RESEARCH STATEMENT

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Introduction

My general research interest lies at the intersection of Riemannian geometry, complex geometry and geometric analysis. More precisely, this topic naturally led me to familiarize myself with the geometry of non-compact rank one symmetric spaces from the Riemannian point of view as well as with the extrinsic geometry of their submanifolds.

Below is a short description of the general setting of my research and of my contributions. Then I will develop my current and future projects.

State-of-the-art and contributions

Non-compact rank one symmetric spaces (NCROSS in short) are geometric objects living at the intersection of differential geometry (they are described by their curvature) and algebra (they are described by the algebraic properties of their isometry group). They are also known as the real, complex, octonionic hyperbolic spaces and the Cayley plane. Their conformal infinity are spheres endowed with a particular geometric structure : for instance, the conformal round sphere is the conformal infinity of the real hyperbolic space, and the odd dimensional Cauchy-Riemann sphere is that of the complex hyperbolic case. There is a tight correspondence between the geometry of NCROSS, which are non-compact, and of their conformal infinities, which are compact. In polar coordinates, besides the real case, their metrics read

$$dr^2 + 4\sinh^2 r\,\eta^2 + 4\sinh^2 \frac{r}{2}\,\gamma \tag{1}$$

where η is a contact form with values in $\mathbb{R}$, $\mathbb{R}^2$ or $\mathbb{R}^7$ and γ is a Carnot-Carathéodory metric defined on its kernel (the sectional curvature is normalized between -1 and $-\frac{1}{4}$). The two tensors η and γ are closely related to the geometry of the boundary. In the complex hyperbolic setting, the boundary at infinity is the CR sphere, which is a strictly pseudoconvex CR manifold.

Riemannian manifolds modeled on NCROSS are of particular interest. Their boundaries at infinity have similar geometric structures, and give birth to new geometric invariant. Anderson and Schoen (1985) have built conformal infinities for simply connected negatively curved manifolds. Bahuaud, Marsh and Gicquaud (2008–2013) have generalized this result to a larger class of complete manifolds, called asymptotically locally hyperbolic manifolds. They have shown that under some natural geometric conditions, their boundary at infinity can be endowed with a canonical conformal structure,

During my PhD, I studied the Kähler counterpart and extended these results to the asymptotically complex hyperbolic case, which is more intricate due to the anisotropy of the metric. In a submitted article (arXiv:2201.12132), I have shown that under analogous geometric conditions, the geometry at infinity of a complete non-compact Kähler manifold whose curvature is asymptotic to that of the complex hyperbolic space is asymptotic to a strictly pseudoconvex CR structure defined on its boundary at infinity. More precisely, the main result is the following.

Theorem. *Let (M, g, J) be a complete, non compact Kähler manifold with essential subset $K \subset M$. Assume that there exists $a > \frac{3}{2}$ such that $\|R - R^0\|_g, \|\nabla R\| = \mathcal{O}(e^{-ar})$, where R is the Riemann curvature tensor. Then ∂K is endowed with a contact 1-form η and a Carnot-Carathéodory metric γ, positive definite on $H = \ker \eta$, of class $\mathcal{C}^1$, such that the metric reads*

$$E^* g = dr^2 + e^{2r}\eta^2 + e^r\gamma + \text{lower order terms.} \tag{2}$$

Moreover, there exists an integrable almost-complex structure J_H on the contact distribution $H = \ker \eta$ such that $d\eta(\cdot, J_H\cdot) = \gamma$. In particular, $(\partial K, \eta, J_H)$ is a strictly pseudoconvex CR manifold of class $\mathcal{C}^1$.

The curvature tensor tensor R^0 is that of the complex hyperbolic space, and the existence of such a subset K is the above mentioned geometric condition. Its boundary ∂K is identified with the boundary at infinity of M and the analogy with (1) is obvious.

Current and future projects

Generalizations of the above work. I am currently working on the possible generalizations of the above result in different settings. There are different questions that naturally arise in view of my previous work.

First, I would like to find how much the assumptions on the curvature can be relaxed. The mentioned condition is about all sectional curvatures. It is a strong assumption, and it is unknown if it is necessary. It is then natural to ask whether this condition can be replaced with a condition on the holomorphic bisectional curvature, which is a function that encodes much geometric information of a Kähler manifold without being too rigid. The case of the holomorphic sectional curvature would also be a very pleasant result [1] but seems a lot more intricate.

Secondly, the current proof of the above Theorems use the Kähler assumption in the following form : the almost-complex structure is parallel and a linear isometry. This in particular implies that the tensor R^0 is parallel. However, no particular tool of complex geometry, such as holomorphic coordinates or Kähler potential, is needed. I believe that the Kähler condition can be weakened with an asymptotic assumption on the almost-complex structure, of the form $\|\nabla J\| \to 0$ fastly enough, and I am currently working on that aspect.

A third direction of generalization is the quaternionic hyperbolic case. The quaternionic hyperbolic space is a NCROSS whose geometry is controlled by three almost complex structure J_1, J_2 and J_3. This case is analogous in many aspects. However, the main difference is that the almost complex structures, although spanning a parallel distribution, are not parallel in themselves.

CR properties of the boundary. My main contribution is to highlight a strictly pseudoconvex CR structure on the boundary at infinity of some Kähler manifolds. It is natural to study the CR properties of that boundaries and how they are related to the inner geometry of the original manifold. I am first planning to study under what condition one can ensure the boundary at infinity to be CR-spherical.

A positive mass Theorem for asymptotically complex hyperbolic manifolds. The study of the asymptotic geometry of NCROSS also led me to study the positive mass conjectures and spin techniques. They are a family of results, originally coming from General relativity, about (semi-)Riemannian manifolds whose metric tensors are strongly asymptotic to that of some non-compact symmetric space. They have become an important field of research in the last decades. Coarsely, they state the following. Assume the complement of a compact in a complete, non-compact manifold is diffeomorphic to the complement of a ball in some non-compact symmetric space, and if in the induced coordinates, the coefficients of the metric and some of their derivatives are asymptotic to that of the model. Suppose finally that the scalar curvature is greater than that of the model. Then there exists an object, called the mass, that is greater (in some sense) than that of the model space, with equality if and only if the original manifold is equal to the model space.

These conjectures are known to be true in many cases (asymptotically flat, Minkowski, real hyperbolic, in dimension lesser or equal to 7 or if the manifold is spin), the mass being a scalar number or a vector-valued functional (Schoen-Yau, Witten, Herzlich). A definition of the mass in the complex hyperbolic setting has been proposed (Minerbe-Maerten), but it seems that it us not sufficient in order to prove the positive mass conjecture in that case with optimal assumptions. I am planning to study in depth a recent preprint by Cap and Gover which brings new perspectives on the mass in the real hyperbolic setting in order to study the complex hyperbolic one.

Isoperimetric inequality in the complex hyperbolic space. In an orthogonal direction, my previous work had lead me to familiarize with the extrinsic geometry of hypersurfaces in NCROSS. These techniques are at the heart of different proofs of the isoperimetric inequalities in the Euclidean and hyperbolic spaces. Moreover, they strongly rely on the fact that in that cases, geodesic spheres are the only closed submanifolds which are totally umbilical. This characterization of geodesic spheres by their curvature is no longer true in the complex hyperbolic space. Whilst not totally umbilical, they however have constant mean curvature, and no other closed hypersurface is known to share this property.

I am planning to try to study whether or not geodesic spheres can be characterized by their constant mean curvature [2], which could eventually lead to an isoperimetric inequality in the complex hyperbolic space. I think the techniques I have used during my PhD thesis (namely, the study of the normal exponential maps of hypersurfaces in the complex hyperbolic space) is a good approach to tackle this problem.

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1. Recall that the complex hyperbolic space is uniquely determined by (i) the fact it is Kähler and (ii) of constant -1 holomorphic sectional curvature.
2. The problem has been addressed by Fornari, Frensel and Ripoll (1993), but their proof is false, as mentionned in an *erratum*.