Sard's Theorem and vector bundles

Exercise 1.

Let M be a submanifold of $\mathbb{R}^n$ of dimension m with $2m < n$.

1. Define
$$f\colon \begin{array}{ccc} M \times M & \longrightarrow & \mathbb{R}^n \\ (x,y) & \longmapsto & x-y \end{array}$$
f is smooth, and since $\dim(M\times M) = 2m < n$, all points are critical. By Sard's Lemma, $f(M\times M)$ has measure zero in $\mathbb{R}^n$, and is therefore nowhere dense. But we have
$$f(M\times M) = \{v \in \mathbb{R}^n \mid \exists (x,y) \in M\times M, v = x-y\} = \{v\in\mathbb{R}^n \mid (M+v)\cap M \neq \varnothing\}$$
It follows that $\forall \varepsilon > 0$, there exists $v \in B(0;\varepsilon)$ such that $v \notin f(M\times M)$, and $(M+v)\cap M = \varnothing$.

2. Consider $\mathbb{S}^1 \subset \mathbb{R}^2$. If $\|v\| < 1$, then $\mathbb{S}^1 \cap (\mathbb{S}^1 + v) \neq \varnothing$ and the result is false in that case.

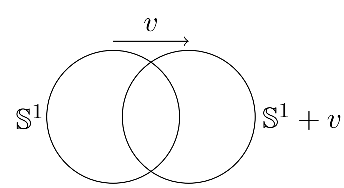

Figure 1: $\mathbb{S}^1 \cap (\mathbb{S}^1 + v) \neq \varnothing$

Exercise 2.

1. The smooth map
$$F\colon \begin{array}{ccc} V \times M & \longrightarrow & \mathbb{R} \\ (f,x) & \longmapsto & f(x) \end{array}$$
has differential
$$\forall (f,x) \in V\times M, \forall (h,v) \in T_{(f,x)}(V\times M) = V\times T_xM, \quad \mathrm{d}F(f,x)\cdot(h,v) = \mathrm{d}f(x)\cdot v + h(x)$$
Since V contains the constant functions, then $\mathrm{d}F(f,x)\cdot(c,0) = c$ for all constant $c\in\mathbb{R}$, and there is no critical point. It follows that 0 is a regular value and $\Sigma = F^{-1}(\{0\})$ is a smooth hypersurface.
Moreover, $T_{(f,x)}\Sigma = \ker \mathrm{d}F(f,x) = \{(h,v) \in V\times T_xM \mid h(x) = -\mathrm{d}f(x)\cdot v\}$.

2. In this question, $M = \mathbb{R}$.
   (a) The map $G = (f,x) \in V\times M \mapsto f(x) \in \mathbb{R}$ is smooth. Its partial differential with respect to the second variable is given by $D_2G(f,x) = (t \mapsto f'(x)t)$. If $f'(x) \neq 0$, D_2G is an isomorphism, and the implicit function Theorem provides the existence the desired map.

(b) $\mathbb{R}_d[X]$ is a finite dimensional vector space containing the constant functions. Apply 2.(a).

(c) The map $t \in (-\varepsilon, \varepsilon) \mapsto P_t = X^2 - t$ is a smooth path in $\mathbb{R}_2[X]$ but there is no smooth function $t \mapsto \varphi(t)$ such that $\varphi(t)$ is a root of P_t.

3. Let p_V and p_M be the projections from Σ to V and M.

(a) Let $\gamma(t) = (\gamma_V(t), \gamma_M(t))$ be a smooth path in Σ with $\gamma(0) = (f, x)$, $\gamma'(0) = (h, v) \in T_{(f,v)}\Sigma$. Then $p_M(\gamma(t)) = \gamma_M(t)$, and therefore,

$$\mathrm{d}p_M(f, x) \cdot (g, v) = \frac{\mathrm{d}}{\mathrm{d}t} p_M(\gamma(t))|_{t=0} = \frac{\mathrm{d}}{\mathrm{d}t} \gamma_M(t)|_{t=0} = v$$

Since V contains constant functions, for all $v \in T_xM$, the constant function $h \in V$ with $\forall y \in X,\ h(y) = -\mathrm{d}f(x) \cdot v$ is in V, thus $(h, v) \in T_{(f,x)}\Sigma$, and $\mathrm{d}p_M(f, x) \cdot (h, v) = v$. It follows that $\mathrm{d}p_M(f, g)$ is surjective and p_M is a submersion.

(b) Similarly, we have

$$\forall (h, v) \in T_{(f,x)}\Sigma, \quad \mathrm{d}p_V(f, x) \cdot (h, v) = h$$

A function $h \in V$ is in the image of $\mathrm{d}p_V$ is and only if there exists $v \in T_xM$ such that $h(x) = -\mathrm{d}f(x) \cdot v$:

- If x is a critical point of f, then $\mathrm{d}f(x) = 0$ and the only functions in the image of $\mathrm{d}p_V(f, x)$ are those vanishing at x. Thus, $\mathrm{d}p_V(f, x)$ is not surjective.
- If x is not a critical point of f, then $\mathrm{d}f(x)$ is surjective. For any $h \in V$, there exists a vector $v \in T_xM$ such that $h(x) = -\mathrm{d}f(x) \cdot v$. It follows that $(h, v) \in T_{(f,x)}\Sigma$ with $\mathrm{d}p_V(f, x) \cdot (h, v) = h$, and $\mathrm{d}p_V(f, x)$ is surjective.

Finally, (f, x) is a critical point of p_V if and only if x is a critical point of f in $f^{-1}(\{0\})$.

(c) From Sard's Theorem, $p_M(\mathrm{Crit}p_M)$ is of measure zero in V. But $p_M(\mathrm{Crit}p_M)$ is the set of functions $f \in V$ such that there exists $x \in f^{-1}(\{0\})$ which is a critical point of f. Hence, it is the set of functions f in V such that 0 is a critical value. The result follows.

Exercise 3.

1. Let us show that f^*E is a submanifold of $M \times E$. Consider the map

$$F : M \times E \to N \times N$$

defined by $F(x, e) = (f(x), \pi(e))$, which is smooth. Consider the diagonal of $N \times N$ defined by $\Delta = \{(y, y) \in N \times N \mid y \in N\}$. Let us show that F is transverse to Δ, so that $f^*E = F^{-1}(\Delta)$ is a submanifold. If $(x, e) \in M \times E$, we have

$$\begin{array}{rccc} \mathrm{d}F(x,e)\colon & T_{(x,e)}M \times E \simeq T_xM \times T_eE & \longrightarrow & T_{F(x,e)}(N \times N) \simeq T_{f(x)}N \times T_{\pi(e)}N \\ & (u, v) & \longmapsto & (\mathrm{d}f(x)u, \mathrm{d}\pi(e)v) \end{array}$$

and

$$T_{(y,y)}\Delta = \{(w, w) \mid w \in T_yN\}$$

so that the transversality condition

$$\forall (x,e) \in F^{-1}(\Delta), \quad \mathrm{d}F(x,e)T_{(x,e)}(M \times E) + T_{F(x,e)}\Delta = T_{F(x,e)}(N \times N)$$

reads

$$\forall (x,e) \in f^*E, \forall (w_1,w_2) \in T_{f(x)}N \times T_{\pi(e)}N, \exists (u,v) \in T_xM \times T_eE, \exists w \in T_{f(x)}N \\ (\mathrm{d}f(x)u + w, \mathrm{d}\pi(e)v + w) = (w_1,w_2)$$

We use without proof that a vector bundle projection is a submersion (show it in local trivialization!). Choose $u = 0$, $w = w_1$ and v such that $\mathrm{d}\pi(e)v = w_2 - w_1$, which does exist because $\mathrm{d}\pi(e)$ is surjective. Then the transversality condition is satisfied.

From the transversality Theorem, $F^{-1}(\Delta) = f^*E$ is a submanifold of $M \times E$, of codimension $\mathrm{codim}_{N\times N}\Delta = \dim N$.

Let us now show that it is a vector bundle. The projection map $\overline{\pi}_M \colon M \times E \to M$ is smooth and restricts to a smooth surjective map $\pi_M \colon f^*E \to M$. Moreover, if $x \in M$, then

$$\pi_M^{-1}(\{x\}) = E_{f(x)}$$

is a vector space. Consider $U \subset N$ an open subset containing $f(x)$ such that we have the local trivialization

$$\chi \colon \pi^{-1}(U) \xrightarrow{\sim} U \times \mathbb{R}^r$$

If $\mathrm{proj}_2 U \times \mathbb{R}^r \to \mathbb{R}^r$ denotes the projection on the second factor, then

$$\begin{array}{rccc} \chi_f \colon & \pi_M^{-1}\left(f^{-1}(U)\right) & \longrightarrow & f^{-1}(U) \times \mathbb{R}^r \\ & (x,e) & \longmapsto & \left(x, \mathrm{proj}_2(\chi(e))\right) \end{array}$$

is a local trivialization for f^*E. It follows that f^*E is a vector bundle over M.

Remark. If the transition functions of E are $g_{U,V} \colon U \cap V \to GL_r(\mathbb{R})$, then transition functions for E, are given by $g_{f^{-1}(U),f^{-1}(V)} \colon f^{-1}(U \cap V) \to GL_r(\mathbb{R})$, where $g_{f^{-1}(U),f^{-1}(V)}(x) = g_{U,V}(f(x))$. It follows that in fact, the pullback bundle f^*E is given by **composing** with f, which is smooth: it is not surprising that f^*E is a smooth object.

2. Let us first show that νM is a manifold by finding local charts. To do so, consider an adapted chart $\varphi \colon V \subset \mathbb{R}^n \to \mathbb{R}^n$ with $\varphi(V \cap (\mathbb{R}^m \times \{0\})) = U \cap M$ (φ is an embedding). If $(e_1, \dots, e_n)$ is the canonical basis of $\mathbb{R}^n$, then $(\mathrm{d}\varphi(x)e_1, \dots \mathrm{d}\varphi(x)e_n)$ is a basis of $T_{\varphi(x)}\mathbb{R}^n \simeq \mathbb{R}^n$ with $(\mathrm{d}\varphi(x)e_1, \dots, \mathrm{d}\varphi(x)e_m)$ a basis of $T_{\varphi(x)}M$.

   From the Gram-Schmidt algorithm, one can thus define smooth functions $v_i \colon V \to \mathbb{R}^n$ such that $(v_1(x), \dots, v_m(x))$ is an orthonormal basis of $T_{\varphi(x)}M$, and $(v_{m+1}(x), \dots, v_n(x))$ is an orthonormal basis of $T_{\varphi(x)}M^\perp$. Therefore, the map

$$\begin{array}{ccc} V \times \mathbb{R}^{m-n} & \longrightarrow & \nu M \\ (x, \lambda_{m+1}, \dots, \lambda_n) & \longmapsto & \left(\varphi(x), \sum_{j=m+1}^n \lambda_i v_i(x)\right) \end{array}$$

   is a smooth local parametrization of νM, and νM is a submanifold of $M \times \mathbb{R}^n$, thus a manifold. It follows that the projection $\pi(x,v) \in \nu M \mapsto x$ is smooth as the restriction of a smooth map onto a submanifold, and the fibers are $T_xM^\perp$, which are indeed vector spaces.

We have shown above that locally, we can find $m-n$ smooth maps $u_i\colon x \in U \subset M \to \nu_x M = T_xM^\perp$ such that $(u_{m+1}(x), \ldots, u_n(x))$ is an orthonormal basis of $\nu_x M$. Therefore, the maps

$$\begin{array}{rcc} \pi^{-1}(U) & \longrightarrow & U \times \mathbb{R}^{m-n} \\ (x,v) & \longmapsto & (x, (\langle v, u_{m+1}(x)\rangle, \ldots, \langle v, u_n(x)\rangle)) \end{array}$$

are local trivializations.

Remark. The local trivializations here are the inverse maps of the local parametrizations we have constructed.

3. The two projections $\pi_M\colon M \times N \to M$ and $\pi_N\colon M \times N \to N$ are smooth. Consider the map
$$\widetilde{\pi}\colon TM \times TN \longrightarrow M \times N$$
defined by $\widetilde{\pi}\,((p,v),(q,w)) = (p,q)$. It is smooth and endows $TM \times TN$ with the structure of a smooth rank $\dim M + \dim N$ vector bundle over $M \times N$ (easy exercise).

We are in the situation
$$\begin{array}{ccc} T\,(M \times N) & & TM \times TN \\ \downarrow & & \downarrow \\ M \times N & \xrightarrow{\text{id}} & M \times N \end{array}$$

Let us show that these vector bundles are isomorphic. The map $F\colon T(M \times N) \to TM \times TN$ defined by

$$F\big((p,q),V\big) = \big((p, \mathrm{d}\pi_M(p,q)V), (q, \mathrm{d}\pi_N(p,q)V\big)$$

is well defined, smooth, linear in the fibers which are of the same finite dimension. To conclude, it suffices to show that it induces linear isomorphisms in the fibers. The linearity is clear. If $v \in T_pM$ and $w \in T_qN$, choose two paths γ_M and γ_N such that v and w are their velocity vectors at 0. Define $\gamma = (\gamma_M, \gamma_N)$. Its velocity vector at zero, say $V \in T_{(p,q)}(M \times N)$ satisfies $F((p,q),V) = \big((p,v),(q,w)\big)$. Finally F induces a surjective linear maps in the fibers which are of the same finite dimension, and thus induces linear isomorphisms in the fibers. This conclude the proof.