Quotient topology & some classic manifolds
Correction

Let us recall a few definitions.

Definitions.

Let X be a topological space.

- X is *compact* if any open cover has a finite subcover, that is:

$$\forall \{U_i\}_{i\in I} \text{ open subsets such that } X = \bigcup_{i\in I} U_i, \exists i_1, \dots, i_n \in I \text{ such that } X = \bigcup_{k=1}^{n} U_{i_k}.$$

- $Y \subset X$ is compact if it is compact for the induced topology.
- X is *Hausdorff* if any two distinct points have disjoint neighbourhoods:

$$\forall x, y \in X, x \neq y, \exists U_x, U_y \text{ open such that } x \in U_x, y \in U_y \text{ and } U_x \cap U_y = \varnothing.$$

- X is *locally compact* if for any point $x \in X$ and any open neighbourhood V of x, there exists a smaller neighbourhood of x whose closure is compact and contained in U_x :

$$\forall x \in X, \forall V \text{ open neighb. of } x, \exists U \subset V \text{open with } \overline{U} \text{ compact}, x \in U \subset \overline{U} \subset V.$$

  If X is in addition Hausdorff, the following is equivalent: any point has a compact neighbourhood.

Let us also recall a few properties of topological spaces and continuous maps.

Lemma 1. *Let $f\colon K \to Y$ be a continuous map with K compact. Then $f(K)$ is compact.*

Proof. Suppose $f(K) \subset \bigcup_{i\in I}$ is an open cover. Then $K \subset f^{-1}(f(K)) \subset \bigcup_{i\in I} f^{-1}(U_i)$ is an open cover (because f is continuous) of K, which is compact. Therefore, there exists $i_1, \dots, i_n$ with $K = \bigcup_{j=1}^{n} f^{-1}\left(U_{i_j}\right)$. It follows that $f(K) \subset \bigcup_{j=1}^{n} U_{i_j}$ is a finite subcover, and that $f(K)$ is compact. □

Lemma 2. *Let X be Hausdorff and $K \subset X$ be compact. Then K is closed.*

Proof. Let us show that the complement $X \setminus K$ is open. Let $x \in X \setminus K$. As X is Hausdorff, for any $y \in K$, one can find open neighbourhoods $U_y \ni x$ and $V_y \ni y$ such that $U_y \cap V_y = \varnothing$. As $K \subset \bigcup_{y\in K} V_y$ and K is compact, there exists a finitly many points $y_1, \dots, y_n$ with $K \subset \bigcup_{j=1}^{n} V_{y_j}$. It follows that $U = \bigcap_{j=1}^{n} U_{y_j}$ is an open neighbourhood of x with $U \subset X \setminus K$, and $X \setminus K$ is open. □

Lemma 3. *Suppose X is compact and $F \subset X$ is closed, then F is compact.*

Proof. Suppose $F \subset \bigcup_{i\in I} U_i$ is an open cover. Then $X = F \cup (X \setminus F) = \left(\bigcup_{i\in I} U_i\right) \cup (X \setminus F)$ is an open cover of X. By compactness, there exists $i_1, \dots, i_n$ such that we have a finite subcover $X = \left(\bigcup_{i=1}^{n} U_{i_j}\right) \cup (X \setminus F)$. It follows that $F \subset \bigcup_{j=1}^{n} U_{i_j}$ is a finite subcover, and F is compact. □

Exercise 1 (Quotient topology).

1. $U \subset X/\sim$ is open $\iff p^{-1}(U) \subset X$ is open.
2. Let $f\colon X/\sim \longrightarrow Y$. Then:

$$\begin{aligned} f \text{ is continuous } &\iff \forall U \subset Y \text{ open }, f^{-1}(U) \subset X/\sim \text{ is open} \\ &\iff \forall U \subset Y \text{ open}, p^{-1}\left(f^{-1}(U)\right) \subset X \text{ is open} \\ &\iff \forall U \subset Y \text{ open}, (f\circ p)^{-1}(U) \subset X \text{ is open} \\ &\iff f\circ p \text{ is continuous.} \end{aligned}$$

3. Consider $\overline{x} \neq \overline{y} \in X/G$ and fix $x \in p^{-1}(\{\overline{x}\})$, $y \in p^{-1}(\{\overline{y}\})$. Note that for all $g \in X$, $x \neq g \cdot y$.

   First, X is locally compact, so that there exists $x \in U_x \subset K_x$ and $y \in U_y \subset K_y$ with U_x and U_y open and K_x and K_y compact. Define $K = K_x \cup K_y$. As $G \curvearrowright X$ is properly discontinuous, the set
$$\{g \in G \mid K \cap gK \neq \varnothing\}$$
   is finite, and so is the set
$$\{g \in G \mid U_x \cap gU_y \neq \varnothing\}.$$
   Call its elements $g_1, \dots, g_n$. The two G-invariant open sets
$$\bigcup_{g\in G} gU_x \text{ and } \bigcup_{g\in G} gU_y$$
   may overlaps precisely because of the elements finitely many elements $g_1, \dots, g_n$: indeed, if $g, h \in G$ are such that $gU_x \cap hU_y \neq \varnothing$, then $U_x \cap \left(g^{-1}h\right) U_y \neq \varnothing$. It follows that $K \cap \left(g^{-1}h\right) K \neq \varnothing$ and thus, $g^{-1}h \in \{g_1, \dots, g_n\}$. Let us correct these overlaps.

   The set X is Hausdorff, and for all $j \in \{1, \dots, n\}$, $x \neq g_j \cdot y$; thus, there exists $V_j \ni x$ and $W_j \ni g_j \cdot y$ open neighbourhoods with $V_j \cap W_j = \varnothing$. Define
$$V = U_x \cap \left(\bigcap_{j=1}^{n} V_j\right) \text{ and } W = U_y \cap \left(\bigcap_{j=1}^{n} {g_j}^{-1} W_j\right).$$
   By construction, V and W are open neighbourhoods of x and y respectively such that
$$\forall g \in G, V \cap gW = \varnothing.$$
   Denoting by $\widetilde{V} = \bigcup_{g\in G} gV$ and $\widetilde{W} = \bigcup_{g\in G} gW$, they are disjoint G-invariant open subsets by construction, with $x \in \widetilde{V}$ and $y \in \widetilde{W}$, so that $p(\widetilde{V}) \cap p\left(\widetilde{W}\right) = \varnothing$. But
$$p^{-1}\left(p\left(\widetilde{V}\right)\right) = \widetilde{V}, \qquad p^{-1}\left(p\left(\widetilde{W}\right)\right) = \widetilde{W},$$
   so that $p(\widetilde{V})$ and $p(\widetilde{W})$ are open in X/G. It follows that they are disjoint open neighbourhoods of $\overline{x}$ and $\overline{y}$, and X/G is Hausdorff.

4. The action $\mathbb{Z}^n \curvearrowright \mathbb{R}^n$ defined by $v \cdot x = x + v$ is properly discontinuous: indeed, if $K \subset \mathbb{R}^n$ is compact with $D = \operatorname{diam}_{\|\cdot\|_\infty} K < +\infty$, and if $v \in \mathbb{Z}^n$ is such that $\|v\|_\infty \geqslant D + 1$, then $K \cap (K + v) = \varnothing$. It follows that

$$\{v \in \mathbb{Z}^n \mid K \cap (K + v) \neq \varnothing\} \subset \overline{B}(0, D + 1) \cap \mathbb{Z}^n$$

is a discret subset of the compact $\overline{B}(0, D + 1)$: it is therefore finite. By 3., $\mathbb{T}^n$ is Hausdorff.

Note that the restriction $p|_{[0,1]^n} : [0,1]^n \to \mathbb{T}^n$ is surjective and continuous. It follows that $\mathbb{T}^n$ is compact as the continuous image of a compact.

Finally, if U is open in $\mathbb{R}^n$, then $p^{-1}(p(U)) = \cup_{v \in \mathbb{Z}^n} (U + v)$ is open as a union of open subsets ($U + v$ is homeomorphic to U). Therefore p is an open map.

5. In order to show that f is an homeomorphism, let us show that $g = f^{-1}$ is continuous. We will use the characterisation by closed subsets: g is continuous if and only if $\forall F \subset K$ closed, $g^{-1}(F) \subset K$ is closed.

Fix $F \subset K$ a closed subset. Then F is compact as a closed subset of K compact. It follows that $g^{-1}(F) = f(F)$ is compact as the continuous image of F compact by f. Now, recall that Y is Hausdorff, so that $F \subset Y$ is closed. The result follows.

A counterexample is given by $\mathrm{id} \colon (X, \tau_1) \to (X, \tau_2)$, with $X = \{0, 1\}$, $\tau_1 = \mathcal{P}(X)$ and $\tau_2 = \{\varnothing, X\}$.

6. The map $f \colon t \in \mathbb{R} \mapsto e^{2i\pi t} \in \mathbb{S}^1$ is a surjective group homomorphism. Moreover, its kernel is $\mathbb{Z}$, which acts properly discontinuously on $\mathbb{R}$. It thus induces a continuous bijection

$$\overline{f} \colon \mathbb{T}^1 \to \mathbb{S}^1.$$

Recall that $\mathbb{T}^1$ is compact Hausdorff by 4., and that $\mathbb{S}^1$ is Hausdorff as a subspace of $\mathbb{C}$. Thus, it is a homeomorphism by 5.

More generally, the exact same study with $f_n \colon \mathbb{R}^n \to \left(\mathbb{S}^1\right)^n$ defined by $f_n(t_1, \ldots, t_n) = \left(e^{2i\pi t_1}, \ldots, e^{2i\pi t_n}\right)$ yields a homeomorphism $\overline{f_n} \colon \mathbb{T}^n \to \left(\mathbb{S}^1\right)^n$.

7. Apparently, we do not have any properly discontinuous action of a discrete group. We have to find one.

Consider the inclusion map $i \colon \mathbb{S}^n \to \mathbb{R}^{n+1} \setminus \{0\}$. It is clearly continuous. Consider now the composition

$$q = p \circ i \colon \mathbb{S}^n \longrightarrow \mathbb{R}^{n+1} \setminus \{0\} \longrightarrow \mathbb{RP}^n.$$

By 1., q is continuous. Moreover, it is surjective as any (linear) line in $\mathbb{R}^{n+1}$ intersects $\mathbb{S}^n$. It follows that $\mathbb{RP}^n = q(\mathbb{S}^n)$. The unit sphere being compact (for example, it is bounded and closed in the finite dimensional linear space $\mathbb{R}^{n+1}$), so is $\mathbb{RP}^n$.

Define the antipodal action $\{\pm 1\} \curvearrowright \mathbb{S}^n$ to be $k \cdot x = kx$. As $\{\pm 1\}$ is finite, it is clearly a properly discontinuous action, and by 2., the quotient $\mathbb{S}^n / \{\pm 1\}$ is Hausdorff. It is also compact, being the image of the quotient map $\pi \colon \mathbb{S}^n \to \mathbb{S}^n / \{\pm 1\}$.

Notice that if $x \in \mathbb{S}^n$, then $q^{-1}(\{q(x)\}) = \{\pm x\}$, so that q descends as a quotient map

$$\overline{q} \colon \mathbb{S}^n / \{\pm 1\} \longrightarrow \mathbb{RP}^n$$

such that $q = \overline{q} \circ \pi$. By 1., $\overline{q}$ is continuous. The map $\overline{q}$ is hence a continuous bijection between $\mathbb{S}^n/\{\pm 1\}$, which is compact Hausdorff, and $\mathbb{RP}^n$, which is compact. By 5., it is a homeomorphism, and finally, $\mathbb{RP}^n$ is compact Hausdorff.

Let us show that p is an open map. Let $U \subset \mathbb{R}^{n+1} \setminus \{0\}$ be an open subset. Then

$$p^{-1}(p(U)) = \bigcup_{r \neq 0} rU$$

where $rU = \{rx \mid x \in U\}$. But rU is the preimage of U by the continuous map $h_r \colon \mathbb{R}^n \setminus \{0\} \to \mathbb{R}^n \setminus \{0\}$ defined by $h_r(x) = \frac{x}{r}$. It follows that rU is open and therefore, $p^{-1}(p(U))$ is open as a union of open subsets. Finally, $p(U)$ is open, and p is an open map.

Exercise 2 (The sphere)**.**

1. The unit sphere is Hausdorff and second-countable as a subspace of $\mathbb{R}^{n+1}$ which is both. Let $N = (0, \dots, 0, 1) \in \mathbb{S}^n$ and $S = (0, \dots, 0, -1) \in \mathbb{S}^n$ be the north and south pole of $\mathbb{S}^n$ and $U_N = \mathbb{S}^n \setminus \{N\}$, $U_S = \mathbb{S}^n \setminus \{S\}$. $\{U_N, U_S\}$ is an open cover of $\mathbb{S}^n$. Consider the stereographic projections

$$\begin{aligned} p_N \colon \mathbb{S}^n \setminus \{N\} &\longrightarrow \mathbb{R}^n \\ (X,t) &\longmapsto \frac{X}{1-t} \end{aligned} \quad , \quad \begin{aligned} p_S \colon \mathbb{S}^n \setminus \{S\} &\longrightarrow \mathbb{R}^n \\ (X,t) &\longmapsto \frac{X}{1+t} \end{aligned}$$

where (X,t) refers to a point of $\mathbb{S}^n$ seen as a subset of $\mathbb{R}^n \times \mathbb{R}$.

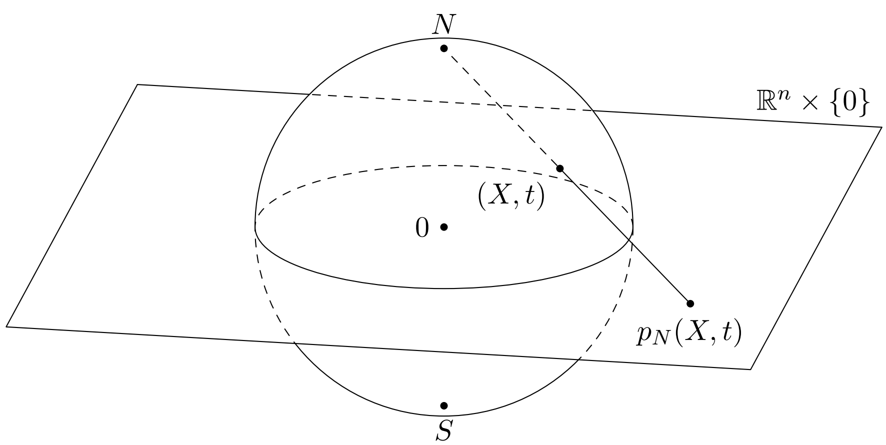

Figure 1: The stereographic projection from the north pole

They are homeomorphisms: indeed, one can check that they are continuous and that they have inverse

$$\begin{aligned} {p_N}^{-1} \colon \mathbb{R}^n &\longrightarrow \mathbb{S}^n \setminus \{N\} \\ x &\longmapsto \left(\frac{2x}{\|x\|^2+1}, \frac{\|x\|^2-1}{\|x\|^2+1} \right) \end{aligned} \quad , \quad \begin{aligned} {p_S}^{-1} \colon \mathbb{R}^n &\longrightarrow \mathbb{S}^n \setminus \{S\} \\ x &\longmapsto \left(\frac{2x}{\|x\|^2+1}, \frac{1-\|x\|^2}{\|x\|^2+1} \right) \end{aligned}$$

which are continuous.

Moreover, as $U_N \cap U_S = \mathbb{S}^n \setminus \{N, S\}$, it holds that $p_N(U_N \cap U_S) = p_S(U_N \cap U_S) = \mathbb{R}^n \setminus \{0\}$, and the transition map $p_N \circ p_S^{-1}$ is given by

$$p_N \circ p_S{}^{-1}(x) = p_N\left(\frac{2x}{\|x\|^2+1}, \frac{1-\|x\|^2}{\|x\|^2+1}\right) = \frac{\frac{2x}{\|x\|^2+1}}{1-\frac{1-\|x\|^2}{\|x\|^2+1}} = \frac{x}{\|x\|^2},$$

which is indeed smooth. Note that it is involutive so that $p_S \circ p_N{}^{-1}$ is also smooth. It follows that $\{(p_N, U_S), (p_S, U_S)\}$ is a smooth atlas. Considering the maximal atlas containing it endows $\mathbb{S}^n$ with a smooth manifold structure.

2. **Here is a slighlty different proof than the one given orally.**

   Let us show that the two stereographic projections are restrictions of two ambiant charts in $\mathbb{R}^{n+1}$. Consider the two geometric inversions

$$\begin{aligned} I_{N,\sqrt{2}} \colon \mathbb{R}^{n+1} \setminus \{N\} &\longrightarrow \mathbb{R}^{n+1} \setminus \{N\} \\ x &\longmapsto N + 2\frac{x-N}{\|x-N\|^2} \end{aligned} \quad , \qquad \begin{aligned} I_{S,\sqrt{2}} \colon \mathbb{R}^{n+1} \setminus \{S\} &\longrightarrow \mathbb{R}^{n+1} \setminus \{S\} \\ x &\longmapsto S + 2\frac{x-S}{\|x-S\|^2} \end{aligned}$$

   They are the geomoetric inversions in the spheres of radius $\sqrt{2}$ and centers N and S respectively. They are two charts for $\mathbb{R}^{n+1}$ and their restrictions to $\mathbb{S}^n$ are p_N and p_S. It follows that the stereographic projections p_N and p_S are induced by ambiant charts, and that the differentiable structure of $\mathbb{S}^n$ induced from the ambiant space $\mathbb{R}^{n+1}$ is compatible with the atlas $\{(U_N, p_N), (U_S, p_S)\}$. They thus define the same differentiable structure on the sphere.

Exercise 3 (Product manifolds)**.**

First, a product of two Hausdorff (resp. second-countable) spaces is Hausdorff (resp. second-countable), so $M \times N$ is Hausdorff (resp. second-countable). Let us find a smooth atlas on $M \times N$.

Let $\{(U_\alpha, \varphi_\alpha)\}_{\alpha\in A}$ (respectively $\{(V_\beta, \psi_\beta)\}_{\beta\in B}$) be the maximal smooth atlas of M^m (respectively of N^n). Let us show that $\{(U_\alpha \times V_\beta, \varphi_\alpha \times \psi_\beta)\}_{(\alpha,\beta)\in A\times B}$ is a smooth atlas of $M \times N$.

First, for all $(\alpha, \beta) \in A \times B$, $U_\alpha \times V_\beta$ is open in $M \times N$ by definition of the product topology, and moreover:

$$\begin{aligned} \bigcup_{(\alpha,\beta)\in A\times B} U_\alpha \times V_\beta &= \bigcup_{\alpha\in A}\bigcup_{\beta\in B} U_\alpha \times V_\beta \\ &= \bigcup_{\alpha\in A}\left(U_\alpha \times \left(\bigcup_{\beta\in B} V_\beta\right)\right) \\ &= \bigcup_{\alpha\in A} U_\alpha \times N \\ &= \left(\bigcup_{\alpha\in A} U_\alpha\right) \times N \\ &= M \times N \end{aligned}$$

so that it is indeed an open cover.

In addition, if $(\alpha, \beta) \in A \times B$, then

$$\begin{aligned} \varphi_\alpha \times \psi_\beta \colon U_\alpha \times V_\beta &\longrightarrow \varphi_\alpha (U_\alpha) \times \psi_\beta (V_\beta) \subset \mathbb{R}^m \times \mathbb{R}^n \\ (x, y) &\longmapsto (\varphi_\alpha(x), \psi_\beta(y)) \end{aligned}$$

is continuous by definition of the product topology, bijective with inverse $\varphi_\alpha^{-1} \times \psi_\beta^{-1}$, which is also continuous. It is thus a homeomorphism.

Finally, if (α_1, β_1) and (α_2, β_2) are elements of $A \times B$, then

$$(\varphi_{\alpha_1} \times \psi_{\beta_1}) \circ (\varphi_{\alpha_2}, \psi_{\beta_2})^{-1} = \left(\varphi_{\alpha_1} \circ \varphi_{\alpha_2}^{-1}\right) \times \left(\psi_{\beta_1} \circ \psi_{\beta_2}^{-1}\right)$$

is defined from $\varphi_{\alpha_2} (U_{\alpha_1} \cap U_{\alpha_2}) \times \psi_{\beta_2} (V_{\beta_1} \cap V_{\beta_2})$ to $\varphi_{\alpha_1} (U_{\alpha_1} \cap U_{\alpha_2}) \times \psi_{\beta_1} (V_{\beta_1} \cap V_{\beta_2})$, is also smooth.

Hence, $M^m \times N^n$ is a smooth manifold of dimension $m + n$.

Exercise 4 (The torus)**.**

Recall that we have shown in Exercise 1.4. that $p \colon \mathbb{R}^n \to \mathbb{T}^n$ is an open map.

1. Let us first consider the 1 dimensional case. We have an open map

$$p \colon \mathbb{R} \to \mathbb{T}^1$$

so that if $\widetilde{U}_1 = (0, 1)$ and $\widetilde{U}_2 = \left(-\frac{1}{2}, \frac{1}{2}\right)$, then $U_1 = p\left(\widetilde{U}_1\right)$ and $U_2 = p\left(\widetilde{U}_2\right)$ are open in $\mathbb{T}^1$. Note that

$$U_1 \cup U_2 = p\left(\widetilde{U}_1\right) \cup p\left(\widetilde{U}_2\right) = p\left(\widetilde{U}_1 \cup \widetilde{U}_2\right) = p\left(\left(-\frac{1}{2}, 1\right)\right) = \mathbb{T}^1,$$

so that $\{U_1, U_2\}$ is an open cover of $\mathbb{T}^1$. Consider the two maps :

$$\varphi_1 \colon U_1 \longrightarrow \widetilde{U}_1, \qquad\qquad \varphi_2 \colon U_2 \longrightarrow \widetilde{U}_2$$

defined so that $\varphi_j(\overline{x})$ is the only element of $\widetilde{U}_j \cap p^{-1}(\{\overline{x}\})$. As p is an open map, φ_j are continuous. Being (local) right inverse to p, they are homeomorphisms. Let us show that $\{(U_j, \varphi_j)\}_{j \in \{1,2\}}$ is a smooth atlas.

We have

$$U_1 \cap U_2 = \mathbb{T}^1 \setminus \left\{\overline{0}, \overline{\frac{1}{2}}\right\}, \quad \varphi_1 (U_1 \cap U_2) = (0, 1) \setminus \left\{\frac{1}{2}\right\}, \quad \varphi_2 (U_1 \cap U_2) = \left(-\frac{1}{2}, \frac{1}{2}\right) \setminus \{0\}.$$

Therefore, the transition functions are:

$$\begin{aligned} \varphi_1 \circ \varphi_2^{-1} \colon \quad \left(-\tfrac{1}{2}, \tfrac{1}{2}\right) \setminus \{0\} \quad &\longrightarrow \quad (0, 1) \setminus \left\{\tfrac{1}{2}\right\} \\ x \quad &\longmapsto \quad \begin{cases} x + 1 & \text{if } x < 0, \\ x & \text{if } x > 0, \end{cases} \end{aligned}$$

and

$$\begin{array}{rccc} \varphi_2 \circ \varphi_1^{-1} \colon & (0,1) \setminus \left\{\frac{1}{2}\right\} & \longrightarrow & \left(-\frac{1}{2}, \frac{1}{2}\right) \setminus \{0\} \\ & x & \longmapsto & \begin{cases} x & \text{if } x < \frac{1}{2}, \\ x-1 & \text{if } x > \frac{1}{2}. \end{cases} \end{array}$$

These two transition functions are both smooth. We thus have constructed a smooth atlas on $\mathbb{T}^1$.

Now, for the n-dimensional case, consider the open subsets

$$\forall\,(i_1, \ldots, i_n) \in \{1,2\}^n, U_{i_1 \cdots i_n} = U_{i_1} \times \cdots \times U_{i_n} \text{ and } \widetilde{U}_{i_1 \cdots i_n} = \widetilde{U}_{i_1} \times \cdots \times \widetilde{U}_{i_n},$$

where $\widetilde{U}_k$ and U_k are defined in the one dimensional case. Similarly to the one dimensional case, we show that they form an open cover of $\mathbb{T}^n$. Define the functions

$$\begin{array}{rccc} \varphi_{i_1 \cdots i_n} \colon & U_{i_1 \cdots i_n} & \longrightarrow & \widetilde{U}_{i_1 \cdots i_n} \\ & \overline{(x_1, \ldots, x_n)} & \longmapsto & \left(\varphi_{i_1}\left(\overline{x_1}\right), \ldots, \varphi_{i_n}\left(\overline{x_n}\right)\right). \end{array}$$

Check that $\{(U_{i_1 \cdots i_n}, \varphi_{i_1 \cdots i_n})\}_{(i_1, \ldots, i_n) \in \{1,2\}^n}$ is a smooth atlas on $\mathbb{T}^n$.

2. Check that the map $\overline{f} \colon \mathbb{T}^n \to \left(\mathbb{S}^1\right)^n$ defined in Exercise 1.6 is smooth while seen in the constructed charts. To see that $\overline{f}^{-1}$ is smooth, use the inverse function theorem (in charts).

Exercise 5 (The projective space)**.**

This Exercices has not been covered in class (at least for one group). Please try to solve it by yourself before checking the correction!

1. Recall from Exercise 1.7. that $p \colon \mathbb{R}^{n+1} \setminus \{0\} \to \mathbb{RP}^n$ is open. Notice that the subset $V_i = \left\{(x_0, \ldots, x_n) \in \mathbb{R}^{n+1} \setminus \{0\} \mid x_i \neq 0\right\}$ is open in $\mathbb{R}^{n+1} \setminus \{0\}$. Therefore, $U_i = p(V_i)$ is open in $\mathbb{RP}^n$. Define

$$\begin{array}{rccc} \Phi_i \colon & V_i & \longrightarrow & \mathbb{R}^n \\ & (x_0, \ldots, x_n) & \longmapsto & \frac{1}{x_i}\left(x_0, \ldots, \widehat{x_i}, \ldots, x_n\right). \end{array}$$

It is clearly continuous and surjective. Moreover, we have

$$\forall t \neq 0, \forall X \in V_i, \Phi_i(tX) = \Phi_i(X).$$

In fact, we have $\Phi_i(X) = \Phi_i(Y) \iff \exists t \neq 0, X = tY$ (check that in that case, $t = \frac{x_i}{y_i}$). It thus induces a continuous bijective map

$$\begin{array}{rccc} \varphi_i \colon & U_i & \longrightarrow & \mathbb{R}^n \\ & [x_0 : \cdots : x_n] & \longmapsto & \frac{1}{x_i}\left(x_0, \ldots, \widehat{x_i}, \ldots, x_n\right) \end{array}$$

Let us show that φ_i is a homeomorphism : to do so, let us find a continuous inverse to ϕ_i. Consider

$$\begin{array}{rccc} \Psi_i \colon & \mathbb{R}^n & \longrightarrow & V_i \\ & (x_0, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n) & \longmapsto & (x_0, \ldots, x_{i-1}, 1, x_{i+1}, \ldots, x_n). \end{array}$$

Then Ψ_i is continuous. It follows that $\psi_i = p \circ \Psi_i$ is continuous. But $\psi_i \circ \varphi_i = \mathrm{id}_{\mathbb{R}^n}$ and $\varphi_i \circ \psi_i = \mathrm{id}_{U_i}$, so that $\varphi_i \colon U_i \to \mathbb{R}^n$ is a homeomorphism.

2. First, $\bigcup_{i=0}^n V_i = \mathbb{R}^{n+1} \setminus \{0\}$ so that $p\left(\bigcup_{i=0}^n V_i\right) = \bigcup_{i=0}^n U_i = \mathbb{RP}^n$, and $\{U_i\}_{i\in\{0,\dots,n\}}$ is an open cover of $\mathbb{RP}^n$. Moreover, if $i \neq j$, $U_i \cap U_j = \{[x_0 : \dots : x_n] \mid x_i, x_j \neq 0\}$ and

$$\varphi_i \circ \varphi_j^{-1}(x_0, \dots, x_{j-1}, x_{j+1}, \dots, x_n) = \left(\frac{x_0}{x_i}, \dots, \frac{x_{i-1}}{x_i}, \frac{x_{i+1}}{x_i}, \dots, \frac{x_{j-1}}{x_i}, \frac{1}{x_i}, \frac{x_{j+1}}{x_i}, \dots, \frac{x_n}{x_i}\right)$$

is a rational function, and hence is smooth. It follows that $\{(U_i, \varphi_i)\}_{i\in\{0,\dots,n\}}$ is a smooth atlas for $\mathbb{RP}^n$.

We already know that $\mathbb{RP}^n$ is Hausdorff from Exercice 1.7.

Finally, it is second-countable as the image of $\mathbb{R}^{n+1} \setminus \{0\}$, which is second-countable, by p, which is continuous and open.

We thus have shown that $\mathbb{RP}^n$ is a smooth manifold.

3. Let us show that p is smooth, that is, for all $i \in \{0, \dots, n\}$, the map

$$p_i = \varphi_i \circ p \colon p^{-1}(U_i) = \mathbb{R}^{n+1} \setminus \{x_i = 0\} \to \varphi_i(U_i) = \mathbb{R}^n$$

is smooth. Its expression is given by

$$p_i(x_0, \dots, x_n) = \frac{1}{x_i}(x_0, \dots, \widehat{x_i}, \dots, x_n)$$

which is indeed smooth as a rational function.

4. Let us show that p is a local diffeomorphism near the north pole N. Let D_N be the open upper hemisphere of $\mathbb{S}^n$, that is

$$D_N = \{(x_0, \dots, x_n) \in \mathbb{S}^n \mid x_n > 0\}$$

and consider the stereographic projection from the south pole p_S (see exercise 2.). Then $p_S(D_N) = B_0(1) \subset \mathbb{R}^n$. Thus, $p_S|_{D_N}$ takes value in U_n and we can consider the composition

$$f = \varphi_n \circ p \circ p_S^{-1} \colon B_0(1) \longrightarrow \mathbb{R}^n$$

which has the expression, for $x = (x_1, \dots, x_n) \in B_0(1)$:

$$f(x) = \varphi_n \circ p\left(\frac{2x}{\|x\|^2+1}, \frac{1-\|x\|^2}{\|x\|^2+1}\right) = \varphi_n\left(\left[\frac{2x}{\|x\|^2+1} : \frac{1-\|x\|^2}{\|x\|^2+1}\right]\right) = \frac{2x}{1-\|x\|^2}.$$

Let us compute its differential at 0. We have, for h small enough:

$$f(h) = \frac{2h}{1-\|h\|^2} = 2h(1+o(h)) = f(0) + 2h + o(h)$$

So that $\mathrm{d}f(0) = 2\,\mathrm{id}$ is invertible. The inverse function Theorem then shows that f is a local diffeomorphism near 0. This means that p is a local diffeomorphism near the north pole N.

The exact same study near any point $P \in \mathbb{S}^n$, considering the stereographic projection from $-P$, shows that the projection p is a local diffeomorphism near P, and thus, p is a local diffeomorphism near any point. It is then a local diffeomorphism.

5. We have the stereographic atlas for $\mathbb{S}^1$ given by $\left\{\left(p_N, \mathbb{S}^1 \setminus \{N\}\right), \left(p_S, \mathbb{S}^1 \setminus \{S\}\right)\right\}$, with transition function

$$\begin{array}{rccc} p_N \circ p_S^{-1} \colon & \mathbb{R}^* & \longrightarrow & \mathbb{R}^* \\ & t & \longmapsto & \frac{1}{t} \end{array}$$

and the affine atlas for $\mathbb{RP}^1$ given by $\{(U_0, \varphi_0), (U_1, \varphi_1)\}$, with transition function

$$\begin{array}{rccc} \varphi_0 \circ \varphi_1^{-1} \colon & \mathbb{R}^* & \longrightarrow & \mathbb{R}^* \\ & t & \longmapsto & \frac{1}{t}. \end{array}$$

Heuristically, these two manifolds are given by the same construction : they are two disjoint copies of $\mathbb{R}$ where we identify the two disjoint $\mathbb{R}^*$ with the inverse map. This allows us to construct a diffeomorphism. Define

$$\begin{array}{rccc} f\colon & \mathbb{RP}^1 & \longrightarrow & \mathbb{S}^1 \\ & [x:y] & \longmapsto & \begin{cases} p_N^{-1} \circ \varphi_0\left([x:y]\right) & \text{if } [x:y] \in U_0 \\ p_S^{-1} \circ \varphi_1\left([x:y]\right) & \text{if } [x:y] \in U_1 \end{cases} \end{array}$$

Check that f is well-defined (that is, its two expressions on $U_0 \cap U_1$ give the same point in $\mathbb{S}^1$) and that f is a diffeomorphism.