Submanifolds, Tangent spaces and Differentials
Critical values, Sard's Theorem

Definitions (Tangent bundle of a submanifold, vector fields, parallelizability)**.**

Let $M^m \subset \mathbb{R}^n$ be a submanifold. Recall that in that case, $T_pM \subset \mathbb{R}^n$ is a linear subspace of dimension m. We define

$$TM = \bigcup_{p \in M} \{p\} \times T_pM \subset \mathbb{R}^n \times \mathbb{R}^n$$

and we call TM the *tangent bundle* of M. It is a submanifold of $\mathbb{R}^n \times \mathbb{R}^n$ (admitted).

A *vector field* on $M^m \subset \mathbb{R}^n$ is a smooth function $X \colon M^n \to \mathbb{R}^n$ such that for all $p \in M$, $X(p) \in T_pM$.

$M^m \subset \mathbb{R}^n$ is said to be *parallelizable* if there exists m vector fields $X_1, \ldots, X_m$ on M such that for all $p \in M$, $\{X_1(p), \ldots, X_m(p)\}$ is a linearly independant family ($i.e$ a basis of T_pM).

Exercise 1 (Tangent space of a submanifold)**.**

Let $M \subset \mathbb{R}^m$ and $N \in \mathbb{R}^n$ be submanifolds of $\mathbb{R}^m$ and $\mathbb{R}^n$ respectively.

1. Describe the tangent space $T_pM \subset \mathbb{R}^m$ of the submanifold M of $\mathbb{R}^m$ at a point p, for each of the four characterizations of a submanifold. Why is there no ambiguity in identifying the tangent space $\subset \mathbb{R}^m$ at a point p of M seen as a submanifold with its tangent space at p where M is seen as a manifold (endowed with the differentiable structure naturally induced)?

2. Let $U \subset \mathbb{R}^m$ and $V \subset \mathbb{R}^n$ be open neighborhoods. Let $\widetilde{f} : U \to V$ be a smooth map such that $f := \widetilde{f}|_{M \cap U} : M \cap U \to N \cap V$. Show that f is a smooth map between manifolds and that
$$\mathrm{d}f_p = \left(\mathrm{d}\widetilde{f}_p\right)\Big|_{T_pM} : T_pM \to T_{f(p)}N.$$

Exercise 2 (Veronese embedding)**.**

1. Let $f \colon M \to N$ be a proper injective immersion between two manifolds. Show that f is an embedding. Is it a necessary condition?

2. Consider the map $f \colon \mathbb{RP}^2 \to \mathbb{RP}^5$ defined in homogeneous coordinates by the relation $f\left(x : y : z\right) = \left(x^2 : y^2 : z^2 : xy : yz : zx\right)$. Show that f is well defined and that it is an embedding.

Exercise 3 (Tangent space of the torus)**.**

1. Find an embedding $\mathbb{T}^2 \to \mathbb{R}^4$ and show that $\mathbb{T}^2$ is parallelizable.

2. Show that $\mathbb{T}^2$ can be embedded in $\mathbb{R}^3$. "Draw" a parallelization of $\mathbb{T}^2 \subset \mathbb{R}^3$.

3. Is $\mathbb{T}^n$ parallelizable?

Exercise 4 (Tangent space of spheres)**.**

1. Show that $\mathbb{S}^1$ is parallelizable. Is $\mathbb{S}^2$ parallelizable?

2. A Lie group is smooth manifold G endow withed a group structure such that the multiplication $\mu : G \times G \longrightarrow G$ and the inversion $\eta : G \longrightarrow G$ are smooth maps. In this exercise, we consider Lie group that are submanifolds of $\mathbb{R}^n$ for some n.

   (a) Show that if $G \subset \mathbb{R}^n$ is a Lie group, then G parallelizable.

   (b) Show that $SU(2)$ is a Lie group diffeomorphic to $\mathbb{S}^3$.

   (c) Deduce that $\mathbb{S}^3$ is parallelizable.

   (d) *(bonus)* Looking at $\mathbb{S}^3$ as the unit sphere of $\mathbb{C}^2$, find 3 linearly independant vector fields on $\mathbb{S}^3$.

Exercise 5 (Computation of a differential)**.**

Compute the differential of $\overline{F} : \mathbb{T}^2 \to \mathbb{S}^2$ defined as the quotient of the map from $\mathbb{R}^2$ to $\mathbb{S}^2$:

$$F : (x, y) \mapsto (\cos(2\pi x)\cos(2\pi y), \cos(2\pi x)\sin(2\pi y), \sin(2\pi x)).$$

On which set is $\overline{F}$ a local diffeomorphism? Is $\overline{F}$ restricted to this domain a global diffeomorphism?

Exercise 6 (Extending smooth function)**.**

Let $M \subset \mathbb{R}^p$ and $N \subset \mathbb{R}^q$ be two submanifolds and $f \colon M \to N$ a smooth function. Show that there exists an open neighbourhood of M in $\mathbb{R}^p$ and a smooth function $g \colon U \to \mathbb{R}^q$ such that $g|_M = f$. *Hint:* use a partition of unity.

Exercise 7 (Critical points VS critical values)**.**

1. Let $F \subset \mathbb{R}$ be a closed subset. Show that there exists a smooth function $f \colon M \to \mathbb{R}_+$ such that $f(x) = 0 \iff x \in K$.

2. Let $K \subset [0, 1]$ be a *fat Cantor set*[1], of measure $\lambda(K) \in (0, 1)$. Show that there exists a smooth homeomorphism $f \colon \mathbb{R} \to \mathbb{R}$ whose critial set is exactly K. What is the Lebesgue measure of $f(K)$?

Exercise 8 (Change of variable)**.**

The change of variable Theorem says that if you have a (at least) $\mathcal{C}^1$ diffeomorphism φ between two open sets of $\mathbb{R}^n$, you have a relation between the measure of a borelian set B, and that of $\varphi(B)$. Using Sard's Theorem, show that the result is still valid if φ is an homeomorphism of class $\mathcal{C}^1$.

Exercise 9 (Introduction to Morse Theory)**.**

Using some nice embeddings of $\mathbb{S}^2$ and $\mathbb{T}^2$, show that there exists smooth real functions on these manifolds with a very few critical points. *Hint:* consider projections onto some axis.

Draw what happens when you "go through" a critical value.

[1] The important fact is that K is closed, with emply interior