non-increasing order. One should think of it as a diagram whose  $i$ th line is made of  $\lambda_i$  empty boxes. We denote by  $|\lambda| = \sum_i \lambda_i$  the total number of boxes of the shape  $\lambda$ .

In the sequel, when we have a shape  $\lambda = (\lambda_1, \lambda_2, \dots)$  and an integer  $N$  greater than the number of lines of  $\lambda$  having a strictly positive length, we will define a sequence  $l$  associated to  $\lambda$  and  $N$ , which is an  $N$ -uple of integers  $l_i = \lambda_i + N - i$ . In particular we have that  $l_1 > l_2 > \dots > l_N \geq 0$  and  $l_i - l_{i+1} \geq 1$ .

- for some fixed  $N \in \mathbb{N}$ , a Young tableau will be any filling of the Young shape above with integers from 1 to  $N$  which is non-decreasing on each line and (strictly) increasing on each column. For each such filling, we define the content of a Young tableau as the  $N$ -uple  $(\mu_1, \dots, \mu_N)$  where  $\mu_i$  is the number of  $i$ 's written in the tableau.

Notice that, for  $N \in \mathbb{N}$ , a Young shape can be filled with integers from 1 to  $N$  if and only if  $\lambda_i = 0$  for  $i > N$ .

- for a Young shape  $\lambda$  and an integer  $N$ , the Schur polynomial  $s_\lambda$  is an element of  $\mathbb{C}\langle x_1, \dots, x_N \rangle$  defined by

$$s_\lambda(x_1, \dots, x_N) = \sum_T x_1^{\mu_1} \dots x_N^{\mu_N}, \quad (2)$$

where the sum is taken over all Young tableaux  $T$  of fixed shape  $\lambda$  and  $(\mu_1, \dots, \mu_N)$  is the content of  $T$ . Note that  $s_\lambda$  is positive whenever the  $x_i$ 's are and, although it is not obvious from this definition (cf for example [21] for a proof),  $s_\lambda$  is a symmetric function of the  $x_i$ 's.

If  $A$  is a matrix in  $\mathcal{M}_N(\mathbb{C})$ , then define  $s_\lambda(A) \equiv s_\lambda(A_1, \dots, A_N)$ , where the  $A_i$ 's are the eigenvalues of  $A$ .

Now the point is that we shall see in Theorem 2.2, whose derivation is the object of section 2, that we can write  $Z_N(\Phi)$  as

$$Z_N(\Phi) = c_N \sum_\lambda s_\lambda(A_N) s_\lambda(B_N) Z_N(\Phi, \lambda)$$

where the sum runs over Young tableaux  $\lambda = (\lambda_1 \geq \lambda_2 \dots \geq \lambda_N)$  and  $Z_N(\Phi, \lambda)$  is a positive function of the shape  $\lambda$  which depends ‘almost continuously’ on the empirical measure

$$\hat{\mu}_\lambda^N := \frac{1}{N} \sum_{i=1}^N \delta_{\frac{\lambda_i + N - i}{N}} \in \mathcal{P}(\mathbb{R}^+)$$

where  $\mathcal{P}(\mathbb{R}^+)$  denotes the set of probability measures on  $\mathbb{R}^+$ . Therefore, to study the asymptotic behaviour of  $Z_N(\Phi)$  we are lead to estimate the deviations of more general measures and establish the following

**Theorem 1.2** *Let  $F : \mathcal{P}(\mathbb{R}^+) \rightarrow \mathbb{R}$  be a bounded continuous function, and  $c : \mathbb{R}^+ \rightarrow \mathbb{R}$  be a continuous function such that  $\liminf_{x \rightarrow +\infty} x^{-1} c(x) > 0$ . Let  $(A_N, B_N)_{N \geq 0}$  be two sequences of matrices with eigenvalues taking their values in  $[\epsilon, 1]$  for some  $\epsilon > 0$  and such that the spectral measures of  $A_N$  and  $B_N$  converge towards  $\mu_A$  and  $\mu_B$  respectively. Let  $a, b \geq 0$  and consider the positive measure on  $\mathcal{P}(\mathbb{R}^+)$  given, for any measurable subset  $M \in \mathcal{P}(\mathbb{R}^+)$ , by*

$$\Pi^N(M) = \sum_\lambda 1_{\hat{\mu}_\lambda^N \in M} s_\lambda(A_N)^a s_\lambda(B_N)^b e^{N^2 F(\hat{\mu}_\lambda^N) - N^2 \int c(x) d\hat{\mu}_\lambda^N(x)}. \quad (3)$$

*Then, if we equip  $\mathcal{P}(\mathbb{R}^+)$  with the standard weak topology,  $(\Pi^N)_{N \geq 0}$  satisfies large deviation bounds with a rate function  $H$  which is infinite on  $\mathcal{L}^c$  where*

$$\mathcal{L} := \left\{ \nu \in \mathcal{P}(\mathbb{R}^+) : d\nu(x) \ll dx, \quad \frac{d\nu(x)}{dx} \leq 1 \right\}$$

*and otherwise given by*

$$H(\nu) = \int c(x) d\nu(x) - \frac{a+b}{2} \Sigma(\nu) - F(\nu) - aI(\log \sharp \mu_A, \nu) - bI(\log \sharp \mu_B, \nu) - \frac{a}{2} S(\mu_A) - \frac{b}{2} S(\mu_B)$$

*where*

- $I(\mu, \nu)$  will be defined in subsection 3.2,

- $\Sigma(\nu) = \iint \log|x - y|d\nu(x)d\nu(y),$
- $S(\mu) = \iint \log(s(x, y))d\mu(x)d\mu(y),$  with

$$s(x, y) = \int_0^1 (\alpha x + (1 - a)y)^{-1} \text{ if } x \neq y, \quad s(x, x) = x^{-1} \text{ otherwise.}$$

- and for  $\mu \in \mathcal{P}(\mathbb{R})$  and any measurable function  $f : \mathbb{R} \rightarrow \mathbb{R},$  we denote by  $f\sharp\mu$  the probability measure such that, for any bounded measurable function  $g$  on  $\mathbb{R},$   $f\sharp\mu(g) = \int g(f(x))d\mu(x).$

More precisely,

1.  $H$  has compactly supported level sets, i.e  $\{\nu \in \mathcal{P}(\mathbb{R}^+) : H(\nu) \leq M\}$  is compact for all  $M < \infty.$
2. For any closed set  $F \in \mathcal{P}(\mathbb{R}^+)$

$$\limsup_{N \rightarrow \infty} \frac{1}{N^2} \log \Pi^N(F) \leq -\inf\{H(\nu), \nu \in F\}$$

3. For any open set  $O \in \mathcal{P}(\mathbb{R}^+)$

$$\liminf_{N \rightarrow \infty} \frac{1}{N^2} \log \Pi^N(O) \geq -\inf\{H(\nu), \nu \in O\}$$

In particular,

$$\lim_{N \rightarrow \infty} \frac{1}{N^2} \log \Pi^N(\mathcal{P}(\mathbb{R}^+)) = -\inf\{H(\nu)\}$$

and the infimum is achieved.

Theorem 4.3 would be a direct consequence of Theorem 1.2 (with  $a = b = 1$  and  $\log Z_N(\Phi, \lambda) = N^2 F(\hat{\mu}_\lambda^N) - N^2 \int c(x) d\hat{\mu}_\lambda^N(x)$ ) if  $Z_N(\Phi, \lambda)$  was indeed a continuous function of  $\hat{\mu}_\lambda^N$  and decayed sufficiently fast as the size of the tableau goes to infinity. Although it is not exactly the case, most of the technicalities are already contained in the proof Theorem 1.2, which, as we shall see in section 6, is of independent interest. Its proof relies on techniques developed in [1] in a continuous setting, the relation of Schur functions with spherical integrals (see section 2) and on [10] where the asymptotics of such integrals were obtained. However, the proof remains rather technical for various reasons, the most severe being that we need to define the spherical integrals in a broader set than what was studied in [10]. In section 3, we prove Theorem 1.2 in details. We precise the strategy used to show the Theorem 1.2 at the beginning of section 3, just after the precise statement of the theorem. We outline how to adapt the proofs to obtain Theorem 4.3 in section 4. Section 5 is devoted to the study of the minimizers of the rate function associated with the asymptotics of  $Z_N(\Phi).$  They are reminiscent of [14] since they are described in terms of an additional measure describing the optimal shape of the Young tableau. They involve also, following [9] and [16], the solutions of an Euler equation for isentropic flow with negative pressure  $p(\rho) = -\frac{\pi^2}{3}\rho^3.$

Finally, we comment our result, other applications of our techniques, and their relations with the problem of the enumeration of maps in section 6.

## 2 Formulation of the matrix model as a sum over characters

Before going into the details of the large deviation principles we have announced in the introduction, we devote this section to show the character expansion for  $Z_N(\Phi)$  (see Theorem 2.2). This will be useful in section 4 and can also be seen as a justification for the definition of  $\Pi^N$  we introduced above and therefore as a motivation to prove such a result like Theorem 1.2.

Since we shall later also be interested by the Gibbs measure associated with such a model we more generally define, after (1), if  $X$  is a measurable subset of  $\mathcal{P}(\mathbb{R})$

$$Z_N(\Phi)(X) \equiv \int_{\hat{\mu}_M^N \in X} dM e^{-\frac{N}{2} \text{tr} M^2 - \text{tr} \otimes \text{tr} \log(I \otimes I - B_N \otimes \Phi(M) A_N)}, \quad (4)$$

where, for an Hermitian matrix  $M \in \mathcal{H}_N(\mathbb{C})$  with eigenvalues  $(M_1, \dots, M_N) \in \mathbb{R}^N$ , we shall denote  $\hat{\mu}_M^N$  the spectral measure of  $M$  given by

$$\hat{\mu}_M^N = \frac{1}{N} \sum_{i=1}^N \delta_{M_i}.$$

$\hat{\mu}_M^N$  is an element of the space  $\mathcal{P}(\mathbb{R})$  of probability measures on the real line. We endow  $\mathcal{P}(\mathbb{R})$  with its usual weak topology (i.e  $\mu_n \in \mathcal{P}(\mathbb{R})$  converges towards  $\mu$  iff  $\mu_n(f) = \int f d\mu_n$  converges to  $\mu(f)$  for all  $f$  in the space  $\mathcal{C}_b(\mathbb{R})$  of bounded continuous functions).

We shall assume that

### Hypothesis 2.1

1. If  $\|\cdot\|_N$  denotes the operator norm in  $\mathcal{M}_N(\mathbb{C})$ ,  $\sup_{N \in \mathbb{N}} \|A_N\|_N$  and  $\sup_{N \in \mathbb{N}} \|B_N\|_N$  are finite and  $\Phi$  is bounded. Without loss of generality, we will assume hereafter that

$$\sup_{N \in \mathbb{N}} \|A_N\|_N \leq 1, \quad \sup_{N \in \mathbb{N}} \|B_N\|_N \leq 1$$

This amounts to multiply  $\Phi$  by  $\sup_{N \in \mathbb{N}} \|A_N\|_N \cdot \sup_{N \in \mathbb{N}} \|B_N\|_N$ .

2. For all  $N \in \mathbb{N}$ ,  $A_N$  and  $B_N$  are non-negative and  $\Phi$  takes its value in  $\mathbb{R}^+$ .
3. If we define  $\rho_\Phi := -\log \|\Phi\|_\infty$ , we assume that

$$e^{-\rho_\Phi} := \|\Phi\|_\infty < 1. \quad (5)$$

Note that this assumption insures that for each  $N$ ,  $I \otimes I - B_N \otimes \Phi(M) A_N$  has positive eigenvalues, so that its logarithm is well defined and  $\text{tr} \otimes \text{tr} \log(I \otimes I - B_N \otimes \Phi(M) A_N)$  is bounded so that the partition function itself is well defined.

The goal of this section is to express the partition function  $Z_N(\Phi)(X)$  in terms of spherical integrals, where a spherical integral  $I_N$  over the unitary group is given, for two real diagonal matrices  $D_N, E_N$ , by

$$I_N(D_N, E_N) := \int \exp\{N \text{tr}(U D_N U^* E_N)\} dm_N(U),$$

where  $m_N$  denote the Haar measure on the unitary group  $\mathcal{U}_N$ . In the sequel, we will denote  $\Delta$  the VanderMonde determinant given, for any diagonal matrix  $A_N = \text{diag}(a_1, \dots, a_N)$ , by  $\Delta(A_N) =$

$$\Delta(a) = \prod_{i < j} |a_i - a_j|.$$

The main result of this section is

**Theorem 2.2** *When Hypothesis 2.1 is satisfied, we have that*

$$Z_N(\Phi)(X) = c_N \sum_{\lambda} s_{\lambda}(A_N) s_{\lambda}(B_N) Z_N(\Phi, \lambda)(X) \quad (6)$$

where :

- $\mathcal{U}_N$  is the unitary group of dimension  $N$ ,
- the sum holds over all Young shapes,
- $s_{\lambda}$  is the Schur polynomial corresponding to a Young shape  $\lambda$ ,

$$- Z_N(\Phi, \lambda)(X) = \int_{\hat{\mu}_M^N \in X} I_N \left( \log \Phi(M), \frac{l}{N} \right) \frac{\Delta(\log \Phi(M))}{\Delta(\Phi(M))} \Delta(M)^2 e^{-\frac{N}{2} \sum_{i=1}^N M_i^2} \prod_{i=1}^N dM_i,$$

where  $l$  is the sequence associated to  $\lambda$  and  $N$ ,

- $c_N$  is a constant which only depend on  $N$ .

Denoting  $|\lambda| = \sum_i \lambda_i$ , we can rewrite (6) into

$$Z_N(\Phi)(X) = c_N \sum_{\lambda} s_{\lambda}(A_N) s_{\lambda}(B_N) Z_N(\Psi, \lambda)(X) e^{-\rho_{\Phi} |\lambda|} \quad (7)$$

where  $\Psi = (\|\Phi\|_{\infty})^{-1} \Phi$  and  $c_N$  is a constant which only depend on  $N$ .

**Proof.**

1. *Expansion along Young tableaux*

By definition, if  $(B_{N,i})_{1 \leq i \leq N}$  and  $((\Phi(M)A_N)_i)_{1 \leq i \leq N}$  are respectively the eigenvalues of  $B_N$  and  $\Phi(M)A_N$ , we can rewrite :

$$e^{-tr \otimes tr \log(I \otimes I - B_N \otimes \Phi(M)A_N)} = \prod_{i,j=1}^N \frac{1}{1 - B_{N,i}(\Phi(M)A_N)_j}, \quad (8)$$

where condition (5) ensures the existence of the right hand side.

The Cauchy formula (for a reference and a proof, see for example formula 4.8.4 in the book of Sagan [21]) gives us that

$$\prod_{i,j=1}^N \frac{1}{1 - B_{N,i}(\Phi(M)A_N)_j} = \sum_{\lambda} s_{\lambda}(B_N) s_{\lambda}(\Phi(M)A_N), \quad (9)$$

where  $\lambda$  is the shape of a Young tableau and  $s_{\lambda}$  is the Schur polynomial corresponding to this shape.

Note that  $s_{\lambda}(B_N) \geq 0$  since  $B_N \geq 0$  as well as  $s_{\lambda}(\Phi(M)A_N) = s_{\lambda}(A_N^{\frac{1}{2}} \Phi(M) A_N^{\frac{1}{2}}) \geq 0$ . Hence, the above series converges absolutely and we can use Fubini's theorem to write our partition function

$$Z_N(\Phi)(X) = \sum_{\lambda} s_{\lambda}(B_N) \int_{\hat{\mu}_M^N \in X} e^{-\frac{N}{2} tr M^2} s_{\lambda}(\Phi(M)A_N) dM. \quad (10)$$

2. *Formulating  $Z_N(\Phi)(X)$  in terms of Schur polynomials*

It is useful to recall now the result of Weyl which establishes that  $s_\lambda$  coincides with the character of the unitary group associated to the shape  $\lambda$  (this is contained in theorem 7.5.B of [23]). This allows us to apply to our  $s_\lambda$ 's a key fact about characters: the well known property of orthogonality. More precisely, if  $V$  and  $W$  are two unitary matrices of size  $N$ , this property reads, for any shape  $\lambda$ ,

$$\int s_\lambda(UVU^*W)dm_N(U) = \frac{1}{d_\lambda}s_\lambda(V)s_\lambda(W), \quad (11)$$

where  $dm_N$  is the Haar measure on the unitary group  $\mathcal{U}_N$  normalized to have mass one and  $d_\lambda = s_\lambda(1, 1, \dots, 1)$ . Its explicit form is

$$d_\lambda = \frac{\Delta(l)}{\prod_{i=1}^{N-1} i!}, \quad (12)$$

with  $l = \text{diag}(l_1, \dots, l_N)$  where we recall that  $l_i = \lambda_i + N - i$ .

A proof of formula (11) can be easily deduced from proposition II.4.2 of [2] (see also exercise 3 p.84 therein) whereas the explicit expression of  $d_\lambda$  given in (12) appears in [23].

As a consequence, with the notations introduced above,

$$\int s_\lambda(U\Phi(M)U^*A_N)dm_N(U) = \frac{1}{d_\lambda}s_\lambda(\Phi(M))s_\lambda(A_N). \quad (13)$$

Combining equations (10) and (13), we can rewrite our partition function

$$Z_N(\Phi)(X) = c'_N \sum_\lambda \frac{1}{d_\lambda} s_\lambda(A_N) s_\lambda(B_N) \int_{\hat{\mu}_M^N \in X} s_\lambda(\Phi(M)) e^{-\frac{N}{2} \text{tr} M^2} \Delta(M)^2 \prod_{i=1}^N dM_i, \quad (14)$$

where  $\prod_{i=1}^N dM_i$  is the product Lebesgue measure on  $\mathbb{R}^N$  and  $c'_N$  some normalizing constant, only depending on  $N$ .

3. *Relation between Schur polynomials and spherical integrals*

We can now recall the following determinantal formula for  $s_\lambda$ , that can be found for example in corollary 4.6.2 of [21]:

$$s_\lambda(\mathbf{x}) = \frac{\det(x_i^{l_j})_{i,j}}{\Delta(\mathbf{x})}, \quad (15)$$

where  $\Delta$  is the VanderMonde determinant,  $\mathbf{x} = (x_i)_{1 \leq i \leq N}$  and  $l$  is the tableau associated to  $\lambda$  (that is to say  $l_j = \lambda_j + N - j$  for  $1 \leq j \leq N$ ).

We then use a formula due to Harish-Chandra (see [18]): if  $C_N$  and  $D_N$  are two  $N \times N$  matrices whose eigenvalues  $C_N(i)$  and  $D_N(j)$  are distinct, we have that

$$I_N(C_N, D_N) = \frac{\det(\exp NC_N(i)D_N(j))_{i,j}}{\Delta(C_N)\Delta(D_N)}. \quad (16)$$



This last equation together with the determinantal formula (15) allows us to rewrite for any  $M \in \mathcal{H}_N(\mathbb{C})$  with non negative distinct eigenvalues:

$$s_\lambda(M) = I_N \left( \log M, \frac{l}{N} \right) \Delta \left( \frac{l}{N} \right) \frac{\Delta(\log M)}{\Delta(M)}, \quad (17)$$

Note that under the measure  $e^{-\frac{N}{2} \text{tr} M^2} dM$ , the eigenvalues of the matrix  $M$  are almost surely distinct, and therefore so are the eigenvalues of the two matrices  $\Phi(M)$  and  $\log \Phi(M)$  by hypothesis 2.1.3. Note however that (17) extends readily to any non negative matrix by extending by continuity the definition

$$\frac{\Delta(\log M)}{\Delta(M)} = e^{\sum_{i < j} s(\lambda_i, \lambda_j)},$$

with  $s$  as defined in Theorem 1.2.

From (17), we conclude that there exists a constant  $c_N$  depending only on  $N$  such that,

$$\begin{aligned} Z_N(\Phi)(X) &= c_N \sum_{\lambda} s_\lambda(A_N) s_\lambda(B_N) \\ &\times \int_{\hat{\mu}_M^N \in X} I_N \left( \log \Phi(M), \frac{l}{N} \right) \frac{\Delta(\log \Phi(M))}{\Delta(\Phi(M))} \Delta(M)^2 e^{-\frac{N}{2} \sum_{i=1}^N M_i^2} \prod_{i=1}^N dM_i, \end{aligned}$$

which completes the proof of Theorem 2.2 except from formula (7) which is easily obtained by dividing the  $\Phi$  by its norm before beginning the expansion. ■

**Remark 2.3** *If we denote by*

$$\text{vol}(\mathcal{U}_N) := \frac{\int e^{-\frac{N}{2} \text{tr} M^2} dM}{\int e^{-\frac{N}{2} \text{tr} M^2} \Delta(M)^2 \prod_{i=1}^N dM_i},$$

*we can easily deduce from equations (12), (14) and (17) above, that our normalizing constant  $c_N$  is given by*

$$c_N = \text{vol}(\mathcal{U}_N) \left( \frac{\prod_{i=1}^{N-1} i!}{N^{\frac{N(N-1)}{2}}} \right).$$

### 3 Large deviations estimates for the empirical distribution of Young tableaux following the law $\Pi^N$

The object of this section is to prove Theorem 1.2.

From the definition (3) and following (17), we get that  $\Pi^N$  is the positive measure given, for any measurable subset  $M$  of  $\mathcal{P}(\mathbb{R}^+)$ , by :

$$\begin{aligned} \Pi^N(M) &= e^{\frac{a}{2} N^2 S_N(\hat{\mu}_A^N) + \frac{b}{2} N^2 S_N(\hat{\mu}_B^N)} \\ &\times \sum_{\lambda: \hat{\mu}_\lambda^N \in M} \Delta \left( \frac{l}{N} \right)^{a+b} I_N \left( \log A_N, \frac{l}{N} \right)^a I_N \left( \log B_N, \frac{l}{N} \right)^b e^{N^2 F(\hat{\mu}_\lambda^N) - N^2 \int c(x) d\hat{\mu}_\lambda^N(x)} \end{aligned}$$

where

$$e^{\frac{N^2}{2} S_N(\hat{\mu}_A^N)} := \frac{\Delta(\log(A_N))}{\Delta(A_N)}.$$

Let us denote

$$\tilde{\Pi}^N(M) = \sum_{\lambda} 1_{\hat{\mu}_\lambda^N \in M} \Delta \left( \frac{l}{N} \right)^{a+b} I_N \left( \log A_N, \frac{l}{N} \right)^a I_N \left( \log B_N, \frac{l}{N} \right)^b e^{N^2 F(\hat{\mu}_\lambda^N) - N^2 \int c(x) d\hat{\mu}_\lambda^N(x)}.$$

We shall prove in this section

**Theorem 3.1** *Let  $(F, c, (A_N, B_N), a, b)$  be as in Theorem 1.2.*

*Then  $(\tilde{\Pi}^N)_{N \geq 0}$  satisfies large deviation bounds with rate function  $\tilde{H}$  which is infinite on  $\mathcal{L}^c$  and otherwise given by*

$$\tilde{H}(\nu) = \int c(x) d\nu(x) - \frac{a+b}{2} \Sigma(\nu) - F(\nu) - aI(\log \sharp \mu_A, \nu) - bI(\log \sharp \mu_B, \nu).$$

More precisely,

1.  $\{\nu \in \mathcal{P}(\mathbb{R}^+) : \tilde{H}(\nu) \leq M\}$  is compact for all  $M < \infty$ .
2. For any closed set  $F \in \mathcal{P}(\mathbb{R}^+)$ ,

$$\limsup_{N \rightarrow \infty} \frac{1}{N^2} \log \tilde{\Pi}^N(F) \leq -\inf\{\tilde{H}(\nu), \nu \in F\}$$

3. For any open set  $O \in \mathcal{P}(\mathbb{R}^+)$ ,

$$\liminf_{N \rightarrow \infty} \frac{1}{N^2} \log \tilde{\Pi}^N(O) \geq -\inf\{\tilde{H}(\nu), \nu \in O\}$$

Theorem 1.2 is easily deduced from Theorem 3.1 since

$$S_N(\hat{\mu}_A^N) = \frac{2}{N^2} \sum_{i < j} s(A_i, A_j). \quad (18)$$

Hence, since  $s$  is a bounded continuous function on  $[\epsilon, 1]^2$ , we deduce (see Lemma 7.3.12 in [4]) that, as  $\hat{\mu}_A^N$  converges to  $\mu_A$ ,

$$\lim_{N \rightarrow \infty} S_N(\hat{\mu}_A^N) = S(\mu_A)$$

and similarly for  $B_N$ .

The proof of Theorem 3.1 is heuristically simple since it amounts to perform a Laplace method and notice that the uniform measure on Young shape will not produce any entropy on the scale  $N^2$ . On a rigorous ground, it becomes a bit technical, for mainly the two following reasons :

- The law of  $\hat{\mu}_\lambda^N$  is discrete so that the arguments developed in [1] to obtain large deviation principles in similar scales and potentials have to be adapted. In particular, the discrete nature of the Young tableaux implies that  $\tilde{H}$  is infinite on  $\mathcal{L}^c$ .

- More cumbersome is the fact that the natural space where the empirical measure of the Young tableaux lives is  $\mathcal{P}_1(\mathbb{R}^+) := \{\nu \in \mathcal{P}(\mathbb{R}^+) : \int x d\nu(x) < \infty\}$ . Hence, all the limiting spherical integrals appearing are of the type  $I(\mu, \nu)$  with  $\mu$  in the set  $\mathcal{P}_\infty(\mathbb{R})$  of compactly supported probability measures but  $\nu \in \mathcal{P}_1(\mathbb{R}^+)$ . Such limits were not proved to exist in [10] (where  $\nu(x^2) < \infty$  was assumed), the formula obtained in [10] is not valid, and continuity statements for  $I$  are lacking a priori.

The proof nevertheless follows the usual scheme :

1. In subsection 3.1 we study the rate function and prove that its level sets are compact.
2. In subsection 3.2 we show that the family of measures  $(\tilde{\Pi}^N)_{N \in \mathbb{N}}$  is exponentially tight. More precisely, if we let  $\mathcal{K}_L$  be the compact subset

$$\mathcal{K}_L = \left\{ \nu \in \mathcal{P}(\mathbb{R}^+) : \int x d\nu(x) \leq L \right\}$$

we prove that

$$\limsup_{L \rightarrow \infty} \limsup_{N \rightarrow \infty} \frac{1}{N^2} \log \tilde{\Pi}^N(\mathcal{K}_L^c) = -\infty.$$

3. In subsection 3.3 we prove the upper bound for arbitrarily small balls, i.e if  $d$  is a metric on  $\mathcal{P}(\mathbb{R})$  compatible with the weak topology such as the Dudley's metric  $d$  given by

$$d(\mu, \nu) = \sup \left| \int f d\mu - \int f d\nu \right|,$$

where the supremum is taken over all Lipschitz functions  $f$  with Lipschitz norm less than 1 (note that this distance is compatible with the weak topology), and if we set

$$B(\nu, \delta) = \{\mu \in \mathcal{P}(\mathbb{R}^+); d(\mu, \nu) < \delta\}$$

we show that for any  $\nu \in \cup_{L \in \mathbb{N}} \mathcal{K}_L$ ,

$$\limsup_{\delta \rightarrow \infty} \limsup_{N \rightarrow \infty} \frac{1}{N^2} \log \tilde{\Pi}^N(B(\nu, \delta)) \leq -\tilde{H}(\nu).$$

4. In subsection 3.4 we prove the lower bound for arbitrarily small balls, i.e that for any  $\nu \in \cup_{L \in \mathbb{N}} \mathcal{K}_L$ ,

$$\liminf_{\delta \rightarrow \infty} \liminf_{N \rightarrow \infty} \frac{1}{N^2} \log \tilde{\Pi}^N(B(\nu, \delta)) \geq -\tilde{H}(\nu).$$

By Theorem 4.1.11 in [4], the above results prove Theorem 3.1.

### 3.1 $\tilde{H}$ has compact level sets

To prove that  $\tilde{H}$  has compact level sets, we shall first define it properly, that is define appropriately the limit of the spherical integrals.

### 3.1.1 Definition and properties of I

Let us remind that it was proved in theorem 1.1 of [10] that

$$I(\mu_D, \mu_E) := \lim_{N \rightarrow \infty} \frac{1}{N^2} \log I_N(D_N, E_N) \quad (19)$$

exists for all sequences of diagonal matrices  $(D_N, E_N)_{N \in \mathbb{N}}$  with spectral measures converging towards  $\mu_D$  and  $\mu_E$  respectively and such that  $\sup_N \|D_N\|_N$  and  $\sup_N \hat{\mu}_E^N(x^2)$  are finite. A formula for  $I$  is given in [10] when either  $\Sigma(\mu_E)$  or  $\Sigma(\mu_D)$  are finite. If they are not, the limit still exists since spherical integrals are uniformly continuous (see Lemma 3.2.4) and the measures with finite  $\Sigma$  are dense, but its formula is far from being clear (see a discussion in [11]). However, let us remark that since the spherical integrals under considerations are always bounded, the rate function  $\tilde{H}(\nu)$  is infinite unless  $\nu$  has finite entropy  $\Sigma$  (see the end of section 3.1) so that we can always use the formula given in [10] when  $\int x^2 d\nu(x) < \infty$ .

Since  $\tilde{H}(\nu)$  is infinite if  $\int x d\nu(x) = +\infty$  (see section 3.1) and  $\mu_A$  and  $\mu_B$  are supposed to be supported on  $[\epsilon, 1]$ , it is enough to extend the definition of  $I(\mu, \nu)$  to compactly supported measures  $\mu$  with support in  $\mathbb{R}^-$  but  $\nu \in \mathcal{P}_1(\mathbb{R}^+)$ . We shall prove

**Lemma 3.2** *Let  $R \in \mathbb{R}^+$  and  $\mu$  be a probability measure on  $[-R, 0]$  and  $\nu \in \mathcal{P}_1(\mathbb{R}^+)$ . Then*

1. *Let  $\phi_M(x) = x \wedge M$ .  $I(\mu, \phi_M \# \nu)$  is well defined and decreases towards a limit*

$$I(\mu, \nu) := \lim_{M \rightarrow \infty} I(\mu, \phi_M \# \nu).$$

*Moreover, for any  $M \geq 0$ ,*

$$I(\mu, \phi_M \# \nu) - R\nu(x - \phi_M(x)) \leq I(\mu, \nu) \leq I(\mu, \phi_M \# \nu).$$

2. *Let  $\mathcal{P}^R(\mathbb{R}) = \{\mu \in \mathcal{P}(\mathbb{R}) : \mu([-R, R]^c) = 0\}$  and  $\mathcal{P}_q(\mathbb{R}^+) = \{\mu \in \mathcal{P}(\mathbb{R}^+) : \mu(|x|^q) \leq R\}$ . Then there exists a function  $\kappa(\delta, R)$  such that for any  $R < \infty$ ,  $\kappa(\delta, R)$  goes to zero as  $\delta$  goes to zero and for any  $(\mu, \mu') \in \mathcal{P}^R(\mathbb{R})$  any  $(\nu, \nu') \in \mathcal{P}_2(\mathbb{R})$ , such that  $d(\mu, \mu') + d(\nu, \nu') < \delta$ ,*

$$|I(\mu, \nu) - I(\mu', \nu')| \leq \kappa(\delta, R).$$

3. *For any  $\mu \in \mathcal{P}(\mathbb{R}^-)$  and  $\nu \in \mathcal{P}_1(\mathbb{R}^+)$ ,*

$$\mu(x)\nu(x) \leq I(\mu, \nu) \leq 0.$$

4. *For any sequence  $(D_N, E_N)$  of diagonal Hermitian matrices with  $D_N \leq 0$  and  $E_N \geq 0$ , for any  $M \in \mathbb{R}^+$ ,*

$$I_N(D_N, \phi_M(E_N)) e^{-N\|D_N\|_N \text{tr}(E_N - \phi_M(E_N))} \leq I_N(D_N, E_N) \leq I_N(D_N, \phi_M(E_N)). \quad (20)$$

*Moreover there exists a function  $g : [0, 1] \times \mathbb{R}^+ \mapsto \mathbb{R}^+$ , depending on the limiting measures  $\mu_E, \mu_D$  only, such that  $g(\delta, M)$  goes to zero as  $\delta$  does for any  $M \in \mathbb{R}^+$ , and so that*

$$\left| \frac{1}{N^2} \log \frac{I_N(\hat{D}_N, \phi_M(\hat{E}_N))}{I_N(D_N, \phi_M(E_N))} \right| \leq g(\delta, M). \quad (21)$$

for any  $N \in \mathbb{N}$  and any diagonal matrices  $(D_N, E_N, \hat{D}_N, \hat{E}_N)$  such that  $E_N, \hat{E}_N$  are non-negative and

$$d(\hat{\mu}_{D_N}^N, \hat{\mu}_{\hat{D}_N}^N) + d(\hat{\mu}_{E_N}^N, \hat{\mu}_{\hat{E}_N}^N) < \delta, \quad \hat{\mu}_{E_N}^N(x^2) + \hat{\mu}_{\hat{E}_N}^N(x^2) \leq M.$$

**Proof.**

- We first prove the last point. If we denote  $D_N = \text{diag}(d_1, \dots, d_N)$  and  $E_N = \text{diag}(e_1, \dots, e_N)$ ,

$$\begin{aligned} I_N(D_N, E_N) &= \int e^{N \text{tr}(D_N U E_N U^*)} dm_N(U) \\ &= \int e^{N \sum_{i,j=1}^N d_i e_j |u_{ij}|^2} dm_N(U) \\ &\leq \int e^{N \sum_{i,j=1}^N d_i \phi_M(e_j) |u_{ij}|^2} dm_N(U) \end{aligned}$$

where we used that  $d_i \leq 0$ . The opposite inequality of (20) is also trivial since

$$\begin{aligned} I_N(D_N, E_N) &\geq e^{N \|D_N\|_N \sum_{j=1}^N (e_j - \phi_M(e_j))} \int e^{N \sum_{i,j=1}^N d_i \phi_M(e_j) |u_{ij}|^2} dm_N(U) \\ &= e^{-N \|D_N\|_N \text{tr}(E_N - \phi_M(E_N))} I_N(D_N, \phi_M(E_N)) \end{aligned}$$

The continuity statement (21) is a direct consequence of Lemma 5.1 in [10] since  $\phi_M(E_N)$  is uniformly bounded by  $M$  and  $d(\phi_M \sharp \mu, \phi_M \sharp \mu') \leq d(\mu, \mu')$  for any  $\mu, \mu' \in \mathcal{P}(\mathbb{R})$ .

- We can now prove the first point. From (20), we deduce that for any  $M \in \mathbb{R}^+$ , any  $E_N \geq 0$  with spectral measure converging towards  $\mu_E$  and any sequence of bounded non-positive diagonal matrices  $D_N$  with spectral measure converging towards  $\mu_D$

$$\begin{aligned} \limsup_{N \rightarrow \infty} \frac{1}{N^2} \log I_N(D_N, E_N) &\leq \limsup_{N \rightarrow \infty} \frac{1}{N^2} \log I_N(D_N, \phi_M(E_N)) \\ &= I(\mu_D, \phi_M \sharp \mu_E), \end{aligned} \tag{22}$$

where the last equality comes from the observation that  $(\phi_M(D_N), E_N)$  are uniformly bounded by hypothesis so that the convergence holds by theorem 1.1 in [10]. With  $\mu_E = \phi_L \sharp \nu$  for some  $L \geq M$  and  $E_N$  chosen so that  $\hat{\mu}_{E_N}^N(|x| > L) = 0$ , the left hand side of (22) converges towards  $I(\mu_D, \phi_L \sharp \nu)$  showing that  $M \rightarrow I(\mu_D, \phi_M \sharp \mu_E)$  is non-increasing. Hence, it converges towards some limit (maybe infinite at this stage). Now, we choose a special sequence  $(E_N)_{N \in \mathbb{N}}$  such that

$$\lim_{N \rightarrow \infty} \frac{1}{N} \text{tr}(E_N - \phi_M(E_N)) = \mu_E(x - \phi_M(x)).$$

We can construct it as follows; assume first that  $\mu_E$  has no atoms and set

$$\begin{aligned} E_{1,N} &= \inf \left\{ x / \mu_E((-\infty, x]) \geq \frac{1}{N+1} \right\} \\ E_{i+1,N} &= \inf \left\{ x \geq E_{i,N} / \mu_E((E_{i,N}, x]) \geq \frac{1}{N+1} \right\}. \end{aligned}$$

Then it is not hard to see that  $\hat{\mu}_{E_N}^N = \frac{1}{N} \sum_{i=1}^N \delta_{E_{i,N}}$  converges towards  $\mu_E$ . Moreover,

$$\begin{aligned} \hat{\mu}_{E_N}^N(x - \phi_M(x)) &= \frac{1}{N} \sum_{E_{i,N} \geq M} (E_{i,N} - M) \\ &\leq \frac{N+1}{N} \sum_{E_{i,N} \geq M} (E_{i,N} - M) \mu_E([E_{i,N}, E_{i+1,N}]) \leq \frac{N+1}{N} \mu_E((x - M)1_{x \geq M}). \end{aligned}$$

If  $\mu_E$  has atoms, we consider a finite collection of atoms  $\{a_1, \dots, a_K\}$  such that each of the remaining atoms has mass smaller than  $(N+1)^{-1}$ . Then,  $E_N$  has  $\lfloor N \mu_E(\{a_i\}) \rfloor$  eigenvalues equal to  $a_i$  for  $1 \leq i \leq K$ . The remaining eigenvalues are chosen as above.

Inequality (20) yields with this choice

$$\frac{1}{N^2} \log I_N(D_N, E_N) \geq I_N(D_N, \phi_M(E_N)) e^{-N(N+1) \sup_N \|D_N\|_N \mu_E((x-M)1_{x \geq M})}$$

and therefore

$$\liminf_{N \rightarrow \infty} \frac{1}{N^2} \log I_N(D_N, E_N) \geq -\sup_N \|D_N\|_N \mu_E((x-M)1_{x \geq M}) + I(\mu_D, \phi_M \sharp \mu_E) \quad (23)$$

(22) and (23) shows that for such a sequence

$$\begin{aligned} -\sup_N \|D_N\|_N \mu_E((x-M)1_{x \geq M}) + I(\mu_D, \phi_M \sharp \mu_E) &\leq \liminf_{N \rightarrow \infty} \frac{1}{N^2} \log I_N(D_N, E_N) \\ &\leq \limsup_{N \rightarrow \infty} \frac{1}{N^2} \log I_N(D_N, E_N) \\ &\leq I(\mu_D, \mu_E) \end{aligned} \quad (24)$$

This completes the proof of the first point.

• The second point is a direct consequence of the fourth too. Indeed, let  $(\mu, \mu', \nu, \nu')$  be such that

$$d(\mu, \mu') + d(\nu, \nu') < \delta.$$

Then, we choose a sequence  $(D_N, E_N)$  (resp.  $(\hat{D}_N, \hat{E}_N)$ ) of matrices with spectral measure converging towards  $(\mu, \nu)$  (resp.  $(\mu', \nu')$ ) such that

$$\max\{d(\hat{\mu}_{D_N}^N, \mu), d(\hat{\mu}_{\hat{D}_N}^N, \mu'), d(\hat{\mu}_{E_N}^N, \nu), d(\hat{\mu}_{\hat{E}_N}^N, \nu')\} < \delta$$

which implies

$$d(\hat{\mu}_{D_N}^N, \hat{\mu}_{\hat{D}_N}^N) < 2\delta, \quad d(\hat{\mu}_{E_N}^N, \hat{\mu}_{\hat{E}_N}^N) < 2\delta$$

so that 4. implies, by taking the limit as  $N$  goes to infinity (here  $M = R$ ), that

$$|I(\mu, \nu) - I(\mu', \nu')| \leq g(2\delta, R).$$

• In point 3., the upper bound on  $I$  is trivial and the lower bound comes from Jensen's inequality which yields

$$\begin{aligned}
I_N(D_N, E_N) &= \int e^{N \sum_{i,j=1}^N e_i d_j |u_{ij}|^2} dm_N(U) \\
&\geq e^{N \sum_{i,j=1}^N e_i d_j} \int |u_{ij}|^2 dm_N(U) \\
&= e^{\sum_{i,j=1}^N e_i d_j} = e^{N^2 \hat{\mu}_{E_N}^N(x) \hat{\mu}_{D_N}^N(x)}
\end{aligned}$$

The result is then obtained by letting  $N$  going to infinity. ■

### 3.1.2 $\tilde{H}$ has compact level sets

In this section, we prove Theorem 3.1.1 by proving first that  $\tilde{H}$  is lower semi-continuous and then that its level sets are compact.

•  $\tilde{H}$  is lower semi-continuous, i.e  $\{\nu \in \mathcal{P}(\mathbb{R}^+) : \tilde{H}(\nu) \leq M\}$  is closed for any  $M \in \mathbb{R}^+$ . We recall that  $\mathcal{L}$  is the set of probability measures which are absolutely continuous with respect to Lebesgue measure and with density bounded by one and note that  $\{\tilde{H} \leq M\} = \mathcal{L} \cap \{\tilde{H}^1 \leq M\}$  where  $\tilde{H}^1(\nu)$  is given by the same formula than  $\tilde{H}(\nu)$  even for  $\nu \in \mathcal{L}^c$ . We first check that  $\mathcal{L}$  is closed and then show that  $\tilde{H}^1$  is lower semi-continuous, these two points proving that  $\{\tilde{H} \leq M\}$  is closed.

To show that  $\mathcal{L}$  is closed, take a sequence  $(\nu_n)_{n \in \mathbb{N}}$  of measures in  $\mathcal{L}$  converging weakly to a measure  $\nu$ . For any  $c$  and  $d$ , the function  $\mathbf{1}_{[c,d]}$  is upper semi-continuous so that

$$|d - c| \geq \limsup_{n \rightarrow \infty} \nu_n([c, d]) \geq \nu([c, d]).$$

so that  $\nu$  is in  $\mathcal{L}$ .

We now show that  $\tilde{H}^1$  is a supremum of continuous functions which we define as follows: we let, with  $\phi_M(x) = x \wedge M$  for  $M \geq 0$  as in Lemma 3.2, and for  $\nu \in \mathcal{P}(\mathbb{R}^+)$ ,

$$\tilde{H}^M(\nu) := -aI(\log \sharp \mu_A, \phi_M \sharp \nu) - bI(\log \sharp \mu_B, \phi_M \sharp \nu) + \iint g(x, y) \wedge M d\nu(x) d\nu(y) - F(\nu)$$

with

$$g(x, y) = \left( \frac{a+b}{2} \right) \log |x - y|^{-1} + \frac{1}{2}c(x) + \frac{1}{2}c(y) \quad (25)$$

We claim that for any finite  $M$ ,  $\tilde{H}^M$  is continuous on  $\mathcal{P}(\mathbb{R}^+)$ . Indeed, by Lemma 3.2.2, for  $C = A$  or  $B$ ,  $\nu \in \mathcal{P}(\mathbb{R}^+) \mapsto I(\log \sharp \mu_C, \phi_M \sharp \nu) \in \mathbb{R}$  is continuous since  $\log \sharp \mu_C$  is compactly supported by hypothesis 2.1.1. Moreover, it is not hard to check that  $g$  is bounded below and continuous except when on the diagonal  $\{x = y\}$  where it goes to infinity. Consequently,  $g \wedge M$  is a bounded continuous function on  $\mathbb{R}^2$ . Thus  $\mu \mapsto \iint g(x, y) \wedge M d\mu(x) d\mu(y)$  is bounded continuous.

This last argument finishes to prove that  $\tilde{H}^M$  is a continuous function on  $\mathcal{P}(\mathbb{R}^+)$ . To deduce that  $\tilde{H}^1$  is lower semi-continuous, it is therefore enough to prove that

$$\tilde{H}^1(\nu) = \sup_{M \geq 0} \{\tilde{H}^M(\nu)\}. \quad (26)$$

But this is straightforward since monotone convergence theorem asserts that for any  $f$  bounded below

$$\lim_{M \uparrow \infty} \iint f(x, y) \wedge M d\mu(x) d\mu(y) = \iint f(x, y) d\mu(x) d\mu(y)$$

and by Lemma 3.2.1,  $I(\mu, \phi_M \sharp \nu)$  decreases towards its limit  $I(\mu, \nu)$ .

• As a consequence of the last point, for any  $M \geq 0$ ,  $\{\nu \in \mathcal{P}(\mathbb{R}^+) : \tilde{H}(\nu) \leq M\}$  is closed. We now check that it is compact by showing that it is contained in a compact set. In fact, by Lemma 3.2.3,

$$\tilde{H}(\nu) \geq \iint g(x, y) d\nu(x) d\nu(y) - \sup_{\nu \in \mathcal{P}(\mathbb{R}^+)} F(\nu) \quad (27)$$

and it is not hard to check that, since we assumed  $\liminf x^{-1}c(x) > 0$ , there exists a finite constant  $C$  and  $\rho > 0$  such that for any  $(x, y) \in (\mathbb{R}^+)^2$

$$g(x, y) \geq \frac{\rho}{2}x + \frac{\rho}{2}y + C \quad (28)$$

yielding with (27) that for any  $M \in \mathbb{R}^+$ , if  $C' = C - \sup_{\nu \in \mathcal{P}(\mathbb{R}^+)} F(\nu)$ ,

$$\{\nu \in \mathcal{P}(\mathbb{R}^+) : \tilde{H}(\nu) \leq M\} \subset \left\{ \nu \in \mathcal{P}(\mathbb{R}^+) : \int x d\nu(x) \leq \frac{2}{\rho}(M - C') \right\} := \mathcal{K}_{M, \rho}.$$

Since  $\mathcal{K}_{M, \rho}$  is a compact subset of  $\mathcal{P}(\mathbb{R}^+)$ , the proof is completed.

Note that since  $\iint g(x, y) d\nu(x) d\nu(y) = \int c(x) d\nu(x) - \Sigma(\nu)$  and  $c$  is bounded below, we also see from (27) that  $\tilde{H}(\nu) < \infty$  implies  $|\Sigma(\nu)| < \infty$ . ■

### 3.2 $\tilde{\Pi}^N$ is exponentially tight

The goal of this section is to prove that

**Lemma 3.3**  *$\tilde{\Pi}^N$  is exponentially tight, and more precisely if we set*

$$\mathcal{K}_L := \left\{ \nu \in \mathcal{P}(\mathbb{R}^+) : \int x d\nu(x) \leq L \right\},$$

then

$$\limsup_{L \rightarrow \infty} \limsup_{N \rightarrow \infty} \frac{1}{N^2} \log \tilde{\Pi}^N(\mathcal{K}_L^c) = -\infty.$$

**Proof.** Since the spherical integrals under consideration are uniformly bounded above by one and  $F$  is uniformly bounded by a constant  $\|F\|_\infty$ ,

$$\tilde{\Pi}^N(X) \leq e^{N^2 \|F\|_\infty} \sum_{\lambda: \hat{\mu}_\lambda^N \in X} e^{-N^2 \int_{x \neq y} g(x, y) d\hat{\mu}_\lambda^N(x) d\hat{\mu}_\lambda^N(y)},$$

Choosing  $X = \mathcal{K}_L^c$ , we get by (28) that

$$\tilde{\Pi}^N(\mathcal{K}_L^c) \leq e^{N^2 \|F\|_\infty + N^2 C} \sum_{\lambda} 1_{\hat{\mu}_\lambda^N \in \mathcal{K}_L^c} e^{-N^2 \rho \int x d\hat{\mu}_\lambda^N(x)} \Delta \left( \frac{l}{N} \right)^{a+b} \quad (29)$$



It remains to consider the sums over Young shapes. Let us recall that

$$\hat{\mu}_\lambda^N(x) = \frac{1}{N^2} \sum_{i=1}^N l_i = \frac{1}{N} \sum_{i=1}^N \left( \frac{\lambda_i}{N} - \frac{i}{N} \right) + 1 \leq N^{-2} |\lambda|_N + 1$$

where  $|\lambda|_N = \sum_{i \leq N} \lambda_i$ . Therefore, for any  $L \geq 0$ ,

$$\begin{aligned} \sum_{\lambda: \hat{\mu}_\lambda^N(x) \geq L} e^{-\rho|\lambda|} \Delta \left( \frac{l}{N} \right)^{a+b} &\leq \sum_{\lambda: |\lambda|_N \geq N^2(L-1)} e^{-\rho|\lambda|} \Delta \left( \frac{l}{N} \right)^{a+b} \\ &\leq e^{-\frac{1}{2}\rho N^2(L-1)} \sum_{\lambda: |\lambda|_N \geq N^2(L-1)} e^{-\frac{1}{2}\rho|\lambda|} \Delta \left( \frac{l}{N} \right)^{a+b}. \end{aligned}$$

For any  $j$ ,

$$\prod_{j < i} \left| \frac{l_i}{N} - \frac{l_j}{N} \right| \leq \left( \frac{l_j}{N} \right)^{N-j},$$

therefore, for any shape,

$$\Delta \left( \frac{l}{N} \right)^{a+b} e^{-\frac{1}{2}\rho|\lambda|} \leq e^{(a+b) \sum_j (N-j) \log \frac{l_j}{N} - \frac{1}{4} N \rho \frac{l_j}{N}} \leq e^{N^2 C''},$$

where  $C'' = \sup_x \left\{ (a+b) \log x - \frac{1}{4} \rho x \right\} - \frac{1}{8}$ .

Now the number of Young shapes  $\lambda$  such that  $|\lambda|_N = m$  is bounded by  $C_m^N$  so that we conclude

$$\begin{aligned} \sum_{\lambda: |\lambda|_N \geq N^2(L-1)} e^{-\rho|\lambda|} \Delta \left( \frac{l}{N} \right)^{a+b} &\leq e^{N^2 C} e^{-\frac{1}{2}\rho N^2(L-1)} \frac{1}{N!} \sum_{m \geq N^2(L-1)} m(m-1) \cdots (m-N+1) e^{-\frac{1}{4}\rho m}, \\ &\leq e^{N^2 C} e^{-\frac{1}{2}\rho N^2(L-1)} \frac{1}{N!} \sum_{m \geq N^2(L-1)} e^{N \log m} e^{-\rho m} \\ &\leq e^{-\frac{1}{2}(\rho-\delta)N^2(L-1)} \end{aligned} \tag{30}$$

where in the last line  $\delta$  is any positive number and the inequality holds as soon as  $N$  and  $L$  are big enough. (29) and (30) give Lemma 3.3. ■

### 3.3 $(\tilde{\Pi}^N)_{N \geq 0}$ satisfies a weak large deviation upper bound

In this section, we shall prove the following

**Lemma 3.4**  $\tilde{\Pi}^N$  satisfies a weak large deviation upper bound in the scale  $N^2$  with rate function  $\tilde{H}$  i.e for any  $\nu \in \mathcal{P}(\mathbb{R}^+)$ ,

$$\limsup_{\delta \rightarrow 0} \limsup_{N \rightarrow \infty} \frac{1}{N^2} \log \tilde{\Pi}^N(B(\nu, \delta)) \leq -\tilde{H}(\nu).$$

**Proof.** We first prove that for any  $\epsilon > 0$ , if  $\nu$  is such that there exists two positive real numbers  $\alpha$  and  $\beta$  ( $\alpha < \beta$ ) such that  $\nu([\alpha, \beta]) \geq (1 + \epsilon)(\beta - \alpha)$ , then,

$$\limsup_{\delta \rightarrow 0} \limsup_{N \rightarrow \infty} \frac{1}{N^2} \log \tilde{\Pi}^N(B(\nu, \delta)) = -\infty. \quad (31)$$

The main remark is that, for any shape  $\lambda$ , as the  $l_i$  are (strictly) decreasing we have that, for any  $c < d$ ,

$$\begin{aligned} \hat{\mu}_\lambda^N([c, d]) &= \frac{1}{N} \#\left\{i : \frac{l_i}{N} \in [c, d]\right\} \leq \frac{1}{N}(\lfloor N(d - c) \rfloor + 1) \\ &\leq \left(1 + \frac{\epsilon}{2}\right)(d - c), \end{aligned} \quad (32)$$

where the last inequality holds for  $N$  large enough.

Let be  $\eta > 0$  and consider the function  $f : \mathbb{R} \rightarrow \mathbb{R}$  such that

$$f(x) = \begin{cases} 0, & \text{if } x < \alpha - \eta \text{ or } x > \beta + \eta, \\ \frac{1}{2}(x - \alpha - \eta), & \text{if } \alpha - \eta \leq x \leq \alpha, \\ \eta, & \text{if } \alpha < x < \beta, \\ \frac{1}{2}(-x + \beta + \eta), & \text{if } \beta \leq x \leq \beta + \eta. \end{cases}$$

Note that, for  $\eta$  small enough, the Lipschitz norm of  $f$  is bounded by 1.

And we have, for any shape  $\lambda$ ,

$$\int f d\nu - \int f d\hat{\mu}_\lambda^N = \int_{\alpha - \eta}^{\alpha} f(d\nu - d\hat{\mu}_\lambda^N) + \int_{\beta}^{\beta + \eta} f(d\nu - d\hat{\mu}_\lambda^N) + \int_{\alpha}^{\beta} f(d\nu - d\hat{\mu}_\lambda^N).$$

Using (32) twice, we get that, for any shape  $\lambda$  and  $N$  large enough,

$$\int_{\alpha - \eta}^{\alpha} f d\hat{\mu}_\lambda^N \leq \frac{\eta^2}{2},$$

(and the same thing for  $\beta$ ) and that

$$\int_{\alpha}^{\beta} f d\nu - \int_{\alpha}^{\beta} f d\hat{\mu}_\lambda^N \geq \eta \frac{\epsilon}{2}(\beta - \alpha),$$

so that, if we choose  $\eta = \frac{\epsilon}{4}(\beta - \alpha)$ , we get that

$$\int f d\nu - \int f d\hat{\mu}_\lambda^N \geq \left[\frac{\epsilon}{4}(\beta - \alpha)\right]^2.$$

And we conclude that, if we take  $\delta < [\epsilon 4(\beta - \alpha)]^2$ , the set  $\{\lambda : d(\hat{\mu}_\lambda^N, \nu) < \delta\}$  is empty, which gives (31).

On the other side, by lemma 3.2.4, for any  $M \in \mathbb{R}^+$ ,

$$\tilde{\Pi}^N(B(\nu, \delta)) \leq \sum_{\lambda : d(\hat{\mu}_\lambda^N, \nu) < \delta} I\left(A_N, \phi_M\left(\frac{l}{N}\right)\right)^a I\left(B_N, \phi_M\left(\frac{l}{N}\right)\right)^b \Delta\left(\frac{l}{N}\right)^{a+b} e^{-N^2 \int c(x) d\hat{\mu}_\lambda^N(x) + N^2 F(\hat{\mu}_\lambda^N)}$$

Observe that with  $g$  defined in (25), since  $|\lambda| = \sum \lambda_j = \sum l_j - \sum(N - j) = \sum l_j - 2^{-1}N(N - 1)$ ,

$$\Delta \left( \frac{l}{N} \right)^{a+b} e^{-N^2 \int c(x) d\hat{\mu}_\lambda^N(x)} = e^{-N^2 \int_{y' \neq y} g(y', y) d\hat{\mu}_\lambda^N(y) d\hat{\mu}_\lambda^N(y') - N \int c(x) d\hat{\mu}_\lambda^N(x)},$$

we obtain

$$\begin{aligned} \tilde{\Pi}^N(B(\nu, \delta)) &\leq \sum_{\lambda: d(\hat{\mu}_\lambda^N, \nu) < \delta} I \left( A_N, \phi_M \left( \frac{l}{N} \right) \right)^a I \left( B_N, \phi_M \left( \frac{l}{N} \right) \right)^b \\ &\quad \times e^{-N^2 \int_{y' \neq y} g(y', y) \wedge M d\hat{\mu}_\lambda^N(y) d\hat{\mu}_\lambda^N(y') + N^2 F(\hat{\mu}_\lambda^N) - N \int c(x) d\hat{\mu}_\lambda^N(x)} \\ &\leq e^{NM} \sum_{\lambda: d(\hat{\mu}_\lambda^N, \nu) < \delta} I \left( A_N, \phi_M \left( \frac{l}{N} \right) \right)^a I \left( B_N, \phi_M \left( \frac{l}{N} \right) \right)^b \\ &\quad \times e^{-N^2 \int g(y', y) \wedge M d\hat{\mu}_\lambda^N(y) d\hat{\mu}_\lambda^N(y') + N^2 F(\hat{\mu}_\lambda^N) - N \int c(x) d\hat{\mu}_\lambda^N(x)} \end{aligned} \quad (33)$$

Now, following section 3.1.2, we know that all the functions appearing above are continuous for any finite  $M$  so that for each such  $M$  we find a  $\kappa(\delta, M)$  going to zero as  $\delta$  goes to zero so that

$$\tilde{\Pi}^N(B(\nu, \delta)) \leq e^{-N^2(\tilde{H}^M(\nu) + \kappa(\delta, M))} e^{N(M+C)} \sum_{\lambda: d(\hat{\mu}_\lambda^N, \nu) < \delta} e^{-N \rho \int y d\hat{\mu}_\lambda^N(y)} \quad (34)$$

where we used again (28). We now show that the last entropy term will not contribute in the scale  $N^2$ . We have indeed,

**Lemma 3.5**

$$\frac{1}{N^2} \log \#\{\lambda / d(\hat{\mu}_\lambda^N, \nu) < \delta\} \rightarrow_{N \rightarrow \infty} 0,$$

By (34), and lemma 3.5 we conclude that, for all  $M \geq 0$ ,

$$\limsup_{N \rightarrow \infty} \frac{1}{N^2} \log \tilde{\Pi}^N(B(\nu, \delta)) \leq -\tilde{H}^M(\nu) + \kappa(\delta, M).$$

Letting  $\delta$  going to zero and then  $M$  going to infinity (since we saw in section 3.1.2 that  $\tilde{H}^M$  converges towards  $\tilde{H}$ ) finishes the proof. ■

We now go back to the **proof of lemma 3.5**:

We first show a lower bound for the number of tableaux  $\lambda$  whose empirical measure is such that, for a given  $\epsilon > 0$  and a given  $\nu \in \mathcal{P}(\mathbb{R}^+)$ ,  $d(\frac{1}{N} \sum_{j=1}^N \delta_{l_j}, \nu) < \epsilon$ .

As this number is an integer, we just need to show that this set is non-empty. This is true thanks to two facts: first the set  $\{\nu\}$  is tight so that we choose a convex compact  $K$  such that  $\nu(K) \geq 1 - \frac{\epsilon}{3}$  and then the set  $\mathcal{P}(K)$  of all probability measures on  $K$  endowed with the weak topology is a compact in the locally convex space of measures with mass less than 1, so that the Krein-Milman theorem tells us that  $\mathcal{P}(K)$  is the closure of the convex envelope of its extremal points, which are the Dirac measures. We have the approximation announced above: for  $\epsilon > 0$ , there exists an integer

$N(\epsilon)$  and some real number that we order  $a_{1,N(\epsilon)} > a_{2,N(\epsilon)} > \dots$  such that  $d(\frac{1}{N} \sum_{j=1}^N \delta_{a_{j,N}}, \nu) < \frac{\epsilon}{2}$ . Then for each  $j$  between 1 and  $N$ , we choose for  $l_j$  the integer for which  $\frac{l_j}{N}$  is the closest from  $a_{j,N}$ . This gives us that, for  $N$  large enough

$$\# \left\{ \lambda/d \left( \frac{1}{N} \sum_{j=1}^N \delta_{\frac{l_j}{N}}, \nu \right) < \epsilon \right\} \geq 1.$$

For the upper bound, we first find a compactly supported measure  $\nu'$  (with support  $K = [0, M]$ ) such that  $d(\nu, \nu') < \frac{\epsilon}{2}$ . This gives us that

$$\{\lambda/d(\hat{\mu}_\lambda^N, \nu) < \epsilon\} \subset \left\{ \lambda/d(\hat{\mu}_\lambda^N, \nu') < 3\frac{\epsilon}{2} \right\}.$$

Let us consider the function  $f_2$  given by

$$f_2(x) = \begin{cases} 0, & \text{if } x \leq M \\ x - M, & \text{if } M \leq x \leq M + 2N\epsilon \\ 2N\epsilon & \text{if } x \geq M + 2N\epsilon. \end{cases}$$

$f_2$  is a bounded Lipschitz function whose Lipschitz norm is bounded by 1 and such that  $\int f_2 d\nu' = 0$ . But, if there exists an  $l_j$  greater or equal  $2N^2\epsilon + NM$  then  $\frac{1}{N} \sum_{i=1}^N f_2\left(\frac{l_i}{N}\right) \geq 2\epsilon \geq 3\frac{\epsilon}{2}$ , so that we have the inclusion

$$\{\lambda/d(\hat{\mu}_\lambda^N, \nu) < \epsilon\} \subset \{\lambda/\forall j, l_j \leq 2N^2\epsilon + NM\}$$

and we get the upper bound as we know that

$$\#\{\lambda/\forall j, l_j \leq 2N^2\epsilon + NM\} \leq (2N^2\epsilon + NM)^N.$$

Upper and lower bound together give the result announced in lemma 3.5. ■

### 3.4 $(\tilde{\Pi}^N)_{N \geq 0}$ satisfies a large deviation lower bound

In this part we show that

**Lemma 3.6**  $\tilde{\Pi}^N$  satisfies a large deviation lower bound, i.e for any  $\nu \in \mathcal{P}(\mathbb{R}^+)$ ,

$$\liminf_{\delta \rightarrow 0} \liminf_{N \rightarrow \infty} \frac{1}{N^2} \log \tilde{\Pi}^N(B(\nu, \delta)) \geq -\tilde{H}(\nu).$$

**Proof.** To prove this lower bound, we follow [1] and consider discrete approximations of the probability measures  $\nu \in \{\tilde{H} < \infty\}$  as follows. First note that  $\tilde{H} < \infty$  implies that for any  $\alpha < \beta$ ,  $\nu([\alpha, \beta]) \leq (\beta - \alpha)$ .

Recall that we saw at the end of Lemma 3.3 that  $\tilde{H}(\nu) \leq M$  implies that for some universal constant  $C$  and  $\rho > 0$ ,

$$\rho \int x d\nu(x) \leq M + C \text{ and } \Sigma(\nu) > -\infty. \quad (35)$$

The last condition in particular implies that  $\nu$  have no atoms. We now construct the following approximations.

If  $\nu^L = \phi_L \sharp \nu$ , by Chebychev inequality,

$$d(\nu, \nu^L) \leq \int_{x>L} d\nu \leq \rho^{-1} L^{-1} (M + C),$$

and if  $\nu$  is in  $\mathcal{L}$ , so is  $\nu^L$ .

We then consider

$$\begin{aligned} a_{N,N} &= \inf \left\{ x / \nu^L([0, x]) \geq \frac{1}{N} \right\} \\ a_{i-1,N} &= \begin{cases} \inf \left\{ x \geq a_{i,N} / \nu^L((a_{i,N}, x]) \geq \frac{1}{N} \right\}, & \text{if } a_{i,N} < L \\ L + \frac{1}{N}, & \text{otherwise.} \end{cases} \end{aligned}$$

It is easy to check that since  $\nu$  has no atoms, for  $N \geq N(\eta)$ ,

$$d\left(\nu, \frac{1}{N} \sum_{i=1}^N \delta_{a_{i,N}}\right) < \eta + \rho^{-1} L^{-1} (M + C). \quad (36)$$

Now, for  $N, L$  large enough so that the right hand sides of (36) is smaller than  $2^{-1}\delta$ ,

$$\bigcap_{i=1}^N \left\{ \left| \frac{l_i}{N} - a_{i,N} \right| < \frac{\delta}{2} \right\} \subset \left\{ d\left(\hat{\mu}_\lambda^N, \frac{1}{N} \sum_{i=1}^N \delta_{a_{i,N}}\right) < \frac{\delta}{2} \right\} \subset \{d(\hat{\mu}_\lambda^N, \nu) < \delta\}$$

Therefore

$$\begin{aligned} \tilde{\Pi}^N(B(\nu, \delta)) &\geq \tilde{\Pi}^N\left(\bigcap_{i=1}^N \left\{ \left| \frac{l_i}{N} - a_{i,N} \right| < \frac{\delta}{2} \right\}\right) \\ &\geq \tilde{\Pi}^N\left(\bigcap_{i=1}^N \left\{ \left| \frac{l_i}{N} - a_{i,N} \right| < \epsilon \right\}\right) \end{aligned}$$

for any  $\epsilon \in (0, \frac{\delta}{2}]$ . We now show that for any fixed  $L$ ,

$$\liminf_{\epsilon \downarrow 0} \liminf_{N \rightarrow \infty} \frac{1}{N^2} \log \tilde{\Pi}^N\left(\bigcap_{i=1}^N \left\{ \left| \frac{l_i}{N} - a_{i,N} \right| < \epsilon \right\}\right) \geq -\tilde{H}(\nu^L). \quad (37)$$

Observe first that  $\frac{1}{N} \sum_{i=1}^N \delta_{a_{i,N}}$  is supported in  $[-L - 1, L + 1]$  so that all the spherical integrals are well defined and uniformly continuous by Lemma 3.2. Therefore, we find a  $\kappa(\epsilon)$ , going to zero with  $\epsilon$  such that for  $N$  sufficiently large,

$$\begin{aligned} \tilde{\Pi}^N\left(\bigcap_{i=1}^N \left\{ \left| \frac{l_i}{N} - a_{i,N} \right| < \epsilon \right\}\right) &\geq e^{N^2(aI(\log \sharp \mu_A, \nu^L) + bI(\log \sharp \mu_B, \nu^L) + F(\nu) - \kappa(\epsilon))} \\ &\quad \times \sum_{\left| \frac{l_i}{N} - a_{i,N} \right| < \epsilon} \Delta\left(\frac{l}{N}\right)^{a+b} e^{-N^2 \int c(x) d\hat{\mu}_\lambda^N(x)} \end{aligned} \quad (38)$$

Notice that

$$\begin{aligned} \sum_{\left| \frac{l_i}{N} - a_{i,N} \right| < \epsilon} \Delta \left( \frac{l}{N} \right)^{a+b} e^{-N^2 \int c(x) d\hat{\mu}_\lambda^N(x)} &= \sum_{\left| \frac{l_i}{N} - a_{i,N} \right| < \epsilon} e^{N^2 \left( \frac{a+b}{2} \iint_{x \neq y} \log |x-y| d\hat{\mu}_\lambda^N(x) d\hat{\mu}_\lambda^N(y) - \int c(x) d\hat{\mu}_\lambda^N(x) \right)} \\ &\geq e^{-N \sum_{j=1}^N \sup_{|x-a_{j,N}| \leq \frac{\delta}{2}} c(x) + \frac{a+b}{2} N^2 \iint_{x \neq y} \log |x-y| d\hat{\mu}_\lambda^N(x) d\hat{\mu}_\lambda^N(y)}. \end{aligned}$$

where  $\lambda$  is a Young shape defined by  $l_i := \lfloor N a_{i,N} \rfloor$ .

Note that such a tableau exists since according to the definition of the  $a_{i,N}$ 's since we have that

$$\frac{1}{N} \leq \nu^L([a_{i+1,N}, a_{i,N}]) \leq a_{i,N} - a_{i+1,N},$$

so that

$$N(a_{i,N} - a_{i+1,N}) \geq 1,$$

which insures that  $l_i - l_{i+1} \geq 1$  and so  $\lambda_i \geq \lambda_{i+1}$  for all  $i \in \mathbb{N}$ . Note that  $\left| \frac{l_i}{N} - a_{i,N} \right| < \frac{1}{N}$  is smaller than  $\epsilon$  for  $N$  large enough.

Furthermore, we also get the estimate

$$a_{i+1,N} \leq \frac{l_i}{N} \leq a_{i,N}.$$

Therefore, for  $i, j$  such that  $i < j - 1$ , we have the lower bound

$$\left| \frac{l_i}{N} - \frac{l_j}{N} \right| \geq |a_{i,N} - a_{j-1,N}|,$$

so that we get

$$\begin{aligned} \sum_{\left| \frac{l_i}{N} - a_{i,N} \right| < \epsilon} \Delta \left( \frac{l}{N} \right)^{a+b} e^{-N^2 \int c(x) d\hat{\mu}_\lambda^N(x)} &\geq \exp \left( N^2 \left( -\frac{1}{N} \sum_{j=1}^N (c(a_{j,N}) + C(L, \delta)) \right. \right. \\ &\quad \left. \left. + \frac{a+b}{2} \frac{1}{N^2} \sum_{i+1 < j} \log |a_{i,N} - a_{j,N}| + \frac{a+b}{4N^2} \sum_{i=1}^{N-1} \log \left| \frac{l_{i+1}}{N} - \frac{l_i}{N} \right| \right) \right) \end{aligned}$$

where we  $C(L, \delta)$  is going to zero as  $\delta$  goes to infinity for any given  $L$ . With our choice of the  $a_{j,N}$ 's, we have that

$$\lim_{N \rightarrow \infty} \frac{1}{N} \sum_{j=1}^N c(a_{j,N}) = \int x d\nu^L(x),$$

and

$$\begin{aligned} &\frac{1}{N^2} \sum_{i < j} \log |a_{i,N} - a_{j+1,N}| + \frac{1}{2N^2} \sum_{i=1}^{N-1} \log |a_{i,N} - a_{i+1,N}| \\ &= \sum_{1 \leq i \leq j \leq N-1} \log |a_{i,N} - a_{j+1,N}| \nu^L \otimes \nu^L(a_{i,N} \leq x \leq a_{i+1,N}; a_{j,N} \leq y \leq a_{j+1,N}) \\ &\geq \int_{a_{1,N} \leq x < y \leq a_{N,N}} \log |x - y| d\nu^L(x) d\nu^L(y) \end{aligned} \tag{39}$$

Let's turn our attention to the last term: for any choice of the  $l_i$ 's, as the  $l_i$  are distinct integers, the difference of a pair of them is at least 1, so that we have

$$\prod_{i=1}^{N-1} \left| \frac{l_{i+1}}{N} - \frac{l_i}{N} \right| \geq \left( \frac{1}{N} \right)^{N-1},$$

which gives

$$\liminf_{N \rightarrow \infty} \frac{1}{N^2} \log \sum_{i=1}^{N-1} \log \left| \frac{l_{i+1}}{N} - \frac{l_i}{N} \right| = 0.$$

Putting everything together, we can conclude,

$$\liminf_{\epsilon \downarrow 0} \liminf_{N \rightarrow \infty} \frac{1}{N^2} \log \sum_{\left| \frac{l_i}{N} - a_{i,N} \right| < \epsilon} \Delta \left( \frac{l}{N} \right)^{a+b} e^{-N^2 \int c(x) d\hat{\mu}_\lambda^N(x)} \geq -\frac{a+b}{2} \Sigma(\nu^L) - \int c(x) d\nu^L(x)$$

(39) and (38) prove (37). To finish the proof, we take the supremum over  $L$  to obtain the lower bound thanks to Lemma 3.2.2 and monotone convergence theorem.

## 4 Laplace method for $Z_N(\Phi)(X)$

Let  $\mu_\phi^N$  be the measure on  $\mathcal{P}(\mathbb{R})$  given, for any measurable set  $X$  of  $\mathcal{P}(\mathbb{R})$ , by

$$\mu_\phi^N(X) = \frac{Z_N(\Phi)(X)}{Z_N(\Phi)}.$$

The goal of this section is to prove a large deviation theorem for  $\mu_\phi^N$ .

We first need some definitions.

**Definition 4.1** *With  $\mathcal{L}$  as defined in Theorem 1.2 and  $\rho_\Phi$  given by (5), we let*

$$\mathcal{G}_\Phi(\nu) = \begin{cases} -I(\log \sharp \mu_A, \nu) - I(\log \sharp \mu_B, \nu) - \Sigma(\nu) + \rho_\Phi \cdot \int x d\nu(x), & \text{if } \nu \in \mathcal{L}, \\ +\infty & \text{otherwise,} \end{cases}$$

and if  $\Psi = \|\Phi\|_\infty^{-1} \Phi$ ,

$$J_\Phi(\nu, \mu) := \begin{cases} -I(\log \Psi \sharp \mu, \nu) - \frac{1}{2} S(\Psi \sharp \mu) - \Sigma(\mu) + \frac{1}{2} \int x^2 d\mu(x), & \text{if } \nu \in \mathcal{L}, \\ +\infty & \text{otherwise.} \end{cases}$$

The rate function governing our large deviation principle is then given, for  $\mu \in \mathcal{P}(\mathbb{R})$ , by

$$I_\Phi(\mu) := \inf_{\nu \in \mathcal{P}(\mathbb{R}^+)} (\mathcal{G}_\Phi(\nu) + J_\Phi(\nu, \mu)) - \inf_{\mu' \in \mathcal{P}(\mathbb{R})} \inf_{\nu' \in \mathcal{P}(\mathbb{R}^+)} (\mathcal{G}_\Phi(\nu') + J_\Phi(\nu', \mu')).$$

To prove the large deviation principle, we shall make the following additional hypothesis

### Hypothesis 4.2

The cut-off function  $\Phi$  is bounded below:

$$\exists \epsilon > 0 \text{ s.t. } \forall x \in \mathbb{R}, \Phi(x) \geq \epsilon. \tag{40}$$

The two sequences of matrices  $(A_N)_{N \in \mathbb{N}}$  and  $(B_N)_{N \in \mathbb{N}}$  and their spectral measures  $\hat{\mu}_{A_N}$  and  $\hat{\mu}_{B_N}$  are such that

- there exists an  $\alpha > 0$  so that for all  $N$ ,  $A_N$  and  $B_N$  are bounded below by  $\alpha I$ . Hence, with  $\mathcal{K}$  the compact set  $[\alpha, 1]$ ,  $\text{supp } \hat{\mu}_{A_N} \subset \mathcal{K}$  and  $\text{supp } \hat{\mu}_{B_N} \subset \mathcal{K}$ .
- $\mu_{A_N}$  and  $\mu_{B_N}$  converge weakly respectively to  $\mu_A$  and  $\mu_B$ .

We shall then prove that

**Theorem 4.3** *Under Hypotheses 2.1 and 4.2,*

1.  $I_\Phi$  is a good rate function on  $\mathcal{P}(\mathbb{R})$ , i.e.  $I_\Phi$  is non-negative and for any  $M \in \mathbb{R}^+$ ,  $\{\nu \in \mathcal{P}(\mathbb{R}) : I_\Phi(\nu) \leq M\}$  is compact.
2.  $(\mu_\Phi^N)_{N \in \mathbb{N}}$  satisfies a large deviation principle in the scale  $N^2$  with good rate function  $I_\Phi$ , i.e.
  - For any closed subset  $F$  of  $\mathcal{P}(\mathbb{R})$ ,

$$\limsup_{N \rightarrow \infty} \frac{1}{N^2} \log \mu_\Phi^N(F) \leq - \inf_F I_\Phi,$$

- For any open subset  $O$  of  $\mathcal{P}(\mathbb{R})$ ,

$$\liminf_{N \rightarrow \infty} \frac{1}{N^2} \log \mu_\Phi^N(O) \geq - \inf_O I_\Phi.$$

3. Under Hypothesis 4.2,  $S(\hat{\mu}_{A_N}^N)$  converges towards  $S(\mu_A)$  and idem for  $B_N$ , and

$$\lim_{N \rightarrow \infty} \frac{1}{N^2} \log \frac{Z_N(\Phi)}{Z_N(0)} = - \inf_{\mu \in \mathcal{P}(\mathbb{R})} \inf_{\nu \in \mathcal{P}(\mathbb{R}^+)} (\mathcal{G}_\Phi(\nu) + J_\Phi(\nu, \mu)) + \frac{1}{2} S(\mu_A) + \frac{1}{2} S(\mu_B) + \frac{1}{2} \rho_\Phi.$$

The proof of this theorem is deduced from a large deviation principle obtained for the law of the couple  $(\hat{\mu}_\lambda^N, \hat{\mu}_M^N)$  given by the Gibbs measure defined, for  $X = (X_1, X_2) \in \mathcal{P}(\mathbb{R}^+) \times \mathcal{P}(\mathbb{R})$ , by

$$\Pi_\Phi^N(X) = \frac{1}{Z_N(\Phi)} \sum_{\lambda: \hat{\mu}_\lambda^N \in X_1} s_\lambda(A_N) s_\lambda(B_N) Z_N(\Psi, \lambda)(X_2) e^{-\rho_\Phi |\lambda|} \quad (41)$$

that we can formulate as follows :

**Theorem 4.4**

1. For  $(\nu, \mu) \in \mathcal{P}(\mathbb{R}^+) \times \mathcal{P}(\mathbb{R})$ , we set

$$\mathcal{I}_\Phi(\nu, \mu) := \begin{cases} +\infty & \text{if } \nu \notin \mathcal{L} \text{ or } \int x^2 d\mu(x) = +\infty, \\ J_\Phi(\nu, \mu) + \mathcal{G}_\Phi(\nu) - \inf_{(\nu', \mu') \in \mathcal{P}(\mathbb{R}^+) \times \mathcal{P}(\mathbb{R})} \{J_\Phi(\nu', \mu') + \mathcal{G}_\Phi(\nu')\} & \text{otherwise.} \end{cases}$$

Then  $\mathcal{I}_\Phi$  is a good rate function.

2.  $(\Pi_\Phi^N)_{N \in \mathbb{N}}$  satisfies a full large deviation principle in the scale  $N^2$  with rate function  $\mathcal{I}_\Phi$ .

Theorem 4.3.1 and .2 are direct consequences of Theorem 4.4 and the contraction principle since the application  $(\nu, \mu) \in \mathcal{P}(\mathbb{R}^+) \times \mathcal{P}(\mathbb{R}) \rightarrow \nu \in \mathcal{P}(\mathbb{R})$  is clearly continuous.

**Proof of Theorem 4.4:** This proof follows rather closely that of Theorem 3.1. Let us briefly outline it.

1. To prove that  $\mathcal{I}_\Phi$  is a good rate function, we note first that it is clearly non negative. To show that its level sets are compact, we proceed exactly as in section 3.1;  $\mathcal{G}_\Phi$  has compact level



sets by direct application of Theorem 3.1.1 whereas for  $J_{\Phi}$  we can proceed similarly once we notice that  $\mu \rightarrow S(\Psi \sharp \mu)$  is continuous since  $\Psi$  is bounded below by a positive constant and

$$S(\Psi \sharp \mu) = \iint \log \left( \int_0^1 (a\Psi(x) + (1-a)\Psi(y))^{-1} da \right) d\mu(x) d\mu(y)$$

and introducing the function

$$j(x, y) = \log |x - y|^{-1} + \frac{1}{4}x^2 + \frac{1}{4}y^2,$$

we can treat it as  $g$  to show that  $\mu \mapsto \iint j(x, y) d\mu(x) d\mu(y)$  is lower semicontinuous on  $\mathcal{P}(\mathbb{R})$  and infinite unless  $\int x^2 d\mu(x)$  and  $\Sigma(\mu)$  are finite.

Thus, we see that  $\mathcal{I}_{\Phi}(\nu, \mu)$  is infinite unless

$$\nu \in \mathcal{L}, \quad \int x d\nu(x) < \infty, \quad \Sigma(\nu) > -\infty, \quad \int x^2 d\mu(x) < \infty, \quad \Sigma(\mu) > -\infty.$$

This in particular shows that we can apply the formula for the spherical integral obtained in [10] when  $\int x^2 d\nu(x) < \infty$  and hence the formula for  $\mathcal{I}_{\Phi}$  is then explicit.

2. To prove that  $\Pi_{\Phi}^N$  is exponentially tight, we consider a compact

$$K_L := \left\{ \nu \in \mathcal{P}(\mathbb{R}^+) : \int x d\nu(x) \leq L \right\} \times \left\{ \mu \in \mathcal{P}(\mathbb{R}) : \int x^2 d\mu(x) \leq L \right\}.$$

It is not hard to bound below  $Z_{\Phi}^N$  by some estimate of order  $e^{-N^2 C}$  (for instance by proving the lower bound estimate as below). Then, using the fact that  $S(\Psi \sharp \mu)$  is bounded uniformly as well as the spherical integrals, we find a finite constant  $C'$  such that

$$\Pi_{\Phi}^N(K_L^c) \leq e^{C'N^2} \left( \tilde{\Pi}^N(K_L^c) + \int_{\sum x_i^2 \geq NL} \Delta(x)^2 e^{-\frac{N}{2} \sum_{i=1}^N x_i^2} \prod_{i=1}^N dx_i \right).$$

Following [1] (or the arguments of section 3.2) we easily see that for sufficiently large  $L$

$$\limsup_{N \rightarrow \infty} \frac{1}{N^2} \log \int_{\sum x_i^2 \geq NL} \Delta(x)^2 e^{-\frac{N}{2} \sum_{i=1}^N x_i^2} \prod_{i=1}^N dx_i \leq -\frac{1}{4}L$$

so that we can conclude again by section 3.2.

3. To prove the weak large deviation upper bound, we proceed as in section 3.3 by considering the functions  $g$  (with  $c(x) = \rho_{\Phi} x$  and  $a = b = 1$ ) and  $j$ . We then impose a cutoff on both functions and on the spherical integrals as in (33) to obtain a large deviation upper bound estimate, and then proceed again by optimizing over the cutoff.
4. For the large deviation lower bound, we restrict the sum and the integral also to configurations contained in small neighborhoods of well chosen values  $(a_{i,N})_{1 \leq i \leq N}$  and  $(x_{i,N})_{1 \leq i \leq N}$  and show convergence. This strategy works as well in the continuous setting as can be seen in [1].

## 5 Comments on the minimizers of $\mathcal{I}_\Phi$

In this last section, we wish to give some weak description of the minimizers of  $\mathcal{I}_\Phi$ . We have not been able to prove uniqueness of such minimizers. In [9], uniqueness of the minimizers of the rate function was deduced from convexity arguments which were actually lacking for instance for the  $q$ -Potts model. In fact, the spherical integrals are expressed as the sum of a convex complicated function and the entropies  $\Sigma$  which are concave. Hence, if the full rate function does not contain some term to kill these  $\Sigma$  terms, the convexity of the full rate function becomes unclear. The same phenomenon appears here and despite our efforts we could not overcome this difficulty. It is unclear here whether the minimizer should be unique or not. We here meet the additional difficulty that the formula obtained in [10] for the limit of the spherical integral concerned the case where both probability measures had finite covariance, which is not the case here (one of the argument has only a first moment which is finite, even if the other one is compactly supported).

In this section, we show that the minimizers of  $\mathcal{I}_\Phi$  are compactly supported. We then characterize the minimizers.

**Proposition 5.1** *Assume that  $\Sigma(\log \sharp \mu_A) > -\infty$ ,  $\Sigma(\log \sharp \mu_B) > -\infty$ . Then*

1. *There exists a real number  $M \geq 0$  such that any minimizer  $(\nu, \mu) \in \mathcal{P}(\mathbb{R}^+) \times \mathcal{P}(\mathbb{R})$  of  $\mathcal{I}_\Phi$  satisfies  $\text{supp}(\nu) \subset [0, M]$ .*
2. *If we additionally assume that there exists  $A < B$  in  $\mathbb{R}$  such that for  $L$  large enough  $\Phi$  satisfies*

$$\max_{|x| \geq L} \Phi(x) \leq \inf_{x \in [A, B]} \Phi(x) \quad (42)$$

*then there exists a real number  $M$  such that for any minimizer  $(\nu, \mu) \in \mathcal{P}(\mathbb{R}^+) \times \mathcal{P}(\mathbb{R})$  of  $\mathcal{I}_\Phi$ ,  $\mu$  satisfies  $\text{supp}(\mu) \subset [-M, M]$ .*

3.  *$\mathcal{I}_\Phi$  achieves its minimal value (which is zero). Let  $(\bar{\nu}, \bar{\mu})$  be a minimizer. Then*

*– There exists 3 flows  $(\rho^i, u^i)_{1 \leq i \leq 3}$  such that*

- *$\mu_t^i(dx) = \rho_t^i(x)dx$  is a probability measure for all  $t \in (0, 1)$ .  $t \in [0, 1] \rightarrow \mu_t^i \in \mathcal{P}(\mathbb{R})$  is continuous.*

$$\lim_{t \rightarrow 0} \mu_t^1 = \log \sharp \mu_A, \quad \lim_{t \rightarrow 0} \mu_t^2 = \log \sharp \mu_B, \quad \lim_{t \rightarrow 0} \mu_t^3 = \log \Psi \sharp \mu,$$

$$\lim_{t \rightarrow 1} \mu_t^i = \nu, \quad 1 \leq i \leq 3.$$

- *For  $i \in \{1, 2, 3\}$ ,  $(\rho^i, u^i)$  satisfies the Euler equation for isentropic flow described by the equations, for  $t \in (0, 1)$ ,*

$$\partial_t \rho_t^i(x) = -\partial_x(\rho_t^i(x)u_t^i(x)) \quad (43)$$

$$\partial_t(\rho_t^i(x)u_t^i(x)) = -\partial_x\left(\rho_t^i(x)u_t^i(x)^2 - \frac{\pi^2}{3}\rho_t^i(x)^3\right) \quad (44)$$

*in the sense of distributions that for all  $f \in C_c^{\infty, \infty}(\mathbb{R} \times [0, 1])$ ,*

$$\int_0^1 \int \partial_t f(t, x) d\mu_t^i(x) dt + \int_0^1 \int \partial_x f(t, x) u_t^i(x) d\mu_t^i(x) dt = 0$$

and, for any  $f \in C_c^{\infty, \infty}(\Omega)$  with  $\Omega_i := \{(x, t) \in \mathbb{R} \times [0, 1] : \rho_t^i(x) > 0\}$ ,

$$\int_0^1 \int (2u_t^i(x) \partial_t f(x, t) + (u_t^i(x)^2 - \pi^2 \rho_t^i(x)^2) \partial_x f(x, t)) \rho_t^i(x) dx dt = 0, \quad (45)$$

where  $C_c^{\infty, \infty}(\mathcal{A})$  is the space of functions which are infinitely differentiable on both variables on the open set  $\mathcal{A}$  and compactly supported.

$(\rho^i, u^i)$  are smooth in the interior of  $\Omega_i$ , which guarantees that (43) and (44) hold everywhere in the interior of  $\Omega_i$ . Moreover,  $\Omega_i$  is bounded in  $\mathbb{R} \times [0, 1]$ .

• Let  $\bar{\rho}$  be the density of  $\bar{\nu}$  and  $\bar{\Omega} = \{x : \bar{\rho}(x) > 0\}$ . Then, for any continuously differentiable test function  $\phi$  which is supported in the interior of  $\bar{\Omega}$ ,

$$\int \left( \rho_{\Phi} x - \frac{1}{2} x^2 + \int \log|x-y| d\bar{\nu}(y) \right) \partial_x \phi(x) dx = \sum_{i=1}^3 \int \phi(x) u_1^i(x) dx.$$

• For any  $\phi \in C^1(\text{Im}(\log \Psi)^c \cap \text{supp}(\bar{\mu}))$ ,

$$\int \partial_x \phi(x) \left( \frac{1}{2} x^2 - 2 \int \log|x-y| d\bar{\mu}(y) \right) dx = 0$$

To simplify, we shall assume that  $\log \Psi$  is one to one from  $\mathbb{R}$  into its image  $\text{Im}(\log \Psi)$ . Then, in a very weak sense of distribution, for any  $\phi \in C^1(\text{Im}(\log \Psi) \cap \text{supp}(\bar{\mu}))$

$$\begin{aligned} \int \partial_x \phi \left( -\frac{1}{2} x^2 + \frac{1}{2} (\log \Psi)^{-1}(x)^2 - 2 \int \log|(\log \Psi)^{-1}(x) - y| d\bar{\mu}(y) \right. \\ \left. + \int \log|e^x - \Psi(y)| d\bar{\mu}(y) \right) dx = - \int \phi(x) u_0^3(x) dx. \end{aligned}$$

If  $\bar{\mu}$  has a density with respect to Lebesgue measure, we obtain the usual sense of distribution in the interior of  $\text{Im}(\log \Psi) \cap \text{supp}(\bar{\mu})$ .

The additional assumption is needed to be able to use [9] results which required it.

**Proof.** • We first prove the first point, that is for any minimizer  $(\nu, \mu) \in \mathcal{P}(\mathbb{R}^+) \times \mathcal{P}(\mathbb{R})$  of  $\mathcal{I}_{\Phi}$ ,  $\nu$  is compactly supported. In [9], such a result was obtained by going back to the matrix model. We shall here provide a new proof based on the study of  $\mathcal{I}_{\Phi}$ . The only property of the spherical integral we shall use is the following: Let  $\nu$  and  $\nu^*$  in  $\mathcal{P}(\mathbb{R}^+)$  be such that there exists a coupling  $\pi \in \mathcal{P}(\mathbb{R}^+ \times \mathbb{R}^+)$  of  $(\nu, \nu^*)$  such that  $\pi(x \in \cdot) = \nu(x \in \cdot)$ ,  $\pi(y \in \cdot) = \nu^*(y \in \cdot)$ , and

$$\pi(x \leq y) = 1. \quad (46)$$

Then, for any  $\mu \in \mathcal{P}(\mathbb{R}^-)$  which is compactly supported,

$$I(\nu^*, \mu) \leq I(\nu, \mu). \quad (47)$$

This is a direct consequence of the definition of the spherical integral; indeed, by the above, we can construct discrete approximations  $(l_i, 1 \leq i \leq N)$  and  $(l_i^*, 1 \leq i \leq N)$  such that  $N^{-1} \sum_{i=1}^N \delta_{\frac{l_i}{N}}$  (resp.  $N^{-1} \sum_{i=1}^N \delta_{\frac{l_i^*}{N}}$ ) converges towards  $\nu$  (resp.  $\nu^*$ ) and  $l_i \leq l_i^*$ . Therefore, if  $N^{-1} \sum_{i=1}^N \delta_{\lambda_i}$  approximates  $\mu$  with  $\lambda_i \leq 0$ , it is clear that

$$I_N \left( \frac{l_i}{N}, \lambda_i \right) \geq I_N \left( \frac{l_i^*}{N}, \lambda_i \right)$$

yielding (47) at the limit  $N \rightarrow \infty$ .

Let now  $(\nu^*, \mu^*)$  be a minimizer and  $\nu$  satisfying (46) belonging to  $\mathcal{L}$ . By definition,

$$\mathcal{I}_\Phi(\nu, \mu^*) \geq \mathcal{I}_\Phi(\nu^*, \mu^*),$$

and therefore by (47), since  $\log \sharp \mu_A$ ,  $\log \sharp \mu_B$  and  $\log \Psi \sharp \mu$  are supported in  $\mathbb{R}^-$ ,

$$-\Sigma(\nu) + \rho_\Phi \int x d\nu(x) \geq -\Sigma(\nu^*) + \rho_\Phi \int x d\nu^*(x). \quad (48)$$

We shall use this inequality for a well chosen  $\nu$  which is a modification of  $\nu^*$ . We construct it as follows: recall that  $\nu^* \in \mathcal{L}$  implies that  $\nu^*(dx) = \rho^*(x)dx$  with  $\rho^* \leq 1$ . We assume that  $\nu^*([0, M]) < 1$  and are going to show a contradiction for  $M$  large enough. Observe that  $A := \int_0^3 1_{\{x: \rho^*(x) \leq \frac{1}{2}\}} dx \geq 1$  since  $\int_0^\infty \rho^*(x) dx = 1$ . Set for  $M \geq 3$ ,

$$\nu = \nu_M = 1_{[0, M]} \nu^* + \frac{\alpha_M}{A} 1_{\{\rho^* \leq \frac{1}{2}, x \in [0, 3]\}} dx,$$

with  $\alpha_M = \nu^*([M, \infty])$ .

We have on one side that

$$\begin{aligned} -\Sigma(\nu^*) &= -\Sigma(1_{[0, M]} \nu^*) + 2 \int_{\substack{x < M \\ y > M}} \log |x - y|^{-1} d\nu^*(x) d\nu^*(y) + \int_{\substack{x > M \\ y > M}} \log |x - y|^{-1} d\nu^*(x) d\nu^*(y) \\ &\geq -\Sigma(1_{[0, M]} \nu^*) + 2 \int_{\substack{x < M, y > M \\ |x - y| > 1}} \log |x - y|^{-1} d\nu^*(x) d\nu^*(y) \\ &\quad + \int_{\substack{x > M, y > M \\ |x - y| > 1}} \log |x - y|^{-1} d\nu^*(x) d\nu^*(y) \end{aligned}$$

Using that for all  $a \in (0, 1]$  there exists a finite constant such that for all  $x \geq 0$ ,

$$\log(1 + x) \leq C_a x^a$$

we deduce

$$\begin{aligned} -\Sigma(\nu^*) &\geq -\Sigma(1_{[0, M]} \nu^*) - 2C_a \int_{\substack{x < M, y > M \\ |x - y| > 1}} (|x - y| - 1)^a d\nu^*(x) d\nu^*(y) \\ &\quad - C_a \int_{\substack{x > M, y > M \\ |x - y| > 1}} (|x - y| - 1)^a d\nu^*(x) d\nu^*(y) \\ &\geq -\Sigma(1_{[0, M]} \nu^*) - (2 + \alpha_M) C_a \int_{y > M} y^a d\nu^*(y) \\ &\geq -\Sigma(1_{[0, M]} \nu^*) - (2 + \alpha_M) C_a M^{a-1} \int_{y > M} y d\nu^*(y) \end{aligned} \quad (49)$$

where we used in the last line Chebyshev inequality.

On the other side,

$$\begin{aligned}
-\Sigma(\nu_M) &= -\Sigma(1_{[0,M]}\nu^*) + 2\frac{\alpha_M}{A} \int_{x < M} \int_0^3 1_{\rho^*(y) \leq \frac{1}{2}} \log|x-y|^{-1} dy d\nu^*(x) \\
&\quad + \left(\frac{\alpha_M}{A}\right)^2 \int_0^3 1_{\rho^*(x) \leq \frac{1}{2}} \int_0^3 1_{\rho^*(y) \leq \frac{1}{2}} \log|x-y|^{-1} dy dx \\
&\leq -\Sigma(1_{[0,M]}\nu^*) + 2\frac{\alpha_M}{A} \int_{x < M} \int_0^3 1_{\rho^*(y) \leq \frac{1}{2}} 1_{|x-y| \leq 1} \log|x-y|^{-1} dy \rho^*(x) dx \\
&\quad + \left(\frac{\alpha_M}{A}\right)^2 \int_0^3 1_{\rho^*(x) \leq \frac{1}{2}} \int_0^3 1_{\rho^*(y) \leq \frac{1}{2}} 1_{|x-y| \leq 1} \log|x-y|^{-1} dy dx \\
&\leq -\Sigma(1_{[0,M]}\nu^*) + \left(2\frac{\alpha_M}{A} + \left(\frac{\alpha_M}{A}\right)^2\right) \int_{x < 4} \int_0^3 1_{|x-y| \leq 1} \log|x-y|^{-1} dy dx \\
&\leq -\Sigma(1_{[0,M]}\nu^*) + 4 \left(2\frac{\alpha_M}{A} + \left(\frac{\alpha_M}{A}\right)^2\right) \tag{50}
\end{aligned}$$

Observe now that  $\nu_M$  in  $\mathcal{L}$  for  $M$  large enough so that  $A^{-1}\alpha_M \leq 2^{-1}$ . Furthermore,  $\nu_M$  satisfies (46) since we have been transporting large values of the  $l_i$ 's to smaller one. Hence, we can apply (48) and together with (49), (50), it gives that

$$\rho_\Phi \left( \int_{x > M} x d\nu^*(x) - \frac{\alpha_M}{A} \int_0^3 x 1_{\rho^* < \frac{1}{2}} dx \right) \leq (2 + \alpha_M) C_a M^{a-1} \int_{y > M} y d\nu^*(y) + 4 \left( 2\frac{\alpha_M}{A} + \left(\frac{\alpha_M}{A}\right)^2 \right),$$

showing that for any  $a \in (0, 1)$ , for  $M$  large enough,

$$(\rho_\Phi - (2 + \alpha_M) C_a M^{a-1}) \int_{x > M} x d\nu^*(x) \leq \frac{15}{A} \alpha_M \leq \frac{15}{AM} \int_{x > M} x d\nu^*(x) \tag{51}$$

which shows that  $\int_{x > M} x d\nu^*(x)$  has to be null when  $\rho_\Phi - (2 + \alpha_M) C_a M^{a-1} - \frac{15}{AM} > 0$  that is for  $M$  large enough.

• We now pass to the proof of the second point of the proposition. Let, with  $\beta_M = \mu^*([-M, M]^c)$ , for  $B > A$ ,

$$\mu_M(dx) = 1_{[-M, M]}\mu^*(dx) + \frac{\beta_M}{B - A} 1_{[A, B]} dx$$

Because of our assumption, we see that if  $M$  is large enough and  $[A, B]$  chosen so that

$$\inf_{[A, B]} \Phi \geq \sup_{[-M, M]^c} \Phi$$

for any  $\nu \in \mathcal{P}(\mathbb{R}^+)$ ,

$$I(\log \Psi \sharp \mu_M, \nu) \geq I(\log \Psi \sharp \mu^*, \nu).$$

Hence, when  $(\mu^*, \nu^*)$  minimize  $\mathcal{I}_\Phi$ , we obtain

$$-\Sigma(\mu^*) + \frac{1}{2} \int x^2 d\mu^*(x) - \frac{1}{2} S(\Psi \sharp \mu^*) \leq -\Sigma(\mu_M) + \frac{1}{2} \int x^2 d\mu_M(x) - \frac{1}{2} S(\Psi \sharp \mu_M) \tag{52}$$

Arguing as above, we find that, for any  $a \in (0, 2)$ , there exists a finite constant  $C_a$  such that

$$\Sigma(\mu^*) - \Sigma(\mu_M) \leq C_a M^{a-2} \int x^2 d\mu^*(x) \quad (53)$$

$$-S(\Psi \sharp \mu_M) + S(\Psi \sharp \mu^*) \leq C\beta_M \quad (54)$$

where we observed in the last line that  $\Psi$  was bounded uniformly above and below. Hence, we arrive at

$$\left(\frac{1}{2} - C_a M^{a-2}\right) \int_{x \geq M} x^2 d\mu^*(x) \leq C'\beta_M \leq C' M^{-2} \int_{x \geq M} x^2 d\mu^*(x) \quad (55)$$

where  $C' = C + B^2$ . This is again a contradiction for sufficiently large  $M$ .

• We finally study the characterization of the minimizers. In [9], the characterization was done by going back to the matrix model description. We shall here tackle this problem by a direct study of the rate function. Note that by point 1., any minimizers  $(\bar{\nu}, \bar{\mu})$  is such that  $\bar{\nu}$  is compactly supported. Moreover  $\log \Psi \sharp \bar{\mu}$ ,  $\log \sharp \mu_A$  and  $\log \sharp \mu_B$  are also compactly supported by our hypotheses so that we can apply Property 2.2 in [9] which says that if  $\mu, \nu$  are two probability measures with finite covariance and such that  $\Sigma(\mu) > -\infty$ ,  $\Sigma(\nu) > -\infty$ ,

$$I(\mu, \nu) = -\frac{1}{2} \inf_{(\rho, u) \in C(\mu, \nu)} \{S(\rho, u)\} - \frac{1}{2} (\Sigma(\mu) + \Sigma(\nu) - \mu(x^2) - \nu(x^2)) + c \quad (56)$$

where

$$S(\rho, u) := \int_0^1 \int u_t(x)^2 \rho_t(x) dx dt + \frac{\pi^2}{3} \int_0^1 \int \rho_t(x)^3 dx dt,$$

$$C(\mu, \nu) = \left\{ \rho. \in L^1(dx dt), \int \rho_t(x) dx = 1 \ \forall t \in [0, 1], \lim_{t \rightarrow 0} \rho_t(x) dx = \mu, \lim_{t \rightarrow 1} \rho_t(x) dx = \nu, \right. \\ \left. \partial_t \rho_t(x) + \partial_x(\rho_t(x) u_t(x)) = 0 \right\},$$

where the last equality is to be understood in the sense of distributions. It was shown in [9] that the infimum defining  $I$  is achieved at a unique  $(u^*, \rho^*) \in C(\mu, \nu)$  which is described by an isentropic Euler equation with negative pressure  $p(\rho) = -\frac{\pi^2}{3} \rho^3$ .  $c$  is a universal constant. As a consequence of this formula, since  $\mathcal{I}_\Phi(\mu, \nu) < \infty$  implies that  $\Sigma(\mu) > -\infty$ ,  $\Sigma(\nu) > -\infty$  and  $\mu(x^2) < \infty$ , for any  $\nu \in \mathcal{P}(\mathbb{R}^+)$  such that  $\nu(x^2) < \infty$ , we find that

$$\begin{aligned} \mathcal{I}_\Phi(\mu, \nu) &= \inf_{((\rho^i, u^i) \in C(\mu^i, \nu))_{1 \leq i \leq 3}} \left\{ \frac{1}{2} \sum_{i=1}^3 S(\rho^i, u^i) + \frac{1}{2} \Sigma(\nu) - \Sigma(\mu) + \frac{1}{2} \Sigma(\Psi \sharp \mu) \right. \\ &\quad \left. + \frac{1}{2} \mu(-\log \Psi(x)^2 + x^2) - \frac{3}{2} \nu(x^2) + \rho_\Phi \nu(x) + K(\mu_A, \mu_B) \right\} \\ &:= \inf_{((\rho^i, u^i) \in C(\mu^i, \nu))_{1 \leq i \leq 3}} \Xi((\rho^i, u^i)_{1 \leq i \leq 3}, \nu, \mu), \end{aligned} \quad (57)$$

where  $\mu^1 = \log \sharp \mu_A$ ,  $\mu^2 = \log \sharp \mu_B$ ,  $\mu^3 = \log \Psi \sharp \bar{\mu}$  and  $K(\mu_A, \mu_B)$  is a constant depending only on  $\mu_A$  and  $\mu_B$ .

We now consider a minimizer  $((\bar{\rho}^i, \bar{u}^i)_{1 \leq i \leq 3}, \bar{\mu}, \bar{\nu})$  of  $\Xi$  in  $\Omega := \{\nu \in \mathcal{L}, \mu \in \mathcal{P}(\mathbb{R}), (\bar{\rho}^i, \bar{u}^i)_{1 \leq i \leq 3} \in C(\log \sharp \mu_A, \nu) \times C(\log \sharp \mu_B, \nu) \times C(\log \Psi \sharp \mu, \nu)\}$ . To characterize this minimizer, we perform a small perturbation. Let  $((\rho_\epsilon^i, u_\epsilon^i)_{1 \leq i \leq 3}, \mu_\epsilon, \nu_\epsilon) \in \Omega$  be given, for compactly supported functions  $(\phi^i)_{1 \leq i \leq 3}$  in  $\mathcal{C}^{1,1}(\mathbb{R} \times [0, 1])$  by

$$\rho_\epsilon^i(t, x) = \bar{\rho}^i(t, x) + \epsilon \partial_x \phi^i(t, x) \quad \text{and} \quad u_\epsilon^i(t, x) \rho_\epsilon^i(t, x) = \bar{u}^i(t, x) \bar{\rho}^i(t, x) - \epsilon \partial_t \phi^i(t, x),$$

with  $\partial_x \phi^i(1, x) = \partial_x \phi(1, x)$  independent of  $i$ ,  $\partial_x \phi^i(0, x) = 0$  for  $i = 1, 2$ .

Note that, once we chose the perturbation for  $\rho^i$ , the form of the perturbation for  $u^i \rho^i$  taken above ensures that the first equation  $\partial_t \rho^i(t, x) = -\partial_x(u^i(t, x) \rho^i(t, x))$  is automatically satisfied.

This implies also

$$\nu_\epsilon = \bar{\nu} + \epsilon \partial_x \phi^i(1, x) dx$$

and

$$\log \Psi \sharp \mu_\epsilon(dx) = \log \Psi \sharp \bar{\mu}(dx) + \epsilon \partial_x \phi^3(0, x) dx.$$

We perturb more generally  $\mu$  by setting

$$\mu_\epsilon(dx) = \bar{\mu}(dx) + \epsilon \partial_x \psi(x) dx$$

with the condition

$$\int f(\log \Psi(x)) \partial_x \psi(x) dx = \int f(x) \partial_x \phi^3(0, x) dx$$

for all bounded continuous functions  $f$ .

We shall assume that

$$L(\phi) = \sum_{i=1}^3 \int_0^1 \int \left( \frac{|\partial_t \phi^i(t, x)|^2}{\bar{\rho}^i(t, x)} \right) dx dt + \sum_{i=1}^3 \sup_{t \in (0, 1)} \left\| \frac{\partial_x \phi^i(t, x)}{\bar{\rho}^i(t, x)} \right\|_\infty < \infty.$$

It is not hard to see that under such conditions,  $\Xi((\rho_\epsilon^i, u_\epsilon^i)_{1 \leq i \leq 3}, \mu_\epsilon, \nu_\epsilon)$  is finite.

By the condition

$$\Xi((\rho_\epsilon^i, u_\epsilon^i)_{1 \leq i \leq 3}, \mu_\epsilon, \nu_\epsilon) \geq \Xi((\bar{\rho}^i, \bar{u}^i)_{1 \leq i \leq 3}, \bar{\mu}, \bar{\nu})$$

we obtain, taking the limit  $\epsilon \rightarrow 0$ , that

$$\begin{aligned} & \int \left( \rho_\Phi x - \frac{3}{2} x^2 \right) \partial_x \phi(1, x) dx - \frac{1}{2} \int x^2 \partial_x \phi^3(0, x) dx + \frac{1}{2} \int x^2 \partial_x \psi(x) dx \\ & + \int \int \log |x - y| d\bar{\nu}(y) \partial_x \phi(1, x) dx - 2 \int \int \log |x - y| d\bar{\mu}(y) \partial_x \psi(x) dx \\ & + \int \int \log |e^x - e^y| d \log \Psi \sharp \bar{\mu}(y) \partial_x \phi^3(0, x) dx \\ & + \frac{1}{2} \sum_{i=1}^3 \int \int_0^1 [-2 \partial_t \phi^i(t, x) \bar{u}^i(t, x) - (\bar{u}^i(t, x))^2 \partial_x \phi^i(t, x) + \pi^2 (\bar{\rho}^i(t, x))^2 \partial_x \phi^i(t, x)] dx dt \geq 0 \end{aligned} \tag{58}$$

Changing  $\phi^i$  (for  $1 \leq i \leq 3$ ) and  $\psi$  respectively into  $-\phi^i$  and  $-\psi$ , we get that the inequality in (58) is in fact an equality.

Applying this result with  $\phi^i(0, x) = \phi^i(1, x) = 0$  shows that  $(\bar{u}^i, \bar{\rho}^i)_{1 \leq i \leq 3}$  satisfies the Euler equation for isentropic flow described in the proposition.

We now turn to the boundary conditions expressed in the last two points of Proposition 5.1. To characterize them, we will try to regularize the densities  $\rho_\epsilon^i(t, \cdot)$ . We remark that by Property 2.8 in [9], since  $\bar{\nu}$  and  $\bar{\mu}$  are compactly supported under our hypothesis, we can find sequences of potentials  $(h^{\epsilon, i}, \epsilon > 0, 1 \leq i \leq 3)$  in  $\mathcal{C}_b^{1,1}(\mathbb{R} \times [0, 1])$  such that if we set

$$\rho_\epsilon^i(t, x) := \pi^{-1}(\max\{\partial_t h^{\epsilon, i}(t, x) + 4^{-1}(\partial_x h^{\epsilon, i}(t, x))^2, 0\})^{\frac{1}{2}}$$

then for any  $\epsilon > 0$ ,

$$\begin{aligned} & \int \left( \bar{u}^i(t, x) - \partial_x \frac{h^{\epsilon, i}(t, x)}{2} \right)^2 \bar{\rho}^i(t, x) dx dt + \frac{\pi^2}{3} \int_0^1 \int (\bar{\rho}^i(t, x) - \rho_\epsilon^i(t, x))^2 (\bar{\rho}^i(t, x) + \rho_\epsilon^i(t, x)) dx dt \\ & + \pi^2 \int_0^1 \int |\partial_t h^{\epsilon, i}(t, x) + 4^{-1}(\partial_x h^{\epsilon, i}(t, x))^2 - \pi^2 \rho_\epsilon^i(t, x)^2| \bar{\rho}^i(t, x) dx dt \leq \epsilon. \end{aligned}$$

From this result, we deduce that

$$\begin{aligned} & \sup_{1 \leq i \leq 3} \left| \int_0^1 \int [-2\partial_t \phi^i(t, x) \bar{u}^i(t, x) - (\bar{u}^i(t, x))^2 \partial_x \phi^i(t, x) + \pi^2 (\bar{\rho}^i(t, x))^2 \partial_x \phi^i(t, x)] dx dt \right. \\ & \left. - \int_0^1 \int [-\partial_t \phi^i(t, x) \partial_x h^{\epsilon, i}(t, x) - \frac{1}{4} (\partial_x h^{\epsilon, i}(t, x))^2 \partial_x \phi^i(t, x) + \pi^2 (\rho_\epsilon^i(t, x))^2 \partial_x \phi^i(t, x)] dx dt \right| \leq C_{L(\phi)} \sqrt{\epsilon} \end{aligned}$$

with  $C(L(\phi)) < \infty$  when  $L(\phi) < \infty$ . Moreover, since  $h^{i, \epsilon} \in \mathcal{C}^{1,1}(\mathbb{R} \times [0, 1])$ , we can integrate by part so that

$$\begin{aligned} & \left| \int_0^1 \int [-\partial_t \phi^i(t, x) \partial_x h^{\epsilon, i}(t, x) - 4^{-1} (\partial_x h^{\epsilon, i}(t, x))^2 \partial_x \phi^i(t, x) + \pi^2 (\rho_\epsilon^i(t, x))^2 \partial_x \phi^i(t, x)] dx dt \right. \\ & \left. - 2 \left[ \int h^{\epsilon, i} \partial_x \phi^i dx \right]_0^1 \right| \leq C'(L(\phi)) \sqrt{\epsilon} \end{aligned}$$

We now can define in the sense of distribution

$$\int \Pi_t^i \partial_x \phi^i dx = - \int u_t^i \phi^i dx$$

and by letting  $\epsilon$  going to zero we get that

$$\int [-2\partial_t \phi^i(t, x) \bar{u}^i(t, x) - (\bar{u}^i(t, x))^2 \partial_x \phi^i(t, x) + \pi^2 (\bar{\rho}^i(t, x))^2 \partial_x \phi^i(t, x)] dx dt = 2 \left[ \int \Pi_t^i \partial_x \phi^i dx \right]_0^1.$$



Thus, we have proved that we can rewrite (58) (which we showed to be an equality) under the form

$$\begin{aligned}
& \int \left( \rho_{\Phi} x - \frac{3}{2} x^2 \right) \partial_x \phi(1, x) dx - \frac{1}{2} \int x^2 \partial_x \phi^3(0, x) dx + \frac{1}{2} \int x^2 \partial_x \psi(x) dx \\
& + \iint \log |x - y| d\bar{\nu}(y) \partial_x \phi(1, x) dx - 2 \iint \log |x - y| d\bar{\mu}(y) \partial_x \psi(x) dx \\
& + \iint \log |\Psi(x) - \Psi(y)| d\bar{\mu}(y) \partial_x \psi(x) dx \\
& + \sum_{i=1}^3 \left( \int \Pi_1^i \partial_x \phi(1, x) dx - \int \Pi_0^i \partial_x \phi^i(0, x) dx \right) = 0
\end{aligned} \tag{59}$$

From that we can deduce the boundary conditions we are seeking for.

As the equality (59) holds for any function  $\partial_x \phi(1, x)$  such that  $L(\phi)$  is finite, we find that

$$A(x, \bar{\nu}) = \rho_{\Phi} x - \frac{3}{2} x^2 + \int \log |x - y| d\bar{\nu}(y) + \sum_{i=1}^3 \Pi_1^i(x) \tag{60}$$

is constant in the sense of distribution.

Furthermore, it is not hard to deduce from the representation of  $\rho_i^i$  as a free Brownian motion given in [9] that for  $t$  close enough to one  $\{x : \rho_i^i(x) \geq \epsilon\} \subset \{x : \bar{\rho}(x) \geq 2\epsilon\}$  with  $\bar{\rho}$  the density of  $\bar{\nu}$  with respect to Lebesgue measure. Therefore, for any  $\mathcal{C}_b^1$  function  $\phi$  with compact support in the interior of  $\{x : \bar{\rho}(x) > 0\}$ ,

$$\int \partial_x \phi(x) A(x, \bar{\nu}) dx = 0.$$

Now only the last point of our proposition is left to establish.

The statement of the result is more obscure when dealing with  $\bar{\mu}$  since we do not a priori know if  $\bar{\mu}$  has a density with respect to Lebesgue measure. What we get from (59) is that :

For any  $\psi \in \mathcal{C}_b^1(\text{Im}(\log \Psi)^c \cap \text{supp}(\bar{\mu}))$

$$\int \partial_x \psi(x) \left( \frac{1}{2} x^2 - 2 \int \log |x - y| d\bar{\mu}(y) \right) dx = 0$$

i.e  $\frac{1}{2} x^2 - 2 \int \log |x - y| d\bar{\mu}(y)$  is constant outside of the image  $\text{Im}(\log \Psi)$  of  $\log \Psi$ .

Inside  $\text{Im}(\log \Psi)$ , if we assume that  $\log \Psi$  is one to one from  $\mathbb{R}$  onto its image, we have that

$$B(x, \bar{\mu}) = -\frac{1}{2} x^2 + \frac{1}{2} (\log \Psi)^{-1}(x)^2 - 2 \int \log |(\log \Psi)^{-1}(x) - y| d\bar{\mu}(y) + \int \log |e^x - \Psi(y)| d\bar{\mu}(y) - \Pi_0^3(x)$$

is constant in the weak sense of distribution that is its integral with respect to  $\partial_x \phi^3(x, 0)$  vanishes. If  $\bar{\mu}$  has a density with respect to Lebesgue measure, we find that  $B(x, \bar{\mu})$  is constant in the sense of distribution inside  $\{x : \frac{d\bar{\mu}}{dx} \neq 0\}$  as above, but it is not clear that a  $\phi^3 \neq 0$  indeed exists in general! ■

## 6 Conclusion and remarks

In this paper, we studied the asymptotics of the model given by the partition function (1). In the course of doing so, we adapted the techniques of [1] to study large deviations of the profiles of Young tableaux with a density given by a Vandermonde determinant and Schur polynomial functions (see Theorem 1.2). We believe that these techniques might be useful to study other problems since these kind of distributions appear in different contexts due to their combinatorial nature. For instance, following Migdal-Witten formula [25, 24], the partition function of two-dimensional Yang Mills theory on a cylinder with gauge group  $U(N)$  is given by the central heat kernel defined, at time  $t = TN^{-1}$ , by

$$\mathcal{Z}_N \left( U_1, U_2; \frac{T}{N} \right) = \sum_{\lambda} s_{\lambda}(U_1) s_{\lambda}(U_2) e^{-\frac{T}{2N} C_2(\lambda)}$$

where  $U_1, U_2 \in U(N)$ , the sum runs over Young tableaux  $\lambda$  and

$$C_2(\lambda) = \sum_{i=1}^N \lambda_i(\lambda_i + 1 - 2i + N) = \sum_{i=1}^N l_i^2 - (N-1) \sum_{i=1}^N l_i + \sum_{i=1}^N (N-i)(i-1)$$

with  $l_i = \lambda_i + N - i$  (see for example [8]).

S. Zelditch [26] asked us if we could study the asymptotics of  $\mathcal{Z}_N(U_1, U_2; TN^{-1})$  when  $U_1, U_2$  are not unitary but real diagonal matrices with converging spectral distributions. Our techniques apply readily to this context and we find

**Theorem 6.1** *Let  $A_N, B_N$  be two sequences of uniformly bounded matrices bounded below by  $\epsilon I$  for some  $\epsilon > 0$  with spectral measures converging towards  $\mu_A, \mu_B$ . Then for any time  $T > 0$*

$$\lim_{N \rightarrow \infty} \frac{1}{N^2} \log \mathcal{Z}_N \left( A_N, B_N; \frac{T}{N} \right) = Z(\mu_A, \mu_B, T)$$

with

$$Z(\mu_A, \mu_B; T) = \sup_{\nu \in \mathcal{L}} \left\{ I(\log \sharp \mu_A, \nu) + I(\log \sharp \mu_B, \nu) + \Sigma(\nu) - \frac{T}{2} \int x^2 d\nu(x) + \frac{T}{2} \int x d\nu(x) \right\} \\ + \frac{1}{2} S(\mu_A) + \frac{1}{2} S(\mu_B) - \frac{T}{12}$$

This theorem is a direct consequence of Theorem 1.2 with  $a = b = 1$  and  $c(x) = x^2 - x$ .

In addition to giving a rigorous basis to the study of such natural asymptotics, we gave a firm ground to begin the study of other matrix models where other problems due for instance to signed series might appear. This step seems necessary since the proofs are already rather involved. Furthermore, we developed new arguments to study the saddle points of our model based on transport of mass.

One of the weakness of our result is apparently the cut-off function  $\Phi$ , since the matrix integral (1) is then hard to relate with the enumeration of maps as in [14]. Let us comment heuristically this point. Observe first that the matrix integral (1) with  $\Phi(x) = x$  considered in [14] is always infinite. Indeed, for instance in the case  $A = 1$ , we are integrating

$$Z_N(Id) = \int_{x_i \in \mathbb{R}} \Delta(x)^2 \prod_{i,j=1}^N \frac{1}{1 - b_i x_j} e^{-N \sum x_j^2} \prod dx_j$$

which is clearly infinite for all  $N \in \mathbb{N}^*$ . Hence, everything should be understood formally. The same problem a priori also arise when one considers random triangulations generated by the one matrix integrals

$$\tilde{Z}_N(\lambda) = \int e^{\lambda N \text{tr}(M^3) - \frac{N}{2} \text{tr}(M^2)} dM$$

which is clearly infinite for  $\lambda \neq 0$ . One way to bypass this problem is for instance to consider

$$\tilde{Z}_N(\lambda, \eta) = \int e^{-\eta N \text{tr}(M^4) + \lambda N \text{tr}(M^3) - \frac{N}{2} \text{tr}(M^2)} dM$$

which is well defined for  $\eta > 0$ . Recall that planar maps are enumerated by

$$C(n) = \lim_{N \rightarrow \infty} \partial_\lambda^n \frac{1}{N^2} \log \tilde{Z}_N(\lambda)|_{\lambda=0} = \lim_{N \rightarrow \infty} \partial_\lambda^n \frac{1}{N^2} \log \tilde{Z}_N(\lambda, \eta)|_{\lambda=0, \eta=0}.$$

In the physics literature, these quantities are implicitly supposed to be given by

$$\tilde{C}(n) = \partial_\lambda^n \lim_{N \rightarrow \infty} \frac{1}{N^2} \log \tilde{Z}_N(\lambda, \eta)|_{\lambda=0, \eta=0}.$$

This is fine for rather general one matrix models according to Theorem 1.1 of [6], but this point is open in general.

Similarly, one could try to regularize the dually weighted graph model by considering  $Z_N(\Phi_{\epsilon, R})$  with

$$\Phi_{\epsilon, R}(x) = \frac{x}{1 + \epsilon x^2} + R$$

with  $\epsilon > 0$  and  $R \geq \sqrt{2\epsilon}^{-1}$ . For  $\|A\|$  and  $\|B\|$  small enough (which we can always assume since again only derivatives at the origin should be of interest), we obtain by our result a limit for  $N^{-2} \log Z_N(\Phi_{\epsilon, R})$ . Assuming that the limit can be extended analytically to  $R, \epsilon$  small, we should be able to enumerate, modulo the above ansatz of interchanging derivation and limit, the enumeration of dually weighted graphs.

There is still a long way toward the rigorous understanding of the use of matrix integrals for the enumeration of maps in physics but we hope that this paper provides some useful steps in this direction.

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## References

- [1] G. Ben Arous and A. Guionnet. Large deviations for Wigner's law and Voiculescu's non-commutative entropy. *Probab. Theory Related Fields*, 108(4):517–542, 1997.
- [2] Theodor Bröcker and Tammo tom Dieck. *Representations of compact Lie groups*, volume 98 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1985.

- [3] Benoît Collins. Moments and cumulants of polynomial random variables on unitary groups, the Itzykson-Zuber integral, and free probability. *Int. Math. Res. Not.*, (17):953–982, 2003.
- [4] Amir Dembo and Ofer Zeitouni. *Large deviations techniques and applications*, volume 38 of *Applications of Mathematics*. Springer-Verlag, New York, second edition, 1998.
- [5] P. Di Francesco and C. Itzykson. A generating function for fatgraphs. *Ann. Inst. H. Poincaré Phys. Théor.*, 59(2):117–139, 1993.
- [6] N. M. Ercolani and K. D. T.-R. McLaughlin. Asymptotics of the partition function for random matrices via Riemann-Hilbert techniques and applications to graphical enumeration. *Int. Math. Res. Not.*, (14):755–820, 2003.
- [7] B. Eynard. An introduction to random matrices. Cours de Physique Théorique de Saclay, november 2000. CEA/SPhT, Saclay.
- [8] David J. Gross and Andrei Matytsin. Some properties of large- $N$  two-dimensional Yang-Mills theory. *Nuclear Phys. B*, 437(3):541–584, 1995.
- [9] Alice Guionnet. First order asymptotics of matrix integrals; a rigorous approach towards the understanding of matrix models. *Comm. Math. Phys.*, 2003.
- [10] Alice Guionnet and Ofer Zeitouni. Large deviations asymptotics for spherical integrals. *J. Funct. Anal.*, 188(2):461–515, 2002.
- [11] Alice Guionnet and Ofer Zeitouni. Addendum to large deviations asymptotics for spherical integrals. *J. Funct. Anal.*, To appear, 2004.
- [12] C. Itzykson and J. B. Zuber. The planar approximation. II. *J. Math. Phys.*, 21(3):411–421, 1980.
- [13] Vladimir Kazakov. Solvable matrix models. In *Random matrix models and their applications*, volume 40 of *Math. Sci. Res. Inst. Publ.*, pages 271–283. Cambridge Univ. Press, Cambridge, 2001.
- [14] Vladimir A. Kazakov, Matthias Staudacher, and Thomas Wynter. Character expansion methods for matrix models of dually weighted graphs. *Comm. Math. Phys.*, 177(2):451–468, 1996.
- [15] Vladimir A. Kazakov and Paul Zinn-Justin. Two-matrix model with  $ABAB$  interaction. *Nuclear Phys. B*, 546(3):647–668, 1999.
- [16] A. Matytsin. On the large- $N$  limit of the Itzykson-Zuber integral. *Nuclear Phys. B*, 411(2-3):805–820, 1994.
- [17] M. L. Mehta. A method of integration over matrix variables. *Comm. Math. Phys.*, 79(3):327–340, 1981.
- [18] Madan Lal Mehta. *Random matrices*. Academic Press Inc., Boston, MA, second edition, 1991.
- [19] Madan Lal Mehta and Gilbert Mahoux. A method of integration over matrix variables. III. *Indian J. Pure Appl. Math.*, 22(7):531–546, 1991.
- [20] A.A. Migdal. Recursion equations in lattice gauge theories. *Sov. Phys. JETP*, 42, 1975.
- [21] Bruce E. Sagan. *The symmetric group*. The Wadsworth & Brooks/Cole Mathematics Series. Wadsworth & Brooks/Cole Advanced Books & Software, Pacific Grove, CA, 1991. Representations, combinatorial algorithms, and symmetric functions.
- [22] Dan Voiculescu. The analogues of entropy and of Fisher’s information measure in free probability theory. I. *Comm. Math. Phys.*, 155(1):71–92, 1993.
- [23] Hermann Weyl. *The Classical Groups. Their Invariants and Representations*. Princeton University Press, Princeton, N.J., 1939.

- [24] Edward Witten. On quantum gauge theories in two dimensions. *Comm. Math. Phys.*, 141(1):153–209, 1991.
- [25] Edward Witten. Two-dimensional gauge theories revisited. *J. Geom. Phys.*, 9(4):303–368, 1992.
- [26] Steve Zelditch. *MacDonald identities and the large  $N$  limit of  $YM_2$  on the cylinder*. Preprint, 2004.
- [27] A. Zvonkin. Matrix integrals and map enumeration: an accessible introduction. *Math. Comput. Modelling*, 26(8-10):281–304, 1997. Combinatorics and physics (Marseilles, 1995).