Multilinear Algebra and Differential forms

Exercise 1.

Let K be a field and V be a finite dimensional vector field over it.

1. Let $n \in \mathbb{N}$, show that $\Phi : \bigotimes^n V^* \to (\bigotimes^n V)^*$ is an isomorphism with Φ defined by the formula :
$$\forall (\alpha^i)_{1\leqslant i\leqslant n} \in (V^*)^n,\ \Phi\left(\bigotimes_{i=1}^n \alpha^i\right) : \bigotimes_{i=1}^n v_i \mapsto \prod_{i=1}^n \alpha^i(v_i).$$

2. Let $R_n(V)$ be the sub vector space of $\bigotimes^n V$ generated by the tensor of the form $w_1 \otimes ... \otimes w_i \otimes v \otimes v \otimes w_{i+1} \otimes ... \otimes w_{n-2}$ for the w_j's and v in V. We recall that $\Lambda^n V$ is the quotient of $\bigotimes^n V$ by $R_n(V)$ i.e. we have the short exact sequence :
$$0 \to R_n(V) \to \bigotimes^n V \xrightarrow{\pi_V^n} \Lambda^n V \to 0.$$
Dualise this sequence and determine the image of $(\Lambda^n V)^*$ in $(\bigotimes^n V)^*$.

3. Let $\Psi : (\Lambda^n V)^* \to \Lambda^n V^*$ be the composition $\pi_{V^*}^n \circ \Phi^{-1} \circ (\pi_V^n)^*$. Show that if $(e_1; ...; e_d)$ is a basis of V and J is a subset of $\{1; ...; d\}$ of n elements then $\Psi((e_J)^*) = n!(e^*)_J$.

4. Deduce that if K is of characteristic 0 then $\frac{1}{n!}\Psi$ is a canonical isomorphism.

Exercise 2.

1. For V a finite dimensional real vector space and $p, q \in \mathbb{N}$, show that the natural tensor product map $\bigotimes^p V \otimes \bigotimes^q V \to \bigotimes^{p+q} V$ gives rise to a well defined product $\wedge : \Lambda^p V \otimes \Lambda^q V \to \Lambda^{p+q} V$.

2. Show that if $\alpha \in \Lambda^p V^*$ and $\beta \in \Lambda^q V^*$ then for all $v_1, ..., v_{p+q} \in V$:
$$\alpha \wedge \beta(v_1, ..., v_{p+q}) = \sum_{c \in C_{p,q}} \varepsilon(c)\alpha(v_{c(1)}, ..., v_{c(p)})\beta(v_{c(p+1)}, ..., v_{c(p+q)}),$$
for $C_{p,q} = \{c \in \mathfrak{S}_{p+q} \mid c|_{\{1,...,p\}} \text{ and } c|_{\{p+1,...,p+q\}} \text{ strictly increasing}\}$.

Exercise 3.

Let M be a n-dimensional manifold and $(U; \varphi)$ one of its charts. Let $(e_1; ...; e_n)$ be the canonical basis of $\mathbb{R}^n$ and $(e_1^*; ...; e_n^*)$ its dual basis.

1. Let V be an open subset of $\mathbb{R}^n$ show that a differential form over V is the same as a smooth map from V to $(\mathbb{R}^n)^*$.

2. Show that the space of differential forms over V is a free module of rank n over $\mathcal{C}^\infty(V; \mathbb{R})$ by exhibiting a basis.

3. Let $\left(\frac{\partial}{\partial x^i}\right)_{1\leqslant i\leqslant n}$ be the constant vector fields on U associated to φ. For all $p \in U$ we set $(dx_p^i)_{1\leqslant i\leqslant n}$ the dual basis of $\left(\frac{\partial}{\partial x^i}(p)\right)_{1\leqslant i\leqslant n}$. Show that for all $1 \leqslant i \leqslant n$ the map $dx^i : p \mapsto (p; dx_p^i)$ is the pullback $\varphi^*(e_i^*)$.

4. Deduce that $dx^i \in \Gamma(T^*M; U)$.

5. Show that dx^i_p is actually the differential at p of the function $x^i : U \to \mathbb{R}$, the i-th projection of φ.

6. Show that for all $\alpha \in \Gamma(T^*M; U)$ there exists a unique n-upple of smooth functions on U $(\alpha_i)_{1\leqslant i\leqslant n}$ such that :

$$\alpha = \sum_{i=1}^{n} \alpha_i dx^i.$$

7. Deduce that for all $f \in \mathcal{C}^\infty(M;\mathbb{R})$ the map $df : M \to T^*M$ that associates p to $(p; df_p)$ is a differential form and that for all $p \in U$ we have :

$$df_p = \sum_{i=1}^{n} \frac{\partial f}{\partial x^i}(p) dx^i_p.$$

8. Show that for all $p \in \mathbb{N}$, $\Gamma(\Lambda^p T^*M; U)$ is a free $\mathcal{C}^\infty(U)$-module and exhibit a basis.

Exercise 4.

Let M be a smooth manifold. And consider the map :

$$\begin{array}{rcl}\Phi : \Gamma(T^*M) & \longrightarrow & \mathrm{Hom}_{\mathcal{C}^\infty(M)}(\Gamma(TM); \mathcal{C}^\infty(M)) \\ \alpha & \longmapsto & [X \mapsto [p \mapsto \alpha_p(X_p)]]\end{array}$$

1. Show that it is a well defined injective morphism of $\mathcal{C}^\infty(M)$-modules.

2. Now we want to show that Φ is surjective. Let $\lambda \in \mathrm{Hom}_{\mathcal{C}^\infty(M)}(\Gamma(TM); \mathcal{C}^\infty(M))$, $(U_i; \varphi_i)_{i\in I}$ be a locally finite atlas of M, $(\theta_i)_{i\in I}$ a subordinate partition of unity and $V_i = \theta_i^{-1}(\mathbb{R}^\times) \subset U_i$.

   (a) For $X \in \Gamma(TM; V_i)$ we set $\lambda_i(X)$ to be the function $\frac{1}{\theta_i}\lambda(\theta_i X)|_{V_i}$ over V_i. Show that it is a well defined smooth function over V_i and that λ_i is $\mathcal{C}^\infty(V_i)$-linear.

   (b) Show that if $X \in \Gamma(TM; M)$ then $\lambda_i(X|_{V_i}) = \lambda(X)|_{V_i}$.

   (c) Show that there is a form α^i over V_i such that $\lambda_i(X) = \alpha^i(X)$ for all vector field X over V_i.

   (d) Conclude.