Sard's Theorem and vector bundles

Exercise 1 (Small perturbations of a submanifold).

Let M be a submanifold of $\mathbb{R}^n$ of dimension m with $2m < n$.

1. Show that for all $\varepsilon > 0$, there exists $v \in \mathbb{R}^n$ with $\|v\| < \varepsilon$ such that $(M + v) \cap M = \varnothing$.
2. What if $n \leqslant 2m$?

Exercise 2 ($\mathcal{C}^\infty$ and regular values).

Let M be a manifold and V be a finite dimensional linear subspace of $\mathcal{C}^\infty(M)$ that contains the constant maps.

1. Prove that $\Sigma = \{(f, x) \in V \times M \mid f(x) = 0\}$ is a hypersurface of $V \times M$, and describe $T_{(f,x)}\Sigma$.
2. In this question, $M = \mathbb{R}$.
   (a) Let $(f, x) \in \Sigma$ such that $f'(x) \neq 0$. Show that there exists U and V, open neighborhoods of f and x and a smooth map $\varphi : U \to V$ such that $\varphi(f) = x$ and $g(\varphi(g)) = 0$ for all $g \in U$.
   (b) Deduce that the simple roots of a polynomial map in $\mathbb{R}_d[X]$ are smooth maps of the coefficients.
   (c) What happens for the multiple roots?
3. Let p_V and p_M be the projections from Σ to V and M.
   (a) Show that p_M is a submersion.
   (b) Find the critical points of p_V as well as its critical values.
   (c) Show that the set of $f \in V$ such that $f^{-1}(0)$ is either empty or a hypersurface of M has full measure.

Exercise 3 (Some vector bundles).

1. Let $\pi \colon E \to N$ be a vector bundle and $f \colon M \to N$ be a smooth function. Define $f^*E = \{(x, e) \in M \times E \mid f(x) = \pi(e)\}$. Show that f^*E is a vector bundle over M.
2. If M is a submanifold of $\mathbb{R}^n$, define its *normal bundle* by
$$\nu M = \{(x, v) \in M \times \mathbb{R}^n \mid v \perp T_x M\}$$
   Show that νM is a vector bundle over M.
3. Show that if M and N are two manifolds, then $T(M \times N) \simeq TM \times TN$.

Exercise 4 ($\mathbb{S}^n \times \mathbb{S}^1$ is parallelizable).

A manifold M^n is said to be parallelizable if its tangent bundle is isomorphic to the trivial vector bundle $TM \simeq M \times \mathbb{R}^n$.
Let $M \subset \mathbb{R}^n$ be a submanifold and $i \colon M \to \mathbb{R}^n$ be the inclusion.

1. Show that $i^*(TM)$ is a trivial vector bundle over M.
2. Show that $TM \oplus \nu M$ is a trivial vector bundle.
3. Recalling that $\mathbb{S}^1$ is parallelizable, show that $\mathbb{S}^n \times \mathbb{S}^1$ is parallelizable.