Vector fields, derivations

Exercise 1.

Lemma. *Let M be a a smooth manifold. Then there exists a smooth proper function $g\colon M \to \mathbb{R}$.*

Proof. Let $(U_i, \varphi_i)_{i\in\mathbb{N}}$ ba a partition of unity subordinate to a locally finite open cover $(U_i)_{i\in\mathbb{N}}$. Then $g = \sum_{i\in\mathbb{N}} i\varphi_i$ is a solution. □

Let X be a vector field on M and $g \in \mathcal{C}^\infty(M;\mathbb{R})$ be a smooth and proper function. Define

$$f\colon \left|\begin{array}{ccc} M & \longrightarrow & \mathbb{R} \\ x & \longmapsto & e^{-(\mathrm{d}g(x)\cdot X(x))^2} \end{array}\right.$$

or, in short, $f = \exp(-(X\cdot g)^2)$. It is a positive function. Let $Y = fX$. Then

$$Y \cdot g = (fX)\cdot g = f(X\cdot g) = e^{-(X\cdot g)^2}(X\cdot g)$$

so that $|(Y\cdot g)(x)| = |e^{-(X\cdot g)^2(x)}(X\cdot g)(x)| \leqslant 1$ (since $y \mapsto ye^{-y^2}$ is bounded by 1). Let γ be an integral curve of Y, defined on a bounded interval (a,b) containing 0. Then $g\circ\gamma$ is smooth on (a,b) and

$$\forall t \in (a,b), \quad \frac{\mathrm{d}}{\mathrm{d}t}(g\circ\gamma)(t) = \mathrm{d}g(\gamma(t))\cdot\gamma'(t) = \mathrm{d}g(\gamma(t))\cdot Y(\gamma(t)) = (Y\cdot g)(\gamma(t))$$

and it follows that $|\frac{\mathrm{d}}{\mathrm{d}t}(g\circ\gamma)| \leqslant 1$ on (a,b). Integrating this last inequality shows that $g\circ\gamma$ is bounded on (a,b), and since g is proper, so is $\gamma\big((a,b)\big)$ is relatively compact in M.

From TD4, exercise 3, (which is also a Lemma taught in the lectures) (a,b) cannot be the maximal interval of definition of γ. It follows that Y is complete.

Remark. The integral curves of Y and X have the same image in M. The only difference is that Y has been reduced small enough so that it takes an infinite time for its integral curves to achieve points that are achieved in finite time by the integral curves of X.

Exercise 2.

1. (a) The map

$$f\colon \left|\begin{array}{ccc} \mathbb{R}\times\mathbb{R} & \longrightarrow & M\times M \\ (s,t) & \longmapsto & (\varphi^{-1}(s), \psi^{-1}(t)) \end{array}\right.$$

is continuous, and $\Delta = \{(x,x) \in M\times M \mid x \in M\}$ is closed sinde M is Hausdorff. It follows that $\Gamma = f^{-1}(\Delta)$ is closed in $\mathbb{R}^2$.

(b) Let $(s,t) \in \mathbb{R}^2$. Then

$$(s,t)\in\Gamma \iff \varphi^{-1}(s) = \psi^{-1}(t) \iff t = (\psi\circ\varphi^{-1})(s)$$

so that Γ is the graph of $\psi\circ\varphi^{-1}$.

(c) Since $\psi\circ\varphi^{-1}|_I$ is a diffeomorphism onto its image and since I is an interval, $\psi\circ\varphi^{-1}$ is monotonous. It follows that it has a limit when $s\to a$. If this limit were a real number, then its graph Γ would not be closed, in contradiction with 1.(b). It follows that $\lim_{s\to a}\psi\circ\varphi^{-1}(s) = \pm\infty$.
Suppose by contradiction that $\varphi(U\cap V)$ has at least three connected components. Then one of them is of the form (a,b) with $a,b\in\mathbb{R}$. It follows that $\psi\circ\varphi^{-1}|_{(a,b)}$ diverges to infinity at the limit points a and b, and it realizes a homeomorphism from (a,b) onto $\mathbb{R}$. But as $\psi\circ\varphi^{-1}$ is injective, $\varphi(U\cap V)$ cannot take any value on the other connected components, which is a contradiction.

(d) If $\varphi(U \cap V)$ has one connected components, then $\psi \circ \varphi^{-1}$ is monotonous so that its derivative has constant sign.

If $\varphi(U \cap V)$ has two connected components, then $\varphi(U \cap V) = I_1 \cup I_2$ with I_1 and I_2 intervals and $-\infty < \sup I_1 = a \leqslant b = \inf I_2 < +\infty$, and $\psi \circ \varphi^{-1}$ is monotonous on both intervals. Since it has infinite limits at a and b and since it is injective, they cannot be equal. It follows that $\psi \circ \varphi^{-1}$ is either increasing on I_1 and I_2, or decreasing on I_1 and I_2 (sketch the situation), and $(\psi \circ \varphi^{-1})'$ has constant sign.

2. Define by induction $\sigma(0) = 0$ and

$$\sigma(n+1) = \min\{k \in \mathbb{N} \setminus \{\sigma(0), \dots, \sigma(n)\} \mid U_k \cap (\bigcup_{j=0}^{n} U_j)\}$$

It is well defined since M is connected, so that $\{k \in \mathbb{N} \setminus \{\sigma(0), \dots, \sigma(n)\} \mid U_k \cap (\bigcup_{j=0}^{n} U_j)\}$ is non-empty. By construction, σ is injective. The surjectivity is again a consequence of the connectedness of M (exercise).

3. Let (U_i, ψ_i) be a countable locally finite atlas, with U_i connected, and up to reordering as in 2., assume that

$$\forall n \in \mathbb{N}, \quad U_{n+1} \cap (\bigcup_{j=0}^{n} U_j) \neq \varnothing$$

By induction, define $\varphi_0 = \psi_0$ and

$$\varphi_{n+1} = \begin{cases} \psi_{n+1} & \text{if } (\psi^{n+1} \circ \varphi_j^{-1})' > 0 \text{ for } j \in \{0, \dots, n\} \text{ s.t } U_j \cap U_{n+1} \neq \varnothing, \\ -\psi_{n+1} & \text{otherwise} \end{cases}$$

Show that the two considered cases are the only one possible (exefcise). Then (U_i, φ_i) satisfies the condition.

Let (θ_i) be a partition of unity subordinate to (U_i, φ_i). Define

$$\forall x \in U_i, \quad X_i(x) = \mathrm{d}\varphi_i(x)^{-1} \cdot 1$$

then $X = \sum_i \theta_i X_i$ is a non-zero vector field on M. It follows that M is parallelizable.

4. Let X be a non-zero vector field on M. By Exercise 1. there exists $f > 0$ such that $Y = fX$ is complete, and is non-zero.

Let $(\phi_t)_{t \in \mathbb{R}}$ be the flow of Y: since Y is complete, it is a one-parameter subgroup of $\mathrm{Diff}(M)$. It induces a smooth action by diffeomorphisms $\mathbb{R} \curvearrowright M$ defined by $t \cdot x = \phi_t(x)$. For $x \in M$, let $\mathcal{O}_x = \{\phi_t(x) \mid t \in \mathbb{R}\}$ be the orbit of x. First, note that $y = \phi_t(x) \iff x = \phi_{-t}(y)$. In particular, $y \in \mathcal{O}_x \iff x \in \mathcal{O}_y$, and it follows that if $x \in M$ is fixed,

$$M = \mathcal{O}_x \sqcup \left(\bigcup_{z \in M \setminus \mathcal{O}_x} \mathcal{O}_z \right) \tag{1}$$

Let us show that any orbit is open. For $z \in M$, define $f \colon \mathbb{R} \to M$ by $f(t) = \phi_t(z)$. Then for all t, $f'(t) = \frac{\mathrm{d}}{\mathrm{d}t}\phi_t(z) = Y(\phi_t(z)) \neq 0$, and by the inverse function Theorem (M is 1-dimensional), its image is open. But $f(\mathbb{R}) = \mathcal{O}_z$, which proves that any orbit is open. By equation (1), any orbit is also closed as the complementary of union of orbits. By connectedness of M, if $x \in M$, $M = \mathcal{O}_x$, and the action is transitive.

Fix $x \in M$ and let $G_x = \{t \in \mathbb{R} \mid \phi_t(x) = x\}$ be the isotropy subgroup of x. G_x is a subgroup of $\mathbb{R}$, and since $G_x = h^{-1}(\{x\})$ with $h\colon t \mapsto \phi_t(x)$ continuous, G_x is closed. It follows that either $G_x = \mathbb{R}$, or $G_x = \alpha\mathbb{Z}$ for some $a \geqslant 0$. The case $G_x = \mathbb{R}$ is impossible since $\mathbb{R} \curvearrowright M$ is transitive. Therefore, we have a surjective smooth map

$$\mathbb{R} \longrightarrow M$$

that descends to the quotient as a diffeomorphism

$$\mathbb{R}/\alpha\mathbb{Z} \xrightarrow{\sim} M$$

If $\alpha = 0$, then $M \simeq \mathbb{R}$. Otherwise, $M \simeq \mathbb{S}^1$.

Remark. For another proof, see *Topology from the differentiable viewpoint*, Milnor.

Exercise 3.

Let $x \in U$ be fixed and φ be a cut-off function with compact support in U with $\varphi(x) = 1$. Then

$$\forall y \in M, \quad (f(y) - g(y))\varphi(y) = 0$$

since either $y \in U$ and $f(y) = g(y)$, or $y \notin U$ and $\varphi(y) = 0$. Hence, $(f - g)\varphi = 0$ and since $D0 = 0$, it follows that

$$0 = D((f - g)\varphi) = \varphi D(f - g) + (f - g)D\varphi$$

Evaluating this last equality at x yields

$$0 = D(f - g)(x)$$

This being true for all $x \in U$, then $(D(f - g))|_U = (Df)|_U - (Dg)|_U = 0$, and the proof is complete.

Exercise 4.

Let f and g be smooth functions on N. Then

$$\begin{aligned}(\varphi_* D)(f \times g) &= D((f \times g) \circ \varphi) \circ \varphi^{-1} \\ &= D\left((f \circ \varphi) \times (g \circ \varphi)\right) \circ \varphi^{-1}\end{aligned}$$

Since D is a derivation of $\mathcal{C}^\infty(M; \mathbb{R})$ and since $f \circ \varphi$ and $g \circ \varphi$ are smooth on M, it follows that

$$D\left((f \circ \varphi) \times (g \circ \varphi)\right) = (g \circ \varphi)D(f \circ \varphi) + (f \circ \varphi)D(g \circ \varphi)$$

Thus

$$\begin{aligned}(\varphi_* D)(fg) &= \big((g \circ \varphi)D(f \circ \varphi) + (f \circ \varphi)D(g \circ \varphi)\big) \circ \varphi^{-1} \\ &= (g \circ \varphi \circ \varphi^{-1})D(f \circ \varphi) \circ \varphi^{-1} + (f \circ \varphi \circ \varphi^{-1})D(g \circ \varphi) \circ \varphi^{-1} \\ &= g(D(f \circ \varphi) \circ \varphi{-}1) + f(D(g \circ \varphi) \circ \varphi^{-1}) \\ &= g((\varphi_* D)f) + f((\varphi_* D)g)\end{aligned}$$

The result follows.

Exercise 5.

Let f and g be smooth functions on M. Then

$$\begin{aligned} D \circ D'(fg) &= D(gD'f + gD'g) && \text{by definition of } D' \\ &= D(gD'f) + D(fD'g) && \text{by linearity of } D \\ &= D'fDg + gD(D'f) + D'gDf + fD(D'g) && \text{by definition of } D \\ &= g(D \circ D')f + f(D \circ D')g + D'fDg + DfD'g \end{aligned}$$

It follows that $D \circ D'(f,g)(x) \neq [g(D \circ D')f + f(D \circ D')g](x)$ as long as $Df(x), D'f(x), Dg(x)$ and $D'g(x)$ are non-zero. But the same computations show that

$$[D \circ D' - D' \circ D](fg) = g[D \circ D' - D' \circ D]f + f[D \circ D' - D' \circ D]g$$

so that $D \circ D' - D' \circ D$ is a derivation of $\mathcal{C}^\infty(M; \mathbb{R})$.