Calcul différentiel, rappels sur les sous-variétés de $\mathbb{R}^n$ Correction

Exercice 1 (Définitions).

1. If $U \subset \mathbb{R}^p$ is an open subset and if $f\colon U \to \mathbb{R}^q$ is a function, then f is differentiable at $x \in U$ if there exists a linear map $\mathrm{d}f(x) \in \mathcal{L}(\mathbb{R}^p, \mathbb{R}^q)$ such that
$$f(x+h) = f(x) + \mathrm{d}f(x)h + o(h)$$
for h small enough. In this case, $\mathrm{d}f(x)$ is unique and is called the differential of f at x.
Note that there is a unique normed vector space topology on $\mathbb{R}^p$ and $\mathbb{R}^q$, so that the notation $o(h)$ refers to $o_{\|\cdot\|'}(\|h\|)$ for any norms $\|\cdot\|$ and $\|\cdot\|'$ on $\mathbb{R}^p$ and $\mathbb{R}^q$.

2. f is said to be of class $\mathcal{C}^1$ if f is differentiable at every $x \in U$ and if moreover, the function
$$\mathrm{d}f\colon x \in U \mapsto \mathrm{d}f(x) \in \mathcal{L}(\mathbb{R}^p, \mathbb{R}^q) \simeq \mathbb{R}^{p\times q}$$
is continuous.
By induction, f is said to be of class $\mathcal{C}^{k+1}$ if f is of class $\mathcal{C}^1$ and of $\mathrm{d}f$ is of class $\mathcal{C}^k$.
It is said to be smooth (or of class $\mathcal{C}^\infty$) if it is of class $\mathbb{C}^k$ for all $k \in \mathbb{N}$.

3. $f\colon U \to V$ is said to be a $\mathcal{C}^k$ diffeomorphism if
   - f is of class $\mathcal{C}^k$,
   - f is bijective,
   - its inverse f^{-1} is of class $\mathcal{C}^k$.

   It is said to be a local $\mathcal{C}^k$ diffeomorphism if for all $x \in U$, there exists an open neighbourhood of x in U, say U_x, and an open neighbourhood of $f(x)$ in V, say V_x, such that $f\colon U_x \to V_x$ is a $\mathcal{C}^k$ diffeomorphism.

4. Three equivalent definitions of a submanifold of $\mathbb{R}^n$. $M \subset \mathbb{R}^n$ is a $\mathcal{C}^k$ submanifold of $\mathbb{R}^n$ of dimension p if:

(submersion) $\forall x \in M, \exists U \subset \mathbb{R}^n$ an open neighbourhood of x in $\mathbb{R}^n$, $\exists f\colon U \to \mathbb{R}^{n-p}$ a $\mathcal{C}^k$-submersion ($i.e$ $\forall y \in U, \mathrm{d}f(y)$ is surjective), such that $M \cap U = f^{-1}(\{0\})$.

(immersion) $\forall x \in M$, $\exists U \subset \mathbb{R}^n$ open neighbourhood of x, $\exists V \subset \mathbb{R}^p$ open neighbourhood of 0 and $\exists f\colon V \to \mathbb{R}^n$ a $\mathcal{C}^k$-immersion ($i.e$ $\forall y \in V, \mathrm{d}f(y)$ is injective) such that $f(0) = x$ and f is an homeomorphism from V to $U \cap M$.

(diffeomorphism) $\forall x \in M$, $\exists U \subset \mathbb{R}^n$ open neighbourhood of x, $\exists V \subset \mathbb{R}^n$ open neighbourhood of 0 and $\exists f\colon U \to V$ a $\mathcal{C}^k$-diffeomorphism with $f(x) = 0$ such that $f(U \cap M) = V \cap (\mathbb{R}^p \times \{0\}^{n-p})$.

Note that, thanks to the following Theorems, these two definitions are equivalent.

Exercice 2 (Théorèmes importants).

1. **The implicit function Theorem**. Let $f\colon U \subset \mathbb{R}^p \times \mathbb{R}^q \to \mathbb{R}^q$ be a $\mathcal{C}^k$ map and $(x_0, y_0) \in U$ such that $f(x_0, y_0) = 0$. Suppose that the partial differential $\partial_y f(x_0, y_0) \in \mathcal{L}(\mathbb{R}^q)$ is invertible, ($\partial_y f(x_0, y_0)$ is the differential at y_0 of the map $y \mapsto f(x_0, y)$). Then there exist $V \subset \mathbb{R}^p$ open neighbourhood of x_0, $W \subset \mathbb{R}^p \times \mathbb{R}^q$ open neighbourhood of (x_0, y_0), and $\varphi\colon V \to \mathbb{R}^p$ a $\mathcal{C}^k$ map such that

$$\forall (x, y) \in W, f(x, y) = 0 \iff x \in V \text{ and } y = \varphi(x).$$

Moreover, we have $\forall x \in V$, $\mathrm{d}\varphi(x) = -\left(\partial_y f(x, \varphi(x))\right)^{-1} \circ \partial_x f(x, \varphi(x))$.

2. **The constant rank Theorem.** Let $f\colon U \subset \mathbb{R}^p \to \mathbb{R}^q$ be a $\mathcal{C}^k$ map such that the rank of its differential is a constant map:

$$\exists r \in \mathbb{N}, \forall x \in U, \mathrm{rank}\left(\mathrm{d}f(x)\right) = r.$$

Then there exists $\varphi\colon V_0 \to V_x$ and $\psi\colon W_{f(x)} \to W_0$ two $\mathcal{C}^k$-diffeomorphisms, where V_x and V_0 are open neighbourhood of x and 0 in $\mathbb{R}^p$, and $W_{f(x)}$ and W_0 are open neighbourhood of $f(x)$ and 0 in $\mathbb{R}^q$, such that the map $\widetilde{f} := \psi \circ f \circ \varphi^{-1}\colon V_0 \to W_0$ is given by the map

$$\widetilde{f}(x_1, \ldots, x_p) = (x_1, \ldots, x_r, 0 \ldots, 0).$$

In other words, f is conjugate (by two $\mathcal{C}^k$ diffeomorphisms) to the restriction of the projection onto the r-first coordinates.

3. **The inverse function Theorem.** Let $f\colon U \subset \mathbb{R}^n \to \mathbb{R}^n$ be a $\mathcal{C}^k$ map. Suppose $\mathrm{d}f(x) \in \mathcal{L}(\mathbb{R}^n)$ is invertible. Then there exists $U_x \subset U$ an open neighbourhood of x in U and $V_{f(x)} \subset \mathbb{R}^n$ an open neighbourhood of $f(x)$ such that the restriction

$$f\colon U_x \to V_{f(x)}$$

is a $\mathcal{C}^k$ diffeomorphism. Moreover, we have $\forall y \in U_x$, $\mathrm{d}f^{-1}(f(y)) = (\mathrm{d}f(y))^{-1}$.

In other words, f is a local $\mathcal{C}^k$-diffeomorphism at x.

Exercice 3 (Inversion par rapport à une sphère)**.**

1. Note that $f \circ f = \mathrm{id}_{\mathbb{R}^n \setminus \{0\}}$ and thus, f is invertible with $f^{-1} = f$. Moreover, f is $\mathcal{C}^1$ as a product of two $\mathcal{C}^1$ maps. It is then a bijective $\mathcal{C}^1$ function with $\mathcal{C}^1$ inverse: it is by definition a $\mathcal{C}^1$-diffeomorphism.

2. Let us compute the differential of f: fix $x \in \mathbb{R}^n \setminus \{0\}$ and h small enough. We have

$$\begin{aligned} f(x+h) &= \frac{x+h}{\|x+h\|^2} \\ &= \frac{x+h}{\|x\|^2 + 2\langle x, h\rangle + \|h\|^2} \\ &= \frac{x+h}{\|x\|^2} \frac{1}{1 + 2\langle \frac{x}{\|x\|^2}, h\rangle + o(h)} \\ &= \frac{x+h}{\|x\|^2}\left(1 - 2\left\langle \frac{x}{\|x\|^2}, h\right\rangle + o(h)\right) \\ &= f(x) + \frac{1}{\|x\|^2}\left(h - 2\left\langle \frac{x}{\|x\|}, h\right\rangle \frac{x}{\|x\|}\right) + o(h). \end{aligned}$$

We thus have

$$\forall x \in \mathbb{R}^n \setminus \{0\}, \forall h \in \mathbb{R}^n, \mathrm{d}f(x)h = \frac{1}{\|x\|^2}\left(h - 2\left\langle \frac{x}{\|x\|}, h\right\rangle \frac{x}{\|x\|}\right).$$

To conclude, one can either:

- invoke the fact that $\mathrm{d}f(x) = \frac{1}{\|x\|^2} s_{x^\perp}$, where $s_{x^\perp}$ is the reflection across the hyperplane $x^\perp = \{y \in \mathbb{R}^n \mid \langle x, y\rangle = 0\}$,
- or do some (easy) computations that yield $\langle \mathrm{d}f(x)u, \mathrm{d}f(x)v\rangle = \frac{\langle u,v\rangle}{\|x\|^4}$.

Exercice 4.

First, f is indeed of class $\mathcal{C}^1$. Moreover, if $(x, y, z) \in \mathbb{R}^3$, we have by a direct computation:

$$\mathrm{Mat}_{\mathrm{can}}\left(\mathrm{d}f(x,y,z)\right) = \begin{pmatrix} 0 & 2e^{2y} & 2e^{2z} \\ 2e^{2x} & 0 & -2e^{2z} \\ 1 & -1 & 0 \end{pmatrix}.$$

This yields $\det\left(\mathrm{d}f(x,y,z)\right) = -4\left(e^{2(x+z)} + e^{2(y+z)}\right) \neq 0$. It follows that $\mathrm{d}f(x,y,z)$ is invertible, and the inverse function Theorem provides the existence of an open neighbourhood of (x, y, z) in $\mathbb{R}^3$, say $V_{(x,y,z)}$, and of an open neighbourhood of $f(x, y, z)$ in $\mathbb{R}^3$, say $W_{(x,y,z)}$, such that the restriction $f\colon V_{(x,y,z)} \to W_{(x,y,z)}$ is a $\mathcal{C}^1$-diffeomorphism. It follows that

$$\mathrm{Im}(f) = \bigcup_{(x,y,z)\in\mathbb{R}^3} W_{(x,y,z)}$$

is open in $\mathbb{R}^3$. Note that the equation $f(x, y, z) = (-1, 0, 0)$ has no solution as $2e^{2y} + 2e^{2z} > 0$ for all (x, y, z), so that $\mathrm{Im}(f) \neq \mathbb{R}^3$.

Exercice 5.

We will use the following fact: if $u\colon U \subset \mathbb{R}^n \to \mathcal{L}(\mathbb{R}^p, \mathbb{R}^q)$ is a continuous family of linear maps, then the function $r \mapsto \mathrm{rank}(u(x))$ is **lower** semi-continuous[1], that is

$$\forall x \in U, \exists V_x \subset U \text{ open neighbourhood of } x, \forall y \in V_x, \mathrm{rank}(u(y)) \geqslant \mathrm{rank}(u(x)).$$

Identifying linear maps in finite dimension with their matrices, this fact follows from a continuity argument about the determinant of some sub-matrix.

Let $A = \{\mathrm{rank}(\mathrm{d}f(x)) \mid x \in U\} \subset \mathbb{N}$. Then $A \neq \varnothing$ and A is bounded above by $\min\{p, q\}$: it follows that $m = \max A$ exists and that $m \leqslant \min\{p, q\}$. Consider

$$W = \{x \in U \mid \mathrm{rank}(\mathrm{d}f(x)) = m\}.$$

Then $W \neq \varnothing$ by definition of m. Let us show that W is open. The map f being $\mathcal{C}^1$, $\mathrm{d}f$ is continuous and the map $x \in U \mapsto \mathrm{rank}\left(\mathrm{d}f(x)\right)$ is lower semi-continuous. Let $x \in W$. There exists an open neighbourhood $V \subset U$ of x such that

$$\forall y \in V, m = \mathrm{rank}(\mathrm{d}f(x)) \leqslant \mathrm{rank}(\mathrm{d}f(y)) \leqslant \max_{z\in U} \mathrm{rank}(\mathrm{d}f(z)) = m,$$

[1] During the class, I made the mistake to say it was **upper** semi-continuous. Apologize!

and $\mathrm{d}f$ has constant rank m on V. In follows that $V \subset W$, and W is open. We are now able to apply the constant rank Theorem on W: if $x \in W$ is fixed, there exists a diffeomorphism $\varphi\colon V_x \to V_0$ (resp. $\psi\colon \widetilde{V}_{f(x)} \to \widetilde{V}_0$) between neighbourhoods of x and 0 in $\mathbb{R}^p$ (resp. of $f(x)$ and 0 in $\mathbb{R}^q$) such that

$$\widetilde{f} = \psi \circ f \circ \varphi^{-1}\colon V_0 \to \widetilde{V}_0$$

is given by $\widetilde{f}\,(x_1, \ldots, x_p) = (x_1, \ldots, x_m, 0, \ldots, 0)$. As f is supposed injective and φ, ψ are diffeomorphisms, then $\widetilde{f}$ is injective. This implies that $m = p$: if not, we would have, for $\varepsilon > 0$ small enough, $\widetilde{f}(0, \ldots, 0, \varepsilon) = \widetilde{f}(0, \ldots, 0, -\varepsilon)$, a contradiction. Recall that by definition, $m \leqslant \min\{p, q\}$, so that $m = p \leqslant q$.

For now, we know that on the open subset W, f is of rank $p \leqslant q$, that is, f is an immersion on W. To conclude, we need to show that W is dense in U; in order to do so, let us show that if $V \subset U$ is an open subset, then $V \cap W \neq \varnothing$. Fix $V \subset U$ open and define $m_V = \max\{\mathrm{rank}(\mathrm{d}f(x)) \mid x \in V\}$ and $W_V = \{x \in V \mid \mathrm{rank}(\mathrm{d}f(x)) = m_V\}$. The exact same study as above shows that $W_V \neq \varnothing$, that $m_V = p$ and finally, we have $W_V \subset V \cap W \neq \varnothing$. This concludes the proof.

Exercice 6.

Let $U = \{(x_1, \ldots, x_n) \in \mathbb{R}^n \mid x_1 < \cdots < x_n\}$, which is open in $\mathbb{R}^n$. Consider the map

$$\begin{array}{rccc} f\colon & M_n(\mathbb{R}) \times U & \longrightarrow & \mathbb{R}^n \\ & (M, (x_1, \ldots, x_n)) & \longmapsto & (\chi_M(x_1), \ldots, \chi_M(x_n))\,, \end{array}$$

where χ_M is the characteristic polynomial of M. It is a smooth map as it is polynomial in every entry. Let M_0 be a matrix with n distincts real eigenvalues $\lambda_1 < \ldots < \lambda_n$: we thus have $f\,(M_0, (\lambda_1, \ldots, \lambda_n)) = 0$. The matrix (in the canonical basis of $\mathbb{R}^n$) of $\partial_2 f\,(M_0, (x_1, \ldots, x_n))$ is given by

$$\mathrm{Mat}_{\mathrm{can}}\,(\partial_2 f(M_0, (x_1, \ldots, x_n)) = \begin{pmatrix} \chi'_{M_0}(x_1) & & \\ & \ddots & \\ & & \chi'_{M_0}(x_n) \end{pmatrix}.$$

Therefore, $\det(\partial_2 f(M_0, (\lambda_1, \ldots, \lambda_n)) = \prod_{i=1}^n \chi'_{M_0}(\lambda_i) \neq 0$ (recall that all λ_i are supposed distincts and χ_{M_0} is of degree n, so that χ_{M_0} and χ'_{M_0} do not share any root). The result now follows from a direct application of the implicit function Theorem: there exists $W \subset M_n(\mathbb{R}) \times U$ an open neighbourhood of $(M_0, (\lambda_1, \ldots, \lambda_n))$, $V \subset M_n(\mathbb{R})$ an open neighbourhood of M_0 and $\varphi\colon V \to U$ a smooth map such that

$$\forall\,(M, (\lambda_1, \ldots, \lambda_n)) \in W, f\,(M, (\lambda_1, \ldots, \lambda_n)) = 0 \iff M \in V \text{ and } (\lambda_1, \ldots, \lambda_n) = \varphi(M),$$

that is, all matrices $M \in V$ have n distinct eigenvalues $\lambda_1(M) < \cdots < \lambda_n(M)$ which are smooth maps of M.