

Positivity Proofs for Linear Recurrences through Cones

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Examples

[Straub-Zudilin 2015]

$$s_n = \sum_{k=0}^n (-27)^{n-k} 2^{2k-n} \frac{(3k)!}{k!^3} \binom{k}{n-k} \geq 0, n \in \mathbb{N}$$

Combinatorics

$$v_n = \binom{2n}{n}^2 - \frac{16^n}{4n} > 0, n > 1$$

Gillis-Reznick-Zeilberger family of sequences

$$u_n^{(k)} = \sum_{j=0}^n (-1)^j \frac{(kn - (k-1)j)! k!^j}{(n-j)!^k j!} \geq 0, n \in \mathbb{N}$$
$$k \geq 4$$

Turàn's inequality

$$u_{n-1}(x) = P_n(x)^2 - P_{n-1}(x) P_{n+1}(x) > 0, x \in (-1, 1)$$

$$P_n(x) = \frac{1}{2^n n!} \frac{d^n}{dx^n} (x^2 - 1)^n$$

Solutions of **Linear Recurrences**

Positivity Problem

Input:
$$\begin{cases} p_d(n)u_{n+d} = p_{d-1}(n)u_{n+d-1} + \cdots + p_0(n)u_n, p_i \in \mathbb{Q}[n] \\ u_0, u_1, \dots, u_{d-1} \in \mathbb{Q} \end{cases}$$


Order d

Output: True if $\forall n \in N, u_n > 0$

Recurrence to vector form

$$p_d(n)u_{n+d} = p_{d-1}(n)u_{n+d-1} + \cdots + p_0(n)u_n$$

Let $U_n = (u_n, u_{n+1}, \dots, u_{n+d-1})^t$, then

$$U_{n+1} = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ \frac{p_0(n)}{p_d(n)} & \frac{p_1(n)}{p_d(n)} & \frac{p_2(n)}{p_d(n)} & \cdots & \frac{p_{d-1}(n)}{p_d(n)} \end{pmatrix} U_n$$

$A(n)$

Assumption: $A := \lim_{n \rightarrow \infty} A(n) \in \text{GL}_d(\mathbb{Q})$

Eigenvalues of the recurrence: Eigenvalues of A

$\lambda_1, \dots, \lambda_\nu \in \mathbb{C}$ are the **dominant eigenvalues** of A if :

$$|\lambda_1| = |\lambda_2| = \cdots = |\lambda_\nu| > |\lambda_{\nu+1}| \geq |\lambda_{\nu+2}| \cdots$$

⚠ If no dominant eigenvalue $\in \mathbb{R}_{>0} \implies u_n \not\rightarrow 0$ (for generic initial conditions) Pringsheim's theorem

Decidability

C-finite	P-finite
$u_{n+d} = c_{d-1}u_{n+d-1} + \cdots + c_0u_n, c_i \in \mathbb{Q}$	$p_d(n)u_{n+d} = p_{d-1}(n)u_{n+d-1} + \cdots + p_0(n)u_n, p_i \in \mathbb{Q}[n]$
$u_n = \sum q_i(n)\lambda_i^n, q_i \in \bar{\mathbb{Q}}[n]$	No general closed form

Decidability	
• Arbitrary $d \in \mathbb{N}$ and [Ouaknine-Worrell 2014] $\lambda_1 > \lambda_2 \geq \cdots$ Generic initial conditions • $d \leq 5$, arbitrary λ_i • $d = 6$: Open problems in diophantine approximation	• $d = 2$ and cases of $d = 3$ with [Kauers-Pillwein 2010] $\lambda_1 > \lambda_2 \geq \cdots$ Generic initial conditions + λ_1 simple Arbitrary $d \in \mathbb{N}$? Several dominant eigenvalues?

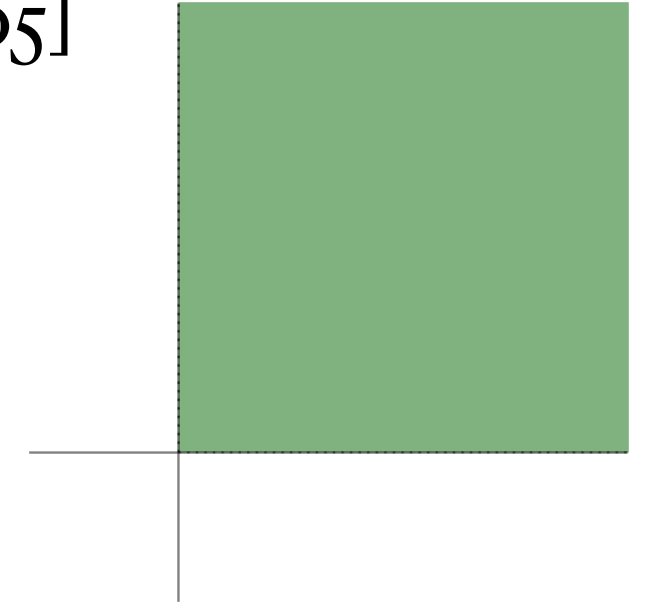
Proofs by induction

$$p_d(n)u_{n+d} = p_{d-1}(n)u_{n+d-1} + \cdots + p_0(n)u_n$$

[Gerhold-Kauers 2005]

By induction: Use **quantifier elimination** to find **iteratively** $m \in \mathbb{N}$ s.t

$$\forall n \geq 0, \forall u_n \geq 0, \dots, \forall u_{n+m} \geq 0 \implies u_{n+m+1} > 0$$

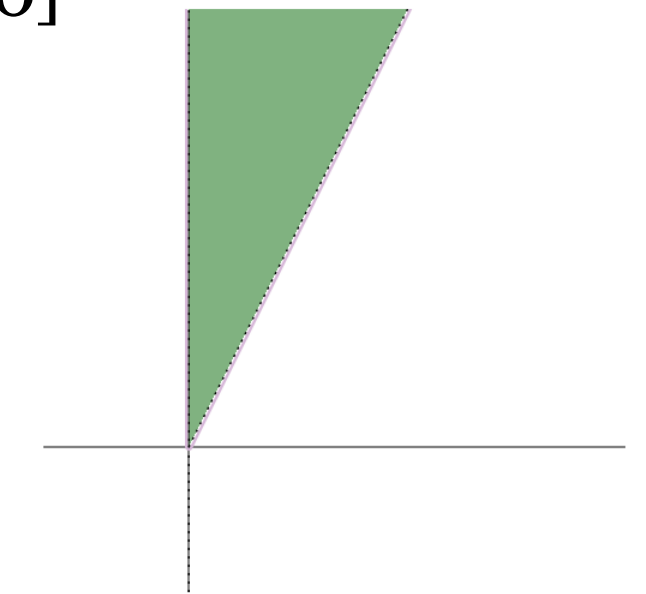


! No guarantee to find m

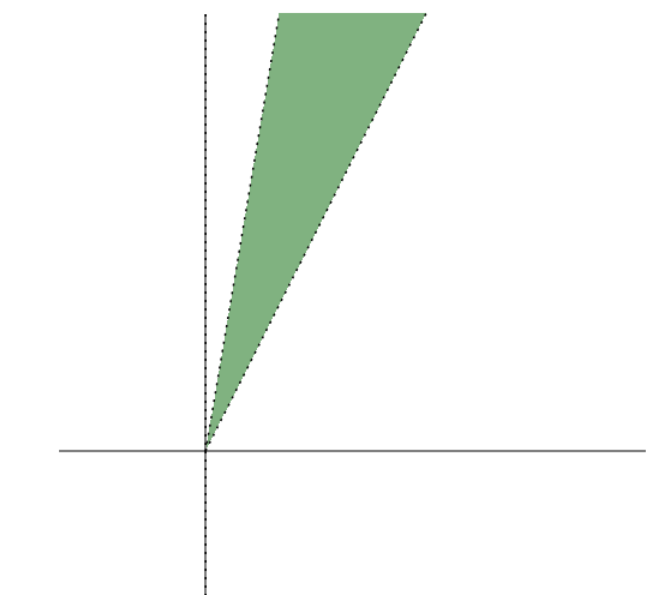
[Kauers-Pillwein 2010]

Variant: Change the induction hypothesis to $\exists \beta > 0, u_{n+1} > \beta u_n > 0$

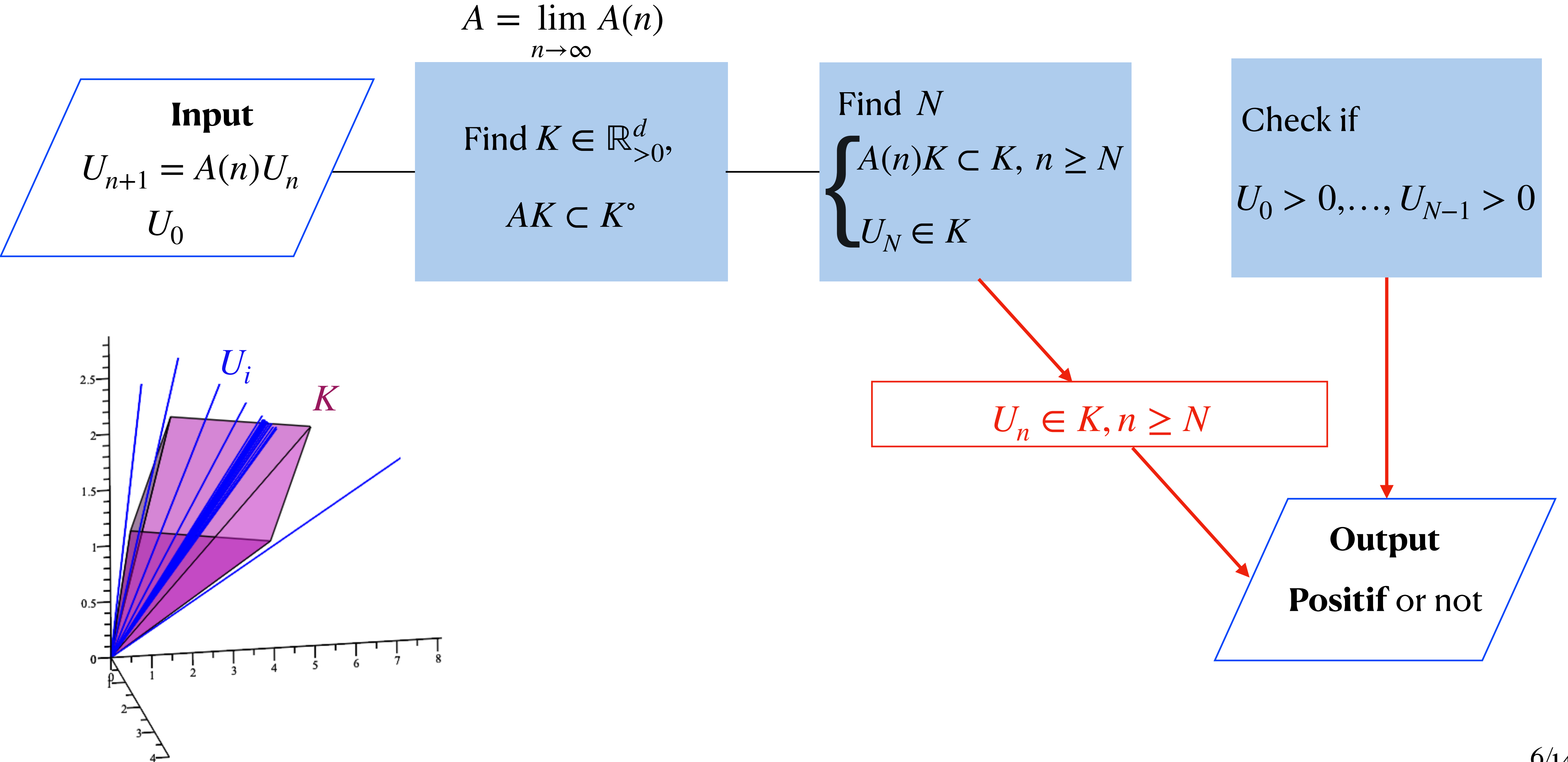
Termination (under assumptions) for $d = 2$ and cases of $d = 3$



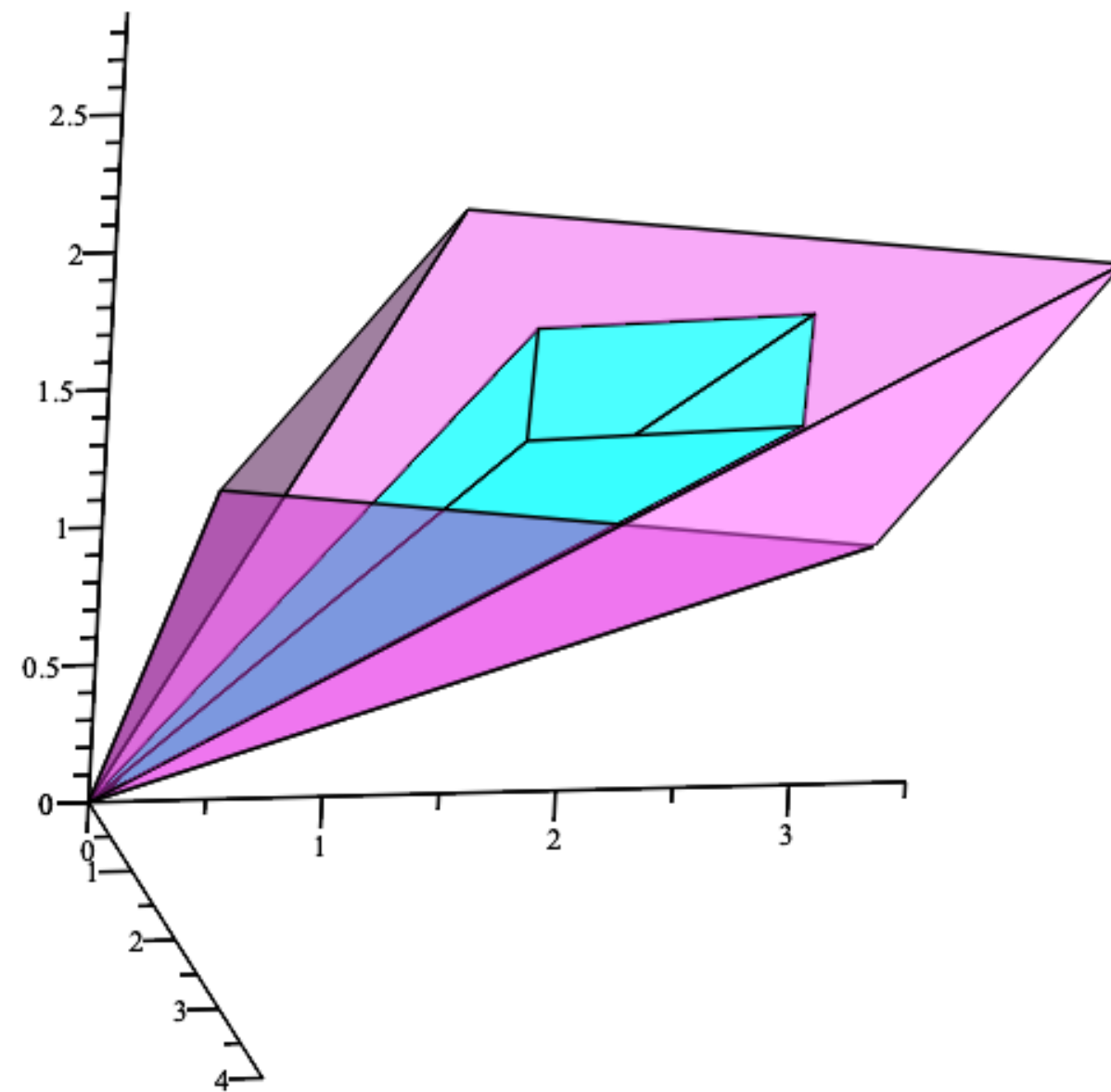
Idea: Find “suitable” cones



Cone-based approach



Contracted cones



$$AK \setminus \{0\} \subset K^\circ$$

A contracts K

$$K \neq \mathbb{R}^d, K^\circ \neq \emptyset$$

Theorem

There exists a **proper** cone K contracted by A if and only if:

$$\lambda_1 > |\lambda_2| \geq \dots$$

λ_1 simple

[Vandergraft 68]

Explicit construction based on the Jordan form of the matrix A

K is **not unique**

Termination: One simple dominant eigenvalue

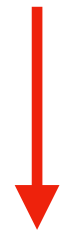
$$U_{n+1} = A(n)U_n, \quad A = \lim_{n \rightarrow \infty} A(n) \in \mathbb{Q}^{d \times d}$$

$$\lambda_1 = |\lambda_2| = \dots = |\lambda_v|?$$

Theorem [I.-Salvy 2024]

Positivity is decidable for $d \in \mathbb{N}$ with $\lambda_1 > |\lambda_2| \geq |\lambda_3| \geq \dots$ + Generic initial conditions + λ_1 simple

Vandergraft Construction



Find $K \in \mathbb{R}_{>0}^d$ s.t.
 $AK \subset K^\circ$

By approximating
the eigenvalues

Convergence of $A(n) \rightarrow A$



Compute n_0 s.t.
 $A(n)K \subset K, n \geq n_0$

Solving polynomial
inequalities in $\mathbb{Q}[n]$

Generic initial
conditions



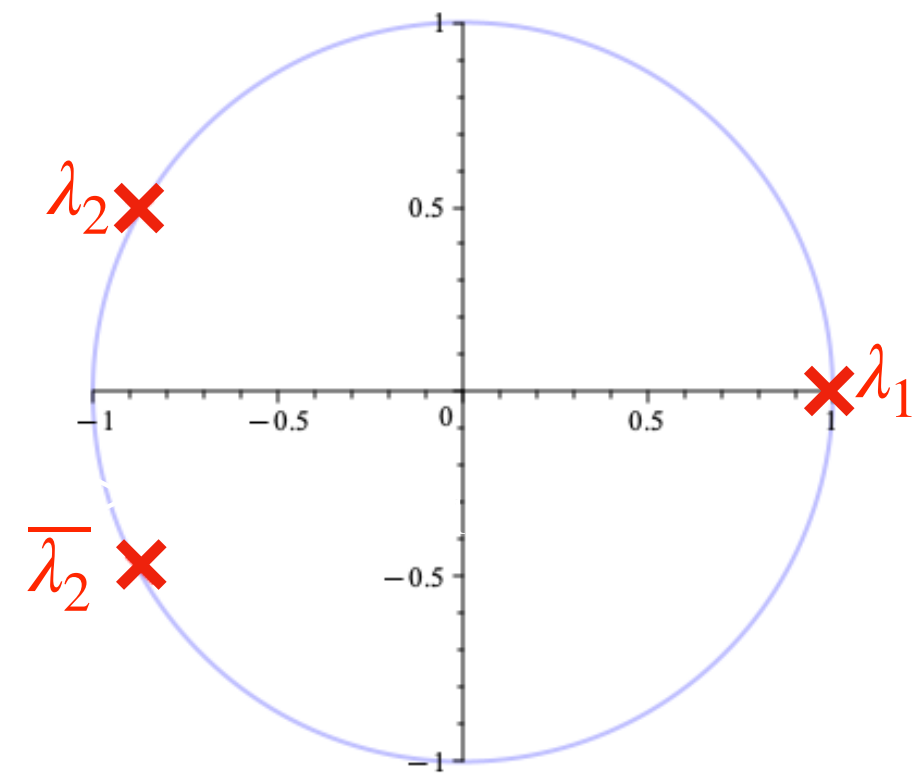
[Friedland 2006]

Find $N \geq n_0$ s.t.
 $U_N \in K$

Computations in $\mathbb{Q}$

Example with no contracted cone

$$u_{n-1} = P_n \left(\frac{1}{4} \right)^2 - P_{n-1} \left(\frac{1}{4} \right) P_{n+1} \left(\frac{1}{4} \right), n \geq 1$$

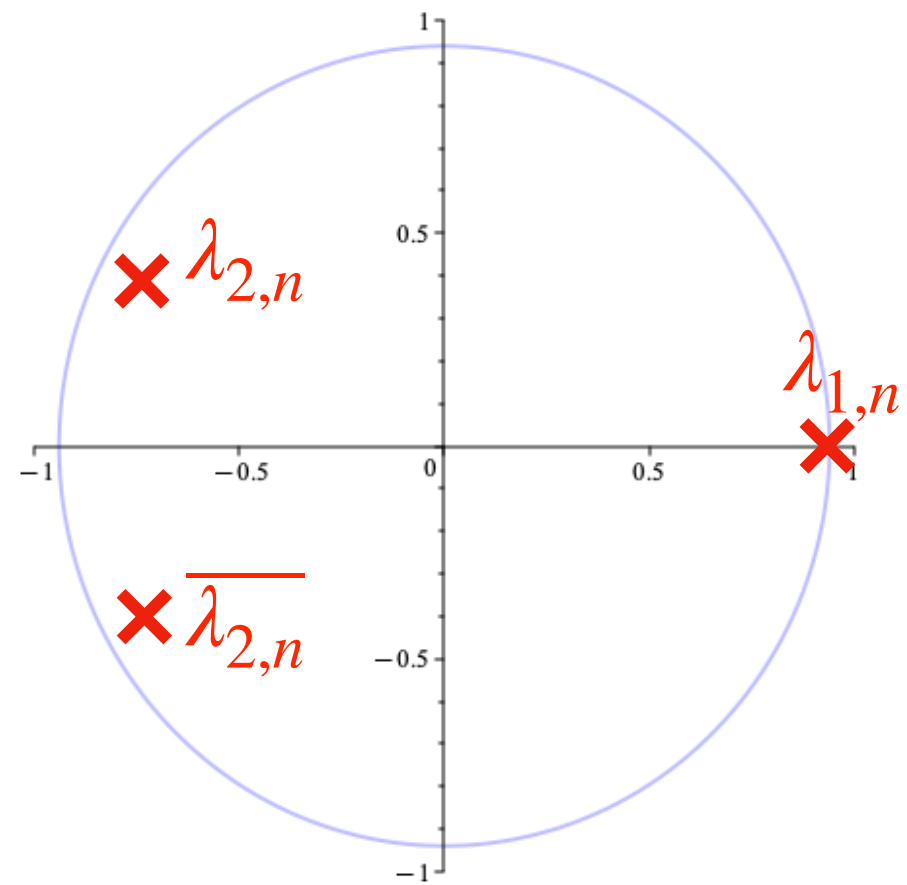


$\lambda_1 > |\lambda_2| \implies$ No cone contracted cone by A

Idea: Find $(K_n)_n \in \mathbb{R}_{>0}^d \mid A(n)K_n \subset K_{n+1}$



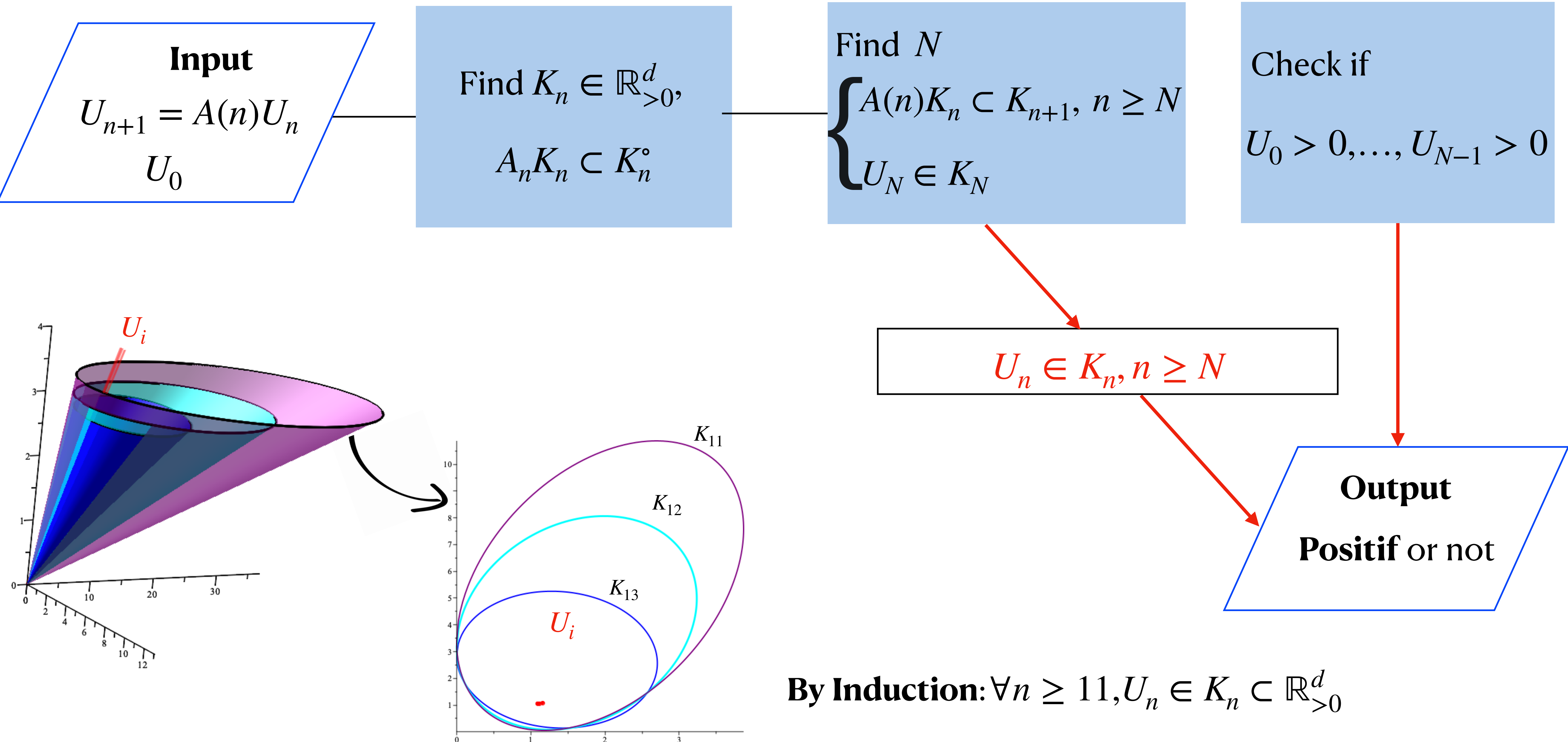
$$U_n \in K_n \implies U_{n+1} \in K_{n+1} \subset \mathbb{R}_{>0}^d$$



$\lambda_{i,n}$: Eigenvalues of $A(n)$

$\lambda_{1,n} > |\lambda_{2,n}| \implies \exists K_n$ contracted by $A(n)$

Cone-based approach: Extension



Several Simple Dominant Eigenvalues

$$U_{n+1} = A(n)U_n, \quad A = \lim_{n \rightarrow \infty} A(n) \in \mathbb{Q}^{d \times d}$$

Theorem [l. 2025]

There exists $(K_n)_n$ such that $A(n)K_n \subset K_{n+1} \subset \mathbb{R}_{>0}^d$ if

$$\cdot \lambda_{1,n} > |\lambda_{2,n}| \geq |\lambda_{3,n}| \geq \dots \geq |\lambda_{k,n}|$$

$$\cdot \lambda_1 = |\lambda_2| = \dots = |\lambda_v| > |\lambda_j|$$

$$\cdot \lambda_{1,n}, \dots, \lambda_{v,n} \text{ simple}$$

$$\cdot \max_{i=1,\dots,k} |\lambda_{i,n} - \lambda_{i,n+1}| = o \left(\lambda_{1,n} - \max_{j=2,\dots,v} |\lambda_{j,n}| \right), \quad n \rightarrow \infty.$$

$$\lambda_i = \lim_{n \rightarrow \infty} \lambda_{i,n}$$
$$\lambda_i \neq \lambda_j$$

**Conditions on the
recurrence**

Corollary

Positivity is **decidable** for recurrences of this class with arbitrary $d \in \mathbb{N}$, if

$$\text{additionally } \lim_{n \rightarrow \infty} \frac{U_n}{\|U_n\|} = V_1 \text{ with } AV_1 = \lambda_1 V_1.$$

**Condition on the
initial conditions**

Examples

Recurrence	Order d	Dominant Eigenvalues	N
$s_n = \sum_{k=0}^n (-27)^{n-k} 2^{2k-n} \frac{(3k)!}{k!^3} \binom{k}{n-k}$	2	One simple	1
Gillis, Reznick, Zeilberger inequality	4 to 24	One simple	< 4 for $d \neq 5$ 546 for $d = 5$
Turán's inequality $x = 1/4$	3	Three simple	11
$v_n = \binom{2n}{n}^2 - \frac{16^n}{4n}$	2	One double	6
${}_2F_1\left(\frac{1}{2}, \frac{1}{2}; 1; x^2\right) \geq (1+x) {}_2F_1\left(\frac{1}{2}, \frac{1}{2}; 1; x\right)$ $x \in (0,1)$	2	One double	4

Timings

[Gillis, Reznick, Zeilberger 83]

$$u_n^{(k)} = \sum_{j=0}^n (-1)^j \frac{(kn - (k-1)j)! k!^j}{(n-j)!^k j!} \geq 0$$


Order k

k	max deg	max $\log_{10}$ coeffs	N	$t(s)$
4	6	10	3	0.1
5	10	20	546	0.2
6	15	32	3	0.2
7	21	52	3	0.4
8	28	72	4	0.5
9	36	100	4	2.4
10	45	129	4	1.7
11	55	175	3	2.8
12	66	202	4	5.3
13	78	270	1	7.3
14	91	310	1	16.5
15	105	366	1	21.1
16	120	433	1	39.8
17	136	531	1	53.9
18	153	569	1	78.6
19	171	699	1	120.4
20	190	739	1	209.1
21	210	850	1	446.5
22	231	960	1	416.9
23	253	1115	1	731.6
24	276	1140	1	1827.2

Conclusion

This cone-based approach gives positivity proofs for a large class of sequences

Ongoing work: Extension to sequences with parameters

$$\forall x \in (-1,1), \quad P_n(x)^2 - P_{n-1}(x)P_{n+1}(x) > 0, \quad n \geq 1.$$

Thank you!