Vector fields, derivations, tangent distributions, Frobenius Theorem

Exercise 1.

Let X and Y be vector fields on a manifold M and write in some local coordinates

$$X = \sum_{i=1}^{n} X^i \frac{\partial}{\partial x^i} \qquad Y = \sum_{j=1}^{n} Y^j \frac{\partial}{\partial x^j}$$

Show that

$$[X, Y] = \sum_{i=1}^{n} \left(\sum_{j=1}^{n} X^j \frac{\partial Y^i}{\partial x^j} - Y^j \frac{\partial X^i}{\partial x^j} \right) \frac{\partial}{\partial x^i}$$

Exercise 2.

Let X be a vector field on the manifold M. Suppose that for all vector fields Y it holds that $[X, Y] = 0$.

1. Show that $\forall f \in \mathcal{C}^\infty(M),\ X \cdot f = 0$.
2. Deduce that $X = 0$.

Exercise 3.

Let $f\colon M^m \to N^n$ be a smooth submersion. We set $\mathcal{K}(f) = \{X \in \Gamma(TM) \mid X \cdot f = 0\}$

1. Show that $\mathcal{K}(f)$ is stable under the Lie bracket.
2. Determine the integral submanifolds of $\mathcal{K}(f)$.

Exercise 4.

Let $X = \frac{\partial}{\partial x} - y\frac{\partial}{\partial z}$ and $Y = \frac{\partial}{\partial y}$ be two vector fields on $\mathbb{R}^3$.

1. Show that at each point $p \in \mathbb{R}^3$, $\{X(p), Y(p)\}$ is linearly independant.
2. Compute $[X, Y]$. Is the spanned distribution integrable?
3. Recover the result withouth using of Frobenius Theorem.

Exercise 5.

Consider the two following vector fields $X = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}$ and $Y = x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}$ on $\mathbb{R}^2$.

1. Compute $[X, Y]$. Are there coordinates (s, t) on some neighbourhood of $(1, 0) \in \mathbb{R}^2$ such that $X = \frac{\partial}{\partial s}$ and $Y = \frac{\partial}{\partial t}$?
2. Compute the flows of X and Y.
3. Build explicit coordinates (s, t) on some neighbourhood of $(1, 0) \in \mathbb{R}^2$ such that $X = \frac{\partial}{\partial s}$ and $Y = \frac{\partial}{\partial t}$.

Exercise 6.

Let M be a manifold and let X and Y be two vector fields on M. Let (φ_t) and (ψ_s) be the local 1-parameter groups of diffeomorphisms generated by X and Y respectively.

1. Show that $[X, Y] = \lim_{t\to 0} \frac{(\mathrm{d}\varphi_{-t})\circ Y \circ \varphi_t - Y}{t} = \lim_{t\to 0} \frac{Y - \mathrm{d}\varphi_t \circ Y \circ \varphi_{-t}}{t}$.
2. Show that $[X, Y] = 0$ if and only if for all (t, s) such that it is defined, $\varphi_t \circ \psi_s = \psi_s \circ \varphi_t$.