Quotient topology & some classic manifolds

Exercise 1 (Quotient topology)**.**

Let X be a topological space and $\sim$ be an equivalence relation on X. We denote by $p : X \to X/\sim$ the canonical projection.

1. Recall the definition of the quotient topology on $X/\sim$.
2. Let $f : X/\sim \to Y$. Show that f is continuous if and only if $f \circ p$ is.
3. Show that if G is a discrete group acting properly discontinuously on a locally compact Hausdorff space X, *i.e.*, for all $g \in G$, $x \mapsto g \cdot x$ is a continuous map and
$$\{g \in G \mid K \cap g \cdot K \neq \varnothing\} \text{ is finite}, \quad \forall K \subset X \text{ compact subset},$$
then the quotient X/G is Hausdorff.
4. Let $\mathbb{T}^n = \mathbb{R}^n/\mathbb{Z}^n$ be the n-dimensional torus. Show that $\mathbb{T}^n$ is compact and Hausdorff, and that p is an open map.
5. Let $f : K \to Y$ be continuous and bijective with K compact Hausdorff and Y Hausdorff. Show that f is a homeomorphism. Give a counter-example if Y is not Hausdorff.
6. We define $\mathbb{S}^1$ as $\{z \in \mathbb{C} \mid |z| = 1\}$. Show that $\mathbb{T}^1$ is homeomorphic to $\mathbb{S}^1$. More generally, show that $\mathbb{T}^n$ is homeomorphic to $(\mathbb{S}^1)^n \subset \mathbb{C}^n$.
7. Let $\mathbb{RP}^n$ be the space defined as the quotient of $\mathbb{R}^{n+1} \setminus \{0\}$ by the equivalence relation "belonging to the same vector line". Show that $\mathbb{RP}^n$ is compact Hausdorff and that p is open.

Exercise 2 (The sphere)**.**

1. Define a differential structure on $\mathbb{S}^n = \left\{x \in \mathbb{R}^{n+1} \;\middle|\; \|x\|_2 = 1\right\}$ by using stereographic projections.
2. The Euclidean sphere $\mathbb{S}^n$ is also a manifold as a submanifold of $\mathbb{R}^{n+1}$. Show that the differentiable structure built in question 1 is the same as the one induced by $\mathbb{R}^{n+1}$.
3. *(bonus)* What happens if the Euclidean norm $\|\cdot\|_2$ is replaced by another norm $\|\cdot\|$?

Exercise 3 (Product manifolds)**.**

Let M and N be smooth manifolds, show that $M \times N$ is a smooth manifold. What is its dimension ?

Exercise 4 (The torus)**.**

1. Build a differentiable structure on $\mathbb{T}^n = \mathbb{R}^n/\mathbb{Z}^n$ such that $p \colon \mathbb{R}^n \to \mathbb{T}^n$ is a local diffeomorphism.
2. Prove that the homeomorphism $\mathbb{T}^n \to (\mathbb{S}^1)^n$ built in Exercise 1.6 is in fact a diffeomorphism.

Exercise 5 (The projective space)**.**

1. Let $i \in \{0, \dots, n\}$. Show that $U_i = \{[x_0 : \dots : x_n] \in \mathbb{RP}^n \mid x_i \neq 0\}$ is open in $\mathbb{RP}^n$, and construct a homeomorphism $\varphi_i \colon U_i \to \mathbb{R}^n$.
2. Show that $\mathbb{RP}^n$ equipped with the atlas $(\varphi_i)_{i \in \{0,\dots,n\}}$ is a smooth manifold.

3. Show that p is smooth.
4. Show that the restriction of p to $\mathbb{S}^n$ is a local diffeomorphism.
5. Show that $\mathbb{RP}^1$ is diffeomorphic to $\mathbb{S}^1$.

Exercise 6 (Klein Bottle)**.**

Let G be the group $\mathbb{Z}^2$ where the group law is defined by $(n,m)\star(p,q) = ((-1)^m p+n, m+q)$. The group $(\mathbb{Z}^2, \star)$ acts on $\mathbb{R}^2$ by the following formula : $(n,m)\cdot(x,y) = ((-1)^m x + n, y+m)$

1. Prove that this action is properly discontinuous (see Exercise 1.3 for the definition).
2. Let K be the quotient $[0,1]\times[0,1]/\sim$, where $\sim$ is the equivalence relation defined by $(x,0)\sim(1-x,1)$ and $(0,y)\sim(1,y)$ for all $x,y\in[0,1]$. Show that K is homeomorphic to the quotient $\mathbb{R}^2/G$. Show that $\mathbb{R}^2/G$ is compact Hausdorff.
3. Build a differential structure on $\mathbb{R}^2/G$ such that the canonical projection $p:\mathbb{R}^2\to\mathbb{R}^2/G$ is a local diffeomorphism.