Vector fields, derivations

Exercise 1.

Let M be a manifold and X be a vector field on M. Show that there exists $f \in \mathcal{C}^\infty(M;\mathbb{R})$ positive such that fX is a complete vector field.

Exercise 2.

In this exercise, we give a classification of 1-dimensional connected smooth manifolds. Let M be such a manifold.

1. Let (U,φ) and (V,ψ) be two charts with U, V connected and $U \cap V \neq \varnothing$. We also assume that $\varphi(U) = \psi(V) = \mathbb{R}$ (which is always possible by composing with a suitable function).
   (a) Let $\Gamma = \{(s,t) \in \mathbb{R}^2 \mid \varphi^{-1}(t) = \psi^{-1}(s)\}$. Show that Γ is closed.
   (b) Show that Γ is the graph of $\psi \circ \varphi^{-1} \colon \varphi(U \cap V) \to \mathbb{R}$.
   (c) Let I be a connected component of $\varphi(U \cap V)$, and assume $a \in \mathbb{R}$ is a boundary point of I. Show that $\lim_{s\to a} \psi \circ \varphi^{-1}(s) = \pm\infty$. Deduce that $U \cap V$ has at most two connected components.
   (d) Show that $(\psi \circ \varphi^{-1})'$ has constant sign.
2. Let $(U_i,\varphi_i)_{i\in\mathbb{N}}$ be a locally finite open cover of M. Show that there exists $\sigma \colon \mathbb{N} \to \mathbb{N}$ a bijection such that
$$\forall j \in \mathbb{N}, \quad U_{\sigma(j+1)} \cap \left(\bigcup_{i=0}^{j} U_{\sigma(i)} \right) \neq \varnothing$$
3. Deduce from 1. and 2. that there exists a locally finite atlas $(U_i,\varphi_i)_{i\in\mathbb{N}}$ of M such that
$$\forall (i,j) \in \mathbb{N}^2, U_i \cap U_j \neq \varnothing, \quad (\varphi_i \circ \varphi_j^{-1})' > 0$$
and deduce that M is parallelizable.
4. Show that there is a one-parameter subgroup of $\mathrm{Diff}(M)$ acting transitively on M. Conclude that M is either diffeomorphic to $\mathbb{R}$ or $\mathbb{S}^1$.

Exercise 3.

Let M be a manifold and D be a derivation of $\mathcal{C}^\infty(M;\mathbb{R})$. Assume f and g are smooth functions on M such that $f = g$ on some open subset $U \subset M$. Show that $D(f) = D(g)$ on U.

Exercise 4.

Let $\varphi \colon M \to N$ be a diffeomorphism and D be a derivation of $\mathcal{C}^\infty(M;\mathbb{R})$. Show that

$$\varphi_* D \colon \left| \begin{array}{ccc} \mathcal{C}^\infty(N;\mathbb{R}) & \longrightarrow & \mathcal{C}^\infty(N;\mathbb{R}) \\ f & \longmapsto & D(f\circ\varphi)\circ\varphi^{-1} \end{array} \right.$$

is a derivation of $\mathcal{C}^\infty(N;\mathbb{R})$.

Exercise 5.

Let D, D' be two derivations of $\mathcal{C}^\infty(M)$. Is $D \circ D'$ a derivation? What about $D\circ D' - D'\circ D$?

Exercise 6.

Let M be a manifold. Show that the ring $\mathcal{C}^0(M;\mathbb{R})$ does not have any non-zero derivation.