TD 8, december 9^{th} 2021

Exercise 1

1) Let $n = \dim M$. Since M is parallelizable, there exists a diffeomorphism

$$\left| \begin{array}{cccc} \phi : & TM & \xrightarrow{\sim} & M \times \mathbb{R}^n \\ & (p,v) & \longmapsto & (p, \varphi(p)\cdot v) \end{array} \right.$$

where $\varphi(p) : T_pM \longrightarrow \mathbb{R}^n$ is a linear isomorphism.

Let $\omega = \phi^*(\mathrm{id}, \det_{can})$, that is,

$\forall p \in M$, $\forall (v_1, .., v_n) \in (T_pM)^n$, $\omega_p(v_1, .., v_n) = \det(\varphi(p)v_1, .., \varphi(p)v_n)$

Then ω is a volume form on M. $\blacksquare$

Alternatively: if $(X_1, .., X_n)$ are n vector fields everywhere linearly independent, define

$$\omega_p(v_1, .., v_n) = \det_{(X_1(p), .., X_n(p))}(v_1, .., v_n)$$

2) Let $m = \dim M$ and $n = \dim N$. Consider the projections $\pi_M : M \times N \longrightarrow M$ and $\pi_N : M \times N \to N$. Since M and N are orientable, then there exist two volume forms $\alpha \in \Omega^m(M)$ and $\beta \in \Omega^n(N)$ respectively. Let $\omega = \pi_M^*(\alpha) \wedge \pi_N^*(\beta) \in \Omega^{m+n}(M \times N)$.

Let us show that ω is nowhere vanishing. Let $(p,q) \in M \times N$, $\{v_1, \dots, v_m\}$ be a basis of T_pM and $\{w_1, \dots, w_n\}$ be a basis of T_qN. Then $\{(v_1,0), \dots, (v_m,0), (0,w_1), \dots, (0,w_n)\}$ is a basis of $T_{p,q} M \times N \simeq T_pM \times T_qN$, and

$$\omega_{p,q}((v_1,0), \dots, (v_m,0), (0,w_1), \dots, (0,w_n)) = \alpha(v_1, \dots, v_m) \times \beta(w_1, \dots, w_n) \neq 0.$$

The result follows. $\blacksquare$

3) Let $\{(U_i, \varphi_i)\}_{i \in I}$ be an atlas of M. Recall that we can build an atlas $\{(TU_i, \Phi_i)\}_{i \in I}$ on TM by

$$\left| \begin{array}{rcl} \Phi_i : TU_i & \longrightarrow & \varphi_i(U_i) \times \mathbb{R}^n \\ (p,v) & \longmapsto & (\varphi_i(p), d\varphi_i(p) \cdot v) \end{array} \right.$$

whose transition functions are given by

$$\left| \begin{array}{rcl} \Phi_i \circ \Phi_j^{-1} : \varphi_j(U_i \cap U_j) \times \mathbb{R}^n & \longrightarrow & \varphi_i(U_i \cap U_j) \times \mathbb{R}^n \\ (x, w) & \longmapsto & (\varphi_i \circ \varphi_j^{-1}(x) ; d(\varphi_i \circ \varphi_j^{-1})(x) \cdot w) \end{array} \right.$$

Let us show that this atlas is always oriented. Fix (i,j) such that $U_i \cap U_j \neq \emptyset$. Then for $(x,w) \in \varphi_j(U_i \cap U_j) \times \mathbb{R}^n$ and $(w_1, w_2) \in \mathbb{R}^n \times \mathbb{R}^n$,

$$d(\Phi_i \circ \Phi_j^{-1})(x,w) \cdot (w_1, w_2) = \left(d(\varphi_i \circ \varphi_j^{-1})(x) \cdot w_1, d(\varphi_i \circ \varphi_j^{-1})(x) \cdot w_2 \right)$$

Therefore, $\det\left(d(\Phi_i \circ \Phi_j^{-1})(x,w)\right) = \left(\det(\varphi_i \circ \varphi_j^{-1})(x)\right)^2 > 0$

And TM is orientable $\blacksquare$

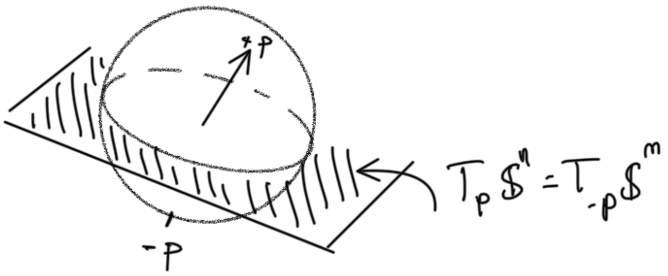
Exercise 2

1) Consider $\mathbb{S}^n$ as embedded in $\mathbb{R}^{n+1}$. Recall that in that case, $\forall p \in \mathbb{S}^n$, $T_p\mathbb{S}^n = \{p\}^\perp$. Let ω be the differential form of degree n defined as

$$\forall p \in \mathbb{S}^n, \forall (v_1, .., v_n) \in T_p\mathbb{S}^n, \ \omega_p(v_1, .., v_n) = \det(p, v_1, ..., v_n)$$

If $\{v_1, ..., v_n\}$ is a basis of $T_p\mathbb{S}^n$, then $\{p, v_1, .., v_n\}$ is a basis of $\mathbb{R}^n$, and therefore, $\omega_p(v_1, .., v_n) \neq 0$. It follows that ω is a volume form on $\mathbb{S}^n$.

If $p \in \mathbb{S}^n$, then $T_p\mathbb{S}^n = \{p\}^\perp = \{-p\}^\perp = T_{-p}\mathbb{S}^n$



If $f : \mathbb{S}^n \longrightarrow \mathbb{S}^n$, $p \longmapsto -p$ is the antipodal map, then with the identification above, $\forall p \in \mathbb{S}^n$, $df(p) = -I_d$.

It follows that the differential form of degree n $f^*\omega$ is given by

$$\forall p \in \mathbb{S}^n, \forall (v_1, .., v_n) \in (T_p\mathbb{S}^n)^n ; \ (f^*\omega)_p(v_1, .., v_n) = \omega_{f(p)}(df(p)v_1, ..., df(p)v_n)$$
$$= \omega_{-p}(-v_1, .., -v_n)$$
$$= \det(-p, -v_1, .., -v_n)$$
$$= (-1)^{n+1} \det(p, v_1, .., v_n)$$

And it follows that $f^*\omega = (-1)^{n+1}\omega$. Therefore, f is orientation preserving if and only if $(-1)^{n+1} = 1$, if and only if n is odd. $\blacksquare$

2) Recall that the quotient map $\pi : \mathbb{S}^n \longrightarrow \mathbb{RP}^n$ is a local diffeomorphism. Suppose that $\mathbb{RP}^n$ is orientable, and let $\omega \in \Omega^n(\mathbb{RP}^n)$ be a volume form. Then $(\pi^*\omega) \in \Omega^n(\mathbb{S}^n)$ is a differential form of degree n on $\mathbb{S}^n$ such that:

- $\forall p \in \mathbb{S}^n$, $(\pi^*\omega)_p = \omega_{\pi(p)}(d\pi(p)\cdot, \dots, d\pi(p)\cdot) \neq 0$
- $\forall p \in \mathbb{S}^n$, $(\pi^*\omega)_{-p} = (\pi^*\omega)_p$.

Hence, $(\pi^*\omega)$ is a volume form on $\mathbb{S}^n$ that is invariant under the antipodal map. It follows from 1) that n is odd.

We have shown that

$$\mathbb{RP}^n \text{ is orientable} \implies n \text{ is odd}.$$

Conversely, suppose that n is odd. Then the volume form defined in 1) is invariant under the antipodal map, and descends to the quotient as a volume form on $\mathbb{RP}^n$, which is thus orientable.

Finally: $\mathbb{RP}^n$ is orientable $\iff$ n is odd. $\blacksquare$

Exercise 3

1) α is clearly linear each variable, and $p \mapsto \alpha_p$ is smooth.

Moreover, $\forall Y, Z \in \Gamma(T\mathbb{R}^3)$, $\alpha(Y,Z) = \frac{1}{3}\omega(X,Y,Z) = -\frac{1}{3}\omega(X,Z,Y) = -\alpha(Z,Y)$.

It follows that $\alpha \in \Omega^2(\mathbb{R}^3)$. In addition, write

$$Y = Y^1 \frac{\partial}{\partial x} + Y^2 \frac{\partial}{\partial y} + Y^3 \frac{\partial}{\partial z} \quad \text{and} \quad Z = Z^1 \frac{\partial}{\partial x} + Z^2 \frac{\partial}{\partial y} + Z^3 \frac{\partial}{\partial z}.$$

then

$$\alpha(X,Y,Z) = \frac{1}{3} dx \wedge dy \wedge dz\,(X,Y,Z) = \frac{1}{3}\det\begin{pmatrix} x & Y^1 & Z^1 \\ y & Y^2 & Z^2 \\ z & Y^3 & Z^3 \end{pmatrix}$$

Therefore,

$$\alpha(X,Y,Z) = \frac{1}{3}\left(x \begin{vmatrix} Y^2 & Z^2 \\ Y^3 & Z^3 \end{vmatrix} - y \begin{vmatrix} Y^1 & Z^1 \\ Y^3 & Z^3 \end{vmatrix} + z \begin{vmatrix} Y^1 & Z^1 \\ Y^2 & Z^2 \end{vmatrix}\right)$$

$$= \frac{1}{3}\left(x\,(Y^2Z^3 - Y^3Z^2) - y\,(Y^1Z^3 - Y^3Z^1) + z\,(Y^1Z^2 - Y^2Z^1)\right)$$

$$= \frac{1}{3}\left(x\, dy \wedge dz - y\, dx \wedge dz + z\, dx \wedge dy\right)(Y,Z).$$

Finally, $\alpha = \frac{1}{3}\left(x\, dy \wedge dz + y\, dz \wedge dx + z\, dx \wedge dy\right)$. $\blacksquare$

2) Recall that if d is the exterior differential, then $d \circ d = 0$.

Suppose by contradiction that $\exists \beta \in \Omega^1(\mathbb{R}^3)$ with $\alpha = d\beta$. Then $d\alpha = d(d\beta) = 0$.

But

$$d\alpha = \frac{1}{3} d\left(x\, dy \wedge dz + y\, dz \wedge dx + z\, dx \wedge dy\right)$$

$$= \frac{1}{3}\left(dx \wedge dy \wedge dz + dy \wedge dz \wedge dx + dz \wedge dx \wedge dy\right)$$

$$= \frac{1}{3} \times 3\, dx \wedge dy \wedge dz = dx \wedge dy \wedge dz \neq 0.$$

contradictio $\blacksquare$

3) We have shown than $d\alpha = dx \wedge dy \wedge dz$

Therefore, $\int_B d\alpha = \int_B dx \wedge dy \wedge dz := \int_B 1\, d\lambda = \mathrm{Vol}(B) = \frac{4}{3}\pi$ $\blacksquare$

4) Let us find a suitable chart in which we can compute the integral.

Consider the parametrization

$$\left| \begin{array}{lcl} F : (0,2\pi)\times\left(-\frac{\pi}{2},\frac{\pi}{2}\right) & \longrightarrow & \mathbb{S}^2 \\ (\theta,\ \varphi) & \longmapsto & (\cos\theta\cos\varphi,\ \sin\theta\cos\varphi,\ \sin\varphi) \end{array}\right.$$

We are in the following situation:

$$\begin{tikzcd} (0,2\pi)\times\left(-\frac{\pi}{2},\frac{\pi}{2}\right) \arrow[r, "F"] \arrow[rr, bend right, "i\circ F"'] & \mathbb{S}^2 \arrow[r, "i"] & \mathbb{R}^3 \end{tikzcd}$$

First, notice that $\mathbb{S}^2 \setminus \mathrm{Im}(F)$ has measure 0, and therefore:

$$\int_{\mathbb{S}^2} i^*\alpha = \int_{\mathrm{Im}(F)} i^*\alpha = \int_{(0,2\pi)\times\left(-\frac{\pi}{2},\frac{\pi}{2}\right)} F^*(i^*\alpha)$$

and note that $F^*(i^*\alpha) = (i\circ F)^*\alpha$. By linearity,

$$(i\circ F)^*\alpha = (i\circ F)^*\frac{1}{3}(x\,dy\wedge dz + y\,dz\wedge dx + z\,dx\wedge dy)$$

$$= \frac{1}{3}\big(x\circ(i\circ F)\,(i\circ F)^*(dy\wedge dz) + y\circ(i\circ F).\,(i\circ F)^*(dz\wedge dx) + z\circ(i\circ F)\,(i\circ F)^*(dx\wedge dy)\big)$$

$$= \frac{1}{3}\big(\cos\theta\cos\varphi.\,(i\circ F)^*(dy\wedge dz) + \sin\theta\cos\varphi\,(i\circ F)^*(dz\wedge dx) + \sin\varphi\,(i\circ F)^*(dx\wedge dy).\big)$$

Recall that:

(i) the pullback commutes with the wedge product: $f^*(\alpha\wedge\beta) = f^*\alpha\wedge f^*\beta$.

(ii) the chain rule gives $(i\circ F)^*dx = dx\circ d(i\circ F) = d(x\circ i\circ F)$
$(i\circ F)^*dy = dy\circ d(i\circ F) = d(y\circ i\circ F)$
$(i\circ F)^*dz = dz\circ d(i\circ F) = d(z\circ i\circ F)$

Then: $(i\circ F)^*dx = d(\cos\theta\cos\varphi) = -\sin\theta\cos\varphi\,d\theta - \cos\theta\sin\varphi\,d\varphi$

$(i\circ F)^*dy = d(\sin\theta\cos\varphi) = \cos\theta\cos\varphi\,d\theta - \sin\theta\sin\varphi\,d\varphi$

$(i\circ F)^*dz = d(\sin\varphi) = \cos\varphi\,d\varphi$

$(i\circ F)^* dz = d(\sin\varphi) = \cos\varphi\, d\varphi$

Therefore:

$$(i\circ F)^*(dy\wedge dz) = [\cos\theta\cos\varphi\, d\theta - \sin\theta\sin\varphi\, d\varphi]\wedge \cos\varphi\, d\varphi = \cos\theta\cos^2\varphi\, d\theta\wedge d\varphi$$

$$(i\circ F)^*(dz\wedge dx) = \cos\varphi\, d\varphi\wedge[-\sin\theta\cos\varphi\, d\theta - \cos\theta\sin\varphi\, d\varphi] = -\sin\theta\cos^2\varphi\, d\varphi\wedge d\theta$$
$$= \sin\theta\cos^2\varphi\, d\theta\wedge d\varphi$$

$$(i\circ F)^*(dx\wedge dy) = [-\sin\theta\cos\varphi\, d\theta - \cos\theta\sin\varphi\, d\varphi]\wedge[\cos\theta\cos\varphi\, d\theta - \sin\theta\sin\varphi\, d\varphi]$$

$$= \sin^2\theta\cos\varphi\sin\varphi\, d\theta\wedge d\varphi - \cos^2\theta\cos\varphi\sin\varphi\, d\varphi\wedge d\theta$$
$$= \cos\varphi\sin\varphi\, d\theta\wedge d\varphi.$$

It follows that

$$(i\circ F)^*\alpha = \frac{1}{3}\left(\cos\theta\cos\varphi\cdot\cos\theta\cos^2\varphi + \sin\theta\cos\varphi\cdot\sin\theta\cos^2\varphi + \sin\varphi\cdot\cos\varphi\sin\varphi\right)d\theta\wedge d\varphi$$

$$= \frac{1}{3}\left(\cos^2\theta\cdot\cos^2\varphi\cdot\cos\varphi + \sin^2\theta\cdot\cos^2\varphi\cdot\cos\varphi + \sin^2\varphi\cos\varphi\right)d\theta\wedge d\varphi$$

$$= \frac{1}{3}\cos\varphi\, d\theta\wedge d\varphi$$

It follows that $\displaystyle\int_{S^2} i^*\alpha = \int_{(0,2\pi)\times(-\frac{\pi}{2},\frac{\pi}{2})} \frac{1}{3}\cos\varphi\, d\theta\wedge d\varphi := \frac{1}{3}\int_0^{2\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos\varphi\, d\theta\, d\varphi$

Finally, $\displaystyle\int_{S^2} i^*\alpha = \frac{1}{3}\, 2\pi\times\left([\sin\varphi]_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\right) = \frac{4}{3}\pi$ $\blacksquare$

Remark: the equality $\displaystyle\int_B d\alpha = \int_{S^2} i^*\alpha$ is not a coincidence: more generally, if M^n is a smooth manifold with boundary, $\alpha\in\Omega^{n-1}(M)$ and $i:\partial M\to M$ is the inclusion map, then $\displaystyle\int_M d\alpha = \int_{\partial M} i^*\alpha.$

Exercise 4

1) We have

$$f^*\left(\frac{dx}{x}\right) = \frac{d(x \circ f(t))}{x \circ f(t)} = \frac{f'(t)\,dt}{f(t)} = \frac{e^t}{e^t}\,dt = dt. \qquad \blacksquare$$

2) It holds that:

$$f^*(dx) = dx \circ df = d(x \circ f) = d(r\cos\theta) = \cos\theta\, dr - r\sin\theta\, d\theta$$

$$f^*(dy) = dy \circ df = d(y \circ f) = d(r\sin\theta) = \sin\theta\, dr + r\cos\theta\, d\theta$$

And therefore:

$$\begin{aligned} f^*(dx \wedge dy) &= (f^* dx) \wedge (f^* dy) = (\cos\theta\, dr - r\sin\theta\, d\theta) \wedge (\sin\theta\, dr + r\cos\theta\, d\theta) \\ &= r\cos^2\theta\, dr \wedge d\theta - r\sin^2\theta\, d\theta \wedge dr \\ &= r\, dr \wedge d\theta \end{aligned} \qquad \blacksquare$$

Exercise 5

1) Recall: $d(f \cdot \alpha) = df \wedge \alpha + f\, d\alpha$.

Write: $\alpha = \dfrac{-y\,dx + y\,dx}{x^2+y^2}$. Then $d\alpha = d\left(\dfrac{1}{x^2+y^2}\right)(-y\,dx + x\,dy) + \dfrac{1}{x^2+y^2}\, d(-y\,dx + x\,dy)$

Since $d\left(\dfrac{1}{x^2+y^2}\right) = \dfrac{-2x\,dx - 2y\,dy}{(x^2+y^2)^2}$ and $d(-y\,dx + x\,dy) = 2\, dx \wedge dy$

Then:
$$d\alpha = \frac{2y^2\, dy \wedge dx - 2x^2\, dx \wedge dy}{(x^2+y^2)^2} + \frac{2\, dx \wedge dy}{x^2+y^2}$$

$$= \frac{-2(x^2+y^2)\, dx \wedge dy}{(x^2+y^2)^2} + 2\frac{dx \wedge dy}{x^2+y^2} = 0 \qquad \blacksquare$$

2) Consider the parametrization

$$\gamma_0 : [0,2\pi] \longrightarrow C_0$$
$$\theta \longmapsto (\cos\theta, \sin\theta)$$

Then $\displaystyle\int_{C_0} i_0^* \alpha = \int_{[0,2\pi]} \gamma_0^* i_0^* \alpha = \int_{[0,2\pi]} \frac{-\sin\theta(-\sin\theta\, d\theta) + \cos\theta(\cos\theta\, d\theta)}{\cos^2\theta + \sin^2\theta}$

$$= \int_0^{2\pi} 1 \, d\theta = 2\pi$$

Similarly, consider $\gamma_1 : [0,2\pi] \longrightarrow C_1$
$$\theta \longmapsto (2+\cos\theta, \sin\theta)$$

Then $\displaystyle\int_{C_1} i_1^* \alpha = \int_{[0,2\pi]} \frac{-\sin\theta(-\sin\theta\, d\theta) + (2+\cos\theta)(\cos\theta\, d\theta)}{(2+\cos\theta)^2 + \sin^2\theta}$

$$= \int_{[0,2\pi]} \frac{(1+2\cos\theta)\, d\theta}{5+4\cos\theta}$$

With messy calculus and analysis: $\displaystyle\frac{1+2\cos\theta}{5+4\cos\theta} = \left[\frac{\theta}{2} + \arctan\left(3\cot\left(\frac{\theta}{2}\right)\right)\right]'$

And

$$\int_{C_1} i_1^* \alpha = \left[\frac{\theta}{2} + \arctan\left(3\cot\left(\frac{\theta}{2}\right)\right)\right]_0^{2\pi} = \pi + \arctan(3\cot(\pi)) - \arctan(3\times\cot 0^+)$$

$$= \pi - \pi = 0.$$