Flow of a vector field

If M is a manifold, X a vector field on M and $x \in M$, we denote by I_x the maximal interval of definition of a solution of $\gamma' = X(\gamma)$ with $\gamma(0) = x$. We denote by $D = \bigcup_{x\in M} I_x \times \{x\}$, which is open in $\mathbb{R} \times M$, and by $\gamma\colon D \to M$ the flow of X, that is $\phi(t,x) = \gamma(t)$ where γ is the maximal solution defined above. A vector field is complete if $D = \mathbb{R} \times M$.

Exercise 1.

Let M be a manifold, $x \in M$ and X be a vector field such that $X(x) \neq 0$. Show that there exist local coordinates $(x^1, \dots, x^n)$ around x with $X = \frac{\partial}{\partial x^1}$.

Exercise 2.

Let M be a manifold, X be a vector field and $\phi\colon D \to M$ be its flow.

1. Show that if there exists $\varepsilon > 0$ with $(-\varepsilon, \varepsilon) \times M \subset D$, then $D = \mathbb{R} \times M$.
2. Show that if M is compact, then X is complete.

Exercise 3.

Let M be a manifold, X a vector field and $\phi\colon D \to M$ is flow. We call $b_x = \sup I_x \in \mathbb{R}_+^* \cup \{+\infty\}$.

1. We suppose that there exists $x \in M$ such that $\phi(\cdot, x)|_{[0,b_x[}\colon [0, b_x[\to M$ has a relatively compact image. Show that $b_x = +\infty$.
2. Deduce that if X has compact support, then X is complete.

Exercise 4.

In this exercise, we will use the result of Exercise 3, question 2: a vector field with compact support is complete.

1. Consider B a ball of radius $r > 0$ in $\mathbb{R}^n$, and $a, b \in B$. Show that there exists a diffeomorphism $f\colon \mathbb{R}^n \to \mathbb{R}^n$ such that $f(a) = b$ and f has compact support in B.
2. Show that the group of diffeomorphisms of a connected manifold M acts transitively on M.
3. Is this action k-transitive for $k \geqslant 2$ if $\dim M \geqslant 2$? If $\dim M = 1$?