Vector fields, derivations, tangent distributions, Frobenius Theorem

Exercise 1.

Fix $f \in \mathcal{C}^\infty(M)$. Then

$$Yf = \sum_{i=1}^n Y^i \frac{\partial f}{\partial x^i}$$

It follows from Leibniz rule that

$$X(Yf) = \sum_{j=1}^n X^j \left(\sum_{i=1}^n \frac{\partial Y^i}{\partial x^j} \frac{\partial f}{\partial x^i} + Y^i \frac{\partial^2 f}{\partial x^j \partial x^i} \right)$$

and similarly,

$$Y(Xf) = \sum_{j=1}^n Y^j \left(\sum_{i=1}^n \frac{\partial X^i}{\partial x^j} \frac{\partial f}{\partial x^i} + X^i \frac{\partial^2 f}{\partial x^j \partial x^i} \right)$$

Since $[X,Y]f = X(Yf) - Y(Xf)$, it follows that

$$\begin{aligned} [X,Y]f = &\sum_{j=1}^n \left(\sum_{i=1}^n X^j \frac{\partial Y^i}{\partial x^j} \frac{\partial f}{\partial x^i} - Y^j \frac{\partial X^i}{\partial x^j} \frac{\partial f}{\partial x^i} \right) \\ &+ \sum_{j=1}^n \left(\sum_{i=1}^n X^j Y^i \frac{\partial^2 f}{\partial x^j \partial x^i} - Y^j X^i \frac{\partial^2 f}{\partial x^j \partial x^i} \right) \end{aligned}$$

Reordering the sum now yields

$$\begin{aligned} [X,Y] \cdot f = &\sum_{i=1}^n \left(\sum_{j=1}^n X^j \frac{\partial Y^i}{\partial x^j} - Y^j \frac{\partial X^i}{\partial x^j} \right) \frac{\partial f}{\partial x^i} \\ &+ \sum_{i,j=1}^n X^i Y^j \left(\frac{\partial^2 f}{\partial x^i \partial x^j} - \frac{\partial^2 f}{\partial x^j \partial x^i} \right) \end{aligned}$$

By Schwarz's Theorem, $\frac{\partial^2 f}{\partial x^i \partial x^j} = \frac{\partial^2 f}{\partial x^j \partial x^i}$, and the last term is zero. Hence

$$[X,Y] \cdot f = \left(\sum_{i=1}^n \left(\sum_{j=1}^n X^j \frac{\partial Y^i}{\partial x^j} - Y^j \frac{\partial X^i}{\partial x^j} \right) \frac{\partial}{\partial x^i} \right) f$$

The result follows.

Exercise 2.

1. Let $f \in \mathcal{C}^\infty(M)$. Then for any vector field Y, $[X, fY] = 0$. Developping this last equality, it follows that

   $$\forall Y \in \Gamma(TM), \quad (X \cdot f)Y + f[X,Y] = 0$$

   Since $[X,Y] = 0$, it follows that

   $$\forall Y \in \Gamma(TM), \quad (X \cdot f)Y = 0$$

   Fix $p \in M$ and let Y be a vector field such that $Y(p) \neq 0$. It follows that $((X \cdot f)Y)(p) = (X \cdot f)(p) \times Y(p) = 0$, and thus, $(X \cdot f)(p) = 0$. Therefore, $X \cdot f$ identically vanishes.

2. From 1., X acts on $\mathcal{C}^\infty(M)$ as the zero derivation. From the identification between vector fields and derivations, it follows that $X = 0$.

Exercise 3.

Watch out: in the papersheet, "$Xf = 0$" has to be understood as $\forall p \in M, X(p) \in \ker \mathrm{d}f(p)$. Of course, since f has range in N, Xf is not a function with range in $\mathbb{R}$ but in TN.

1. Let X and Y be vector fields in $\mathcal{K}(f)$. Then $Xf = Yf = 0$, and it follows that $X(Yf) = 0$ and $Y(Xf) = 0$. Therefore

$$[X, Y]f = X(Yf) - Y(Xf) = 0 - 0 = 0$$

and $[X, Y] \in \mathcal{K}(f)$.

2. For $q \in M$, consider

$$\mathcal{K}_q(f) = \{X(q) \mid X \in \mathcal{K}(f)\} \subset T_qM$$

which is clearly a vector subspace of T_qM. Let $S_p = f^{-1}(\{f(p)\})$ be a level set of f. Since f is a submersion, S_p is a $(\dim M - \dim N) = (m-n)$ dimensional submanifold of M and moreover, for all $q \in S_p, T_q(S_p) = \ker \mathrm{d}f(q)$. It follows that

$$\forall q \in S_p, \quad \mathcal{K}_q(f) \subset T_q(S_p)$$

Let us show the converse inclusion. From the constant rank Theorem, there exists charts φ and ψ around p and $f(p)$ respectively such that

$$\psi \circ f \circ \varphi^{-1}(x^1, \dots, x^m) = (x^1, \dots, x^n)$$

and the level sets of f are locally given in these coordinates by $(x^1, \dots, x^n) = (c_1, \dots, c_n)$. The vectors fields $(\frac{\partial}{\partial x^j})_{j \in \{n+1, \dots, m\}}$ are thus in the kernel of $\mathrm{d}f$ at each point of the coordinate patch. Multiplying them by a suitable bump function leads to $m - n$ vector fields of $\mathcal{K}_q(f)$ that are linearly independent at q, and hence $\mathcal{K}_q(f)$ has dimension $\geqslant m - n$. It follows that $\mathcal{K}_q(f) = T_q(S_p)$. Finally, S_q are $m - n$ dimensional submanifolds tangent to $\mathcal{K}(f)$, a $m - n$ dimensional distribution. They are thus integral submanifold of that distribution.

Exercise 4.

1. In the canonical basis can $= \{e_1, e_2\}$ of $T_p\mathbb{R}^2 = \mathbb{R}^2$, the matrix $\mathrm{mat}_{\mathrm{can}}\{X(p), Y(p)\} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ -y(p) & 0 \end{pmatrix}$ has rank two. The result follows.

2. Recall that $[\frac{\partial}{\partial y}, \frac{\partial}{\partial x}] = [\frac{\partial}{\partial y}, \frac{\partial}{\partial z}] = 0$ and that the Lie bracket is alternating. Therefore,

$$\begin{aligned} [X, Y] &= -[Y, X] = -[\frac{\partial}{\partial y}, \frac{\partial}{\partial x} - y\frac{\partial}{\partial z}] = -[\frac{\partial}{\partial y}, \frac{\partial}{\partial x}] + [\frac{\partial}{\partial y}, y\frac{\partial}{\partial z}] \\ &= 0 + (\frac{\partial}{\partial y} \cdot y)\frac{\partial}{\partial z} + y[\frac{\partial}{\partial y}, \frac{\partial}{\partial z}] = \frac{\partial}{\partial z} \end{aligned}$$

It follows that $[X, Y] \notin \mathrm{Span}\{X, Y\}$, and the distribution $\mathrm{Span}\{X, Y\}$ is not stable under the Lie bracket. It is then non-integrable by Frobenius Theorem.

3. Let (φ_s) and (ψ_t) be the flows of X and Y respectively. By straightforward computations, we have

$$\varphi_s(x,y,z) = (x+s, y, -ys)$$
$$\psi_t(x,y,z) = (x, y+s, z)$$

Write $\mathcal{D} = \operatorname{Span}\{X, Y\}$ be the distribution spanned by X and Y. Suppose by contradiction that $\mathcal{D}$ is integrable and let S_0 be the integral submanifold of $\mathcal{D}$ passing through $(0,0,0)$. Then $T_0S_0 = \mathcal{D}_0 = \operatorname{Span}\{X(0), Y(0)\} = \operatorname{Span}\{(1,0,0);(0,1,0)\}$.

Since X and Y lie in $\mathcal{D}$, the integral curves of X and Y that crosses S_0 stay in S_0, and thus, for any $a,b,c,d \in \mathbb{R}$,

$$\psi_a \circ \varphi_b \circ \psi_c \circ \varphi_d(0,0,0) \in S_0$$

By 2., we have

$$\begin{aligned}\psi_a \circ \varphi_b \circ \psi_c \circ \varphi_d(0,0,0) &= \psi_a \circ \varphi_b \circ \psi_c(d,0,0) \\ &= \psi_a \circ \varphi_b(d,c,0) \\ &= \psi_a(d+b, c, -cb) \\ &= (d+b, a+c, -cb)\end{aligned}$$

Therefore, the path

$$\sigma(t) = \psi_{-t} \circ \varphi_{-t} \circ \psi_t \circ \varphi_t(0,0,0) = (0,0,t^2)$$

lies in S_0. Unluckily, its derivative at zero vanishes and σ does not lead to a contradiction. Here is the trick: consider

$$\forall t \geqslant 0, \quad \gamma(t) = \sigma(\sqrt{t}) = (0,0,t)$$

Then γ is smooth on $\mathbb{R}_+$ and $\gamma'(0) = (0,0,1)$, and $\gamma'(0) \notin \operatorname{Span}\{X(0), Y(0)\}$, which is a contradiction.

Remarks. (a) If you don't like considering smooth paths on $\mathbb{R}_+$ and rather want it to be defined on neighbourhood of the origin, consider

$$\gamma(t) = \begin{cases} \sigma(\sqrt{t}) & \text{if } t \geqslant 0, \\ \sigma(\sqrt{-t}) & \text{if } t \leqslant 0 \end{cases}$$

which is smooth and does the job.

(b) The trick we used is in fact a particular case of the following general statement: let X and Y be two vector fields on M and φ^X, φ^Y be their flows. Then

$$\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \left(\varphi^Y_{-\sqrt{t}} \circ \varphi^X_{-\sqrt{t}} \circ \varphi^Y_{\sqrt{t}} \circ \varphi^X_{\sqrt{t}}\right) = [X,Y]$$

meaning that $[X,Y]$ measures the failure of the path "following X, then Y, then $-X$, then $-Y$" to return to its starting point. Choosing t instead of $\sqrt{t}$ in this path forces to differentiate one more time to recover (twice) the Lie bracket:

$$\frac{1}{2}\left.\frac{\mathrm{d}^2}{\mathrm{d}t^2}\right|_{t=0} \left(\varphi^Y_{-t} \circ \varphi^X_{-t} \circ \varphi^Y_{t} \circ \varphi^X_{t}\right) = [X,Y]$$

Exercise 5.

Consider the two following vector fields $X = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}$ and $Y = x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}$ on $\mathbb{R}^2$.

1. By the formula of Exercise 1. with $X^1 = x$, $X^2 = y$, $Y^1 = -y$ and $Y^2 = x$, we find

$$\begin{aligned}
[X,Y] &= \big(x\frac{\partial(-y)}{\partial x} - (-y)\frac{\partial x}{\partial x} + y\frac{\partial(-y)}{\partial y} - x\frac{\partial x}{\partial y}\big)\frac{\partial}{\partial x} \\
&\quad + \big(x\frac{\partial x}{\partial x} - (-y)\frac{\partial y}{\partial x} + y\frac{\partial x}{\partial y} - x\frac{\partial y}{\partial y}\big)\frac{\partial}{\partial y} \\
&= \big(0 + y - y - 0\big)\frac{\partial}{\partial x} + \big(x + 0 + 0 - x\big)\frac{\partial}{\partial y} \\
&= 0
\end{aligned}$$

Let $p = (1,0)$. Then $\det(X(p), Y(p)) = \det\big(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\big) = 1$, and in a neighbourhood of p, $\{X, Y\}$ are pointwisely linearly independant. Since $[X,Y] = 0$, from Frobenius Theorem, there exists a neighbourhood of p on which X and Y are tangent to coordinates, that is there exists local coordinates (s,t) near p such that $X = \frac{\partial}{\partial s}$ and $Y = \frac{\partial}{\partial t}$.

2. Let $\gamma\colon 0 \in I \to \mathbb{R}$ be a smooth curve, and write $\gamma = (x,y)$ and $\gamma' = x'\frac{\partial}{\partial x} + y'\frac{\partial}{\partial y}$. Then

$$\begin{aligned}
\gamma' = X(\gamma) &\iff x'\frac{\partial}{\partial x} + y'\frac{\partial}{\partial y} = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} \\
&\iff \begin{cases} x' &= x, \\ y' &= y \end{cases} \\
&\iff \forall s \in I, \quad \gamma(s) = (x(0)e^s, y(0)e^s)
\end{aligned}$$

It follows that the flow of X is given by

$$\forall s \in \mathbb{R}, \forall (x,y) \in \mathbb{R}^2, \quad \varphi_s^X(x,y) = e^s(x,y)$$

Similarly,

$$\gamma' = Y(\gamma) \iff \begin{cases} x' &= -y \\ y' &= x \end{cases} \iff \begin{pmatrix} x \\ y \end{pmatrix}' = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$$

Solving this last differential equation shows that the flow of Y is given by

$$\forall t \in \mathbb{R}, \forall (x,y) \in \mathbb{R}^2, \quad \varphi_t^Y(x,y) = ((\cos t)x + (\sin t)y, -(\sin t)x + (\cos t)y)$$

3. Consider the map

$$\Phi\colon (s,t) \in \mathbb{R}^2 \mapsto \varphi_s^X \circ \varphi_t^Y\big((1,0)\big) \in \mathbb{R}^2$$

Explicitly, it is given by

$$\Phi(s,t) = \big(e^s \cos t, -e^s \sin t\big)$$

By the inverse function Theorem, Φ is a local diffeomorphism from a neighbourhood of $(0,0)$ to a neighbourhood of $(1,0)$, and by construction, the tangent vector fields to the coordinates (s,t) are X and Y.