Differential forms, orientability, integration on manifolds

Exercise 1.

1. Show that a parallelizable manifold is orientable.
2. Show that the product of two orientable manifolds is orientable.
3. Show that the tangent bundle TM of a manifold M is an orientable manifold.

Exercise 2.

Let $n \in \mathbb{N}$.

1. Show that the sphere $\mathbb{S}^n$ is orientable. Is the diffeomorphism $x \mapsto -x$ orientation preserving?
2. Is the projective space $\mathbb{R}P^n$ orientable?

Exercise 3.

Let $X = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + z\frac{\partial}{\partial z}$ and $\omega = \mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}z$ on $\mathbb{R}^3$. Define α by

$$\forall Y, Z \in \Gamma(\mathbb{R}^3), \quad \alpha(Y, Z) = \frac{1}{3}\omega(X, Y, Z)$$

1. Show that α is a differential form of degree 2. Give an expression of α in the basis $\{\mathrm{d}x \wedge \mathrm{d}y, \mathrm{d}y \wedge \mathrm{d}z, \mathrm{d}z \wedge \mathrm{d}x\}$.
2. Does there exist $\beta \in \Omega^1(\mathbb{R}^3)$ such that $\alpha = \mathrm{d}\beta$?
3. Let B be the open unit ball of $\mathbb{R}^3$. Compute $\int_B \mathrm{d}\alpha$.
4. Let $i\colon \mathbb{S}^2 \to \mathbb{R}^3$ be the inclusion of the unit sphere. Compute $\int_{\mathbb{S}^2} i^*\alpha$.

Exercise 4.

1. Let $f\colon \mathbb{R} \to \mathbb{R}_+^*$ be defined as $f(t) = e^t$. Compute $f^*(\frac{\mathrm{d}x}{x})$.
2. Let $f\colon (0, +\infty) \times (0, 2\pi) \to \mathbb{R}^2 \setminus \{0\}$ be defined as $f(r, \theta) = (r\cos\theta, r\sin\theta)$. Compute $f^*(\mathrm{d}x \wedge \mathrm{d}y)$.

Exercise 5.

Let $M = \mathbb{R}^2 \setminus \{0\}$. Consider the differential form of degree 1 $\alpha = -\frac{y}{x^2+y^2}\,\mathrm{d}x + \frac{x}{x^2+y^2}\,\mathrm{d}y$.

1. Compute $\mathrm{d}\alpha$.
2. Let C_0 be the circle centered at the origin of radius 1 and C_1 be the circle centered at $(3, 0)$ of radius 2. Let i_0 and i_1 be the respective inclusion maps. Compute $\int_{C_0} i_0^*\alpha$ and $\int_{C_1} i_1^*\alpha$.