The Potts model on random graphs, loop models and random matrices

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Based on joint work with V.Jones, D.Shlyakhtenko, P. Zinn Justin

Loop models, random matrices and planar algebras, to appear in CMP

and E. Maurel Segala

Combinatorial aspects of matrix models, Alea 2006

Outline

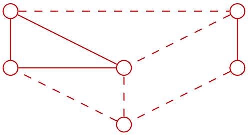
Outline

The Potts model

The partition function of the Potts model on a graph $G = (V, E)$ is given by

$$\begin{aligned} Z_G &= \sum_{\sigma:V\to\{1,\dots,Q\}} \exp\Big(K \sum_{\{i,j\}\in E} \delta_{\sigma_i,\sigma_j}\Big) \\ &= \sum_{\sigma:V\to\{1,\dots,Q\}} \prod_{\{i,j\}\in E} (1 + v\delta_{\sigma_i,\sigma_j}) \\ &= \sum_{E'\subset E} v^{\sharp\text{ bonds}}\, Q^{\sharp\text{ clusters}} \end{aligned}$$

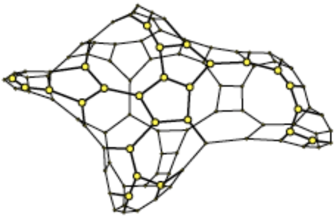
$v = e^K - 1$, bonds=edges in E', clusters=connected components of the subgraph (V, E')

 $\longrightarrow \quad v^4 Q^3$

Definition of maps

We shall consider the Potts model on (random planar maps).
A map is a connected graph which is embedded into a surface in such a way that edges do not cross and faces (obtained by cutting the surface along the edges) are homeomorphic to a disk.
The genus of the map is the minimal genus of a surface in which it can be properly embedded.

By Euler formula :
$2 - 2g = \sharp$ vertices
$+ \sharp$ faces
- $\sharp$ edges.



Potts model on planar maps

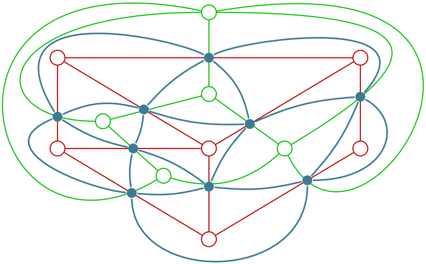
We shall consider the Potts model on random planar maps. We assume these graphs are rooted, that is are given a distinguished oriented edge. It is given by the partition function

$$\begin{aligned}
Z &= \sum_{G=(V,E)} x^{\#E} y^{\#V} Z_G \\
&= \sum_{G=(V,E)} x^{\#E} y^{\#V} \sum_{\sigma: V \to \{1,\dots,Q\}} \exp\Big(K \sum_{\{i,j\}\in E} \delta_{\sigma_i,\sigma_j}\Big) \\
&= \sum_{G=(V,E)} x^{\#E} y^{\#V} \sum_{E' \subset E} v^{\sharp \text{ bonds}} Q^{\sharp \text{ clusters}}
\end{aligned}$$

$v = e^K - 1$, bonds=edges in E', clusters=connected components of the subgraph (V, E')

The Potts model on random planar maps as a loop model

If G is a planar map, there is dual (green) and a medial (blue) planar graph G_m



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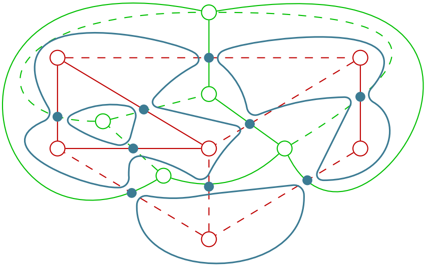
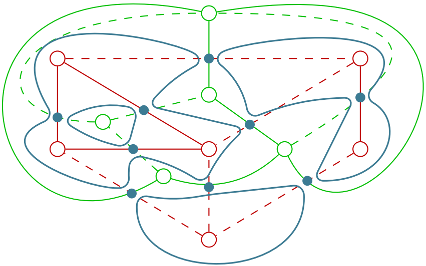


Splitting a vertex

The Potts model on random planar maps as a loop model



If G is a planar map, there is a bijection

$$(G, E') \Leftrightarrow (\text{loops, shaded vertices})$$

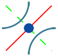
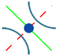
Moreover, writing Euler formula in each cluster gives the relation

$$\sharp\text{loops} = 2\sharp\text{clusters} + \sharp\text{bonds} - \sharp V$$

The Potts model on random graphs as a loop model

The equivalence to the loop model allows to state that

$$Z = \sum_{G=(V,E)} x^{\# E} Q^{-\frac{1}{2}\# V} \sum_{E' \subset E} v^{\sharp \text{ bonds}} \, Q^{\sharp \text{ clusters}}$$

$$= \sum_{\Gamma} \delta^{\# \text{ loops}} \alpha^{\#}$$  $\beta^{\#}$ 

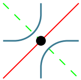
where the summation is restricted to 4-valent rooted planar maps, and

$$\delta = \sqrt{Q} \qquad \frac{\alpha}{\beta} = \frac{v}{\sqrt{Q}} \qquad \beta = x$$

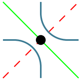
δ is called the fugacity.

The Potts model on planar maps and loop models

Hence, the partition function Z of the Potts model on planar maps is a generating function for the number of possible matchings of the end points of n copies of the vertex

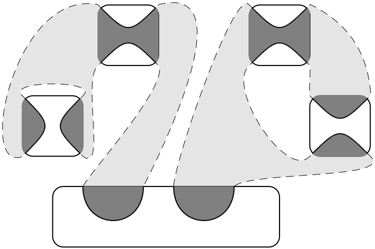


and m copies of the vertex



so that the resulting graph

- is planar, connected,
- has p loops,
- is checkerboard shaded.



Outline

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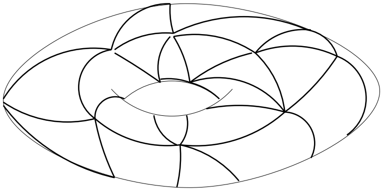
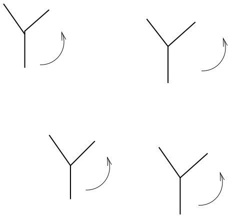


"Baby combinatorial problem"

Be given p vertices of valence d drawn on a surface (that is given an orientation).
Problem : Count the number of maps with genus g that can be obtained by matching the end points of the half-edges of the vertices.

Example :
$g = 0$,
$p = 4$,
$d = 3$.



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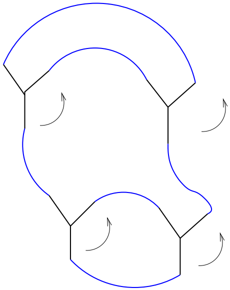

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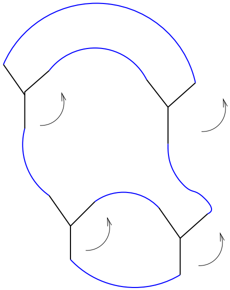
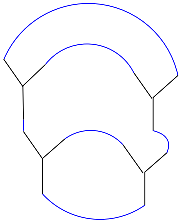


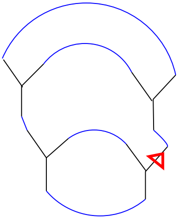
The counting is done up to homeomorphisms. How do you count symmetries ? (between vertices, edges of the vertices etc)

Combinatorial approach

Tutte (60's) : count rooted planar maps.
A root= A distinguished oriented edge.
Prescribing a root reduces the number of symmetries ;



Unrooted map



Rooted map

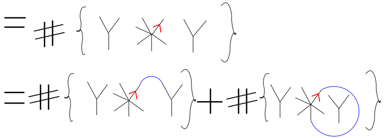
A map M with n edges has $2n/\sharp\text{Automorphism}(M)$ possible roots.
Later we shall count maps with labelled rooted vertices ; a rooted map with $n+1$ vertices has $n!3^n$ possible labellings and rooting of its vertices.

Tutte's Theorem

The number $A((p,3))$ of rooted planar maps with p vertices of valence 3 is equal to $2^{p+1}\frac{(3p)!}{p!(2p+2)!}$

Proof : Surgery on maps =Induction relations on number of maps. Let $A((p,3),(1,n))$ be the number of planar maps with p vertices of degree 3 and one of degree n.

$$A((p,3),(1,n))$$



$$= A((p-1,3),(1,n+1)) + \sum_{k=0}^{n-2}\sum_{\ell=0}^{p} A((\ell,3),(1,k))A((p-\ell,3),(1,n-k-2))$$

More general map enumeration

More general maps could be enumerated by using combinatorial techniques, including surgery but also bijections etc
For instance

- The Ising model on random planar maps [Bousquet-Mélou, Schaeffer](2002)
- The Potts model on random planar maps [Bernardi, Bousquet-Mélou](2009)

However each case is treated separately and is not easy. The physicists have used since the seventies a rather indirect but somehow systematic approach : the matrix integrals.

Outline

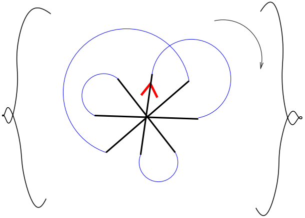
Combinatorics and Gaussian variables

Wick formula : If $(G_1, \cdots, G_{2p})$ is a centered Gaussian vector,

$$\mathbb{E}[G_1 G_2 \cdots G_{2p}] = \sum_{\substack{1 \le s_1 < s_2 .. < s_p \le 2p \\ r_i > s_i}} \prod_{j=1}^{p} \mathbb{E}[G_{s_j} G_{r_j}].$$

Example : If for all i, $G_i = G$ follow the standard Gaussian distribution,

$E[G^{2p}] = \sharp$ { number of rooted graphs with one vertex with valence $2p$ } $= \sharp$



The Gaussian Unitary Ensemble (GUE)

Let $\mathcal{H}_N = \{A \in \mathcal{M}_{N\times N}(\mathbb{C}); A = A^*\}$. The law μ_N of the GUE is the probability measure on $\mathcal{H}_N$

$$d\mu_N(A) = \frac{1}{Z_N} e^{-\frac{N}{2}\mathrm{Tr}\,(A^2)} dA\,.$$

In other words, $A_{lk} = \bar{A}_{kl}$ for $1 \leq k < l \leq N$ and

$$A_{kl} = (2N)^{-\frac{1}{2}}(g_{kl} + i\tilde{g}_{kl}) \text{ for } \quad k < l, \quad A_{kk} = N^{-\frac{1}{2}} g_{kk}$$

where the $(g_{kl}, \tilde{g}_{kl}, k \leq l)$ are i.i.d standard Gaussian variables ;

$$P(d\tilde{g}_{kl}, dg_{kl}, k \leq l) = \prod_{1\leq k\leq l\leq N} e^{-\frac{1}{2}(g_{kl})^2} \frac{dg_{kl}}{\sqrt{2\pi}} \prod_{1\leq k<l\leq N} e^{-\frac{1}{2}(\tilde{g}_{kl})^2} \frac{d\tilde{g}_{kl}}{\sqrt{2\pi}}.$$

The enumeration of maps and one matrix-integrals

For all $d \in \mathbb{N}^*$, all $p \in \mathbb{N}$,

$$\int \left(N\mathrm{Tr}\,(A^d) \right)^p d\mu_N(A) = \sum_{F \geq 0} \frac{1}{N^{\frac{dp}{2} - p - F}} G(d, p, F)$$

$$G(d,p,F) = \sharp\{\text{Union of maps with } F \text{ faces and } p \text{ vertices of valence } d\}.$$

Recall that a connected graph can be embedded into a surface with Euler characteristic

$$\chi = 2 - 2g = \sharp\text{vertices} + \sharp\text{faces} - \sharp\text{edges} = p + F - \frac{pd}{2}.$$

Counting is done with labeled vertices and half-edges.

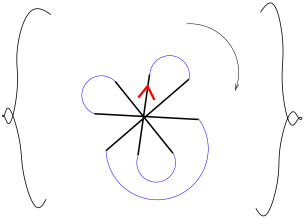
Application

Wigner (1958) already noticed that,

$$\lim_{N\to\infty}\int \frac{1}{N}\mathrm{Tr}(A^d)d\mu_N(A) = G(d,1,\frac{d}{2}+1) = \begin{cases} 0 & \text{if d is odd,} \\ C_{\frac{d}{2}} & \text{otherwise,} \end{cases}$$

where $C_{\frac{d}{2}}$ is the Catalan number, i.e. the number of rooted planar ($g=0$) maps with one vertex of valence d.

(Here, $p = 8, F = 5$)

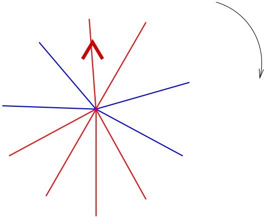
$= \sharp$ 

$G(d,1,\frac{d}{2}+1) = \int x^d d\sigma(x)$, where σ is the semi-circular law

The enumeration of colored maps and several matrices integrals

Let $m \in \mathbb{N}$. To any monomial $q(X_1, \cdots, X_m) = X_{i_1} \cdots X_{i_d}$, we associate (bijectively) a star of type $q =$

oriented vertex with half-edges of color i_1, i_2, ...i_d, ordered clockwise, the first half-edge being marked.



Here $q(X) = X_1{}^2 X_2{}^2 X_1{}^4 X_2{}^2$.

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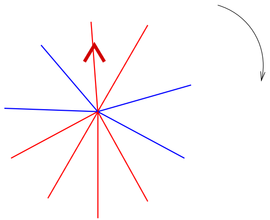


oriented vertex with half-edges of color i_1, i_2, ...i_d, ordered clockwise, the first half-edge being marked.
Here $q(X) = X_1{}^2 X_2{}^2 X_1{}^4 X_2{}^2$.

For any monomial q, all $p \in \mathbb{N}$,

$$\int N\mathrm{Tr}\left(q(A_1, \cdots, A_m)\right) d\mu_N(A_1) \cdots d\mu_N(A_m) = \sum_{g \geq 0} \frac{M(q,g)}{N^{2g-2}},$$

$M(q,g) = \sharp\{$ rooted maps with genus g build on one star of type q $\}$.

The matching is only allowed between half-edges of the same color.

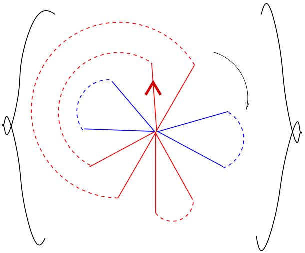
Application

Voiculescu (1984) Let $m \in \mathbb{N}$ and $q(X_1, \cdots, X_m) = X_{i_1} \cdots X_{i_d}$ for $i_1, \cdots, i_d \in \{1, \cdots, m\}$

$$\begin{aligned} \lim_{N\to\infty} \int \frac{1}{N} \mathrm{Tr}\left(q(A_1, \cdots, A_m)\right) d\mu_N(A_1) \cdots d\mu_N(A_m) &= G_c(q, 1, 0) \\ &:= \sigma_m(q) \end{aligned}$$

where $\sigma_m(q)$ is the number of planar maps drawn with a star of type q by gluing half-edges of the same color (Here, $q(\mathbf{X}) = X_1{}^2 X_2{}^2 X_1{}^4 X_2{}^2$)

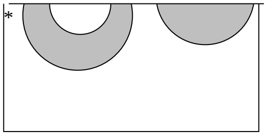
$= \sharp$



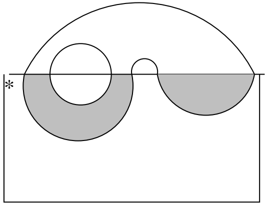
$\sigma_m =$ law of m *free* semi-circular variables.

Application to loop models

The Temperley-Lieb elements are boxes with boundary points connected by non-intersecting strings, equipped with a shading and a marked boundary point.



Problem : Take your favorite Temperley-Lieb element. Count the number of planar matching of the end points of the Temperley-Lieb elements so that there are exactly n loops.

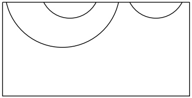


Random matrices and loop enumeration

For a Temperley-Lieb element B, we denote $p \overset{B}{\sim} \ell$ if a string joins the pth boundary point with the ℓth boundary point in B, then we associate to B with k strings the polynomial

$$q_B(X) = \sum_{\substack{i_j = i_p \text{ if } j \overset{B}{\sim} p \\ 1 \le i_j \le n}} X_{i_1} \cdots X_{i_{2k}}.$$

$$q_B(X) = \sum_{i,j,k=1}^{n} X_i X_j X_j X_i X_k X_k \Leftrightarrow$$



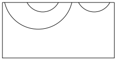
Theorem

If ν^M denotes the law of n independent GUE matrices,

$$\lim_{M\to\infty} \int \frac{1}{M} \mathrm{Tr}\left(q_B(X)\right) \nu^M(dX) = \sum n^{\sharp\text{loops}}$$

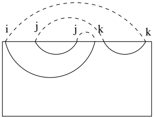
where we sum over all planar maps that can be built on B.

Proof

By Voiculescu's theorem, if $B =$ ,

$$\lim_{M\to\infty} \int \frac{1}{M} \mathrm{Tr}\left(q_B(X)\right) \nu^M(dX)$$

$$= \sum_{i,j,k=1}^{n} \lim_{M\to\infty} \int \frac{1}{M} \mathrm{Tr}\left(X_i X_j X_j X_i X_k X_k\right) \nu^M(dX)$$

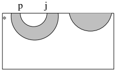
$$= \sum \sum_{i,j,k=1}^{n}$$ 

$$= \sum n^{\sharp \mathrm{loops}}$$

because the indices have to be constant along loops.

Problem ; this only gives the generating function at integer values of the fugacity and therefore does not permit to identify it.

Non integer fugacities, $\beta = 0$ [cf Jones 99']

Recall $p \overset{B}{\sim} j$ if a string joins the pth boundary point with the jth boundary point in the TL element B .

$$q_B(X) = \sum_{i_j = i_p \text{ if } j \overset{B}{\sim} p} X_{i_1} \cdots X_{i_{2k}} \Rightarrow q_B^v(X) = \sum_{e_j = e_p^o \text{ if } j \overset{B}{\sim} p} \sigma_B(w) X_{e_1} \cdots X_{e_{2k}}$$

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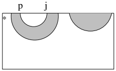
Recall $p \overset{B}{\sim} j$ if a string joins the pth boundary point with the jth boundary point in the TL element B 

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- e_i edges of a bipartite graph $\Gamma = (V = V_+ \cup V_-, E)$ so that the adjacency matrix of Γ has eigenvalue δ with eigenvector $(\mu_v)_{v \in V}$ with $\mu_v \geq 0$ ($\exists$ for any $\delta \in \{2\cos(\frac{\pi}{n})\}_{n \geq 3} \cup [2, \infty[$)

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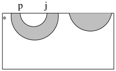
Recall $p \overset{B}{\sim} j$ if a string joins the pth boundary point with the jth boundary point in the TL element B 

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- The sum runs over loops $w = e_1 \cdots e_{2k}$ in Γ which starts at v. $v \in V_+$ iff $*$ is in a white region.

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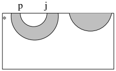
Recall $p \overset{B}{\sim} j$ if a string joins the pth boundary point with the jth boundary point in the TL element B 

.

$$q_B(X) = \sum_{i_j = i_p \text{ if } j \overset{B}{\sim} p} X_{i_1} \cdots X_{i_{2k}} \Rightarrow q_B^v(X) = \sum_{e_j = e_p^o \text{ if } j \overset{B}{\sim} p} \sigma_B(w) X_{e_1} \cdots X_{e_{2k}}$$

• e_i edges of a bipartite graph $\Gamma = (V = V_+ \cup V_-, E)$ so that the adjacency matrix of Γ has eigenvalue δ with eigenvector $(\mu_v)_{v \in V}$ with $\mu_v \geq 0$ ($\exists$ for any $\delta \in \{2\cos(\frac{\pi}{n})\}_{n \geq 3} \cup [2, \infty[$)

• The sum runs over loops $w = e_1 \cdots e_{2k}$ in Γ which starts at v. $v \in V_+$ iff $*$ is in a white region.

•If $\sigma(e) := \sqrt{\frac{\mu_{t(e)}}{\mu_{s(e)}}}$, $e = (s(e), t(e))$, we set $\sigma_B(e_1 \cdots e_{2p})$ to be the product $\prod_{\substack{i \overset{B}{\sim} j \\ i < j}} \sigma(e_i)$ of the $\sigma(e)$ so that each string of B brings $\sigma(e)$ with e the edge which labels the start of the string.

Non integer fugacities

For $e \in E$, $e = (s(e), t(e))$, X_e^M are independent (except $X_{e^o} = X_e^*$) $[M\mu_{s(e)}] \times [M\mu_{t(e)}]$ matrices with i.i.d centered Gaussian entries with variance $1/(M\sqrt{\mu_{s(e)}\mu_{t(e)}})$.

Recall
$$q_B^v(X^M) = \sum_{\substack{w=e_1\cdots e_{2k}\in L_B \\ s(e_1)=v}} \sigma_B(w) X_{e_1}^M \cdots X_{e_{2k}}^M$$

Theorem (G-Jones-Shlyakhtenko 07')

Let Γ be a bipartite graph whose adjacency matrix has δ as Perron-Frobenius eigenvalue. Let B be Temperley-Lieb element so that $*$ is in an unshaded region. Then, for all $v \in V^+$

$$\lim_{M\to\infty} E\left[\frac{1}{M\mu_v}\mathrm{Tr}(q_B^v(X^M))\right] = \sum \delta^{\sharp\text{loops}}$$

where the sum runs above all planar maps built on B.

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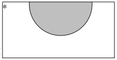
where the sum runs above all planar maps built on B.

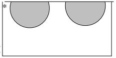
Based on $\sum_{e\in E:s(e)=v} \mu_{t(e)} = \delta\mu_v$.

Proof by examples

If $B =$ , for all $v \in V^+$

$$\mathbb{E}[\frac{1}{M\mu_v}\mathrm{Tr}(\sum_{e:s(e)=v} \sigma(e)X_eX_{e^0}))] = \frac{1}{M\mu_v}\sum_{e:s(e)=v}\sqrt{\frac{\mu_{t(e)}}{\mu_v}}\frac{M\mu_v M\mu_{t(e)}}{M\sqrt{\mu_{t(e)}\mu_{s(e)}}}$$

$$= \frac{1}{\mu_v}\sum_{e:s(e)=v}\mu_{t(e)} = \delta$$

If $B =$ , for all $v \in V^+$

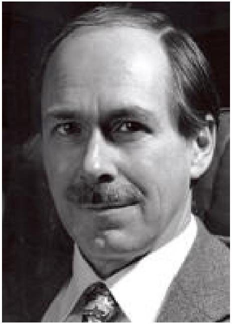
$$\lim_{M\to\infty}\mathbb{E}[\frac{1}{M\mu_v}\mathrm{Tr}(\sum_{\substack{e:s(e)=v\\ s(f)=v}}\sigma(e)\sigma(f)X_eX_{e^0}X_fX_{f^0})]$$

$$= \delta^2 + \frac{1}{\mu_v}\sum_{e=f}\frac{\mu_{t(e)}}{\mu_v}\frac{\mu_v^2\mu_{t(e)}}{\mu_{t(e)}\mu_v} = \delta^2 + \delta$$

More generally, the edge is constant along the loop and brings the contribution $\mu_{t(e)}/\mu_v$ hence leading after summation to δ.

Random matrices and the enumeration of maps with several vertices

't Hooft noticed in 1974 that matrix integrals are generating functions for the enumeration of maps. If μ_N is the Gaussian law on $N \times N$ Hermitian matrices, we have at list formally



$$\log \int e^{-N\mathrm{Tr}\left(\sum_{i=1}^{p} t_i q_i(A_1,\cdots,A_m)\right)} d\mu_N^{\otimes m}(A_1,\cdots,A_m)$$

$$= \sum_{k_1,..,k_p\in\mathbb{N}} \sum_{g\geq 0} \frac{1}{N^{2g-2}} \prod_{j=1}^{p} \frac{(-t_j)^{k_j}}{k_j!} M\left((k_i,q_i)_{1\leq i\leq p}; g\right)$$

where $M\left((k_i,q_i)_{1\leq i\leq p}; g\right)$ counts maps with genus g build on k_i stars of type q_i so that only half-edges of the same color are matched.

The formal derivation

To prove that formally $\log \int e^{-N\mathrm{Tr}\left(\sum_{i=1}^{p} t_i q_i(A_1,\cdots,A_m)\right)} d\mu_N^{\otimes m}(A_1,\cdots,A_m)$ is a generating function for maps : Expand the exponential together with the remarks

- For all monomials $q_1,\ldots,q_p$ we have

$$\int \prod_{i=1}^{p}(N\mathrm{Tr}(q_i(A_1,\cdots,A_m)))d\mu_N^{\otimes m}(A_1,\cdots,A_m) = \sum_{g\geq 0} \frac{M(q_1,\ldots,q_p;g)}{N^{2-2g}}$$

where $M(q_1,\ldots,q_p;g)$ is the number of union of maps with genus g which can be build upon one star of type q_1, ..., one star of type q_p.

- Taking the logarithm reduces the sum to connected graphs.

First order asymptotics

$$F_N(\mathbf{t}) = \frac{1}{N^2} \log \int e^{-N\mathrm{Tr}(V_{\mathbf{t}}(A_1,\cdots,A_m))} d\mu_N(A_1)..d\mu_N(A_m), \; V_{\mathbf{t}} = \sum_{i=1}^{p} t_i q_i.$$

Theorem Hypothesis : $\phi_{V_{\mathbf{t}}} : (A_1(ij), \cdots, A_m(ij))_{i\le j} \to \mathrm{Tr}\,(V_{\mathbf{t}}(\mathbf{A}))$ is real-valued. $\phi_{V_{\mathbf{t}}}$ is convex (or we add a cutoff).
For all $\ell \ge 0$, $\exists\, \varepsilon_\ell > 0$ so that if $|\mathbf{t}| = \sum_{i=1}^{p} |t_i| \le \varepsilon_\ell$,

$$F_N(\mathbf{t}) = \sum_{g=0}^{\ell} \frac{1}{N^{2g}} \sum_{k_1,\cdots,k_p} \prod \frac{(-t_i)^{k_i}}{k_i!} M\left((q_i,k_i)_{1\le i\le p}; g\right) + o\left(\frac{1}{N^{2\ell}}\right)$$

$M\left((q_i,k_i)_{1\le i\le p}; g\right) = \sharp\{$maps with genus g with k_i stars of type $q_i\}$

-$m=1$: Ambjórn et al. (95), Albeverio-Pastur-Scherbina (01), Ercolani-McLaughlin (03)

-$m\ge 2$: G.-Maurel-Segala ($\ell \le 1$ (05)(06)),Maurel-Segala (for all ℓ (06))

Small t_i's expansion, more results

Take $V_{\mathbf{t}} = \sum_{i=1}^{p} t_i q_i$. Let $\hat{\mu}_{\mathbf{A}}^N$ be the (mean) empirical distribution

$$\hat{\mu}_A^N(P) := \int \frac{1}{N} \mathrm{Tr}\left(P(A_1, \cdots, A_m)\right) d\mu_{V_{\mathbf{t}}}^N(A_1, \cdots, A_m)$$

$$d\mu_{V_{\mathbf{t}}}^N(A_1, \cdots, A_m) = e^{-N^2 F_N(\mathbf{t}) - N\mathrm{tr}(V_{\mathbf{t}}(A_1, \cdots, A_m))} d\mu_N(A_1)..d\mu_N(A_m)$$

Then (same hypothesis as before)

$$\hat{\mu}_A^N(P) = \sum_{g=0}^{\ell} \frac{1}{N^{2g}} \tau_g^{(t_i)_{1\le i\le p}}(P) + o(\frac{1}{N^{2\ell}})$$

with

$$\tau_g^{(t_i)_{1\le i\le p}}(P) = \sum \prod \frac{(-t_i)^{k_i}}{k_i!} M((q_i, k_i)_{1\le i\le p}, (P, 1); g).$$

Proof. $g = 0$, $V_{\mathbf{t}} = \sum_{i=1}^{n} t_i q_i$

All the limit points $\tau_{\mathbf{t}}$ of

$$\hat{\mu}_A^N(P) := \int \frac{1}{N} \mathrm{tr}\left(P(A_1, \cdots, A_m)\right) d\mu_{V_{\mathbf{t}}}^N(A_1, \cdots, A_m)$$

satisfy Schwinger-Dyson's (or loop) equation

$$\tau_{\mathbf{t}}(X_i Q) = \tau_{\mathbf{t}} \otimes \tau_{\mathbf{t}}\left(\partial_i Q\right) - \tau_{\mathbf{t}}(D_i V Q) \quad \forall Q \in \mathbb{C}\langle X_1, \cdots, X_m\rangle \quad \forall i \in [1, m]$$

with $\partial_i P = \sum_{P = P_1 X_i P_2} P_1 \otimes P_2$, $D_i P = \sum_{P = P_1 X_i P_2} P_2 P_1$.
By convexity (or cutoff), $\exists R < \infty$ so that $|\tau_{\mathbf{t}}(q)| \leq R^{\deg q}$.
If $|\mathbf{t}| \leq \varepsilon_0$, there is a unique solution

$$\tau_{\mathbf{t}}(q) = \sum_{k_1, \cdots, k_p \in \mathbb{N}} \prod_{i=1}^{p} \frac{(-t_i)^{k_i}}{k_i!} M\left((q, 1), (q_i, k_i)_{1 \leq i \leq p}; 0\right)$$

How to get Schwinger-Dyson's equation

By integration by parts, if $q = q(A_1, \cdots, A_m)$

$$\begin{aligned} \int \mathrm{Tr}(A_i q) d\mu^N_{V_\mathbf{t}} &= \sum_{k,\ell} \int [\frac{1}{N} \partial_{A_i(\ell k)} q_{\ell k} - q_{\ell k} \partial_{A_i(\ell k)} \mathrm{tr}(V)] d\mu^N_{V_\mathbf{t}} \\ &= \int [\frac{1}{N} \mathrm{Tr} \otimes \mathrm{Tr}](\partial_i q) - \mathrm{Tr}(q \mathcal{D}_i V)] d\mu^N_{V_\mathbf{t}} \end{aligned}$$

By concentration of measure for all monomial P

$$\mu^N_{V_\mathbf{t}} \left(|\frac{1}{N} \mathrm{Tr} P - \mu^N_{V_\mathbf{t}}(\frac{1}{N} \mathrm{Tr} P)| \geq \delta \right) \leq C e^{-cN\delta}$$

and therefore as N goes to infinity any limit point $\tau(P)$ of $\mu^N_{V_\mathbf{t}} | \frac{1}{N} \mathrm{Tr} P, P \in \mathbb{C}\langle X_1, \ldots, X_n \rangle$ satisfies the Schwinger equation

$$\tau_\mathbf{t}(X_i q) = \tau_\mathbf{t} \otimes \tau_\mathbf{t} (\partial_i q) - \tau_\mathbf{t}(D_i V q)$$

Schwinger-Dyson are equivalent to Tutte equation

$M((q_1,k_1)\cdots(q_p,k_p),q)$ denotes the number of planar maps build on k_i stars of type q_i and 1 of type $q = X_{i_1}\cdots X_{i_n}$. Tutte surgery shows that $M((q_1,k_1)\cdots(q_p,k_p),q)$ equals

$$\sum_{q=X_{i_1}p_1X_{i_1}p_2}\sum_{\ell_i=0}^{k_i}\frac{k_i!}{\ell_i!(k_i-\ell_i)!}M((q_i,\ell_i)_{1\le i\le p},p_1)M((q_i,k_i-\ell_i)_{1\le i\le p},p_2)$$

$$+\sum_{i=1}^{p}k_i\sum_{q_i=q_i^1X_{i_1}q_i^2}M((q_1,k_1)\cdots(q_i,k_i-1)\cdots(q_p,k_p),(qq_i^2q_i^1,1))$$

Put $\tau_{\mathbf{t}}(q)=\sum\prod\frac{(-t_i)^{k_i}}{k_i!}M((q_1,k_1)\cdots(q_p,k_p),(q,1))$. Then, summation shows that $\tau_{\mathbf{t}}$ satisfies the Schwinger-Dyson equation

$$\tau_{\mathbf{t}}(X_iQ)=\tau_{\mathbf{t}}\otimes\tau_{\mathbf{t}}(\partial_iQ)-\tau_{\mathbf{t}}(D_iVQ)$$

Advantage : we know that $\tau_{\mathbf{t}}$ is a tracial state !

Matrix models for loop models

Let B_i be Temperley Lieb elements with $*$ with color $\sigma_i \in \{+,-\}$, $1 \le i \le p$. Let Γ be a bipartite graph whose adjacency matrix has eigenvalue δ as before. Let ν^M be the law of the previous independent rectangular Gaussian matrices and set

$$d\nu^M_{(B_i)_i}(X_e) = \frac{1_{\|X_e\|_\infty \le L}}{Z^M_B} e^{M\mathrm{tr}(\sum_{i=1}^p \beta_i \sum_{v \in V_{\sigma_i}} \mu_v q^v_{B_i}(X))} d\nu^M(X_e).$$

Theorem (G-Jones-Shlyakhtenko-Zinn Justin 10')

For any $L > 2$, for β_i small enough real numbers, for any Temperley-Lieb element B with color σ, any $v \in V_\sigma$,

$$\lim_{M\to\infty} \int \frac{1}{M\mu_v} tr(q^v_B(X)) d\nu^M_{(B_i)_i}(X) = \sum_{n_i \ge 0} \sum \delta^{\sharp \mathrm{loops}} \prod_{i=1}^p \frac{\beta_i^{n_i}}{n_i!}$$

where we sum over the planar maps build on n_i TL elements B_i and one B.

Matrix model for the Potts model

Let $\delta \in \{2\cos(\frac{\pi}{n})\}_{n\geq 3} \cup [2,\infty[$ and $\Gamma = (V_+ \cup V_-, E)$ be a bipartite graph with eigenvalue δ and Perron-Frobenius eigenvector μ.

$$\nu^M_{\beta_\pm}(dX_e) = \frac{1_{\|X_e\|_\infty \leq L}}{Z^M_{\beta_\pm}} e^{M\mathrm{tr}\left(\sum_{v\in V}\mu_v \sum_{\sigma=\pm}\beta_\sigma 1_{v\in V_\sigma}\left(\sum_{e:s(e)=v}\sigma(e)X_eX_e^*\right)^2\right)}$$

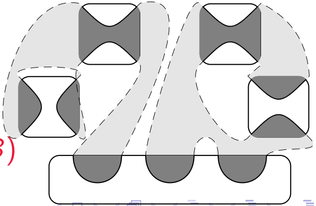
$$\prod_e e^{-\frac{M}{2}(\mu_{s(e)}\mu_{t(e)})^{\frac{1}{2}}\mathrm{tr}(X_eX_e^*)} dX_e dX_e^*$$

Theorem (G-Jones-Shlyakhtenko-Zinn Justin 10')

Then, for L large enough, $\beta_\pm$ small enough, for all TL B, $v \in V_{\sigma_B}$,

$$\lim_{M\to\infty}\frac{1}{M\mu_v}\int tr(q^v_B(X))\nu^M_{\beta_\pm}(dX)$$

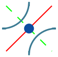
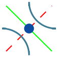
$$= \sum_{n_+,n_-\geq 0}\sum\prod_{i=\pm}\frac{\beta_i^{n_i}}{n_i!}\delta^{\sharp\mathrm{loops}} =: \mathrm{Tr}_{\beta_\pm,\delta}(B)$$



Outline

Recap about the Potts model

Assume for simplicity Γ finite hereafter (which includes $\delta = 2\cos(\pi/n)$, $n \geq 3$). By construction

$$\sum_{\Gamma} \delta^{\#\text{ loops}} \beta_-^{\#}$$

$$\beta_+^{\#}$$


equals

$$\lim_{M\to\infty} \frac{1}{M^2 \sum \mu(v)^2} \log Z^M_{\beta_\pm}$$

with for L large enough $Z^M_{\beta_\pm}$ equals to

$$\int_{\|X_e\|\leq L} e^{M\mathrm{Tr}\left(\sum_{v\in V} \mu_v (\beta_+ 1_{v\in V_+} + \beta_- 1_{v\in V_-})\left(\sum_{e:s(e)=v} \sqrt{\mu_v \mu_{t(e)}} X_e X_e^*\right)^2\right)} \nu^M(dX)$$

if ν^M is the law of X_e, $[M\mu_{s(e)}] \times [M\mu_{t(e)}]$ matrices with iid centered Gaussian entries with covariance $(M^2 \mu_{s(e)} \mu_{t(e)})^{-\frac{1}{2}}$

Solving the matrix model :HS transformation

With G_v $[M\mu_v] \times [M\mu_v]$ independent matrices from the GUE and X_e $[M\mu_{s(e)}] \times [M\mu_{t(e)}]$ matrices with covariance $(M^2\mu_{s(e)}\mu_{t(e)})^{-\frac{1}{2}}$ under ν^M, with $\alpha_\pm = \sqrt{2\beta_\pm}$, $\Delta(\lambda) = \prod_{i \neq j}(\lambda_i - \lambda_j)$, $Z^M_{\beta_\pm} =$

$$
\begin{aligned}
&= \int_{\|X_e\| \leq L} e^{M \mathrm{Tr}\left(\sum_{v \in V} \mu_v (\beta_+ 1_{v \in V_+} + \beta_- 1_{v \in V_-})\left(\sum_{e:s(e)=v} \sigma(e) X_e X_e^*\right)^2\right)} \nu^M(dX) \\
&= \int_{\|X_e\| \leq L} e^{M \mathrm{Tr}\left(\sum_{\sigma=\pm} \sum_{v \in V_\sigma} \alpha_\sigma G_v \left(\sum_{e:s(e)=v} \sqrt{\mu_v \mu_{t(e)}} X_e X_e^*\right)\right)} \nu^M(dX, dG) \\
&\approx \int_{\|G_v\| \leq L'} \prod_{e \in E_+} e^{-\mathrm{Tr} \otimes \mathrm{Tr}(\log(I + \alpha_+ I \otimes G_{s(e)} + \alpha_- G_{t(e)} \otimes I))} \nu^M(dG) \\
&= \int_{|\lambda_i^e| \leq L'} \prod_{e \in E_+} e^{-\sum_{i=1}^{[M\mu_{s(e)}]} \sum_{j=1}^{[M\mu_{t(e)}]} \log(1 + \alpha_+ \lambda_i^e + \alpha_- \eta_j^e)} \\
&\quad \Delta(\eta^e)\Delta(\lambda^e) e^{-\frac{[M\mu_{s(e)}]}{2} \sum_{i=1}^{[M\mu_{s(e)}]} (\lambda_i^e)^2 - \frac{[M\mu_{t(e)}]}{2} \sum_{j=1}^{[M\mu_{t(e)}]} (\eta_j^e)^2} d\lambda^e d\eta^e.
\end{aligned}
$$

The auxiliary matrix model :large deviations

Let $P_\alpha^{M,L}$ be absolutely cont. wrt Lebesgue with density

$$\frac{1_{|\lambda^v|\le L}}{Z_\alpha^{M,L}}\prod_{e\in E_+}\prod_{\substack{1\le i\le[M\mu_{s(e)}]\\1\le j\le[M\mu_{t(e)}]}}\frac{1}{1+\alpha_+\lambda_i^{s(e)}+\alpha_-\lambda_j^{t(e)}}\prod_{v\in V}\Delta(\lambda^v)e^{-\frac{M\mu_v}{2}\sum(\lambda_i^v)^2}.$$

By large deviation analysis,

$$P_\alpha^{M,L}\left(d(\frac{1}{M\mu_v}\sum\delta_{\lambda_i^v},\nu_v)<\epsilon\,\forall\, v\right)\approx e^{-M^2[I(\nu_v,v\in V)-\inf I]}$$

with if $\Sigma(\nu)=\int\log|x-y|d\nu(x)d\nu(y)$

$$I(\nu_v,v\in V)=\sum_v\frac{\mu_v^2}{2}(\int x^2d\nu_v(x)-2\Sigma(\nu_v))$$

$$-\sum_{e\in E_+}\mu_v\mu_{t(e)}\int\log|1+\alpha_+x+\alpha_-y|d\nu_v(x)d\nu_{t(e)}(y).$$

Moreover $\lim_{M\to\infty}M^{-2}\log Z_\alpha^{M,L}=-\inf I$

The auxiliary matrix model (Kostov, 95')

Let Γ be as before and consider the Gibbs measure $P_\alpha^{M,L}$ absolutely cont. wrt Lebesgue with density

$$\frac{1_{|\lambda^v|\leq L}}{Z_\alpha^{M,L}} \prod_{e\in E_+} \prod_{\substack{1\leq i\leq [M\mu_{s(e)}] \\ 1\leq j\leq [M\mu_{t(e)}]}} \frac{1}{1+\alpha_+\lambda_i^{s(e)}+\alpha_-\lambda_j^{t(e)}} \prod_{v\in V} \Delta(\lambda^v) e^{-\frac{M\mu_v}{2}\sum(\lambda_i^v)^2}.$$

Theorem [G-J-S-ZJ 10'] • I achieves its minimal value at a unique set of probability measures $\nu_v, v\in V$.

• $\exists\, \nu_+,\nu_-\in P(\mathbb{R})$, so that $\nu_v=\nu_\pm$ if $v\in V_\pm$.

• For all $L>2$ not too large,

$$\lim_{M\to\infty} E\Big[\frac{1}{M\mu_v}\sum_{i=1}^{[M\mu_v]}(\lambda_i^v)^p\Big] = \int x^p d\nu_v(x) \qquad \forall \quad p\in\mathbb{N}, \quad v\in V.$$

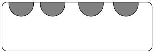
More about the relation between Potts model and its auxiliary model

Let $M(z) = \int \sum_{n \geq 0} z^n x^n d\nu_+(x)$ and put

$$\gamma(z) = \alpha_+ z / (1 - z^2 M(z))$$

and

$$C(z, A, B) = \sum_{n \geq 0} z^n \sum \delta^\ell \frac{\beta_+^{n_+}}{n_+!} \frac{\beta_-^{n_-}}{n_-!}$$

where we sum over the planar maps build over n_+ (resp. n_-) vertices with two strings and two black (resp. white) regions and one vertex of type $B_n =$ 
Then, for small z,

$$C(z, A, B) = \frac{\alpha_+}{z}[1 - \frac{\alpha_+ \gamma^{-1}(z)}{z}].$$

Study of ν_-, ν_+

(ν_-, ν_+) are the unique minimizers of,

$$\sum_{\varepsilon=\pm}\left(\frac{1}{2}\int x^2 d\nu_\varepsilon(x) - \int \log|x-y| d\nu_\varepsilon(x) d\nu_\varepsilon(y)\right)$$

$$+\delta \int \log|1+\alpha_+ x + \alpha_- y| d\nu_+(x) d\nu_-(y).$$

Let p_+ (resp. p_-) be the law of $1+\alpha_+ x$ and $-\alpha_- y$ under ν_+ (resp. ν_-).

•For $\alpha_\pm$ small enough, $p_\pm$ has a connected support $[a_\pm, b_\pm]$ around 1 (resp. 0) and $a_- < b_- < a_+ < b_+$.

• Set $G_\pm(z) = \int (z-x)^{-1} dp_\pm(x)$. Then

$$G_\pm(z+i0) + G_\pm(z-i0) = P_\pm(z) + \delta G_\mp(z) \qquad z \in [a_\pm, b_\pm]$$

with $P_-(z) = z/\alpha_-$, $P_+(z) = (1-z)/\alpha_+$.

• If $q = 2\cos\pi/n$, $n \geq 3$, $G_\pm$ satisfy an algebraic equation.

Exact solution for $G_\pm(z) = \int (z-x)^{-1} d\nu_\pm(z)$

Introduce

$$u(z) = \int_{b_-}^{z} \frac{1}{\sqrt{(v-a_+)(v-a_-)(v-b_+)(v-b_-)}} dv,$$

with inverse $z(u)$. With $\delta = q + q^{-1}$ set

$$\omega_\pm(u) = q^{\pm 1} G_+(z(u)) - G_-(z(u)) \pm \frac{1}{q-q^{-1}} (P_+ + q^{\pm 1} P_-) z(u).$$

Then $\omega_\pm(u+2K) = \omega_\pm(u) \qquad \omega_\pm(u+2iK') = q^{\pm 2}\omega_\pm(u)$

Theorem

$\omega_\pm$ are meromorphic with only poles at $\pm u_\infty$. Set

$$\Theta(u) = 2\sum_{k=0}^{\infty} e^{i\pi \frac{\omega_2}{\omega_1}(k+1/2)^2} \sin(2k+1)\frac{\pi u}{\omega_1}$$

$$\omega_+(u) = c_+ \frac{\Theta(u-u_\infty-\nu\omega_1)}{\Theta(u-u_\infty)} + c_- \frac{\Theta(u+u_\infty-\nu\omega_1)}{\Theta(u+u_\infty)}$$

Conclusion

- Loop models can be represented by matrix models for a continuum of fugacities δ.
- Therefore the computation of the matrix model allows to identify the generating function for all δ by analyticity.
- The matrix model for the shaded $O(n)$ model (or the Potts model on random graphs) can be computed.