TD 9 - Correction

Exercise 1

1) $\mathcal{L}_X$ is a derivation of $\Omega^*(\mathbb{R}^3)$ of degree 0. In particular, $\mathcal{L}_X \omega \in \Omega^3(\mathbb{R}^3)$.

Since $\Omega^3(\mathbb{R}^3)$ is a rank 1 free $\mathcal{C}^\infty(\mathbb{R}^3)$-module with basis ω, there exists a unique function $\operatorname{div} X : \mathbb{R}^3 \to \mathbb{R}$ such that $\mathcal{L}_X \omega = (\operatorname{div} X)\,\omega$. $\blacksquare$

2) By Cartan's magic formula, $\mathcal{L}_X \omega = d(i_X \omega) + i_X d\omega$.

Recall that $d\omega = 0$ since ω is a top form. Moreover, if $Y = Y^1\partial_1 + Y^2\partial_2 + Y^3\partial_3$ and $Z = Z^1\partial_1 + Z^2\partial_2 + Z^3\partial_3$, then

$$(i_X\omega)(Y,Z) = \omega(X,Y,Z) = \begin{vmatrix} X^1 & Y^1 & Z^1 \\ X^2 & Y^2 & Z^2 \\ X^3 & Y^3 & Z^3 \end{vmatrix} = X^1\,(Y^2Z^3 - Y^3Z^2) - X^2(Y^1Z^3 - Y^3Z^1) + X^3(Y^1Z^2 - Y^2Z^1)$$

$$= X^1\, dx^2\wedge dx^3(Y,Z) - X^2\, dx^1\wedge dx^3(Y,Z) + X^3\, dx^1\wedge dx^2(Y,Z)$$

Hence, $i_X\omega = X^1 dx^2\wedge dx^3 - X^2 dx^1\wedge dx^3 + X^3 dx^1\wedge dx^2$, and therefore,

$$d(i_X\omega) = \frac{\partial X^1}{\partial x^1} dx^1\wedge dx^2\wedge dx^3 - \frac{\partial X^2}{\partial x^2} dx^2\wedge dx^1\wedge dx^3 + \frac{\partial X^3}{\partial x^3} dx^3\wedge dx^1\wedge dx^2$$

$$= \left(\frac{\partial X^1}{\partial x^1} + \frac{\partial X^2}{\partial x^2} + \frac{\partial X^3}{\partial x^3}\right) dx^1\wedge dx^2\wedge dx^3$$

Finally, $\operatorname{div} X = \dfrac{\partial X^1}{\partial x^1} + \dfrac{\partial X^2}{\partial x^2} + \dfrac{\partial X^3}{\partial x^3}$ $\blacksquare$

3) Let $M = B(0,r)$. M is a smooth manifold with boundary $\partial M = S(0,r)$.

It follows by Stokes formula that $\displaystyle\int_{S(0,r)} i^*(i_X\omega) = \int_{B(0,r)} d(i_X\omega)$.

By 2), $d(i_X\omega) = (\operatorname{div} X)\, dx^1\wedge dx^2\wedge dx^3$

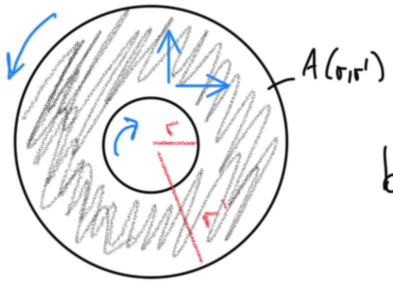
By its expression, $\operatorname{div} X \equiv 3$, hence, it follows that

$$\int i^*(i_X\omega) = \int 3 \cdot dx^1\wedge dx^2\wedge dx^3 = 3 \times \operatorname{Vol}(B(0,r)) = 4\pi r^3.$$ $\blacksquare$

$\partial(0,r)$ $\quad\quad$ $B(0,r)$

Exercice 2

1) For $0 < r < r'$, let $A(r,r') = \left\{(x,y) \in \mathbb{R}^2 \mid r^2 \leq x^2+y^2 \leq r'^2\right\}$, endowed with the



orientation of $\mathbb{R}^2$. It is a smooth manifold with boundary $\partial A(r,r') = \mathscr{C}(0,r')^{+} \sqcup \mathscr{C}(0,r)^{-}$

anti-clockwise $\quad$ clockwise

Hence, by Stokes Theorem, $\displaystyle\int_{\mathscr{C}(0,r)} i^*\omega - \int_{\mathscr{C}(0,r')} i^*\omega = \int_{\partial A(r,r')} i^*\omega = \int_{A(r,r')} d\omega$.

But it holds that

$$
\begin{aligned}
d\omega &= d\left(\frac{-y}{x^2+y^2}\right)\wedge dx + d\left(\frac{x}{x^2+y^2}\right)\wedge dy \\
&= \frac{-(x^2+y^2)+2y^2}{(x^2+y^2)^2}\, dy\wedge dx + \left(\frac{(x^2+y^2)-2x^2}{(x^2+y^2)^2}\right) dx\wedge dy \\
&= \frac{y^2-x^2}{(x^2+y^2)^2}\,(dx\wedge dy - dy\wedge dx) \\
&= 0.
\end{aligned}
$$

The result follows. $\blacksquare$

2) By Cartan's magic formula, $\mathcal{L}_X\omega = d(i_X\omega) + i_X d\omega$

By 1), $d\omega = 0$, and by its expression, $i_X\omega = \frac{-y}{x^2+y^2}\times x + \frac{x}{x^2+y^2}\times y = 0$.

Hence, $\mathcal{L}_X\omega = 0$. $\blacksquare$

Exercise 3

1) $H^0(M) = \{f \in \mathscr{C}^\infty(M) \mid df = 0\} / \{0\} \simeq \{f \in \mathscr{C}^\infty(M) \mid df = 0\}$.

Write $M = M_1 \sqcup \dots \sqcup M_k$ partition in connected components.

$\forall i \in \{1, \dots, k\}$, fix $x_i \in M_i$. Let F be defined as

$$F : H^0(M) \longrightarrow \mathbb{R}^k$$
$$f \longmapsto (f(x_1), \dots, f(x_k))$$

F is clearly linear. Since functions that are constant on each connected component are closed, F is surjective. Finally, F is injective: if $f \in \operatorname{Ker} F$, then $df \equiv 0$, and by the mean value Theorem, f is locally constant, hence constant on each connected component. It follows from $f(x_1) = \dots = f(x_k) = 0$ that $f \equiv 0$. $\blacksquare$

2) Recall that $H^n(M) = \{\omega \in \Omega^n(M) \mid d\omega = 0\} / \{d\alpha \mid \alpha \in \Omega^{n-1}(M)\}$

That is, $[\omega] = [\omega'] \in H^n(M) \iff \exists \alpha \in \Omega^{n-1}(M),\ \omega - \omega' = d\alpha$.

Let $I : \Omega^n(M) \longrightarrow \mathbb{R}$, $\omega \longmapsto \int_M \omega$. Then I is linear. Let us show that I descends to the quotient.

Let $\omega, \omega' \in \Omega^n(M)$ with $[\omega] = [\omega']$. There exists $\alpha \in \Omega^{n-1}(M)$, $\omega - \omega' = d\alpha$.

Hence, $I(\omega) - I(\omega') = \int_M d\alpha = \int_{\partial M} i^*\alpha = 0$ since $\partial M = \emptyset$, and $I(\omega) = I(\omega')$. I descends to the quotient as

$$\tilde{I} : H^n(M) \longrightarrow \mathbb{R}$$
$$[\omega] \longmapsto \int_M \omega.$$

Let us show that $\tilde{I}$ is an isomorphism. Let ω_0 be a volume form.

Then $\tilde{I}([\omega_0]) = \int_M \omega_0 \neq 0$ and $\tilde{I}$ is a non-zero linear form, hence surjective.

Let us show that $\tilde{I}$ is injective.

<u>Lemma:</u> let $\omega \in \Omega^n(\mathbb{R}^n)$ be a top-form with compact support, with $\int_{\mathbb{R}^n} \omega = 0$.
Then there exists $\alpha \in \Omega^{n-1}(\mathbb{R}^n)$ with $\omega = d\alpha$, and with compact support.

Write $\omega = f(x^1, .., x^n)\, dx^1 \wedge .. \wedge dx^n$ with $f \in \mathcal{C}^\infty(\mathbb{R}^n)$ with compact support.

Let $h : \mathbb{R} \longrightarrow \mathbb{R}$ be smooth, with compact support, and $\int_{\mathbb{R}} h = 1$.

For $k \in \{1 \dots n\}$, define

$$f_k : \mathbb{R}^n \longrightarrow \mathbb{R} \text{ by } f_k(x^1, .., x^n) = h(x^1) .. h(x^k) \int_{\mathbb{R}^k} f(y^1, .., y^k, x^{k+1}, .., x^n)\, d\lambda(y)$$

and set $f_0 = f$. By assumption on ω, $f_n = 0$.

Fix $k \in \{1, .., n\}$, and set

$$g_k(x^1, .., x^n) = \int_{t=-\infty}^{x_k} f_k(x^1, .., x^{k-1}, t, x^{k+1}, .., x^n) - f_{k-1}(x^1, .., x^{k-1}, t, x^{k+1}, .., x^n)\, dt.$$

g_k is smooth with compact support. Moreover, $\frac{\partial g_k}{\partial x^k} = f_k - f_{k-1}$

It follows that

$$\begin{aligned}
\omega = f\, dx^1 \wedge .. \wedge dx^n &= (f_0 - f_n)\, dx^1 \wedge .. \wedge dx^n \\
&= -\sum_{k=1}^n (f_k - f_{k-1})\, dx^1 \wedge .. \wedge dx^n \\
&= -\sum_{k=1}^n \frac{\partial g_k}{\partial x^k}\, dx^1 \wedge .. \wedge dx^n \\
&= \sum_{k=1}^n (-1)^k\, d\left(g_k\, dx^1 \wedge .. \wedge \widehat{dx^k} \wedge .. \wedge dx^n\right) \\
&= d\alpha
\end{aligned}$$

with $\alpha = \sum_{k=1}^n g_k\, dx^1 \wedge .. \wedge \widehat{dx^k} \wedge .. \wedge dx^n$

Let us show that this result extends to compact manifolds. Let $\omega \in \Omega^n(M)$ with

$\int_M \omega = 0$. Choose an open cover $M = U_1 \cup U_2 \cup \dots \cup U_N$ where each U_i is diffeomorphic to $\mathbb{R}^n$. Choose an adapted partition of unity $\{\varphi_i\}_{i=1}^N$ (with $\operatorname{supp}\varphi_i \subset U_i$) and write $\omega = \sum_{i=1}^N \varphi_i \omega = \sum_{i=1}^N \omega_i$ with $\omega_i = \varphi_i \omega$ which has compact support in U_i. By induction, it suffices to show the case when $N=2$.

Since $\omega = \omega_1 + \omega_2$, it follows that

$$0 = \int_{U_1 \cup U_2} \omega = \int_{U_1 \cup U_2} \omega_1 + \omega_2 = \int_{U_1} \omega_1 + \int_{U_2} \omega_2 \quad \text{and} \quad \int_{U_1} \omega_1 = -\int_{U_2} \omega_2$$

Let η be a top-form with compact support in $U_1 \cap U_2$, such that $\int_{U_1 \cap U_2} \eta = -\int_{U_1} \omega_1$.

Then $\omega = \omega_1 + \omega_2 = (\omega_1 - \eta) + (\omega_2 + \eta) = \tilde{\omega}_1 + \tilde{\omega}_2$

with $\tilde{\omega}_1$ with compact support in U_1 and $\int_{U_1} \tilde{\omega}_1 = 0$ and

$\tilde{\omega}_2$ with compact support in U_2 and $\int_{U_2} \tilde{\omega}_2 = 0$.

By the Lemma (U_1 and U_2 are $\simeq \mathbb{R}^n$), there exist α_1 and $\alpha_2 \in \Omega^{n-1}(M)$ with compact support in U_1 and U_2 respectively such that $\tilde{\omega}_i = d\alpha_i$

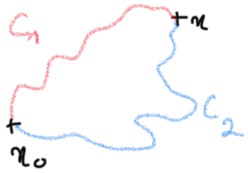
Hence, $\omega = d\alpha_1 + d\alpha_2 = d(\alpha_1 + \alpha_2)$.

By induction, all n-form with $\int_M \omega = 0$ are exact.

It follows that if $[\omega]$ is such that $\bar{I}([\omega]) = 0$, then $[\omega] = 0$, and $\bar{I}$ is injective. Hence, it is an isomorphism. ∎

Exercise 4

1) a) Suppose $c_1, c_2 : [0,1] \longrightarrow M$ are two path with $\underset{= x_0}{c_1(0) = c_2(0)}$, $c_1(1) = c_2(1) = x$.



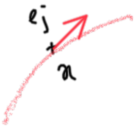
Consider the loop: $\gamma : [0,1] \longrightarrow M$

$$t \longmapsto \begin{cases} c_1(2t) & \text{si } t \in [0, \frac{1}{2}] \\ c_2(1-2t) & t \in [\frac{1}{2}, 1] \end{cases}$$

then by assumption, $0 = \int_{S^1} \gamma^* \omega = \int_{[0,1]} c_1^* \omega - \int_{[0,1]} c_2^* \omega$. It follows that f is well defined

b) Let $x \in M$ and (U, φ) be a chart centered at x. Consider the path

$\sigma : (-\varepsilon, \varepsilon) \to M$ defined by $\sigma(t) = \varphi(\varphi^{-1}(x) + t e_j)$ for $j \in \{1, \dots, n\}$.



Then: $\partial_j f = df(e_j) = \lim_{t \to 0} \frac{f(\sigma(t)) - f(x)}{t} = \lim_{t \to 0} \frac{1}{t} \int_1^{1+t} \omega_{\sigma(s)}(\sigma'(s)) ds$

$= \omega_x(e_j)$

Hence, $df = \omega$. $\blacksquare$

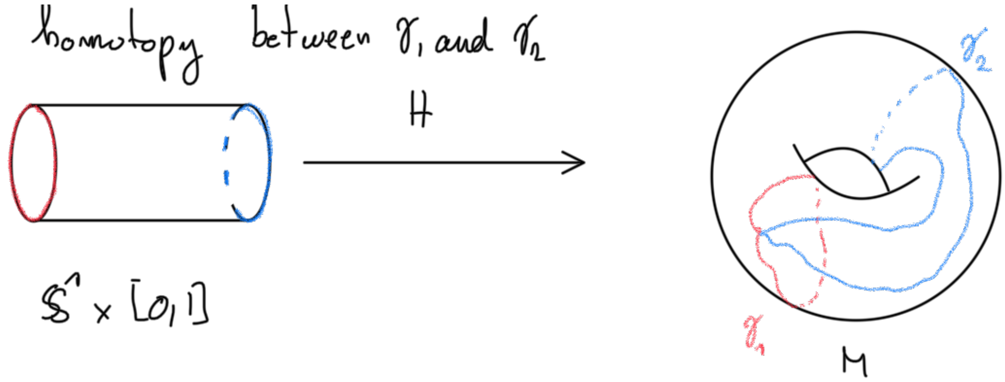
2) Let $\omega \in \Omega^1(M)$. Suppose that $\forall \gamma \in \pi_1(M)$, $\int_\gamma \omega = 0$. By 1), there exits $f \in \mathcal{C}^\infty(M)$ such that $\omega = df$. Hence, $[\omega] = 0 \in H^1_{dR}(M)$.

Conversly, suppose $\omega = df$. Let $\gamma : [0,1] \longrightarrow M$ be a loop (with $\gamma(0) = \gamma(1)$).

then $\int_\gamma \omega = \int_0^1 \omega_{\gamma(t)}(\gamma'(t)) dt = \int_0^1 df(\gamma(t)) \cdot \gamma'(t) dt = f(\gamma(1)) - f(\gamma(0)) = 0$. $\blacksquare$

3) a) Suppose that $\gamma_1 \sim \gamma_2$. Let $H : S^1 \times [0,1] \to M$

be an homotopy between γ_1 and γ_2



Then $S^1 \times [0,1]$ is an oriented manifold with oriented boundary $S^1 \times \{0\} - S^1 \times \{1\}$. Let $\omega \in \Omega^1(M)$, then by Stokes thm:

$$\int_{\gamma_1} \omega - \int_{\gamma_2} \omega = \int_{S^1} \gamma_1^* \omega - \int_{S^1} \gamma_2^* \omega = \int_{S^1 \times \{0\}} H^* \omega - \int_{S^1 \times \{0\}} H^* \omega = \int_{S^1 \times [0,1]} d(H^* \omega) = \int_{S^1 \times [0,1]} H^*(d\omega).$$

Hence, $\int_{\gamma_1} \omega = \int_{\gamma_2} \omega$ as soon as $d\omega = 0$.

It follows that $[\gamma] \longmapsto \int_\gamma \omega$ is welldefined on $\{\omega \in \Omega^1(M) \mid d\omega = 0\}$.

Since S^1 has no boundary, Stokes Theorem shows that $\int_\gamma \omega$ only depends on the cohomology class $[\omega]$, therefore,

$$h : H^1_{dR}(M) \longrightarrow \mathrm{Hom}(\pi_1(M), \mathbb{R})$$
$$[\omega] \longmapsto \left([\gamma] \mapsto \int_\gamma \omega \right)$$

is well defined. Since the integral is linear and $[\lambda\omega + \omega'] = \lambda[\omega] + [\omega']$, h is a linear map. ∎

b) This is a direct consequence of 1) and 2), since

$\ker h = \{ [\omega] \mid \forall \gamma, \int_\gamma \omega = 0 \}$ ∎

c) If M is contractible, then $\pi_1(M) \simeq \{0\}$, hence $\mathrm{hom}(\pi_1(M), \mathbb{R}) = \{0\}$.

Since h is injective, it follows that $H^1_{dR}(M) = \{0\}$. $\blacksquare$