RESEARCH PROJECT

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I started my mathematical research under the supervision of Marc Herzlich and Philippe Castillon at the University of Montpellier. They proposed me to work on a problem lying at the intersection of Riemannian geometry, complex geometry and geometric analysis. Precisely, they asked me to identify what can be said about the asymptotic geometry of a complete non-compact Kähler manifold whose geometry is asymptotically locally modeled on that of the complex hyperbolic space. This naturally led me to familiarize myself with the geometry of non compact rank one symmetric spaces from the Riemannian point of view as well as with the extrinsic geometry of submanifolds in these spaces.

Below is a short description of my previous work and a research project as a natural extension of my doctoral thesis.

Introduction

Asymptotic geometry of hyperbolic and complex hyperbolic spaces. Negatively curved spaces are in the heart of my mathematical research. The hyperbolic space is the most simplest smooth example: it is the unique simply connected Riemannian manifold with constant negative curvature -1. In the unit ball model, it is endowed with the Riemannian metric

$$g_{\mathbf{H}} = \frac{4}{(1-\|x\|^2)^2}\sum_{i=1}^{n} \mathrm{d}x^i \otimes \mathrm{d}x^i = \frac{4}{(1-\rho^2)^2}\big(\mathrm{d}\rho\otimes\mathrm{d}\rho + \rho^2 \mathring{g}\big) \tag{1}$$

where the second equality is given by the polar decomposition $\mathbf{R}^n \simeq \mathbf{R}_+ \times \mathbf{S}^{n-1}$ and the identification $\rho = \|x\| \in [0,1)$. The metric $\mathring{g}$ is the *round metric* of the unit sphere. The asymptotic geometry of this space can be studied through the extrinsic geometry of concentric geodesic spheres. The change of variables $r = \tanh\frac{\varrho}{2}$ shows that geodesic spheres of radius r are isometric to the unit sphere $\mathbf{S}^{n-1}$ endowed with the metric $(\sinh^2 r)\mathring{g}$. All these spheres are *conformal* with each other and share common geometric properties. Moreover, they are totally umbilical hypersurfaces with *shape operator* given by $S = (\coth r)\,\mathrm{Id}_{T\mathbf{S}}$, and their extrinsic geometry is therefore *isotropic*. The asymptotic geometry of the hyperbolic manifold is closely related to the *conformal* properties of the unit sphere $\mathbf{S}^{n-1}$. Namely, the hyperbolic spaces admits a *conformal compactification*, whose boundary at infinity is the conformal sphere. This remark is at the center of the study of conformal infinity

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of asymptotically hyperbolic manifolds [FG85, Lee06], which is of high interest in the so-called AdS/CFT correspondence in theoretical Physics.

The complex counterpart of hyperbolic geometry is the *complex hyperbolic space.* Its geometry reflects the additional structure that this space is given, namely, its complex structure compatible with its metric structure. It is the unique complete simply connected Kähler manifold with constant *holomorphic sectional curvature*[1] -1. However, its full sectional curvature is not constant, and is pinched between -1 and $-1/4$. In the spirit of the decomposition of the hyperbolic metric (1), the complex hyperbolic space of dimension n can be thought of as the unit ball of $\mathbf{C}^n$ endowed with the following Riemannian metric, in polar exponential coordinates

$$g_{\mathbf{C}H} = \mathrm{d}r \otimes \mathrm{d}r + 4\sinh^2 r\, \theta \otimes \theta + 2\sinh^2 \frac{r}{2}\gamma \tag{2}$$

where θ is the standard contact form of the unit sphere $\mathbf{S}^{2n-1} \subset \mathbf{C}^n$ and $\gamma = \mathrm{d}\theta(\cdot, J\cdot)$ is the Levi form defined on the contact structure $H = \ker\theta = T\mathbf{S}^{2n-1} \cap iT\mathbf{S}^{2n-1}$ with J the multiplication by i in H. This expression reveals the main difference with the real case: the geometry is anisotropic. Concentric geodesic spheres are no longer conformal with each other. Instead, they are Berger spheres, and the metric explodes or implodes in the direction of the Reeb vector field X of the contact form θ, depending on whether $r \to +\infty$ or 0. Their tangent spaces split as $TS = \mathbf{R}X \oplus H$ and their shape operator splits diagonally as $S = (\coth r)\,\mathrm{Id}_{\mathbf{R}X} + (\frac{1}{2}\coth\frac{r}{2})\,\mathrm{Id}_H$. It is worth noting that they are still of constant mean curvature. The sphere at infinity $(\mathbf{S}^{2n-1}, H, J)$ is no longer the conformal sphere, but instead is endowed with a richer geometry, namely, a *strictly pseudo-convex CR structure.*

ALH and ALCH manifolds. A *conformally compact* manifold (M, g) is a complete Riemannian manifold which is the interior of a compact manifold $\overline{M}$ endowed with a boundary defining function $\varrho\colon \overline{M} \to \mathbf{R}_+$, and such that the metric $\varrho^2 g$ extends smoothly up to the boundary. The conformal manifold $(\partial\overline{M}, [\varrho^2 g])$ is called the conformal infinity of (M, g). Conformal invariants of $(\partial\overline{M}, [\varrho^2 g])$ give automatically rise to geometric invariants of the Riemannian manifold (M, g)[2]. If $|\mathrm{d}\varrho| = 1$ on the boundary, the sectional curvature of (M, g) approaches -1 near the boundary, and (M, g) is called *asymptotically hyperbolic.* This latter definition is extrinsic. Anderson and Schoen have shown that Hadamard manifolds with negatively pinched curvature admit a conformal infinity whose Hölder regularity is discussed in terms of the pinching [And83, AS85], by solving the Dirichlet problem at infinity. Later on, Bahuaud, Gicquaud and Marsh have extended this result to the non simply connected case, provided the existence of a suitable compact convex subset they call an *essential subset* and an exponential decay of the sectional

[1]These are sectional curvatures of tangent planes which are stable under the almost complex structure.

[2]Remark that the Moebius group $\mathrm{Conf}(\mathbf{S}^{n-1}, [\mathring{g}])$ is equal to the isometry group $\mathrm{Isom}(\mathbf{H}^n, g_{\mathbf{H}})$.

curvature to -1 near infinity [Bah09, BG11, Gic13]. They call these manifolds *asymptotically locally hyperbolic* (ALH). They moreover show that the gluing of the boundary to the original manifold has regularity which is discussed in terms of the decay rate. The techniques use therein are mainly ODE's techniques applied to a system of coupled differential equations in *Fermi coordinates* relating the metric induced on some level hypersurfaces and their shape operator.

The study of the asymptotic geometry of complete non-compact Riemannian manifolds has proven to be fruitful in the understanding of the geometry of complex domains. By building a complete metric in the interior of such a domain, which sends the boundary to infinity, one can read much geometric information in the asymptotic development of the metric [Fef76, FG85, Hir00]. The induced structure on the boundary at infinity leads to geometric invariants of the domain. The Bergman metric and the Kähler-Einstein metric are examples of such metrics and have been at the center of complex geometry for decades. The example of the unit ball in the standard Hermitian space is particularly interesting and leads to the complex hyperbolic space. More generally, these metrics are *asymptotically complex hyperbolic*, and the boundary of the domains is endowed with a CR structure, which is generically strictly pseudoconvex. This CR structure is a geometric invariant of the domain and can be read in the asymptotic development of the metric near infinity. These boundaries are again given as an extrinsic data through their embedding in the standard Hermitian space. Biquard has constructed many examples of asymptotically complex hyperbolic Einstein metrics with prescribed CR infinity [Biq00].

Question 1. *Does there exist intrinsic conditions under which a complete non-compact Kähler manifold admits a boundary at infinity endowed with a (possibly strictly pseudo convex) structure?*

The desired structure at infinity being of contact nature, and the ambient geometry being highly anisotropic, the techniques used by Marsh, Bahuaud and Gicquaud do not generalize. In my PhD thesis, I have answered this question the following way.

Theorem 1. *Let (M, g, J) be a complete non-compact Kähler manifold. Assume there exists $K \subset M$ an* essential subset *with $\sec(M \setminus K) \leqslant 0$, and let $r = d_g(\cdot, K)$. Assume further more that $\|R - R^0\|_g = \mathcal{O}(e^{-ar})$ with $a > 1$. Then there exists a non(vanishing continuous differential form of degree 1 $\eta \in \mathcal{C}^0(M; T^*\partial K)$ and a continuous field of symmetric positive semi-definite bilinear forms $\gamma_H \in \mathcal{C}^0(M; S^2T^*\partial K)$ such that the metric reads*

$$\mathrm{d}r \otimes \mathrm{d}r + e^{2r}\eta \otimes \eta + e^r\gamma_H + \textit{lower order terms} \tag{3}$$

In addition, γ_H is positive definite on $H = \ker \eta$.

I call these manifolds *asymptotically locally complex hyperbolic manifolds.* The existence of the essential subset is a geometric condition ruling out the formation of cusps and allowing to perform a sort of polar decomposition. The exponential decay of the curvature tensor to the constant -1 holomorphic sectional curvature tensor R^0 is a necessary condition[3]. The local assumption on the curvature hence yields a Taylor expansion near infinity similar to that of 3. The analogy does not end here.

Theorem 2. *If furthermore, there exists $b > 1$ such that $\|\nabla R\| = \mathcal{O}(e^{-br})$, the differential form of degree 1 η is of class $\mathcal{C}^1$, and is contact.*

Theorem 3. *If $\min\{a, b\} > \frac{3}{2}$, then γ_H is also of class $\mathcal{C}^1$, and there exists a $\mathcal{C}^1$ integrable almost complex structure J_H on H such that $\gamma_H = \mathrm{d}\eta(\cdot, J_H\cdot)$. In particular, $(\partial K, H, J_H)$ is a strictly pseudoconvex CR manifold of class $\mathcal{C}^1$.*

The CR structure is canonically associated to the embedding $K \subset M$. Moreover, ∂K can be identified with the visual boundary of (M, g), and M admits a compactification $M \subset \overline{M}$ with strictly pseudoconvex CR boundary $\partial_\infty M \simeq (\partial K, H, J_H)$. It is worth noting that the local assumptions on the curvature and its covariant derivative surprisingly allow one to recover very strong information on the geometry at infinity of the considered manifolds, a fact that was not expected at first sight.

Research project

During my previous work, I had to familiarize myself with many different aspects of the geometry of the complex hyperbolic spaces, their submanifolds and the Kähler manifolds that are modeled on them. This naturally led me to ask myself different type of questions, in several directions. Here are set some research projects I would like to pursue during the following years.

Generalization in other settings. The first project I have in mind is the different types of generalization of my previous work. Although the conditions on the decay rates a and b seem to be optimal, it is hoped that several assumptions might be relaxed while still having a similar conclusion. Several directions are then possible.

Holomorphic bisectional curvature comparison. The holomorphic bisectional curvature have first been introduced explicitly by Goldberg and Kobayashi [GK67]. Given X and Y two unit vectors tangent at the same base point, the holomorphic bisectional curvature associated to the two complex lines $\operatorname{span}\{X, JX\}$ and $\operatorname{span}\{Y, JY\}$ it is defined as the value $R(X, JX, Y, JY)$[4]. It is a geometric invariant of the Kähler manifold that contains more information than the holomorphic

[3]There exist complex domains with asymptotically complex hyperbolic metrics with subexponential decays whose boundaries are not strictly pseudoconvex CR manifolds.

[4]The convention is the following. If $\{X, Y\}$ is an orthonormal family, the sectional curvature of $\operatorname{span}\{X, Y\}$ is $R(X, Y, X, Y)$. For the sphere, it is positive.

sectional curvature, and less information than the sectional curvature itself. Several Theorems obtained from assumptions on the sign of the sectional curvature of a Kähler manifold generalize with weaker assumptions on the holomorphic bisectional curvature.

The condition of convergence of the Riemannian curvature tensor R to the constant -1 holomorphic curvature tensor R^0 is a condition imposed on all sectional curvatures at once. It is natural to ask whether or not Theorem 1, 2 and 3 generalize under a weaker assumption on the holomorphic bisectional curvature. Although comparison geometry with the holomorphic sectional curvature is really intricate, beautiful results have been obtained when comparing bisectional curvatures [GK67, SY80, Lot21]. However, the proofs obtained during my PhD do not generalize as they are. Indeed, they all generalize on a sharp estimate of the growth of some natural Jacobi fields defined along geodesics rays. This growth is derived from the Jacobi equation, which involves the tensor $R(\gamma_p', \cdot)\gamma_p'$ with γ_p the above mentioned geodesic rays.

The non-Kähler case. Kähler manifolds live at the intersection of Riemannian geometry, almost complex geometry and symplectic geometry. Their structure is defined from a Riemannian metric g, an almost complex structure J and a symplectic form ω which are all compatible with each other. In particular, this forces the almost complex structure to be integrable, and the underlying manifold to carry a complex structure.

From the chosen point of view, the Kähler conditions are set as $\nabla J = 0$ and $g(J\cdot, J\cdot) = g$. Both conditions seem crucial in the proof of Theorems 1, 2 and 3. They imply in particular that $J\gamma_p'$ is a parallel vector field along the geodesic ray γ_p, and the skew-symmetry of J is of constant use. However, it would be of interest to understand how much relaxing one of the two conditions, for example by setting $\|\nabla J\| = \mathcal{O}(e^{-cr})$ for some $c > 0$ and keeping the isometry condition, leads to similar results. On the contrary, building counter-examples would be instructive to understand the importance of the Kähler condition.

The Einstein case. Einstein metrics are (semi-)Riemannian metrics satisfying the equality $\mathrm{Ric} = \lambda g$, where Ric denotes the Ricci tensor and $\lambda \in \mathbf{R}$ is a constant. Einstein metric are nice generalizations of constant curvature metrics and their Riemann curvature tensor have nice algebraic properties. They are in themselves a beautiful and fruitful field of research [Bes07]. They are also of high interest for their physical meaning in general relativity. In [Biq00] are obtained plenty of Einstein metrics in a neighborhood of the conformal infinity of non-compact rank one symmetric spaces. They are built as deformations of the standard structure, and are in 1-1 correspondence with some Carnot-Caratéodory metrics on the boundary at infinity. It follows that they exist in a large amount in the complex hyperbolic case. Moreover, Einstein metrics satisfy some elliptic regularity, and an appropriate bootstrap technique allows one to control the covariant derivatives of the

Riemann tensor at all orders, see [BG11]. I believe that the Einstein assumption might lead to nice generalizations.

Other non-compact rank one symmetric spaces. Riemannian non-compact rank one symmetric spaces are classified in four families: the real hyperbolic spaces $\mathbf{RH}^n$, the complex hyperbolic spaces $\mathbf{RC}^n$, the quaternionic hyperbolic spaces $\mathbf{HH}^n$, and finally the Cayley hyperbolic plane $\mathbf{OH}^2$. They are all Hadamard manifolds and their sphere at infinity is endowed with a particular geometric structure reflecting their Riemannian geometry. As mentioned earlier, the sphere at infinity of the real hyperbolic space is a conformal sphere, while that of the complex hyperbolic space is a strictly pseudoconvex CR sphere. In the two other cases, the sphere at infinity is endowed with a particular distribution of the tangent bundle, of respective codimension 3 and 7, related to respectively 3 and 7 linearly independent contact structures. Biquard has build Einstein metrics asymptotically modeled on each of that geometries [Biq00] with prescribed contact structures and distribution at infinity.

Let us focus on the quaternionic hyperbolic case in this explanation. The quaternionic hyperbolic space is endowed with three almost complex structures J_1, J_2 and J_3. They induce on its boundary at infinity a contact-quaternionic structure, once again related to a Carnot-Carathéodory metric γ, which is however only defined on a codimension 3 distribution. The difficulty in the generalization relies in the fact that the almost complex structures are not parallel. They however generate a parallel bundle, meaning that ∇J_i is a linear combination of the J_k's. Although there was a gap to cross to generalize Bahuaud and Gicquaud's work from the ALH setting to the ALCH one, due to the anisotropic nature of the latter geometry and the impossibility to apply the ODE's techniques of the cited authors, it appears that the similarity between the complex hyperbolic geometry and the quaternionic hyperbolic one is strong enough to hope to extend the results obtained to the last case.

A positive mass Theorem for ACH manifolds. Working on asymptotic geometry modeled on non-compact rank one symmetric spaces, I have naturally got interested in the construction of asymptotic invariants, the most renowned being the mass. I am therefore planning to study this object and focus on the mass in the complex hyperbolic setting.

The *positive mass conjecture* is a family of statements of the form

Conjecture. *Let M be an asymptotically M_0 manifold, with scalar curvature greater than that of M_0. Then there exists a geometric invariant $m(M)$, called the mass, such that $m(M) \geqslant m_0$, where m_0 is the mass of M_0. Moreover, the equality case is achieved if and only if $M = M_0$.*

The manifold M_0 refers to some model space such as the Euclidean, Minkowski, Schwarzschild, hyperbolic, as well as a non-compact rank one symmetric space.

Some integrability condition on the scalar curvature is usually also required. The inequality has to be understood in a large sense, specific to the situation.

The conjecture is known to be true in several cases. The original proof of the asymptotically flat setting relies on a variational method using minimal surfaces [SY79, SY81], which applies in dimension at most 7[5]. If moreover the manifold is supposed to admit a spin structure, the conjecture is true in all dimensions [Wit81]. The mass is defined here as a real number, and vanishes exactly in the Euclidean case.

Much progress has been made in general cases, and the mass is still an active field of research in itself (see the recent preprint [CG21]). In the hyperbolic case, the mass is defined as a vector [Wan01, CH03]. The Euclidean and hyperbolic spaces being of isotropic nature, the story could have ended there, believing that more complicated symmetric spaces do not share the same properties. Outstandingly, the rigidity part of the conjecture has been shown to be true in the complex hyperbolic case [Her98, BH02] and in several non-compact rank-one symmetric spaces, provided some spin conditions are satisfied [Lis09], under strong asymptotic conditions on the metric. It is natural to believe that there is a common background behind these results.

The nature of the mass is in direct relation with the complexity of the geometric structure of the sphere at infinity of the model, hence the difference of nature between the flat case (a scalar) and the hyperbolic case (a vector). A definition for the mass in the complex hyperbolic case has been proposed in [MM12]. However, it is again of scalar nature, which frustratingly does not reflect the higher complexity of the strictly pseudo convex CR nature of the sphere at infinity. An interesting subject of research would be the study of this object. A deep understanding of the concepts involved may allow a more general definition for the mass, and perhaps a proof of the positive mass conjecture in the asymptotically complex hyperbolic case. The above-mentioned preprint [CG21] brings a new perspective on the mass and deserves to be studied in depth.

Characterizing geodesic spheres of the complex hyperbolic space by their curvature, isoperimetric inequality. The technique used in my PhD thesis involves a sharp understanding of the extrinsic geometry of a family of level hypersurfaces of asymptotically locally hyperbolic manifolds. To that end, I investigated the extrinsic geometry of hypersurfaces of the complex hyperbolic manifold.

In the Euclidean and real hyperbolic spaces, geodesic spheres are characterized among the set of closed hypersurfaces as the unique ones being totally umbilical, and of constant mean curvature. Concentric spheres of the complex hyperbolic

[5]The reason is that in higher dimension, there exists singular minimal surfaces, and the original proof by induction does not apply.

space $\mathbf{CH}^n$ are constant mean curvature (CMC) hypersurfaces whose shape operator have two eigenvalues, namely $\coth r$ and $\frac{1}{2}\coth\frac{r}{2}$, where r is the radius. The multiplicities are respectively 1 and $\frac{n-2}{2}$. Hence, in that case, they are not umbilical. Their geometry is well known [KM15], but surprisingly enough, it is still not known how to characterize them by their curvature. This problem has been addressed in [FFR93], but the proof is false, as mentioned in the erratum [FFR95]. The mean curvature flow is a natural approach (*cf.* the different papers of M. Ritoré, such as [Rit05, Rit09]) and may lead to progress related to that question. The inverse mean-curvature flow is also a possible candidate [Pip19].

This question is somehow related to isoperimetric inequalities, whose standard proofs rely on the characterization of geodesics spheres by their curvature.

Isoperimetric inequality. *Let (M^n, g) be a Riemannian manifold. There exists a constant $\lambda(M, g) > 0$ such that if $\Sigma \subset M$ is a domain with smooth boundary, then:*

$$(\mathrm{Vol}_{n-1}(\partial\Sigma))^n \geqslant \lambda(M, g)(\mathrm{Vol}_n(\Sigma))^{n-1} \tag{4}$$

and the equality is achieved if and only if Σ is a geodesic ball.

They are known to be true in many cases, such as the Euclidean, hyperbolic or Hadamard cases (in dimension 2,3, and 4 in the latter case).

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