Building Prescribed Quantitative Orbit Equivalence

Amandine Escalier

WWU Münster

Plan

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I — Context, motivations

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Def.

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Obstruction?
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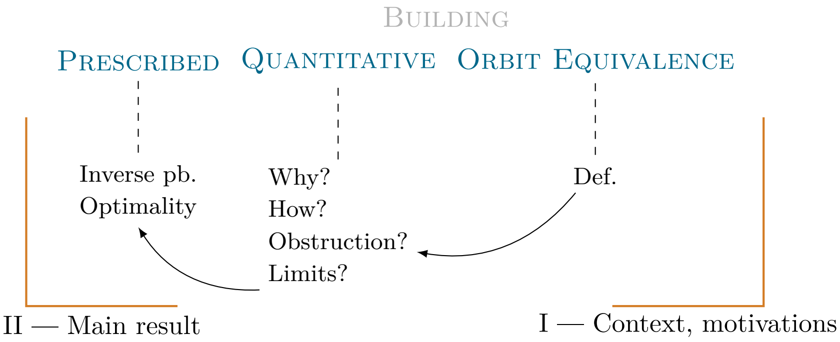
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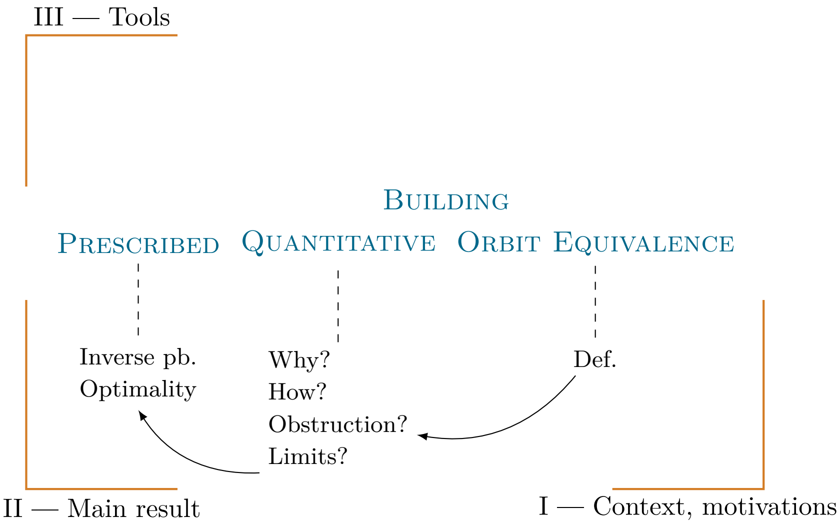
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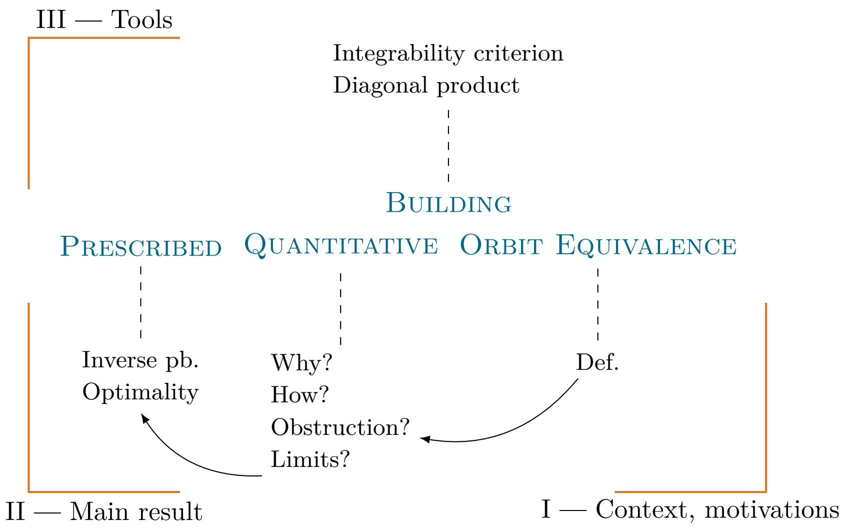
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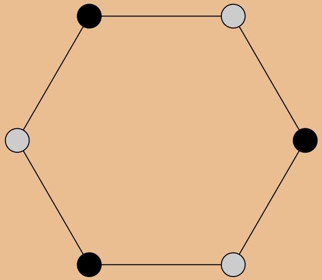
I — Quantitative Orbit Equivalence



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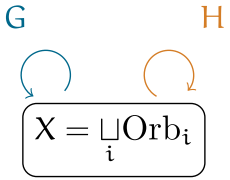

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OE is not responsive to geometry.
$\rightarrow$ **Refine** this relation to **distinguish** amenable groups.

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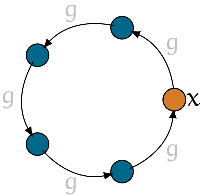

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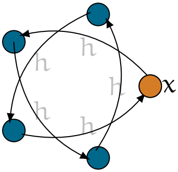

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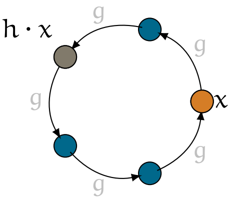

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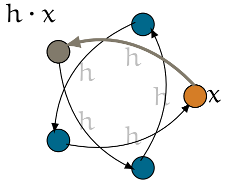

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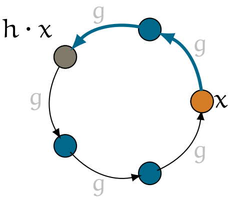

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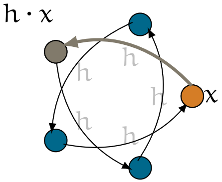

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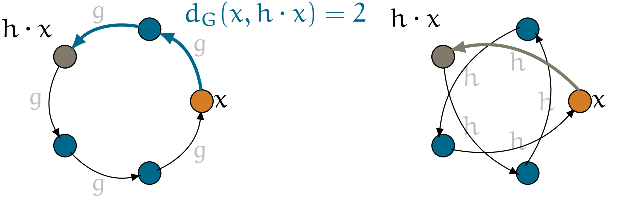

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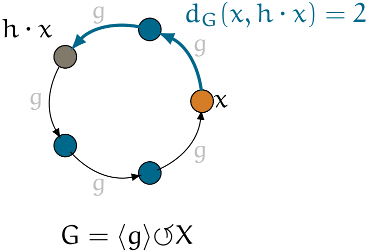

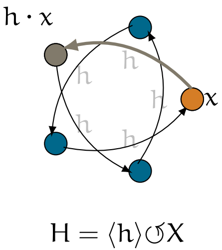

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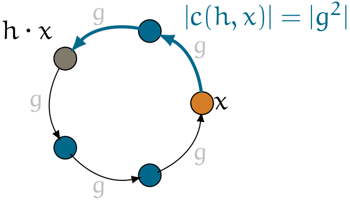

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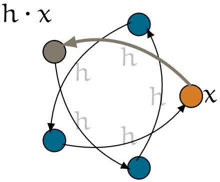

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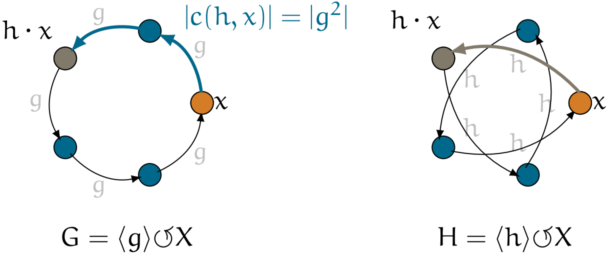

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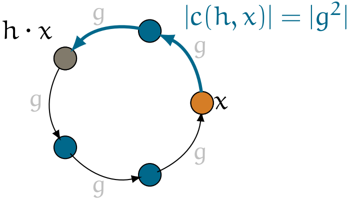

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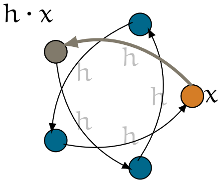

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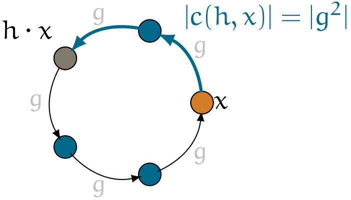

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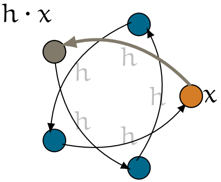

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Is $d_{S_\mathsf{G}}(x, h \cdot x)$ **finite**? $x \mapsto d_{S_\mathsf{G}}(x, h \cdot x)$ **bounded**? in $\mathbf{L}^p$?

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Definition. We say that we have a (L^p, L^q)**-integrable** OE coupling from G to H if for all $g \in G$ and $h \in H$

$$\int_X \left(d_{S_H}(x, g \cdot x)\right)^p d\mu < \infty \quad \int_X \left(d_{S_G}(x, h \cdot x)\right)^q d\mu < \infty.$$

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Let $\varphi, \psi : \mathbb{R}_+^* \to \mathbb{R}_+^*$ be two unbounded increasing functions.

[Delabie, Koivisto, Le Maître, Tessera, '20]

Definition. We say that we have a **(φ, ψ)-integrable** OE coupling from G to H if for all $g \in G$ and $h \in H$

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Abbreviation. (φ, ψ)-OE from G to H.

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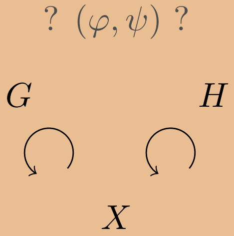
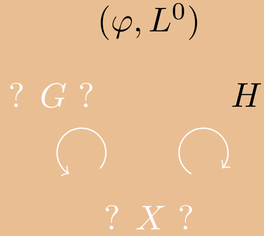
→ Is there an upper bound to the integrability of a coupling between two given groups?

II — Optimality and inverse problem





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Example

- If $\mathsf{G} = \mathbb{Z}^2$ and $\mathsf{H} = \mathbb{Z}$ then $I_{\mathsf{G}}(n) \sim n^{1/2}$, $I_{\mathsf{H}}(n) \sim n$.

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- We have an (L^p, L^0)-OE from $\mathbb{Z}^2$ to $\mathbb{Z}$ for all $p < 1/2$.

More generally: **Do there exist G and H admitting a (φ, L^0)-OE st. $\varphi \circ I_H \sim I_G$?**

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Fix a group H and an increasing map φ.

- Can we find G and a (φ, L^0)-OE from G to H,
- St. G has "optimal" profile, *ie.* $\varphi \circ I_{\mathsf{H}} \sim I_{\mathsf{G}}$?

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[1] $\varphi = \rho \circ \log$ st. $\rho : [1, +\infty) \to [1, +\infty)$ is continuous,

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Theorem. [E. '22] If φ *does not grow too fast*[1] then there exists a group G such that

- there exists a (φ, L^0)-integrable OE from G to $\mathbb{Z}$;
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[1] $\varphi = \rho \circ \log$ st. $\rho : [1, +\infty) \to [1, +\infty)$ is continuous, ρ and $x \mapsto x/\rho(x)$ are non-decreasing [BZ]

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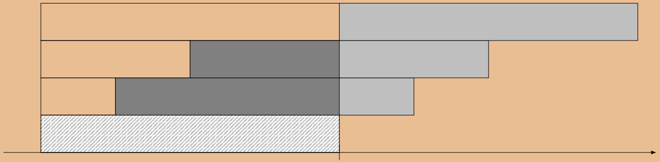
III — Building prescribed orbit equivalences



Reminder of the problem

Given φ and H find a group G and a (φ, L^0)-integrable OE from G to H, st $\varphi \circ I_H \sim I_G$.

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Diagonal products

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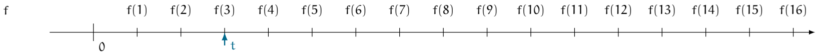
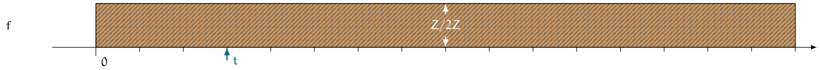
Sofic Approximations

III.2 — Diagonal products

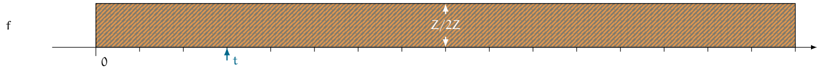


III.2 — Diagonal products



III.2 — Diagonal products

Let $(\Gamma_m)_m$ be a sequence of finite groups and let



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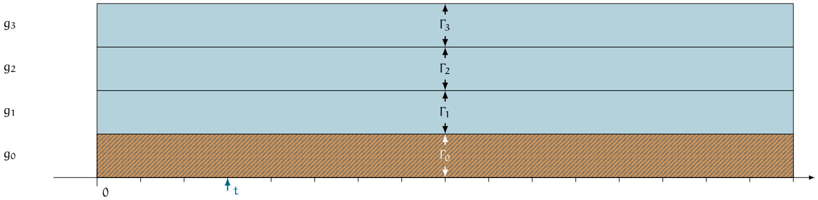

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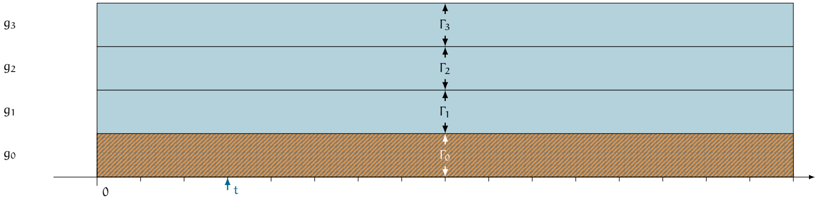
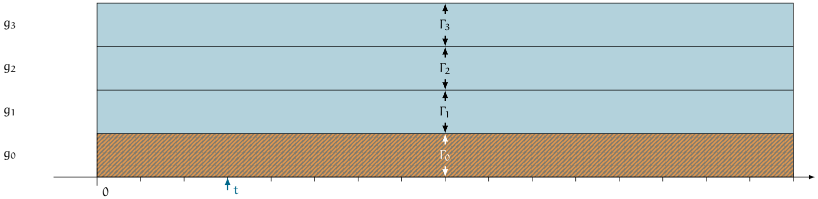


III.2 — Diagonal products

Let A, B be finite groups and $(\Gamma_m)_m$ st. $\Gamma_m = \langle A, B \rangle$ and $\Gamma_0 := A \times B$ and consider



III.2 — DIAGONAL PRODUCTS

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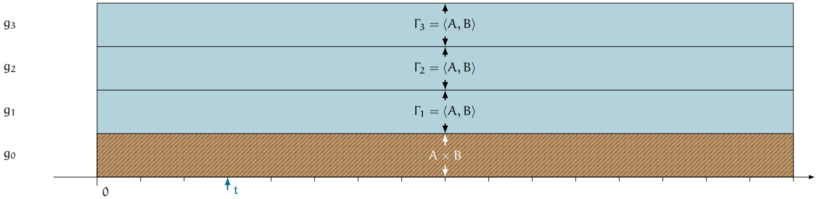
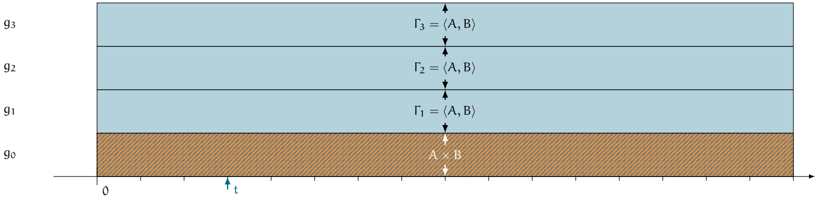


III.2 — Diagonal products

Introducing a parameter (Allow to control I_Δ.)

Let $(k_m)_{m\in\mathbb{N}}$ an increasing sequence

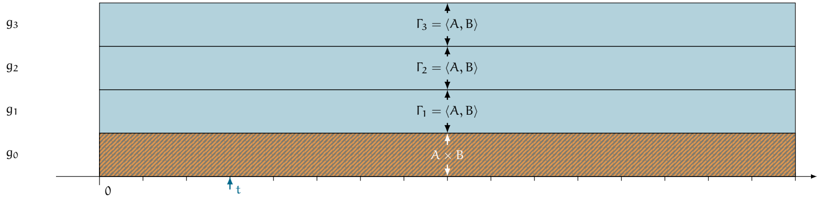


III.2 — Diagonal products

Introducing a parameter (Allow to control I_Δ.)

Let $(k_m)_{m\in\mathbb{N}}$ an increasing sequence

$$\Delta = \left\langle \Big((a\delta_0)_m, 0\Big), \Big((b\delta_{k_m})_m, 0\Big), (0,1) \,\Big|\, a \in A,\ b \in B \right\rangle .$$



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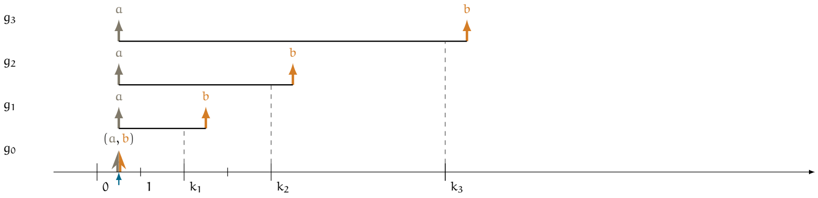

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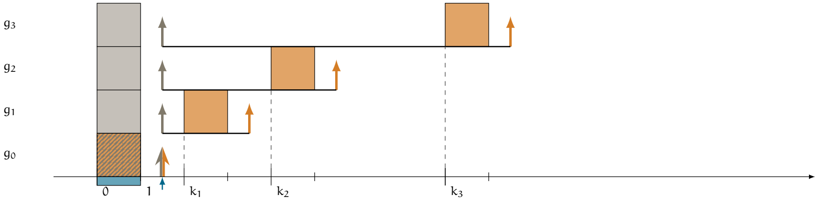

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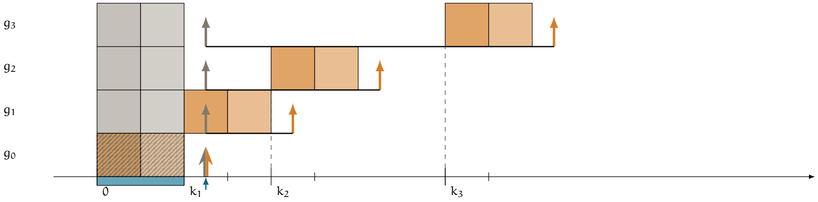

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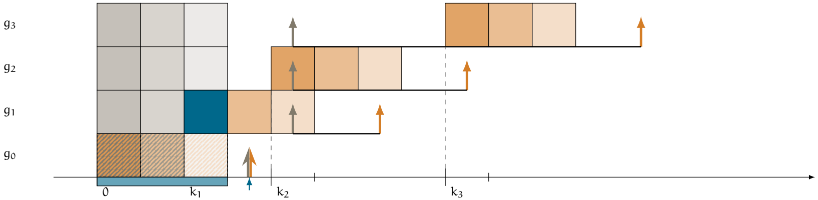

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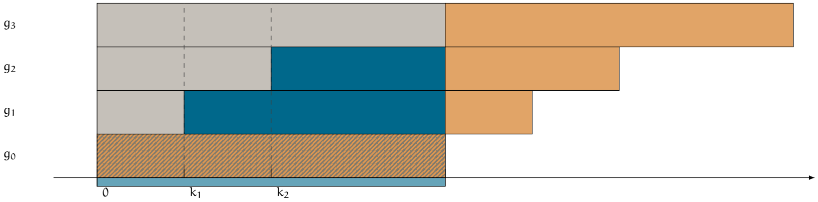

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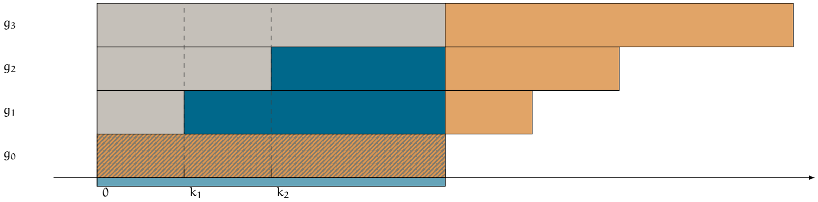
$\mathbf{range}(f,t) = \{\text{visited cursors}\}.$

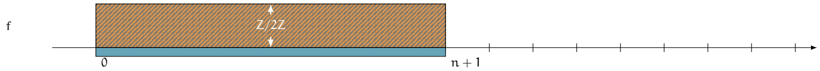


Go to coupling

III.2 — Diagonal products: Følner

Følner in a Lamplighter

$$F_n = \left\{(f,t) \mid t \in [0,n], \operatorname{supp}(f) \subseteq [0,n]\right\}$$



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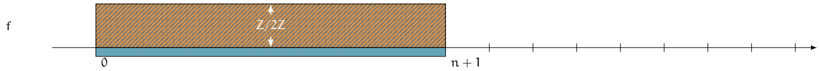

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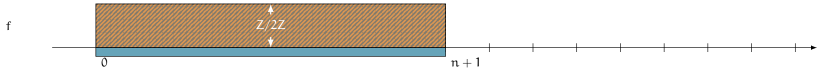
Set of elements obtained by allowing the lamplighter to move between 0 and n.

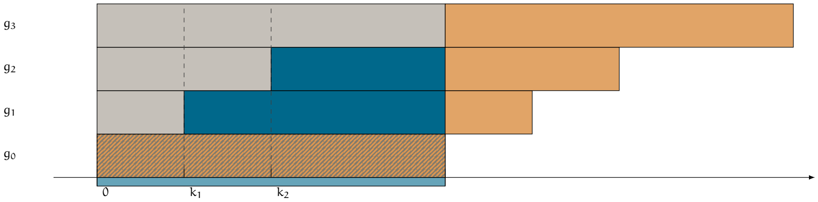


III.2 — Diagonal products: Følner

Følner in a Diagonal product

$$F_n = \left\{(f,t) \in \Delta \mid \text{range}(f,t) \in [0,n]\right\}$$

Set of elements obtained by allowing the lamplighter to move between 0 *and* n*.*



III.2 — Sofic Approximations

Let $(\mathcal{G}_n)_n$ be a Følner sequence for G and $(\mathcal{H}_n)_n$ Følner for H.

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Theorem. [Carderi et al. '23?] If for all n there exists a bijective map $\iota_n \colon \mathcal{G}_n \to \mathcal{H}_n$, and for every $s \in S_{\mathsf{G}}$ there exists $\delta > 0$ such that

$$\lim_{R\to\infty} \sup_n \sum_{r=0}^{R} \varphi(r) \frac{\left|\left\{x \in \mathcal{G}_n^{(1)} \mid d_{\mathcal{H}_n}(\iota_n(x), \iota_n(s\cdot x)) = r\right\}\right|}{|\mathcal{G}_n|} < \infty;$$

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Criterion involving distance and proportion.

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Criterion involving distance and proportion.

Rk One can replace Følner by Sofic approximation

III.3 — Coupling with $\mathbb{Z}$

Goal: Given φ, find G and a (φ, L^0)-OE from G to $\mathbb{Z}$.

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$$
\begin{array}{ccccc}
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 & & & & \downarrow \iota_n \\
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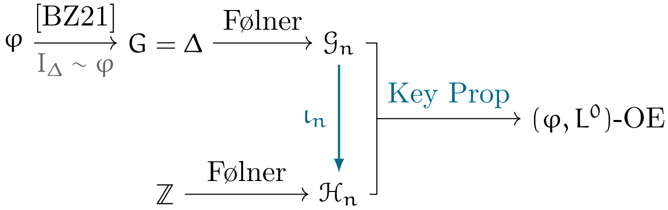
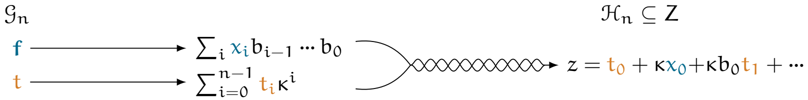
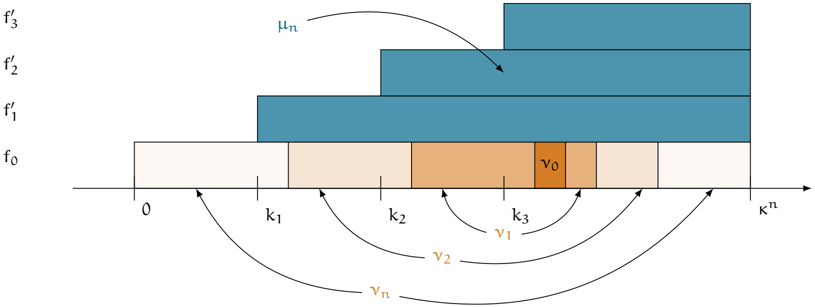


Thank you for your attention!

Numbering: general idea



Numbering: lamps configuration



Numbering: floor zero

Let $t \in [0, \kappa^n - 1]$ and $i \in \{0, \dots, n\}$ and define

$$\mathcal{B}(i)(t) := \left[\sum_{j=i}^{n-1} t_j \kappa^j, \sum_{j=i}^{n-1} t_j \kappa^j + \kappa^i - 1 \right].$$