Stokes' Theorem, Lie derivative, de Rham Cohomology

Exercise 1.

Let X be a vector field on $\mathbb{R}^3$ and $\omega = \mathrm{d}x^1 \wedge \mathrm{d}x^2 \wedge \mathrm{d}x^3$ be the standard volume form.

1. Show that there exists a unique function, called the divergence of X and denoted by $\operatorname{div} X$ such that $\mathcal{L}_X\omega = (\operatorname{div} X)\omega$.
2. Express $\operatorname{div} X$ in terms of the components of X in the basis $\{\partial_1, \partial_2, \partial_3\}$.
3. Assume $X = x^1\partial_1 + x^2\partial_2 + x^3\partial_3$ is the radial vector field. Compute $\int_{S(0,r)} i^*(\iota_X\omega)$ where $i\colon S(0,r) \to \mathbb{R}^3$ is the inclusion of the sphere of radius r centered at 0.

Exercise 2.

Let $\omega = -\frac{y}{x^2+y^2}\,\mathrm{d}x + \frac{x}{x^2+y^2}\,\mathrm{d}y \in \Omega^1(\mathbb{R}^2 \setminus \{0\})$. Let $X = x\partial_x + y\partial_y$.

1. Show that the integral of ω on any circle centered at 0 is independent from its radius.
2. Compute $\mathcal{L}_X\omega$.

Exercise 3.

Let M^n be a closed manifold and $H^*_{dR}(M)$ be its de Rham cohomology.

1. Suppose that M has d connected components. Show that $H^0_{dR}(M) \simeq \mathbb{R}^d$.
2. Suppose that M is connected and orientable. Show that $H^n_{dR}(M) \simeq \mathbb{R}$.

Exercise 4.

Let M be a a connected manifold. A loop in M is a piecewise smooth map $\gamma\colon \mathbb{S}^1 \to M$.

1. Let $\omega \in \Omega^1(M)$. Suppose that for all loops γ, $\int_{\mathbb{S}^1} \gamma^*\omega = 0$. Fix $x_0 \in M$, and for $x \in M$, define $f(x) = \int_{[0,1]} c^*\omega$ where $c\colon [0,1] \to M$ is a smooth path from x_0 to x.
   (a) Show that f is well defined, that is f does not depend on c.
   (b) Show that $\omega = \mathrm{d}f$.
2. Show that $[\omega] = 0 \in H^1(M) \iff \forall\gamma, \quad \int_{\mathbb{S}^1} \gamma^*\omega = 0$.
3. Fix $x_0 \in M$. Let $\pi_1(M, x_0)$ be the fundamental group of M based at x_0, that is the set of all equivalence classes of piecewise smooth loops $\gamma\colon \mathbb{S}^1 \to M$ with $\gamma(1) = x_0$ such that $\gamma_1 \sim \gamma_2 \iff \gamma_1\overline{\gamma_2}$ is null homotopic. The group law is given by $[\gamma_1][\gamma_2] = [\gamma_1\gamma_2]$. We define
$$h\colon \left| \begin{array}{ccc} H^1_{dR}(M) & \longrightarrow & \operatorname{Hom}(\pi_1(M, x_0), \mathbb{R}) \\ [\omega] & \longmapsto & \left([\gamma] \mapsto \int_{\mathbb{S}^1} \gamma^*\omega\right) \end{array} \right.$$
   (a) Show that h is a well-defined group homomorphism.
   (b) Show that h is injective.
   (c) Show that if M is contractible, then $H^1_{dR}(M) = \{0\}$.