Flow of a vector field
Correction

Warning. All the following corrections rely on one simple fact: in geometry, we need to draw things in order to understand what's happening. The reader is invited to draw vector fields, charts and integral lines if willing to understand what follows.

Exercise 1.

We want to find a chart $(\mathcal{O}, \psi)$ around x such that for all $y \in \mathcal{O}$, $\mathrm{d}\psi(y) \cdot X(y)$ is equal to e_1, the first vector of the canonical basis of $\mathbb{R}^n$.

Let $\varphi \colon U \ni x \to \mathbb{R}^n$ be a chart centered at x (that is, $\varphi(x) = 0$). Since $X(x) \neq 0$, X does not vanish in a neighbourhood of x: up to shrinking, we suppose that $X \neq 0$ on U.

Now, up to an affine transformation, we can suppose that at the point x, $\mathrm{d}\varphi(x) \cdot X(x)$ is equal to $e_1 \in T_0\mathbb{R}^n = \mathbb{R}^n$, the first vector of the canonical basis of $\mathbb{R}^n$. We want to build a chart for which this equality is true everywhere. Consider the vector field $Y = \varphi_* X$ on $\varphi(U)$, that is

$$\forall z \in \varphi(U), \quad Y(z) = \mathrm{d}\varphi(\varphi^{-1}(z)) \cdot X(\varphi^{-1}(z))$$

(this is just the vector field X read in the chart $(\mathcal{O}, \phi)$) and let $\phi \colon D \to \varphi(U)$ be its flow. Define

$$F \colon \begin{array}{ccc} V \subset \mathbb{R}^n & \longrightarrow & \varphi(U) \\ (x^1, \dots, x^n) & \longmapsto & \phi\left(x^1, \left(0, x^2, \dots, x^n\right)\right) \end{array}$$

where $V = \left\{\left(x^1, \dots, x^n\right) \mid \left(x^1, \left(0, x^2, \dots, x^n\right)\right) \in D\right\}$, which is open as the preimage of D by the continuous function $\iota \colon \mathbb{R}^n \to \mathbb{R}^{n+1}$ defined by $\iota(x^1, \dots, x^n) = (x^1, 0, x^2, \dots, x^n)$. Notice that $0 \in V$ since $\iota(0) = 0 \in D$.

F is clearly smooth, and we have, by the definition of the flow of a vector field

$$\forall (x^1, \dots, x^n) \in V, \quad \frac{\partial F}{\partial x^1}(z) = \frac{\partial}{\partial x^1}\phi\left(x^1, (0, x^2, \dots, x^n)\right) = Y\left(\phi\left(x^1, (0, x^2, \dots, x^n)\right)\right)$$

which gives, at $(x_1, \dots, x^n) = 0$:

$$\frac{\partial F}{\partial x^1}(0) = Y\left(\phi(0,0)\right) = Y(0) = e_1$$

The time 0 map of a flow being the identity map, we have

$$F(0, x^2, \dots, x^n) = \phi\left(0, (0, x^2, \dots, x^n)\right) = (0, x^2, \dots, x^n)$$

so that $\frac{\partial F}{\partial x^j}(0) = e_j$ for $j \geqslant 2$.

Therefore, $\mathrm{d}F(0) = \mathrm{Id}$. By the inverse function Theorem, there exists $W \subset V$ open subset such that $F \colon W \to F(W) \subset \varphi(U)$ is a diffeomorphism. It moreover satisfies the equality

$$\frac{\partial F}{\partial x^1}(z) = Y(F(z))$$

Recall that by definition of partial derivatives, we have $\frac{\partial F}{\partial x^1} = \mathrm{d}F \cdot e_1$. Hence, it holds that

$$\forall w \in F(W), \quad \mathrm{d}\left(F^{-1}\right)(w) \cdot Y(w) = e_1$$

Recall that $Y = \varphi_* X$, so that the chain rule gives, wherever it makes sense

$$\mathrm{d}\left(F^{-1} \circ \varphi\right)(z) \cdot X(z) = \mathrm{d}\left(F^{-1}\right)(\varphi(z)) \circ \mathrm{d}\varphi(z) \cdot X(z) = \mathrm{d}\left(F^{-1}\right)(\varphi(z)) \cdot Y(\varphi(z)) = e_1$$

and finally, the chart $\psi = F^{-1} \circ \varphi$ defined on $\mathcal{O} = \varphi^{-1}(F(W))$ is a solution.

Exercise 2.

1. Fix $x \in M$. Let us show by induction that $\forall n \geqslant 1$, $\left[-n\frac{\varepsilon}{2}, n\frac{\varepsilon}{2}\right] \subset I_x$.

   - for $n = 1$, it follows from the assumption.
   - suppose $\left[-n\frac{\varepsilon}{2}, n\frac{\varepsilon}{2}\right] \subset I_x$. Let $x^- = \phi(-n\frac{\varepsilon}{2}, x)$ and $x^+ = \phi(n\frac{\varepsilon}{2}, x)$. By assymptions, $(-\varepsilon, \varepsilon) \subset I_{x^-}, I_{x^+}$ so that $\phi(\cdot, x^{\pm})$ are defined on $[-\frac{\varepsilon}{2}, \frac{\varepsilon}{2}]$. The curve

   $$\begin{array}{rcl} \gamma\colon [-(n+1)\frac{\varepsilon}{2}, (n+1)\frac{\varepsilon}{2}] & \longrightarrow & M \\ t & \longmapsto & \begin{cases} \phi\left(t + n\frac{\varepsilon}{2}, x^-\right) & \text{if } t \in [-(n+1)\frac{\varepsilon}{2}, -n\frac{\varepsilon}{2}] \\ \phi(t, x) & \text{if } t \in [-n\frac{\varepsilon}{2}, n\frac{\varepsilon}{2}] \\ \phi\left(t - n\frac{\varepsilon}{2}, x^+\right) & \text{if } t \in [n\frac{\varepsilon}{2}, (n+1)\frac{\varepsilon}{2}] \end{cases} \end{array}$$

     is then smooth (check that) and satisfies $\gamma' = X(\gamma)$, $\gamma(0) = x$. By maximality of the flow, we have that $\left[-(n+1)\frac{\varepsilon}{2}, (n+1)\frac{\varepsilon}{2}\right] \subset I_x$.

   Thus, $\mathbb{R} = \cup_{n \geqslant 1} \left[-n\frac{\varepsilon}{2}, n\frac{\varepsilon}{2}\right] \subset I_x$ and $I_x = \mathbb{R}$. This being true for all $x \in M$, X is a complete vector field.

   Remark. The morality is that if there is a uniform temporal interval on which the solutions of the Cauchy problem

   $$\begin{cases} \dfrac{\mathrm{d}\phi(t,x)}{\mathrm{d}t} & = & X(\phi(t,x)) \\ \phi(0,x) & = & x \end{cases}$$

   are all defined, then they are defined at any time.

2. Recall that D is open in $\mathbb{R} \times M$. Since the product topology is generated by products of open subsets and as $(0, x) \in D$ for all $x \in M$, there exist $\varepsilon_x > 0$ and $W_x \subset M$ open neighbourhood of x with

   $$\forall w \in M, \quad (0, x) \in (-\varepsilon_x, \varepsilon_x) \times W_x \subset D$$

   Now, if M is compact, so is $\{0\} \times M \subset \bigcup_{x \in M} (-\varepsilon_x, \varepsilon_x) \times W_x$, and there exists a finite subcover

   $$\{0\} \times M \subset \bigcup_{i=1}^{n} (-\varepsilon_{x_i}, \varepsilon_{x_i}) \times W_{x_i}$$

   Set $\varepsilon = \min_{i=1,\dots,n} \varepsilon_{x_i}$: it follows that $(-\varepsilon, \varepsilon) \times M \subset D$. By 1., $D = \mathbb{R} \times M$, and finally, X is complete.

Exercise 3.

1. Suppose the image of $\phi(\cdot, x)$ is relatively compact. Then there exists an accumulation point of $\phi(t, x)$ when $t \to b_x$: call it z. Suppose by contradiction that $b_x < +\infty$.

   Because D is open, there exists $\varepsilon > 0$ and $W \subset M$ open neighbourhood of z such that $(-\varepsilon, \varepsilon) \times W \subset D$. Let $b_x - \frac{\varepsilon}{3} < t_0 < b_x$ be such that $y = \phi(t_0, x) \in W$. Then $(-\varepsilon, \varepsilon) \subset I_y$, and thus $\phi(\frac{\varepsilon}{2}, y)$ is defined. But then

   $$\phi\left(\frac{\varepsilon}{2}, y\right) = \phi\left(\frac{\varepsilon}{2}, \phi(t_0, x)\right)$$

and $t_0+\frac{\varepsilon}{2} \in I_x$ by maximality. But by construction, we have $t_0+\frac{\varepsilon}{2} > b_x-\frac{\varepsilon}{3}+\frac{\varepsilon}{2} = b_x+\frac{\varepsilon}{6}$, which is in contradiction with $b_x = \sup I_x$.

It follows that $b_x = +\infty$.

2. Say $\mathrm{supp} X = K$ is compact.

   If $x \notin K$, $X(x) = 0$, then $\gamma(t) = x$ is a solution on $\mathbb{R}$ of $\gamma' = X(\gamma), \gamma(0) = x$, and therefore $I_x = \mathbb{R}$.

   If $x \in K$, then $\forall t \in I_x$, $\phi(t, x) \in K$: if not, there would exist some t_0 with $\phi(t_0, x) \notin K$, and thus, $x = \phi(-t_0, \phi(t_0, x)) = \phi(t_0, x) \notin K$ from the discussion above about initial datum in $M \setminus K$. If follows that $\phi(\cdot, x)$ has image in K, which is compact, and thus, by 1, $\sup I_x = +\infty$. The exact same proof as in 1. shows that we also have $\inf I_x = -\infty$. It follows that $I_x = \mathbb{R}$.

   Finally, for any $x \in M$, $I_x = \mathbb{R}$. It follows that X is complete.

Exercise 4.

1. Up to a translation, we can suppose B is centered at 0.

   Let X be the constant vector field on $\mathbb{R}^n$ defined by $X(p) = b - a$. Its flow is defined by $\phi^X(t, p) = p + t(b - a)$. $\phi(1, \cdot)$ is a diffeomorphism sending a to b but has non compact support.

   Consider a cut-off function $f\colon \mathbb{R}^n \to \mathbb{R}$ such that $f \equiv 1$ on $B(0, r')$, where $a, b \in B(0, r') \subsetneq B(0, r)$, and $f \equiv 0$ on $\mathbb{R}^n \setminus B(0, r)$. Define $\widetilde{X} = fX$. This vector field has compact support so that it is complete: let $\widetilde{\phi}$ be its flow, which is defined for all time. By construction, $X|_{B(0,r')} = \widetilde{X}|_{B(0,r')}$, so that $\widetilde{\phi}(1, a) = b$. But as $\widetilde{X}|_{\mathbb{R}^n \setminus B(0,r)} = 0$ we have $\widetilde{\phi}(1, x) = x$ if $\|x\| \geqslant r$.

   It follows that $\widetilde{\phi}(1, \cdot)\colon \mathbb{R}^n \to \mathbb{R}^n$ is a diffeomorphism with compact support in $\overline{B(0, r)}$ sending a to b.

2. Fix $x \in M$ and define $W_x = \{y \in M \mid \exists f \in \mathrm{Diff}(M), f(x) = y\}$ the orbit of x under the action $\mathrm{Diff}(M) \curvearrowright M$. First, $W_x \neq \varnothing$ since $x \in W_x$: the identity map is a diffeomorphism. Let us show that W_x is both open and closed in M.

   Let $\varphi\colon U \ni x \to \mathbb{R}^n$ be a chart centered at x with $B(0, r) \subset \varphi(U)$. Fix $y \in \varphi^{-1}(B(0, r))$. By 1., there exists a diffeomorphism $u\colon \mathbb{R}^n \to \mathbb{R}^n$ with compact support in $B(0, r)$ such that $u(0) = \varphi(y)$. Then the function

$$f\colon \begin{array}{ccc} M & \longrightarrow & M \\ z & \longmapsto & \begin{cases} \varphi^{-1} \circ u \circ \varphi(z) & \text{if } z \in U \\ z & \text{if } z \notin U \end{cases} \end{array}$$

   is a diffeomorphism of M sending x to y. Therefore, $\varphi^{-1}(B(0, r)) \subset W_x$, and W_x is a neighbourhood of x.

   Note that $x \in W_y \iff y \in W_x$ (because for a diffeomorphism f, $f(x) = y \iff x = f^{-1}(y)$), so that in fact, W_x is a neighbourhood of any of its points. Thus, W_x is open.

   To show that W_x is closed, notice that

$$M \setminus W_x = \bigcup_{z \notin W_x} W_z$$

is open.

But then, if M is connected, W_x is non-empty and both open and closed, so that $W_x = M$, and $\mathrm{Diff}(M) \curvearrowright M$ transitively.

3. • If $\dim M \geqslant 2$.

   The key is to show that if $(x_1, \ldots, x_k)$ are distinct and so are $(y_1, \ldots, y_k)$, then there exist k continuous paths $\gamma_j \colon [0,1] \to M$ with disjoint images with $\gamma_j(0) = x_j$ and $\gamma_j(1) = y_j$: we leave this as an exercise.

   Now, there exists k disjoint open subset U_j with $\gamma_j([0,1]) \subset U_j$. Construct a diffeomorphism $f_j \colon M \to M$ with compact support in U_j such that $f_j(x_j) = y_j$ (To do so, cover $\gamma_j([0,1])$ in U_j with a finite number of charts and apply the technique of 1.).

   Then $f = f_k \circ \cdots \circ f_1$ is a diffeomorphism of M which sends $(x_1, \ldots, x_k)$ to $(y_1, \ldots, y_k)$.

   • If $\dim M = 1$.

   The intermediate value Theorem is an obstruction to the k transitivity, as shows the two following figures.

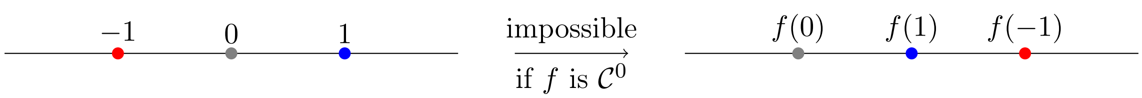

Figure 1: $\mathrm{Diff}(\mathbb{R}) \curvearrowright \mathbb{R}$ is not 3-transitive

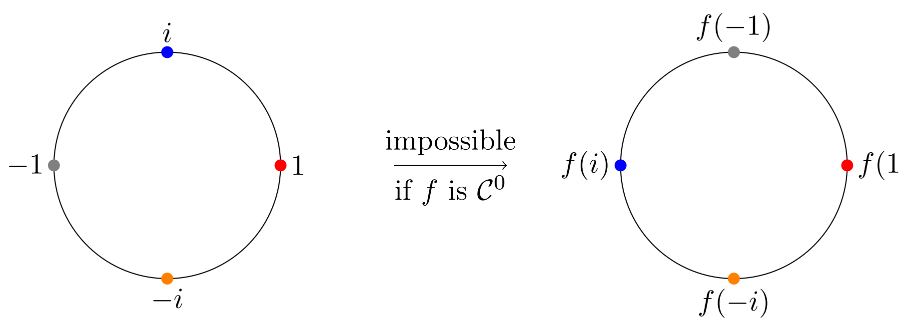

Figure 2: $\mathrm{Diff}(\mathbb{S}^1) \curvearrowright \mathbb{S}^1$ is not 4-transitive