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Géométries locale et asymptotique des groupes : Immeubles et LG-rigidité, équivalences orbitale et mesurée

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Doctoral Thesis in Mathematics

Local and Asymptotic Geometry of Groups

Buildings and LG-rigidity — Measure and Orbit Equivalence

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“

Did I make some mistakes?
Yes.
Did I only make mistakes?
Yes.
But did it all work out?
Kind of.

— **Samuel Barnett as Dirk Gently**
Dirk Gently's Holistic Detective Agency (Season 1, Episode 6), Max Landis

Pour Frédéric Benedetti,
Qui m'a montré la beauté des mathématiques,
Et s'est dit que je pourrais peut-être réussir quelques choses dans ce domaine.

Pour Fabien Caron,
Qui a entretenu mon émerveillement,
Et dans les pas de qui je suis allée sur les bancs de Jussieu.

AVANT-PROPOS

Cette thèse se compose de deux parties, pensées pour pouvoir être lues indépendemment. Elles sont précédées d'une courte introduction rappelant le contexte général. Si la deuxième partie expose les résultats de travaux en cours, la première partie correspond elle à un article intitulé « Rigidité Local-Globale des réseaux de $\mathsf{SL}_n(\mathbb{K})$ » [Esc20] (à paraître dans *Annales de l'Institut Fourier*). En dehors de l'introduction pour laquelle il y a deux versions (française et anglaise), le reste du manuscrit est rédigé en langue anglaise. Signalons enfin qu'un index des notations se trouve en fin de manuscrit à toutes fins utiles.

FOREWORD

This thesis is composed of two parts, thought and written to be read independently. A short introduction recalling the general context precedes them. The second part exposes results of ongoing work while the first corresponds to an article entitled "Local-to-Global Rigidity of lattices in $\mathsf{SL}_n(\mathbb{K})$" [Esc20] (to appear in *Annales de l'Institut Fourier*). The manuscript is written in english but the introduction has a french and an english version. Finally let us mention the presence of a notations index at the end of this manuscript for all practical purposes.

RÉSUMÉ

Ce manuscrit présente les travaux de recherche effectués durant ma thèse sur les questions de rigidité Locale-Globale et les problèmes d'équivalence mesurée.

Après une courte introduction, nous présentons dans la première partie les résultats obtenus sur la LG-rigidité et correspondant à l'article [Esc20].
Un graphe transitif $\mathcal{G}$ est dit *Locale-Globale rigide* s'il existe $R > 0$ tel que tout autre graphe dont les boules de rayon R sont isométriques aux boules de rayon R de $\mathcal{G}$ est revêtu par $\mathcal{G}$. Un example de tel graphe est donné par l'immeuble de Bruhat-Tits de $\mathsf{PSL}_n(\mathbb{K})$ lorsque $n \geqslant 4$ et $\mathbb{K}$ est un corps local non-archimédien de caractéristique nulle. Dans cette première partie nous étendons cette propriété de rigidité à une nouvelle classe de graphes quasi-isométriques à l'immeuble parmi lesquels figurent les réseaux sans-torsion de $\mathsf{SL}_n(\mathbb{K})$.
La preuve est l'occasion de démontrer un résultat sur la structure locale des immeubles. Nous montrons que si l'on y considère une $\mathsf{PSL}_n(\mathbb{K})$-orbite donnée, alors un sommet est uniquement déterminé par les sommets voisins contenus dans cette orbite.

Dans notre deuxième partie nous exposons les travaux (en cours) portant sur les équivalences orbitales et mesurées.
On dit que deux groupes sont *orbite équivalents* si tous deux admettent une action sur un même espace de probabilité qui partagent les mêmes orbites (à ensemble de mesure nulle près). Notamment, le théorème d'Ornstein et Weiss stipule que tout groupe infini moyennable est orbite équivalent au groups des entiers. Delabie, Koivisto, Le Maître et Tessera ont introduit une version quantitative de l'équivalence orbitale et de son pendant mesuré afin d'affiner cette relation au sein des groupes moyennables infinis. Ils obtiennent en outre des obstructions à l'existence de telles équivalences à l'aide du profil isopérimétrique.
Dans cette partie nous proposons de répondre au problème inverse de la quantification (trouver un groupe qui est orbite ou mesure équivalent à un groupe prescrit avec quantification prescrite) dans le cas du groupe des entiers ou du groupe d'allumeur de réverbère. Pour ce faire nous nous basons sur les produits diagonaux introduits par Brieussel et Zheng fournissant des groupes à profil isopérimétrique prescrit.

MOTS-CLÉS : graphe, rigidité, immeubles, corps local, LG-rigidité, équivalence orbitale, équivalence mesurée, couplage, produit diagonaux, profil isopérimétrique, pavage, suite de Følner, approximations Sofiques.

ABSTRACT

This manuscript presents the research work carried out during my thesis concerning LG-rigidity problems and orbit equivalence questions.

After a short general introduction, we expose in the first part of this manuscript the results obtained on LG-rigidity and corresponding to the article [Esc20]. A vertex-transitive graph $\mathcal{G}$ is called *Local-to-Global rigid* if there exists $R > 0$ such that every other graph whose balls of radius R are isometric to the balls of radius R in $\mathcal{G}$ is covered by $\mathcal{G}$. An example of such a graph is given by the Bruhat-Tits building of $\mathsf{PSL}_n(\mathbb{K})$ with $n \geqslant 4$ and $\mathbb{K}$ a non-Archimedean local field of characteristic zero. In this part we extend this rigidity property to a class of graphs quasi-isometric to the building including torsion-free lattices of $\mathsf{SL}_n(\mathbb{K})$.
The proof is the occasion to prove a result on the local structure of the building. We show that if we fix a $\mathsf{PSL}_n(\mathbb{K})$-orbit in it, then a vertex is uniquely determined by the neighbouring vertices in this orbit.

The second part presents (ongoing) work on orbit and measure equivalence.
We say that two groups are *orbit equivalent* if they both admit an action on a same probability space that share the same orbits (up to a set of measure zero). In particular the Ornstein-Weiss theorem implies that all infinite amenable groups are orbit equivalent to the group of integers. Delabie, Koivisto, Le Maître and Tessera introduced a quantitative version of orbit equivalence and its measured couterpart to refine this notion between infinite amenable groups. They furthermore obtain obstructions to the existence of such equivalences using the isoperimetric profile.
In this part we offer to answer the inverse problem (find a group being orbit or measure equivalent to a prescribed group with prescribed quantification) in the case of the group of integers or of the lamplighter group. To do so we use the diagonal products introduced by Brieussel and Zheng giving groups with prescribed isoperimetric profile.

KEYWORDS: graph, rigidity, building, local field, LG-rigidity, orbit equivalence, measure equivalence, couplings, diagonal products, isoperimetric profile, tiling, Følner sequence, Sofic approximation.

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"

Le café, les tapis
Les bongos, les losanges, les années, les lilas
Merci nature d'être là, super sympa.
Merci nature d'être là, super sympa.
Merci nature d'être là, super sympa.

— **Salut c'est cool**
Merci nature

CONTENTS

INTRODUCTION (VERSION FRANÇAISE)

> “
>
> *Une des causes principales de la misère dans les sciences est qu'elles se croient riches, le plus souvent présomptueusement. Leur but n'est pas d'ouvrir une porte à la sagesse infinie mais de poser une limite à l'erreur infinie.*
>
> — **Bertolt Brecht**
> *La vie de Galilée*

Si l'on vous donne un objet, ou disons plutôt un sujet, quel qu'il soit —une petite cuillère, un ficus, un chaton— et que l'on vous demande d'étudier sa structure, plusieurs manières de procéder s'offrent à vous. Une première possibilité est l'approche macroscopique du sujet : on peut chercher à comprendre sa forme, sa couleur, son volume… On considère l'objet dans sa *globalité*. Un autre moyen de procéder serait d'étudier la structure microscopique du sujet, en déterminant par exemple sa composition chimique, sa structure moléculaire, son organisation cellulaire… On considère alors le sujet à une échelle beaucoup plus *locale*. Bien que basées sur deux points de vues différents, ces deux méthodes sont non-seulement complémentaires, mais peuvent aussi s'entrecroiser : si l'on sait que la forme d'un objet est bien définie, on peut en déduire que sa structure moléculaire est plus proche de celle d'un solide que d'un liquide ; si l'on sait que les cellules du sujet comportent une paroi cellulaire, on peut en déduire qu'il s'agit d'une plante verte plutôt que d'un mammifère.

Entrecroiser ces approches est ce que nous proposons de faire dans cette thèse en prenant des *groupes* pour sujets d'observation. À l'échelle macroscopique, nous cherchons à décrire les groupes à travers leur géométrie « asymptotique » ou « à grande échelle » ; à l'échelle microscopique nous verrons que la donnée de l'organisation moléculaire sur quelques millimètres cubes du groupe peut nous permettre de déduire des informations sur la forme de celui-ci. Ces deux points de vue et les outils auxquels nous faisons appel inscrivent ainsi cette thèse à l'interface des théories *géométrique, ergodique* et *mesurée* des groupes.

La première de ces trois théories a pour objet l'étude des groupes *via* leurs actions sur des espaces géométriques ou topologiques. En observant la manière qu'a un groupe d'agir sur les éléments de l'espace, de préserver ou non les distances, nous pouvons en déduire des informations sur la structure algébrique du groupe. Dans le cas fort amène des groupes de type fini on peut de plus voir le groupe lui-même comme espace géométrique en considérant son *graphe de Cayley*. Rappelons que si S_G est une partie génératrice finie (symétrique) de G, le graphe de Cayley (G, S_G) est le graphe dont les sommets sont les éléments de G et dont l'ensemble des arêtes est donné par $\{(g, sg) \mid g \in G, s \in S_G\}$.

Un tel graphe est alors muni d'une distance naturelle fixant à 1 la longueur d'une arête. À l'aune de la précédente définition, le graphe de Cayley obtenu et la métrique correspondantes dépendent donc fortement du choix de partie génératrice S_G considérée. Néanmoins deux graphes de Cayley différents partagent certaines caractéristiques géométriques. Plus exactement, deux graphes de Cayley ont même *géométrie à grande échelle ;* c'est ce que l'on formalise à l'aide de la notion de quasi-isométrie.

Définition 1

> Soient (X, d_X) et (Y, d_Y) deux espaces métriques. On dit qu'une application f définie de X vers Y est une *quasi-isométrie* s'il existe $L \geqslant 1$ et $\varepsilon > 0$ tels que pour tout $x, x' \in X$
>
> $$\frac{1}{L} d_X(x, x') - \varepsilon \leqslant d_Y(f(x), f(x')) \leqslant L d_X(x, x') + \varepsilon,$$
>
> et pour tout $y \in Y$ il existe $x \in X$ tel que $d_Y(y, f(x)) \leqslant L$.

Dans l'un de ses travaux fondateurs de la théorie géométrique des groupes, Gromov [GNR93] soulève le problème de la classification des groupes selon leur géométrie à grande échelle —ou « à quasi-isométrie près ». La recherche et l'étude d'invariants de quasi-isométrie tels que le type de croissance d'un groupe ou le profil isopérimétrique (voir Definition 6.1.12) ont mené à de remarquables résultats tels que le théorème de Gromov sur les groupes à croissance polynomiale ou la classification des réseaux irréductibles dans les groupes de Lie semi-simples (voir [Far97]). Mais en plus de poser les fondements de ce qu'est la théorie géométrique des groupes, Gromov, faisant fi des barrières entre domaines, ouvre un pont vers une autre branche des mathématiques en présentant la notion suivante comme le pendant *mesuré* de la quasi-isométrie.

Définition 2

> Deux groupes de type fini G et H sont *mesure équivalents* s'il existe des actions libres, préservant la mesure et qui commutent de G et H sur un espace mesuré (X, μ) telles que chaque action admette un domaine fondamental de mesure fini.

Nous développerons plus en détails cette notion d'équivalence mesurée dans le chapitre 6, mais poursuivons d'abord notre exploration mathématique et cheminons vers les terres de l'étude dynamique des groupes. Car parallèlement à la théorie mesurée de ces derniers et sous l'impulsion des travaux fondateurs de Dye [Dye59, Dye63], ces contrées ont vu naître l'analogue *ergodique* de l'équivalence mesurée.

Définition 3

> On dit que deux groupes de type fini G et H sont *orbite équivalents* s'il existe un espace de probabilité (X, μ) et des actions libres de G et H sur X préservant la mesure telles que pour presque tout $x \in X$ on ait $G \cdot x = H \cdot x$.

Arrêtons-nous quelques instants et tournons-nous à nouveau vers le rivage de la théorie géométrique des groupes. Nous avons vu que la relation de quasi-isométrie traduisait le fait d'avoir même géométrie à grande échelle et que nombre d'invariants existaient. Qu'il s'agisse du type de croissance ou du profil isopérimétrique, ces fonctions caractérisent certaines propriétés du groupe dont elles sont issues et sont préservées par les quasi-isométries. Mais sur la berge de la théorie ergodique, l'équivalence orbitale procède avec bien moins de délicatesse en écrasant des familles entières de groupes sur une seule et même classe d'équivalence. Une fulgurante illustration de cette rustrerie nous est donnée par le théorème d'Ornstein et Weiss [OW80] qui prouve que tout groupe infini moyennable est orbite équivalent à $\mathbb{Z}$. Dès lors s'impose la nécessité d'affiner cette relation d'équivalence orbitale. Pour ce faire Delabie, Koivisto, Le Maître et Tessera proposent dans [DKLMT20] une définition *quantifiée* des équivalences orbitale et mesurée, en évaluant l'intégrabilité des applications distances (voir ci-dessous) définies sur les graphes de Schreier associés aux actions. Rappelons que si S_G est une partie génératrice finie d'un groupe G agissant sur un espace X, le *graphe de Schreier* associé à cette action est le graphe dont les sommets sont les éléments de X et dont l'ensemble des arêtes est donné par $\{(x, s \cdot x) \mid x \in X, s \in S_G\}$. On peut munir ce graphe de la distance usuelle d_{S_G} fixant à 1 la longueur d'une arête.

Supposons maintenant pour simplifier que nous sommes dans le cas d'une équivalence orbitale entre deux groupes G et H. Afin de mesurer la proximité de ces deux actions, nous pouvons étudier pour tout $g \in S_G$ et tout $h \in S_H$ les *applications distance* associées :

$$x \mapsto d_{S_G}(x, h \cdot x) \quad x \mapsto d_{S_H}(x, g \cdot x).$$

Plus précisément, Delabie et al. [DKLMT20] proposent de mener cette étude en mesurant l'intégrabilité desdites applications distances. Cependant, plutôt que de se contenter d'intégrabilités L^p où $p \in [0, +\infty]$, les quatre auteurs apportent une plus grande finesse dans leur quantification en considérant la (φ, ψ)-*intégrabilité*.

Définition 4

> Soient G et H deux groupes de types finis orbit équivalents et $\varphi, \psi : (0, +\infty) \to (0, +\infty)$ deux fonctions croissantes non-bornées. Notons (X, μ) un espace sur lequel G et H partagent les mêmes orbites. On dit que l'on a un *couplage* (φ, ψ)-*intégrable* si pour tout $g \in S_G$ (resp. $h \in S_H$) il existe c_g (resp. c_h) tel que
>
> $$\int_{X/H} \varphi\left(\frac{1}{c_g} d_{S_H}(x, g \cdot x)\right) d\mu(x) < +\infty$$
> $$\int_{X/G} \psi\left(\frac{1}{c_h} d_{S_G}(x, h \cdot x)\right) d\mu(x) < +\infty.$$

C'est autour de ces notions que s'organise notre Partie ii où nous cherchons à répondre au problème dit « inverse » de la quantification : à groupe H et intégrabilité (φ, ψ) prescrits existe-t-il un groupe G admettant un couplage (φ, ψ)-intégrable avec H ? Pour nous aider dans cette quête, les produits diagonaux introduits par Brieussel et Zheng dans [BZ21] s'avéreront de précieux alliés et nous permettront de répondre à la précédente question lorsque $H = \mathbb{Z}$ (Théorème 9) et $H = \mathbb{Z}/q\mathbb{Z} \wr \mathbb{Z}$ (Théorème 10).

Ainsi étudions nous les groupes à l'échelle macroscopique — pour ne pas dire *asymptotique* — depuis les rives des théories ergodiques et mesurée des groupes, à l'aide des versions quantifiées des équivalences susmentionnées. Mais nous pouvons aussi nous demander ce qu'il advient lorsque l'on adopte le point de vue opposé. En effet, plutôt que d'étudier des objets selon leur géométrie à grande échelle, nous pouvons chercher à savoir si les propriétés *locales* d'un objet peuvent avoir des conséquences *globales* sur sa géométrie. C'est le point de vue adopté dans notre Partie i où notre étude des graphes et groupes se rapproche d'une observation au microscope de ces objets. Plus précisément, nous étudions les graphes en considérant et comparant leurs boules de rayon R, pour un R fixé.

Définition 5

> Soit $R > 0$. Deux graphes transitifs sont dits *R-localement* les mêmes si leurs boules de rayon R sont isométriques.

Il est alors naturel de demander s'il peut y avoir des conséquences à plus large échelle de cette locale similarité. Autrement dit, la trame locale d'un graphe peut-elle contraindre sa structure globale ? Parmi les premiers à s'être intéressés à cette question, Benjamini [Ben13] et Georgakopoulos [Geo17] formalisent cette propriété sous le nom de *rigidité Locale-Globale* ou *LG-rigidité*.

Définition 6

> On dit qu'un graphe transitif X est *Locale-Globale rigide* s'il existe $R > 0$ tel que tout graphe R-localement X est revêtu par X.

De nombreux exemples de tels graphes — détaillés au chapitre 1 — existent, parmi lesquels les fameux et non moins fascinants *immeubles de Bruhat-Tits*. En effet de la Salle et Tessera [dlST16] ont montré que pour tout n plus grand que 4 et tout corps local non-archimédien $\mathbb{K}$ de caractéristique nulle, l'immeuble de Bruhat-Tits de $\mathsf{PSL}_n(\mathbb{K})$ était LG-rigide (voir Chapitre 2 pour les définitions). Dans ce manuscrit nous montrons que la rigidité de ces immeubles va encore plus loin, en prouvant qu'une information locale *partielle* suffit à reconstruire de tels graphes. Nous introduisons la notion *d'empreinte* d'un sommet de l'immeuble qui désigne l'intersection du 1-voisinage du sommet avec une $\mathsf{PSL}_n(\mathbb{K})$-orbite fixée, et montrons que cette empreinte caractérise le sommet (voir Théorème 8). Nous utilisons alors cette propriété pour étendre la LG-rigidité à une nouvelle classe de graphes quasi-isométriques à l'immeuble (Théorème 7), parmi lesquels figurent les réseaux sans-torsion de $\mathsf{SL}_n(\mathbb{K})$.

RÉSULTATS PRINCIPAUX

Nous regroupons ici les résultats principaux démontrés dans cette thèse. Leurs énoncés ainsi qu'une plus complète mise en contexte peuvent-être retrouvés dans les chapitres 1 (pour la rigidité Locale-Globale) et 6 (pour les équivalences orbitale et mesurée).

Rigidité Locale-Globale et immeubles

Le résultat principal de la Partie i est le Théorème 1.2.6 que nous rappelons ci-dessous.

Théorème 7

> Soient $n \neq 3$ et $\mathbb{K}$ un corps local (non nécessairement commutatif) non-archimédien de caractéristique zéro. Soit $\mathcal{X}$ l'immeuble de Bruhat-Tits de $\mathsf{PSL}_n(\mathbb{K})$ et X un graphe transitif. Si
> - il existe un morphisme injectif ρ de $\mathrm{Isom}(X)$ vers $\mathrm{Isom}(\mathcal{X})$ tel que $\rho(\mathrm{Isom}(X))$ est d'indice fini dans $\mathrm{Isom}(\mathcal{X})$;
> - il existe une quasi-isométrie injective et $\mathrm{Isom}(X)$-équivariante de X vers $\mathcal{X}$;
>
> alors X est LG-rigide.

Nous en déduisons en particulier que les réseaux sans torsion de $\mathsf{SL}_n(\mathbb{K})$ sont LG-rigides (voir Théorème 1.2.5). La démonstration du théorème ci-dessus repose sur une étude approfondie de la structure locale des immeubles de Bruhat-Tits. Si $\mathcal{C}$ est une $\mathsf{PSL}_n(\mathbb{K})$-orbite de l'immeuble $\mathcal{X}$, nous définissons *l'empreinte* de type $\mathcal{C}$ d'un sommet x comme

$$\mathcal{P}_{\mathcal{C}}(x) := B_{\mathcal{X}}(x, 1) \cap \mathcal{C}.$$

Nous montrons alors dans la propriété 2.3.4 (que l'on rappelle ci-dessous), que cette empreinte caractérise le sommet.

Propriété 8

> Soit $\mathcal{X}$ l'immeuble de Bruhat-Tits de $\mathsf{PSL}_n(\mathbb{K})$ et $\mathcal{C}$ une $\mathsf{PSL}_n(\mathbb{K})$-orbite de $\mathcal{X}$. Notons x et y deux sommets de $\mathcal{X}$. Si $\mathcal{P}_{\mathcal{C}}(x) = \mathcal{P}_{\mathcal{C}}(y)$ alors $x = y$.

Équivalences orbitale et mesurée

La deuxième partie de cette thèse se focalise sur la construction d'équivalence mesurée ou orbitale à l'intégrabilité prescrite et comporte deux résultats principaux. Le premier (Théorème 6.3.1) concerne l'existence d'une équivalence mesurée avec $\mathbb{Z}$.

Théorème 9

> Pour toute fonction croissante $\rho : [1, +\infty[\to [1, +\infty[$ telle que $\rho(1) = 1$ et $x/\rho(x)$ est croissante, il existe un groupe G tel que
> - $I_G \simeq \rho \circ \log$
> - Il existe une équivalence orbitale de G vers $\mathbb{Z}$ qui est $(\varphi_\varepsilon, \exp \circ \rho)$-intégrable pour tout $\varepsilon > 0$, où $\varphi_\varepsilon(x) := \rho \circ \log(x)/(\log \circ \rho \circ \log(x))^{1+\varepsilon}$.

Le second résultat établit l'existence d'un couplage sous-groupe mesuré (*cf.* Définition 6.1.1) à intégrabilité prescrite avec le groupe d'allumeur de réverbère $L_q := \mathbb{Z}/q\mathbb{Z} \wr \mathbb{Z}$.

Théorème 10

> Pour tout $\alpha > 0$ il existe un groupe G tel que
> - $I_G(n) \simeq \log(n)^{1/(1+\alpha)}$
> - si $\varepsilon > 0$ et l'on définit $\varphi_\varepsilon(x) := x^{\frac{1}{1+\alpha+\varepsilon}}$ alors il existe un couplage sous-groupe mesuré de G vers $\mathbb{Z}$ qui est φ_ε-intégrable.

ORGANISATION DU MANUSCRIT

Après cette brève introduction historique, le manuscrit se découpe en deux parties, prévues pour être lues indépendamment. Chaque partie comporte notamment un introduction détaillée au sujet qu'elle traite et se termine par une conclusion sur des problèmes ouverts. La Partie i concerne les immeubles et problèmes de rigidité Locale-Globale. Les Théorème 7 et Propriété 8 y sont notamment prouvés. Ces résultats ont donné lieu à une publication [Esc20] à paraître dans *Annales de l'Institut Fourier*. La Partie ii se consacre aux équivalences orbitales et mesurées. Nous y montrons en particulier les Théorèmes 9 et 10. Cette partie concerne un travail en cours, les suites espérées de ces travaux sont évoquées en conclusion. Enfin, une annexe regroupe certains résultats sur les produits diagonaux et un index des notations.

INTRODUCTION (ENGLISH VERSION)

> *Once you've decided that something's absolutely true, you've closed your mind on it, and a closed mind doesn't go anywhere. Question everything. That's what education's all about.*
>
> — David Eddings
> *Belgarath the Sorcerer*

If you are given an object, or rather a subject, whatever it may be —a spoon, a ficus, a kitten— and you are asked to study its structure, there are several ways to proceed. A first possibility is to choose the macroscopic approach: you may try to understand the shape, color or volume of the subject. You consider the object *globally*. Another way to proceed would be to study the microscopic structure of the subject by identifying its chemical composition, its molecular structure or its cellular organisation for example. You consider then the subject at a far more *local* scale. Although based on two different points of view, these two methods are not only complementary, they also intertwin: if you know that the shape of an object is well defined you can deduce that its molecular structure is closer to the one of a solid than the one of a liquid; if you know that the cells of your subject have a cell wall you can deduce that the subject is a plant rather than a mammal.

Intertwining these approaches is what we propose to do in this thesis by taking *groups* as studied subjects. At a macroscopic scale we try to describe groups through their "asymptotic" (or "large scale") geometry; at a microscopic scale we will see that the data of the molecular organisation of a few cubic millimetres of the group will allow us to deduce informations about its shape. These two points of view and the tools we use put this thesis at the crossroads of *geometric, ergodic* and *measured* groups theories.

The first of these three theories is devoted to the study of groups *via* their actions on geometric or topological spaces. By observing how a group acts on the elements of the space or if it preserves or not distances, we can deduce informations on the algebric structure of the group. In the notably friendly framework of finitely generated groups we can furthermore see the group itself as a geometric space by considering its *Cayley graph*. Recall that if S_G is a finite (symmetric) generating set of G then the Cayley graph (G, S_G) is the graph whose vertices are the elements of G and whose set of edges is given by $\{(g, sg) \mid g \in G, s \in S_G\}$. Such a graph is endowed with a natural distance fixing to 1 the length of an edge. In the light of the above definition, the obtained Cayley graph and its corresponding metric thus depend on the choice of generating set S_G. Nonetheless, two different Cayley graphs share the same *large scale geometry;* this is what we formalise with the notion of quasi-isometry.

Definition 1

> Let (X, d_X) and (Y, d_Y) be two metric spaces. We say that a map f from X to Y is a *quasi-isometry* if there exist $L \geqslant 1$ and $\varepsilon > 0$ such that for all $x, x' \in X$
>
> $$\frac{1}{L} d_X(x, x') - \varepsilon \leqslant d_Y\left(f(x), f(x')\right) \leqslant L d_X(x, x') + \varepsilon,$$
>
> and for all $y \in Y$ there exists $x \in X$ such that $d_Y(y, f(x)) \leqslant L$.

In one of his seminal works on geometric group theory, Gromov [GNR93] brings the matter of the classification of groups according to their large scale geometry —or "up to quasi-isometry". Research and study of quasi-isometry invariants such as the growth type of a group or the isoperimetric profile (see Definition 6.1.12) lead to remarkable results such as the Gromov theorem on groups of polynomial growth or the classification of irreducible lattices in semi-simple Lie groups [Far97]. But in his work Gromov does not just set the foundations of geometric group theory: ignoring boundaries between mathematical areas, he also opens a bridge to another field by presenting the following notion as the *measured* counterpart of quasi-isometry.

Definition 2

> Two finitely generated groups G and H are *measure equivalent* if there exist two free commuting measure preserving actions of G and H on a measured space (X, μ) such that each action admits a fundamental domain of finite measure.

We will give more details about this measure equivalence notion in Chapter 6 but let us first continue our mathematical exploration and journey to the territory of dynamical group study. Indeed in parallel with the measure group theory and under the impulsion of seminal works of Dye [Dye59, Dye63] these lands saw the emergence of the *ergodic* counterpart of measure equivalence.

Definition 3

> We say that two finitely generated groups G and H are *orbit equivalent* if there exist a probability space (X, μ) and free, measure preserving actions of G and H on X such that for almost every $x \in X$ we have $G \cdot x = H \cdot x$.

Let us stop here for a few moments and turn again to the shore of geometric group theory. We saw that being quasi-isometric could be interpreted as having the same large-scale geometry. We also mentioned the existence of several invariants. Whether it is the growth type or the isoperimetric profile, these maps characterize some properties of the group from which they derive and are both preserved by quasi-isometries. But on the riverside of ergodic theory, proceeds orbit equivalence with far less delicacy: it crushes entire families of groups on one same equivalence class. A dazzling illustration of this ruthlessness is given by the Ornstein-Weiss theorem [OW80] which proves that all infinite amenable group is orbit equivalent to $\mathbb{Z}$. From then arises the need of a refined notion of orbit equivalence. To do so Delabie, Koivisto, Le Maître and Tessera offer in [DKLMT20] a *quantified* version of orbit and measure equivalence. They propose to estimate the integrability of the distance maps (see below) defined on the Schreier graphs associated to the actions. Recall that if S_G is a finite generating set of a group G acting on a space X then the *Schreier graph* associated to this action is the graph whose vertices are the elements of X and whose set of edges is given by $\{(x, s \cdot x) \mid x \in X, s \in S_G\}$. Such a graph is endowed with the usual distance d_{S_G} fixing to 1 the length of an edge.

Assume for simplicity that we are given an orbit equivalence between two finitely generated groups G and H. In order to measure how close the two actions are, we can study for all $g \in S_G$ and $h \in S_h$ the associated *distance maps:*

$$x \mapsto d_{S_G}(x, h \cdot x) \quad x \mapsto d_{S_H}(x, g \cdot x).$$

More precisely, Delabie et al. [DKLMT20] offer to proceed by measuring the integrability of the aforementioned distance maps. However, rather than merely consider L^p-integrabilities with $p \in [0, +\infty]$, the four authors bring a greater precision in their quantification by considering what they call *φ-integrability*.

Definition 4

> Let G and H be two finitely generated, orbit equivalent groups and $\varphi, \psi : (0, +\infty) \to (0, +\infty)$ two non-decreasing unbounded maps. Denote by (X, μ) a space where G and H share the same orbits. We say that we have a (φ, ψ)-*integrable coupling* if for all $g \in S_G$ (resp. $h \in S_H$) there exists c_g (resp. c_h) such that
>
> $$\int_{X/H} \varphi\left(\frac{1}{c_g} d_{S_H}(x, g \cdot x)\right) d\mu(x) < +\infty$$
> $$\int_{X/G} \psi\left(\frac{1}{c_h} d_{S_G}(x, h \cdot x)\right) d\mu(x) < +\infty.$$

This is around these notions that revolves our Part ii. We try yo answer to the "inverse" quantification problem: to a given group H and prescribed integrability (φ, ψ), can one find a group G admitting a (φ, ψ)-integrable coupling with H? To help us during this quest, the diagonal products introduced by Brieussel and Zheng in [BZ21] will be invaluable allies and will allow us to answer the preceding question when $H = \mathbb{Z}$ (Theorem 9) and $H = \mathbb{Z}/q\mathbb{Z} \wr \mathbb{Z}$ (Theorem 10).

We thus study groups at the macroscopic scale —or *asymptotic* scale— from the shore of ergodic and measure group theories using quantified versions of the above-mentioned equivalences. But one can also adopt the opposite perspective. Indeed, instead of asking whether an object is determined by its coarse geometry, one can ask whether *local* properties of an object can have *global* implications for its geometry. This is the point of view we adopt in Part i where our study of graphs and groups is a kind of microscope observation of these objects. To be more precise, we study graphs by considering and comparing their balls of radius R, with fixed $R > 0$.

Definition 5

> Let $R > 0$. Two transitive graphs are said to be *R-locally the same* if their balls of radius R are isometric.

It is then natural to ask whether there are large scale consequences of this local similarity. That is to say, can the local weft of a graph constrain its global structure? Among the first to consider this question, Benjamini [Ben13] and Georgakopoulos [Geo17] formalise this property under the name of *Local-to-Global rigidity* also called *LG-rigidity*.

Definition 6

> A transitive graph X is said to be *Local-to-Global rigid* if there exists $R > 0$ such that any graph being R-locally the same as X is covered by X.

There are numerous examples of such graphs (we detail them in Chapter 1) including the famous and utterly fascinating *Bruhat-Tits buildings*. Indeed de la Salle and Tessera [dlST16] showed that for all n greater than 4 and all non-Archimedean local skew field $\mathbb{K}$ of characteristic zero, the Bruhat-Tits building of $\mathsf{PSL}_n(\mathbb{K})$ is LG-rigid (see Chapter 2 for the definitions). In this manuscript we show that the rigidity of these buildings is even stronger by proving that only a *partial* local information is enough to reconstruct such graphs. We introduce the notion of *print* of a vertex in the building which corresponds to the intersection of the 1-neighbourhood of the vertex with a given $\mathsf{PSL}_n(\mathbb{K})$-orbit and prove that this print characterizes the vertex (see Proposition 8). We then use this property to extend this LG-rigidity property to a new class of graphs quasi-isometric to the building (see Theorem 7) including the torsion free-lattices of $\mathsf{SL}_n(\mathbb{K})$.

MAIN RESULTS

We gather here the main results of this manuscript. Their statements and a more complete contextualisation can be found in Chapter 1 (for the Local-to-Global rigidity) and Chapter 6 (for orbit and measure equivalences).

Local-to-Global rigidity and buildings

The main result of Part i is Theorem 1.2.6 which we recall below.

Theorem 7

> Let $n \neq 3$ and $\mathbb{K}$ be a non-Archimedean local skew field of characteristic zero. Let $\mathcal{X}$ be the Bruhat-Tits building of $\mathsf{PSL}_n(\mathbb{K})$ and X a transitive graph. If X verifies that
> - There is an injective homomorphism ρ from $\mathrm{Isom}(X)$ to $\mathrm{Isom}(\mathcal{X})$ such that $\rho(\mathrm{Isom}(X))$ is of finite index in $\mathrm{Isom}(\mathcal{X})$;
> - There is a $\mathrm{Isom}(X)$-equivariant injective quasi-isometry q from X to $\mathcal{X}$;
>
> then X is SLG-rigid.

In particular we deduce from this result the LG-rigidity of torsion free lattices in $\mathsf{SL}_n(\mathbb{K})$ (see Theorem 1.2.5). The proof of the above theorem relies on a detailed study of the local structure of buildings. If $\mathcal{C}$ is a $\mathsf{PSL}_n(\mathbb{K})$-orbit in $\mathcal{X}$, we define the *print* of type $\mathcal{C}$ of a vertex x as

$$\mathcal{P}_{\mathcal{C}}(x) := B_{\mathcal{X}}(x, 1) \cap \mathcal{C}.$$

We show in Proposition 2.3.4 (which we recall below) that this print characterizes the vertex.

Proposition 8

> Let $\mathcal{X}$ be the Bruhat-Tits building of $\mathsf{PSL}_n(\mathbb{K})$ and $\mathcal{C}$ be a $\mathsf{PSL}_n(\mathbb{K})$-orbit of $\mathcal{X}$. Denote by x and y two vertices of $\mathcal{X}$. If $\mathcal{P}_{\mathcal{C}}(x) = \mathcal{P}_{\mathcal{C}}(y)$ then $x = y$.

Orbit and measure equivalences

The second part of this memoir deals with the contruction of orbit or measure equivalences with prescribed integrability and contains two main results. The first one (Theorem 6.3.1) concerns the existence of a measure equivalence with $\mathbb{Z}$.

Theorem 9

> For all non-decreasing function $\rho : [1, +\infty[\to [1, +\infty[$ such that $\rho(1) = 1$ and $x/\rho(x)$ is non-decreasing, there exists a group G such that
> - $I_G \simeq \rho \circ \log$;
> - there exists an orbit equivalence coupling from G to $\mathbb{Z}$ that is $(\varphi_\varepsilon, \exp\circ\rho)$-integrable for all $\varepsilon > 0$, where $\varphi_\varepsilon(x) := \rho \circ \log(x)/(\log\circ\rho \circ \log(x))^{1+\varepsilon}$.

The second result gives the existence of a measure subgroup coupling (see Definition 6.1.1) with prescribed integrability with the lamplighter group $L_q := \mathbb{Z}/q\mathbb{Z} \wr \mathbb{Z}$.

Theorem 10

> For all $\alpha > 0$ there exists a group G such that
> - $I_G(x) \simeq (\log(x))^{1/(1+\alpha)}$;
> - for all $\varepsilon < 0$ if we define $\varphi_\varepsilon(x) := x^{\frac{1}{1+\alpha+\varepsilon}}$ then there exists a φ_ε-integrable measure subgroup coupling from G to L_q.

ORGANISATION OF THE MANUSCRIPT

After this short historical introduction, this manuscript is composed of two parts which have been thought and written to be read independently. Each part has its own detailed introduction and ends with a conclusion on open problems. Part i concerns buildings and problems of Local-to-Global rigidity: this is where Theorem 7 and Proposition 8 are proved. These results lead to an article [Esc20] to appear in *Annales de l'Institut Fourier*. Part ii is devoted to orbit and measure equivalences. In particular we prove Theorems 9 and 10. This second part concerns an ongoing work; the expected outcomes are detailed in conclusion. Finally an appendice gather some results on diagonal products and a notations index.

Part I

LOCAL-TO-GLOBAL RIGIDITY AND BUILDINGS

> *En essayant continuellement on finit toujours par réussir. Donc plus ça rate, plus on a de chance que ça marche.*
>
> — **Proverbe Shadock**

1 AN INTRODUCTION TO LOCAL-TO-GLOBAL RIGIDITY

> “
>
> *The door was the way to... to... The Door was The Way. Good. Capital letters were always the best way of dealing with things you didn't have a good answer to.*
>
> — **Douglas Adams**
> *Dirk Gently's Holistic Detective Agency*

A recurring theme in geometric group theory is that *local* properties of an object can have *global* implication for its geometry. A classical example is given by Lie groups and their locally defined Lie algebras. Another striking illustration is provided by the work of Tits [Tit81] who gave a local characterization of a particular family of graphs called "buildings of type $\tilde{A}_{d-1}$" (see Section 2.1 for a definition). The example that inspires our work here is given by Riemannian geometry. Indeed, a well known fact in this field stipulates that a complete Riemannian manifold which is locally isometric to a symmetric space is covered by a symmetric space. In this Part i we focus on a discrete version of that property. Precisely, *graphs* and their local-to-global properties are the objects we focus on. All graphs will be equipped with the usual metric, fixing the length of an edge to one.

A natural local condition to impose on a graph is to be d-regular for some $d \in \mathbb{N}$, which means that all the vertices must have degree d. A well-known result about such a graph is that the d-regular tree is its universal convering. This is a first example of a global information deduced only by a local knowledge of the graph.

One can now ask what happens if we impose a local condition which is stronger than d-regularity. We formalize this in the next definition.

Definition 1.0.1

> Let $R > 0$ and let X and Y be two graphs.
> We say that Y is **R-locally X** if every ball of radius R in Y is isometric to a ball of radius R in X.
> If Y is R-locally X and X is R-locally Y then we say that they are **R-locally the same.**
> We will say that Y **locally the same** as X (resp. Y and X are locally the same) if there exists R such that Y is R-locally X (resp. Y and X are R-locally the same).

Example 1.0.2. In Figure 1.1 $B_X(x_0, 2)$ is isometric to $B_Y(y_0, 2)$.

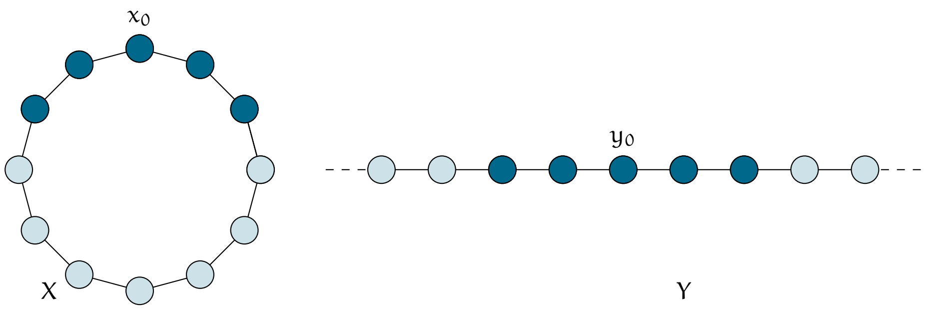


Figure 1.1.: Two graphs 2-locally the same.

Since being d-regular means being 1-locally the same as the d-regular tree, we can thus say that every graph that is 1-locally the same as a regular tree is covered by it. Hence we only need information on a very small scale to know if we can cover a graph with a regular tree: it reflects a form of *rigidity*, something we formalize in the next paragraph.

1.1 RIGIDITIES

1.1.1 Local-to-Global rigidity

We saw that for a graph, being 1-locally the same as the d-regular tree is enough to be covered by it. Now if we are allowed to draw information from a larger scale —that is to say from balls of radius $R > 1$— can we have a smilar covering result? In other words: are there graphs that cover all graph R-locally the same as them? We formalise that property by defining *Local-to-Gobal rigidity*, also named *LG-rigidity*.

Definition 1.1.1

> Let $R > 0$.We say that X is **Local-to-Global-rigid at scale** R (or R-**LG-rigid** for short) if every graph Y which is R-locally X is covered by X.
> We say that a graph X is **LG-rigid** if there exists $R > 0$ such that X is R-LG-rigid.

Example 1.1.2. Benjamini and Ellis [BE16] showed that for any $d \geqslant 2$ the Cayley graph of $\mathbb{Z}^d$ endowed with its usual generating set is 3-LG-rigid. They also proved that 3 was optimal showing that $\mathbb{Z}^3$ is not LG-rigid at scale 2.

Example 1.1.3. De la Salle et Tessera [dlST19, Theorem C] proved that every graph quasi-isometric to a tree is LG-rigid.

Benjamini [Ben13] and Georgakopoulos [Geo17] conjectured that any Cayley graph of a finitely presented group is LG-rigid at some scale $R > 0$. That conjecture was proven to be false in [dlST19, Theorem B], where the authors built counter-examples using groups with *torsion* elements.

Counter-example 1.1.4. The groups $F_2 \times F_2 \times \mathbb{Z}/2\mathbb{Z}$ and $SL_4(\mathbb{Z})$ admit Cayley graphs that are not LG-rigid.

Remark here that we do not state that every Cayley graph of these groups is non-LG-rigid, but that each group *admits* a non-LG-rigid Cayley graph. Indeed, in [dlST19, Theorem J] the authors also showed that every finitely presented group with an element of infinite order has a Cayley graph which is LG-rigid. Hence, LG-rigidity for a Cayley graph depends on the generating set. In particular LG-rigidity is not invariant under quasi-isometries.

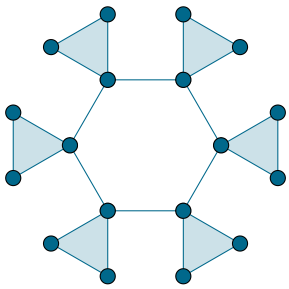
With a little bit more of material, we will be able to give a topological interpretation of Local-to-Global rigidity. To do so we need to define the notion of *large scale simple connectedness.*

1.1.2 Large scale simple connectedness

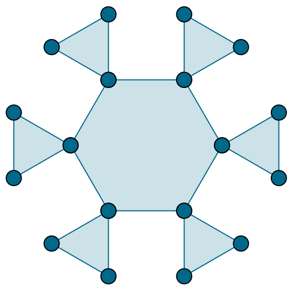
For a graph $\mathcal{G}$ and $k \in \mathbb{N}$ we define a 2-complex noted $P_k(\mathcal{G})$ such that:

- Its 1-skeleton is given by $\mathcal{G}$;
- Its 2-skeleton is composed of m-gons (for $m \in [0, k]$) defined by the simple loops of length m in $\mathcal{G}$ (up to cyclic permutations).

Example 1.1.5. Figure 1.2 represents a complex $P_k(\mathcal{G})$ for a graph $\mathcal{G}$ composed of a hexagon and six triangles. On the left $P_3(\mathcal{G})$ is represented: only the triangles are filled. On the right triangles and hexagons are filled. Remark that in that case $P_3(\mathcal{G})$ is the same as $P_4(\mathcal{G})$ and $P_5(\mathcal{G})$ since the graph has no squares or pentagons.



(a) $P_3(\mathcal{G})$



(b) $P_6(\mathcal{G})$

Figure 1.2.: Examples of $P_k(\mathcal{G})$ for $k = 3$ and $k = 6$

Definition 1.1.6

> We say that $\mathcal{G}$ is **k-simply connected** or simply connected at scale k if $P_k(\mathcal{G})$ is simply connected.

Example 1.1.7. The graph represented in Figure 1.2 is 6-simply connected, since the complex $P_6(\mathcal{G})$ (on the right of the figure) is simply connected. However, it is not k-simply connected for $k < 6$, since $\mathcal{G}$ is composed of an hexagon.

Example 1.1.8. Let G be a finitely generated group and T a finite symmetric generating set. The Cayley graph (G, T) is simply connected at scale k if and only if G has a presentation $\langle T, \mathcal{R} \rangle$ with relations of length at most k.

Example 1.1.9. Let $\mathbb{K}$ be a non-Archimedean local skew field. The one skeleton of the Bruhat-Tits building of $PSL_n(\mathbb{K})$ is simply connected at scale 3.

Remark 1.1.10. If $k \leqslant k'$, then every k-simply connected graph is k'-simply connected.

The following proposition allows us to restrict the study of the LG-rigidity of a graph $\mathcal{G}$ to some smaller class of graphs.

Proposition 1.1.11 (*de la Salle, Tessera, [dlST16, Proposition 1.5]*)

> Let $k \in \mathbb{N}$ and $\mathcal{G}$ be a k-simply connected graph, with cocompact isometry group. Then $\mathcal{G}$ is LG-rigid if and only if there exists R such that every k-simply connected graph which is R-locally $\mathcal{G}$ is isometric to $\mathcal{G}$.

To apply this result to our proof we will need to show that the studied graph X is simply connected. The following proposition shows that being simply connected is invariant under quasi-isometry.

Proposition 1.1.12 (*de la Salle, Tessera, [dlST16, Theorem 2.2]*)

> Let $k \in \mathbb{N}^*$ and let $\mathcal{G}$ be a k-simply connected graph. If $\mathcal{H}$ is quasi-isometric to $\mathcal{G}$, then there exists $k' \in \mathbb{N}^*$ such that $\mathcal{H}$ is simply connected at scale k'.

Before moving to the next section, let us mention a consequence of that last property. Indeed, this result allows us to look at the LG-rigidity notion with a topological point of view. Let's denote $\mathfrak{G}_k$ the set of isometry classes of locally finite k-simply connected graphs. We can define a distance on this set by:

$$d_{\mathfrak{G}_k}(X, Y) := \inf\left\{2^{-r} \ : \ X \text{ and } Y \text{ are } r\text{-locally the same}\right\},$$

which endows $\mathfrak{G}_k$ with a topology. The above proposition implies that a graph is LG-rigid if and only if its isometry class in $\mathfrak{G}_k$ is isolated for this topology.

1.1.3 Strong-Local-to-Global rigidty

Our rigidity notion can be refined in what is called the *Strong Local-to-Global rigidity*, also named *SLG-rigidity*.

Definition 1.1.13

> Let $r, R > 0$. We say that X is **SLG-rigid at scale** (r, R) if for all Y which is R-locally X and for all isometry f from $B_X(x, R)$ to $B_Y(y, R)$, the restriction of f to $B_X(x, r)$ extends to a covering of Y by X.
> We say that X is **SLG-rigid** if there exist two radii r and R such that X is SLG-rigid at scale (r, R).

Such a refinement is far more than just a subtlety: it actually proves necessary to obtain our main result (see page 42 for more details).

The following proposition gives us many examples of SLG-rigid graphs.

Proposition 1.1.14 (*de la Salle, Tessera [dlST19, Proposition 3.8]*)

> A graph with cocompact isometry group is LG-rigid if and only if it is SLG-rigid.

For example, any LG-rigid Cayley graph is actually SLG-rigid. In the same article, de la Salle and Tessera proved a powerful condition relating to the isometry group of a Cayley graph. We will refer to the isometry group of a Cayley graph (Γ, S) as $\mathrm{Isom}(\Gamma, S)$.

Theorem 1.1.15 (*de la Salle, Tessera [dlST19, Theorem E]*)

> Let Γ be a finitely presented group and S be a symmetric generating set and denote by (Γ, S) the corresponding Cayley graph. If $\mathrm{Isom}(\Gamma, S)$ is discrete, then (Γ, S) is SLG-rigid.

As stated in [dlST19, Corollary F], we can deduce two new classes of examples from the above theorem. But before, let us introduce what we call *LG-rigid groups*.

Definition 1.1.16

> We say that a finitely presented group is **LG-rigid** (resp. **SLG-rigid**) if all its Cayley graphs are LG-rigid (resp. SLG-rigid).

Example 1.1.17. Torsion-free groups of polynomial growth are SLG-rigid.

Example 1.1.18. Torsion-free, non-virtually free lattices in connected simple real Lie groups are SLG-rigid.

1.2 TOWARDS OUR MAIN RESULTS

Let us now turn to our main results. We state our main theorem and discuss the hypothesis in the second subsection and we detail the structure of the proof in the third one. But first, let us start by giving more context and motivations.

1.2.1 From buildings to quasi-buildings

So far the graphs chosen as examples are mostly Cayley graphs, but these are not the only LG-rigid ones. Indeed, besides the case of quasi-trees seen above, another interesting example is given by *Bruhat-Tits buildings* (see Section 2.1 for a definition).

Theorem 1.2.1 (*de la Salle, Tessera, [dlST16, Theorem 0.1]*)

> Let $\mathbb{K}$ be a non-Archimedean local skew field. If $\mathbb{K}$ has positive characteristic and $n \geqslant 3$, then the Bruhat-Tits building of $\mathrm{PSL}_n(\mathbb{K})$ is not LG-rigid.
> If $\mathbb{K}$ has characteristic zero and $n \geqslant 4$, then the Bruhat-Tits building of $\mathrm{PSL}_n(\mathbb{K})$ is SLG-rigid.

Keeping in mind the above theorem, consider the following question asked in [dlST19].

Question 1.2.2. *Among lattices in semi-simple Lie groups, which ones are LG-rigid?*

This question concerns *real* Lie groups but one can also wonder what happens for the p-*adic* case. Indeed, by a well known result of Svarc and Milnor, any lattice of $\mathrm{SL}_n(\mathbb{K})$ is quasi-isometric to the associated building (see Lemma 4.1.1 for more details). The fact that such a lattice is "almost" a building encouraged us to study the p-adic version of Question 1.2.2.

Question 1.2.3. *Among lattices in p-adic Lie groups, which ones are LG-rigid?*

De la Salle and Tessera showed [dlST16] that if $\mathbb{K}$ has positive characteristic, then there exist p-adic lattices that are torsion-free, cocompact but not LG-rigid.

Counter-example 1.2.4. Let $n \geqslant 3$. There exists in $\mathrm{PGL}_n(\mathbb{F}_p((T)))$ a torsion-free cocompact lattice that is not LG-rigid.

But when $\mathbb{K}$ has characteristic zero, the situation is quite different...

1.2.2 Main results

When $\mathbb{K}$ is a non-Archimedean local skew field of characteristic zero, an element of response to Question 1.2.3 is provided by our first result hereunder.

Theorem 1.2.5

> Let $n \neq 3$ and $\mathbb{K}$ be a non-Archimedean local skew field of characteristic zero. The torsion-free lattices of $\mathrm{SL}_n(\mathbb{K})$ are SLG-rigid.

This result is actually a corollary of our main theorem below which goes beyond the lattices framework and gives a rigidity result in a more general case.

Theorem 1.2.6

> Let $n \neq 3$ and $\mathbb{K}$ be a non-Archimedean local skew field of characteristic zero. Let $\mathcal{X}$ be the Bruhat-Tits building of $\mathrm{PSL}_n(\mathbb{K})$ and X be a transitive graph. If X verifies that
>
> - There is an injective homomorphism ρ from Isom(X) to Isom$(\mathcal{X})$ such that $\rho($Isom$(X))$ is of finite index in Isom$(\mathcal{X})$;
> - There is a Isom(X)-equivariant injective quasi-isometry q from X to $\mathcal{X}$;
>
> then X is SLG-rigid.

Let us discuss the hypothesis, starting with the assumption made on n. If $n = 2$ then $\mathcal{X}$ is the $(p+1)$-regular tree, thus by Example 1.1.3 any graph quasi-isometric to $\mathcal{X}$ is LG-rigid which proves the theorem. Now, as we will see in the sketch of the proof, the main tool of our demonstration is the LG-rigidity of the building. But if $n = 3$ the question of the rigidity of $\mathcal{X}$ is still open. Indeed in that case a lot of flexibility seems to be allowed (see [BP07]). Thus our demonstration deals mainly with the case where $n \geqslant 4$.

Then, let us look at the hypothesis made on the characteristic of $\mathbb{K}$. According to Theorem 0.4 of [dlST16] and more precisely according to its *proof*, we get Counter-example 1.2.4 above. It implies in particular that if we omit the characteristic zero hypothesis, then Theorems 1.2.5 and 1.2.6 are not true.

Finally, before moving to the sketch of the proof let us discuss the hypothesis made on the torsion in Theorem 1.2.5. First, introducing torsion in a group is in some case a useful way to build non-LG-rigid graphs. Indeed the Counter-example 1.1.4 is built this way. Second, in order to link (Γ, S) to $\mathcal{X}$ we will need an injection of Isom(Γ, S) into Isom$(\mathcal{X})$. Using a famous result of Kleiner and Leeb we will show that Isom(Γ, S) acts on the buildings by isometries. The injection into Isom$(\mathcal{X})$ will then be allowed by the following proposition.

Proposition 1.2.7 (*de la Salle, Tessera [dlST19, Proposition 6.2]*)

> Let Γ be an infinite, torsion-free, finitely generated group and let S be a finite symmetric generating subset of Γ. Then the isometry group of (Γ, S) has no non-trivial compact normal subgroup.

For more details on how we use this proposition, see the proof of Lemma 4.1.2.

1.2.3 Idea of the proof and structure of Part i

As stated in the discussion below Theorem 1.2.6, the proof deals mainly with the case where $n \geqslant 4$. So, Let $n \geqslant 4$ and $\mathbb{K}$ be non-Archimedean local skew field of characteristic zero and denote by $\mathcal{X}$ the Bruhat-Tits building of $\mathrm{PSL}_n(\mathbb{K})$. Let X be the studied graph and Y be a graph R-locally the same as X and denote by q a quasi-isometry from X to $\mathcal{X}$. The main idea of the proof is to use the rigidity of $\mathcal{X}$ to build the wanted covering from X to Y (see Figure 1.3), thus we need to build a graph locally the same as $\mathcal{X}$. We will denote such a graph $\mathcal{Y}$.

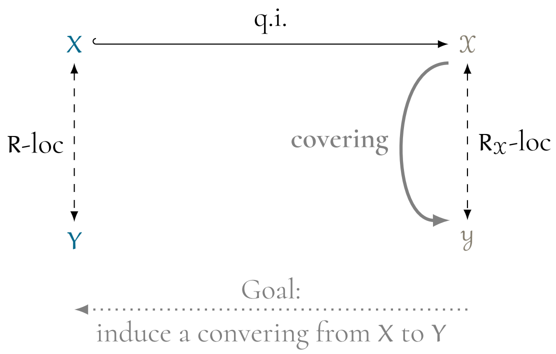


Figure 1.3.: Sketch of the proof

Moreover, for the rigidity of the building to induce a covering between X and Y, we want $\mathcal{Y}$ to contain a copy of the vertices of Y. Hence the goal is to define the vertices of $\mathcal{Y}$ to be composed of the vertices of Y and a copy of each vertex in $\mathcal{X}\backslash q(X)$ and define the edges to correspond to edges in X. With such a description $\mathcal{Y}$ is a "hybrid" graph and to define its edges we might need to understand how to link a vertex coming from Y to a vertex coming from $\mathcal{X}$. Hence, to avoid such a hybridation we chose to define the vertices only with informations encoded in Y. That is why we introduce the notion of *print* in the building (see Definition 2.3.1). It allows us to characterize a vertex in $\mathcal{X}$ by a set of neighbouring vertices in $\mathrm{im}(q)$ and, using a well chosen set of isometries from Y to X, to transfer this print notion to Y. Each print in Y corresponds to a vertex in $\mathcal{X}\backslash q(X)$. The vertices of the wanted graph $\mathcal{Y}$ will be composed of the vertices of Y and of prints in Y. It will now be easier to build edges between these vertices; the key argument to construct such edges is presented in Section 3.1.1.

Using the rigidity of the building we will obtain an isometry between $\mathcal{X}$ and $\mathcal{Y}$. To conclude the proof we will show that this isometry induces the wanted covering between Y and X.

ORGANIZATION OF PART 1 After that first introducing chapter, the second one is devoted to Bruhat-Tits buildings. We recall all the necessary material and study the aforementioned *prints*. The third section is devoted to the proof of Theorem 1.2.6. We develop in it the necessary engineering to build a graph locally the same as the building and conclude using the rigidity of the building to prove the rigidity of the studied graph. We prove Theorem 1.2.5 in the fourth chapter where we check that the lattice verifies the hypothesis of our main theorem.

BRUHAT-TITS BUILDINGS

2

En mathématiques, les noms sont arbitraires. Libre à chacun d'appeler un opérateur auto-adjoint un « éléphant » et une décomposition spectrale une « trompe ». On peut alors démontrer un théorème suivant lequel « tout éléphant a une trompe ». Mais on n'a pas le droit de laisser croire que ce résultat a quelque chose à voir avec de gros animaux gris.

— **Ivar Ekeland**
Le Calcul, l'Imprévu

2.1 DEFINITION OF BUILDINGS

Let $n \geqslant 2$. We recall here the description of the Bruhat-Tits building associated to $\mathsf{PSL}_n(\mathbb{K})$ for some non-Archimedean local skew field $\mathbb{K}$. See [AB18] for more details.

2.1.1 Non-Archimedean local skew fields

Let $\mathbb{K}$ be a field (not necessarily commutative). A *discrete valuation* on $\mathbb{K}$ is a surjective homomorphism $\nu : \mathbb{K}^* \to \mathbb{Z}$ satisfying $\nu(x+y) \geqslant \min\{\nu(x), \nu(y)\}$ for all $x, y \in \mathbb{K}^*$ such that $x+y \neq 0$. If $\mathbb{K}$ is endowed with such a valuation, we can extend ν on all $\mathbb{K}$ by setting $\nu(0) = +\infty$. We say that $\mathbb{K}$ is a *non-Archimedean local skew field* if it is locally compact for the topology associated to a discrete valuation.

Example 2.1.1. If $\mathbb{K} = \mathbb{Q}$ and p is a prime, then every $x \in \mathbb{K}$ can be written as $x = p^n a/b$ where a and b are integers non-divisible by p. The map defined by $\nu(p^n a/b) := n$ is a discrete valuation over $\mathbb{K}$. The field $\mathbb{Q}_p$ is the completion of $\mathbb{Q}$ with respect to the *p-adic absolute value* defined by $|x|_p = p^{-\nu(x)}$.

Example 2.1.2. Let $\mathbb{K} = \mathbb{F}_p((T))$, the field of *formal Laurent series* over $\mathbb{F}_p$. Denote by $f = \sum_{k\in\mathbb{Z}} a_k T^k$ an element in $\mathbb{F}_p((T))$ then the map defined by $\nu(f) := \min\{k : a_k \neq 0\}$ is a valuation over $\mathbb{K}$.

Let $\mathcal{O}$ denote the *ring of integers* of $\mathbb{K}$ with respect to ν, that is to say $\mathcal{O} := \{x \in \mathbb{K} : \nu(x) \geqslant 0\}$. This ring has a unique *prime ideal* $\mathfrak{m} := \{x \in K : \nu(x) > 0\}$. Finally, let π be a generator of $\mathfrak{m}$ as an $\mathcal{O}$-module.

Example 2.1.3. If $\mathbb{K} = \mathbb{Q}_p$ then its ring of integers is $\mathcal{O} = \mathbb{Z}_p$. Moreover $\mathfrak{m} = p\mathbb{Z}_p$ and $\pi = p$.

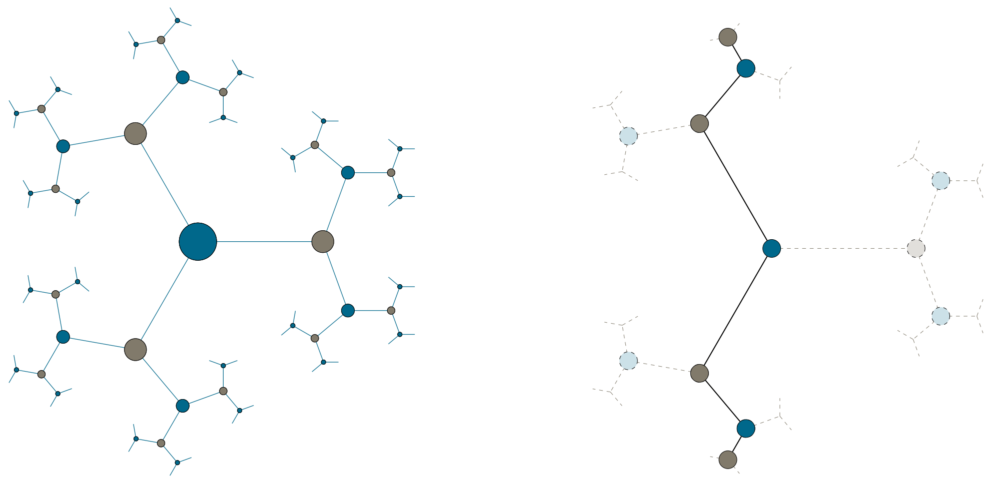
Example 2.1.4. If $K = \mathbb{F}_p((T))$ then $\mathcal{O} = \mathbb{F}_p[[T]]$. Moreover $\mathfrak{m} = X\mathbb{F}_p[[T]]$ and $\pi = X$.

2.1.2 Buildings

Let $\mathbb{K}$ be a non-Archimedean local skew field endowed with a valuation ν. An $\mathcal{O}$-lattice of $\mathbb{K}^n$ is an $\mathcal{O}$-submodule which generates $\mathbb{K}^n$ as a $\mathbb{K}$ vector space. Such a lattice can be written as $\mathcal{O}e_1 + \dots + \mathcal{O}e_n$ for a basis $(e_1, \dots, e_n)$ of $\mathbb{K}^n$. Since for any $a \in \mathbb{K}^*$ and any lattice L, the module aL is also a lattice, we can define the equivalence relation of *lattices modulo homothety*. We denote by $[L]$ the class of a lattice L.
The Bruhat-Tits building of $\mathrm{PSL}_n(\mathbb{K})$ is a simplicial complex of dimension $n-1$ denoted by $\hat{\mathcal{X}}$ whose 1-skeleton (denoted by $\mathcal{X}$) is described as follows. The vertices are the classes of $\mathcal{O}$-lattices modulo homothety. Two vertices x_1 and x_2 are linked by an edge if there exists representatives L_1 of x_1 and L_2 of x_2 such that:

$$pL_1 \subset L_2 \subset L_1.$$

Example 2.1.5. One can show that the building of $\mathrm{PSL}_2(\mathbb{Q}_p)$ is a $(p+1)$-regular tree. Figure 2.1a gives a representation of the building when $p = 2$.



(a) The building has two $\mathrm{SL}_2(\mathbb{Q}_2)$-orbits

(b) Representation of one apartment

Figure 2.1.: The building of $\mathrm{PSL}_2(\mathbb{Q}_2)$

A representation of the building of $\mathrm{PSL}_3(\mathbb{Q}_2)$ can be found in the Building Gallery developped by Bekker and Solleveld. For details on the employed method see [BS20].

2.2 STRUCTURAL PROPERTIES

2.2.1 Orbits and types

The building is endowed with a natural action of $\mathrm{GL}_n(\mathbb{K})$. Indeed let $L = \oplus_{i=1}^n \mathcal{O}e_i$ be a lattice in $\mathbb{K}^n$, the action of $g \in \mathrm{GL}_n(\mathbb{K})$ on L is defined by $gL := \oplus_{i=1}^n \mathcal{O}g(e_i)$. Since $\mathrm{GL}_n(\mathbb{K})$

acts transitively on the bases, the aforedefined action is transitive. Now let x be a vertex in $\mathcal{X}$ and L be a representative of x and define for $g \in GL_n(\mathbb{K})$

$$g \cdot x := [gL] .$$

Since $g \cdot (aL) = a(g \cdot L)$ for all $a \in \mathbb{K}^*$, the above definition does not depend on the choice of representative L. Thus $GL_n(\mathbb{K})$ acts transitively on $\mathcal{X}$ by isometry. Moreover this action induces a transitive action of $PGL_n(\mathbb{K})$ on $\mathcal{X}$ by isometry.

If $L = \oplus_i \mathcal{O} e_i$ is a lattice we define its *type* to be $\nu(\det(e_1, \dots, e_n))$. Since:

$$\forall a \in \mathbb{K}^* \quad \nu(\det(ae_1, \dots, ae_n)) = \nu(\det(e_1, \dots, e_n)) \mod n,$$

one can define the *type of a vertex* x in $\mathcal{X}$ to be the value modulo n of the type of one of its representatives. We denote by $\tau(x)$ the type of x.
If L' is a second lattice, we can choose our basis $e_1, \dots, e_n$ for L in such a way that L' admits a basis of the form $a_1 e_1, \dots, a_n e_n$ for some $a_i \in \mathbb{K}^*$. The scalars a_i can be taken to be powers of π. The incidence relation defined above implies that if the classes of L and L' are linked by an edge in $\mathcal{X}$, then they have different types.

Remark 2.2.1. Remark that if $L = \oplus_i \mathcal{O} e_i$ and

$$L' = \mathcal{O}\pi e_1 \oplus \cdots \oplus \mathcal{O}\pi e_j \oplus e_{j+1} \oplus \cdots \oplus e_n,$$

then $\tau([L']) = \tau([L]) + j \mod n$.

The action of $SL_n(\mathbb{K})$ on $\mathcal{X}$ preserves the determinant and is transitive on the pairs of vertices of the same type. So there are exactly n orbits under the action of $SL_n(\mathbb{K})$ (see Figure 2.1a and Figure 2.2 for examples).

2.2.2 Apartments

If $\boldsymbol{e}$ is a basis of $\mathbb{K}^n$ then the set of vertices $\{\oplus_{i=1}^n \mathcal{O}\pi^{k_i} e_i \mid k_i \in \mathbb{Z}\}$ induces a sub-complex denoted by $\mathcal{A}$, which is isometric to a $(n-1)$-dimensional Euclidean space tiled by regular $(n-1)$-simplices. We call such sub-complexes *apartments*. For example an apartment in the building of $PSL_2(\mathbb{Q}_2)$ is isometric to $\mathbb{R}^1$ tiled with segments of length 1 (see Figure 2.1b), whereas for $PSL_3(\mathbb{Q}_2)$ the apartment are isometric to $\mathbb{R}^2$ and tiled with triangles (see Figure 2.2).

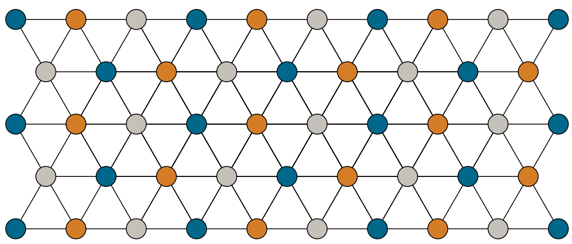


Figure 2.2.: Apartment in the building of $PSL_3(\mathbb{Q}_2)$. Colors correspond to $SL_3(\mathbb{Q}_2)$-orbits.

For any two points in $\hat{\mathcal{X}}$ there exists an apartment containing them. If $x, y \in \hat{\mathcal{X}}$ let $\mathcal{A}$ be an apartment containing x and y and define $d_{\hat{\mathcal{X}}}(x, y)$ to be equal to the euclidean distance $d_{\mathcal{A}}(x, y)$. This definition does not depend on the choice of apartment $\mathcal{A}$ and thus endows $\widehat{\mathcal{X}}$ with a well defined distance. Moreover, this distance verifies the *negative curvature inequality:* for all $x, y, z \in \widehat{\mathcal{X}}$ and $t \in [0, 1]$

$$d^2_{\widehat{\mathcal{X}}}(z, tx + (1-t)y) \leqslant td^2_{\widehat{\mathcal{X}}}(z, x) + (1-t)d^2_{\widehat{\mathcal{X}}}(z, y) - t(1-t)d^2_{\widehat{\mathcal{X}}}(x, y). \tag{2.1}$$

Denote by $d_{\mathcal{X}}$ the distance on the 1-skeleton $\mathcal{X}$ assigning length 1 to an edge. Then $d_{\mathcal{X}}(x, y)$ is greater than $d_{\widehat{\mathcal{X}}}(x, y)$ for all vertices x and y in $\mathcal{X}$.

2.2.3 Contractibility

Using the above inequality one can show that the building is contractible (see [AB18] for more details). We can actually show that *convex sets* in $\widehat{\mathcal{X}}$ are themselves contractible.

Claim 2.2.2. Let $r > 0$. Any convex set in $\widehat{\mathcal{X}}$ is contractible.

Proof. Let $r > 0$ and $\mathcal{C}$ a convex set in $\widehat{\mathcal{X}}$ and endow it with the distance induced by $d_{\widehat{\mathcal{X}}}$. Take $x_0 \in \mathcal{C}$ and define,

$$\mathcal{H} : \begin{cases} [0, 1] \times \mathcal{C} & \to \mathcal{C}, \\ (t, x) & \mapsto tx + (1-t)x_0. \end{cases}$$

Since $\mathcal{C}$ is convex, the map $\mathcal{H}$ is well-defined. Moreover $\mathcal{H}(0, \cdot) = \mathrm{id}_{\mathcal{C}}$ and $\mathcal{H}(1, x) = x_0$ for all x in $\mathcal{C}$. Let us show that $\mathcal{H}$ is continuous. Take $x, x' \in \mathcal{C}$ and $t, t' \in [0, 1]$ and let $z = t'x' + (1-t')x_0$. By eq. (2.1)

$$d^2_{\widehat{\mathcal{X}}}(z, tx + (1-t)x_0) \leqslant td^2_{\widehat{\mathcal{X}}}(z, x) + (1-t)d^2_{\widehat{\mathcal{X}}}(z, x_0) - t(1-t)d^2_{\widehat{\mathcal{X}}}(x, x_0). \tag{2.2}$$

But if $\mathcal{A}$ is a an apartment containing z and x_0, then by property of the Euclidean distance $d_{\mathcal{A}}$

$$d_{\widehat{\mathcal{X}}}(z, x_0) = d_{\mathcal{A}}(t'x' + (1-t')x_0, x_0) = t'd_{\mathcal{A}}(x', x_0) = t'd_{\widehat{\mathcal{X}}}(x', x_0),$$

which tends to $td_{\widehat{\mathcal{X}}}(x, x_0)$ as (t', x') tends to (t, x). Similarly

$$\begin{aligned} d_{\widehat{\mathcal{X}}}(z, x) \leqslant d_{\widehat{\mathcal{X}}}(z, x') + d_{\widehat{\mathcal{X}}}(x', x) &= d_{\widehat{\mathcal{X}}}(t'x' + (1-t')x_0, x') + d_{\widehat{\mathcal{X}}}(x', x), \\ &= (1-t')d_{\widehat{\mathcal{X}}}(x', x_0) + d_{\widehat{\mathcal{X}}}(x', x), \end{aligned}$$

which converges to $(1-t)d_{\widehat{\mathcal{X}}}(x, x_0) + d_{\widehat{\mathcal{X}}}(x', x)$ as (t', x') tends to (t, x). Thus the right term of eq. (2.2) converges to 0 as (t', x') tends to (t, x). Hence the continuity of $\mathcal{H}$ and the contractibility of $\mathcal{C}$. □

2.3 PRINTS

In this section we show that a vertex in the building $\mathcal{X}$ can be determined by a part of its 1-neighbourhood. More precisely, if i belongs to $\{0, \ldots, n\}$ we prove that a vertex in the building is entirely determined by its type and the vertices in its 1-neighbourhood having type i.

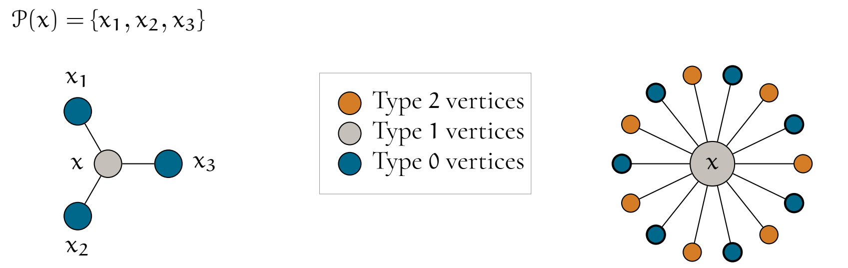
2.3.1 Definition and examples

Recall that if $\mathcal{C}$ is an $SL_n(\mathbb{K})$-orbit in $\mathcal{X}$ then $\mathcal{C}$ is exactly the set of vertices having type i for some $i \in \{1, \dots, n\}$, *viz.* there exists $i \in \{0, \dots, n\}$ such that $\mathcal{C} = \tau^{-1}(i)$.

Definition 2.3.1

> Let x be a vertex of $\mathcal{X}$ and $\mathcal{C}$ be an $SL_n(\mathbb{K})$-orbit in $\mathcal{X}$. We define the **print of type** $\mathcal{C}$ of x, denoted by $\mathcal{P}_{\mathcal{C}}(x)$ (or $\mathcal{P}(x)$ if there is no ambiguity), to be the intersection of the 1-neighbourhood of x with $\mathcal{C}$, *viz.* $\mathcal{P}(x) := B_{\mathcal{X}}(x, 1) \cap \mathcal{C}$.

Example 2.3.2. Figure 2.3 represents a ball of radius 1 in two different cases. The case when $n = 2$ and $|\mathcal{O}/\pi\mathcal{O}| = 2$ (for example when $\mathbb{K} = \mathbb{Q}_2$) is represented on the left figure. The case when $\mathbb{K} = \mathbb{Q}_2$ and $n = 3$ is represented on the right figure. In each case, the print of type 0 of x corresponds to the set of blue vertices. In the second case, the print of type 2 of x corresponds to the orange vertices.



$B(x, 1)$ for $n = 2$ and $\mathbb{K} = \mathbb{Q}_2$ $\qquad$ $B(x, 1)$ for $\mathbb{K} = \mathbb{Q}_2$ and $n = 3$

Figure 2.3.: Prints and 1-neighbourhood of a vertice in $\mathcal{X}$

Remark 2.3.3. If $x \in \mathcal{C}$ then $B_{\mathcal{X}}(x, 1) \cap \mathcal{C} = \{x\}$ thus $\mathcal{P}_{\mathcal{C}}(x) = \{x\}$.

2.3.2 Tracking vertices through their prints

The following result proves that a vertex in $\mathcal{X}$ is uniquely determined by its print.

Proposition 2.3.4

> Let $\mathcal{C}$ be a $SL_n(\mathbb{K})$-orbit and x and y be two vertices in $\mathcal{X}$.
> If $\mathcal{P}_{\mathcal{C}}(x) = \mathcal{P}_{\mathcal{C}}(y)$ then $x = y$.

Before showing the above property, let us recall (and prove) a useful fact concerning the choice of representative of a vertex.

Claim 2.3.5. For any vertex in $\mathcal{X}$, we can always find a representative $\oplus_i \mathcal{O}\pi^{k_i} e_i$ of the vertex such that

$$\begin{cases} \forall i \in \{1, \dots, n\} & k_i \geqslant 0, \\ \exists i_0 \in \{1, \dots, n\} & k_{i_0} = 0. \end{cases} \tag{2.3}$$

Proof of the claim. Indeed, let $x \in \mathcal{X}$ and let $(l_1, \dots, l_n)$ be a representative of x and let i_0 be such that $l_{i_0} = \min_i l_i$, then

$$\left[\oplus_{i=1}^n \mathcal{O}\pi^{l_i} e_i\right] = \pi^{-l_{i_0}} \left[\oplus_{i=1}^n \mathcal{O}\pi^{l_i - l_{i_0}} e_i\right] = \left[\oplus_{i=1}^n \mathcal{O}\pi^{l_i - l_{i_0}} e_i\right].$$

Thus $(l_1 - l_{i_0}, \dots, l_1 - l_{i_0})$ is a representative of x and verifies eq. (2.3). $\square$

Now, let us prove that the print determines the vertex.

Proof of Proposition 2.3.4. Let $\mathcal{C}$ be a $\mathsf{SL}_n(\mathbb{K})$-orbit and denote by $\mathcal{P}(z)$ the print of type $\mathcal{C}$ of a vertex z. Let x and y be two vertices in $\mathcal{X}$ such that $\mathcal{P}(x) = \mathcal{P}(y)$.

First remark that if $x \in \mathcal{C}$ then $\mathcal{P}(x) = \{x\}$ which implies that $\mathcal{P}(y) = \mathcal{P}(x) = \{x\}$. But then y has only one neighbour of type $\mathcal{C}$, which is something that is only possible if y belongs to $\mathcal{C}$. Thus $\{y\} = \mathcal{P}(y) = \{x\}$ and so $x = y$.

Now assume that x does not belong to $\mathcal{C}$ and take $\mathcal{A}$ to be an apartment containing x and y. Let $t \in \{0, \dots, n-1\}$ such that $\mathcal{C}$ is exactly the set of vertices of type t. Define $P := \mathcal{P}(x) \cap \mathcal{A}$ and let $\boldsymbol{e}$ be a basis such that

$$\mathcal{A} = \left\{\oplus_{i=1}^n \mathcal{O}\pi^{k_i} e_i \mid k_i \in \mathcal{O}\right\} \quad \text{and} \quad x = (0, \dots, 0).$$

By Claim 2.3.5, we can chose a representative $(k_1, \dots, k_n)$ of y such that $k_i \geqslant 0$ for all i and there exists i_0 such that $k_{i_0} = 0$. Now define the sequence $i_1, \dots, i_n$ such that $k_{i_1} \geqslant \dots \geqslant k_{i_n} = 0$ and let

$$l_{i_1} = \dots = l_{i_{\tau(x)-t}} = 0 \qquad l_{i_{\tau(x)-t+1}} = \dots = l_{i_n} = 1.$$

In other words we want the vertex defined by $(l_1, \dots, l_n)$ to have coordinates zero where the coordinates of y take their greatest values and have a coordinate 1 where y has a zero (namely in position i_n). The number of coordinates equal to 1 in $(l_1, \dots, l_n)$ is then determined by the condition that the vertex has to belong to $\mathcal{C}$.

By remark 2.2.1 the vertex $z = (l_1, \dots, l_n)$ has type t, indeed

$$\tau(z) = \tau(x) + n - (\tau(x) - t) = n + t = t \mod n.$$

That is to say z belongs to $\mathcal{C}$. Moreover it is at distance 1 from x, so z belongs to P. But if $k_{i_1} > 0$, then $d(z, y) > 1$ thus z cannot belong to $\mathcal{P}(y)$. Hence $k_{i_1} \leqslant 0$, that is to say $k_i = 0$ for all i and thus $y = x$. $\square$

This proves that a vertex in $\mathcal{X}$ is uniquely determined by its print. Thus, we can introduce the following definition without ambiguity.

Definition 2.3.6

Let x to be a vertex in $\mathcal{X}$. We say that x is the **source** of $\mathcal{P}(x)$.

To conclude this section, let us look at the behaviour of prints under the action of $\mathsf{PSL}_n(\mathbb{K})$. Let $x \in \mathcal{X}$ and let $\alpha \in \mathsf{PSL}_n(\mathbb{K})$. Since α is an isometry and is type-preserving, we get

$$\alpha\Big(\mathcal{P}_{\mathcal{C}}(x)\Big) = \alpha\Big(B(x, 1) \cap \mathcal{C}\Big) = \alpha\big(B(x, 1)\big) \cap \alpha(\mathcal{C}) = B\left(\alpha(x), 1\right) \cap \mathcal{C}.$$

We deduce the following lemma.

Lemma 2.3.7

Let $x \in \mathcal{X}$. If α belongs to $\mathsf{PSL}_n(\mathbb{K})$, then $\alpha\left(\mathcal{P}(x)\right) = \mathcal{P}\left(\alpha(x)\right)$.

3 QUASI-BUILDINGS

> *Ma thèse n'est qu'une succession d'erreurs mathématiques.*
>
> — **Mario Gonçalves**

Reinforced with all the knowledge on buildings exposed in the last chapter, we can now turn to the study of graphs quasi-isometric to a building (or *quasi-buildings* for short). Indeed, the aim of this chapter is to prove Theorem 1.2.6 which we recall below.

Theorem 1.2.6

Let $n \neq 3$ and $\mathbb{K}$ be a non-Archimedean local skew field of characteristic zero. Let $\mathcal{X}$ be the Bruhat-Tits building of $\mathsf{PSL}_n(\mathbb{K})$ and X be a transitive graph. If X verifies that

- There is an injective homomorphism ρ from $\mathrm{Isom}(X)$ to $\mathrm{Isom}(\mathcal{X})$ such that $\rho(\mathrm{Isom}(X))$ is of finite index in $\mathrm{Isom}(\mathcal{X})$;
- There is a $\mathrm{Isom}(X)$-equivariant injective quasi-isometry q from X to $\mathcal{X}$;

then X is SLG-rigid.

As stated in Chapter 1, the proof —and subsequently this chapter— deals mainly with the case where $n \geqslant 4$. Recall that the main idea of the proof is to define a "hybrid" graph that will be locally the same as the building $\mathcal{X}$, in order to exploit the LG-rigidity of $\mathcal{X}$. This construction is done in the second section of this chapter and relies on the notion of print introduced in Chapter 2. We then show in the third section that the covering from the building to our hybrid graph induces a covering from X to Y. But before entering the heart of the matter, we show and recall some useful material concerning extension of isometries and the isometry group of our studied graph $\mathrm{Isom}(X)$.

3.1 PRELIMINARY RESULTS

Before diving into the core of the proof, let us state some necessary material.

3.1.1 Isometries extension

To build the "hybrid" graph mentionned above, we will first define vertices and edges locally. That is to say we will first define a (finite) graph that will be isometric to a ball in $\mathcal{X}$ and part of which will be composed of the vertices of a ball of radius R in Y. In order to extend that definition of a graph locally the same as the building to the definition of one globally the same as $\mathcal{X}$, we will need to be able to extend local isometries. We recall here the result of de la Salle and Tessera [dlST19, Lemma 4.1] that will serve our purpose.

Proposition 3.1.1 (*de la Salle, Tessera*)

> Let $\mathcal{G}$ be a graph with cocompact discrete isometry group. Given some $r_1 \geqslant 0$, there exists $r_2 \geqslant r_1$ such that: for every $g \in \mathcal{G}$, the restriction to $B_{\mathcal{G}}(g, r_1)$ of an isometry $f : B_{\mathcal{G}}(g, r_2) \to \mathcal{G}$ coincides with the restriction of an element of $\mathrm{Isom}(\mathcal{G})$.

We illustrate the above proposition in Figure 3.1. The local isometry f is defined on $B_{\mathcal{G}}(g, r_2)$ which is represented by the dark blue disc. When restricted to $B_{\mathcal{G}}(g, r_1)$ (represented by the light blue disc), the isometry f coincides with a global isometry of $\mathcal{G}$ denoted by a.

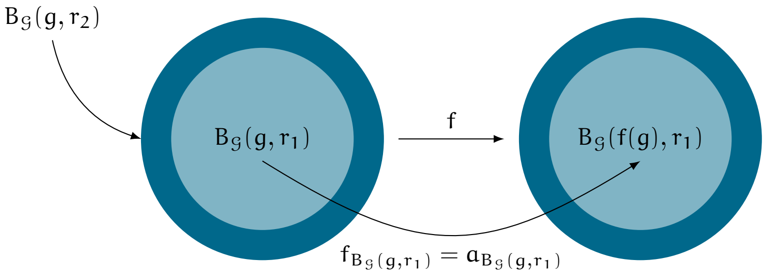


Figure 3.1.: Extension of a local isometry

It is however not necessarily true that f coincides on the whole $B(g, r_2)$ with an isometry of $\mathcal{G}$. Indeed, truncating the entire graph to some ball might allow some kind of flexibility near the boundary of the ball (see Example 3.1.2 and Figure 3.2). Hence, in order to coincide with a global isometry we need to restrict the local isometry f to a smaller ball which do not contain the flexible area.

Counter-example 3.1.2. Let $\mathcal{G}$ be the Cayley graph of $\mathbb{Z}^2$ endowed with its usual generating part. We consider in Figure 3.2 an isometry f defined on $B\big((0,0), 1\big)$ such that f fixes $(0,0)$, $(-1,0)$ and $(0,-1)$ (represented by the blue vertices) and exchange $(1,0)$ with $(0,1)$ (the orange and brown vertices). Then f is an isometry from $B\big((0,0),1\big)$ to $B\big((0,0),1\big)$, but can not coincide with a global isometry of $\mathcal{G}$ on that ball. Indeed, if such a global isometry existed, then it should send the vertex $(-1,1)$ (represented by the light brown vertex on the left part of the figure) at distance 1 from both $f(-1,0) = (-1,0)$ and $f(0,1) = (1,0)$. Which is impossible since the only point at distance 1 from $(1,0)$ and $(-1,0)$ is $(0,0)$ and it is already the image of $(0,0)$.

3.1.2 Preliminary results on X

We saw that the building can be partitioned into types (or $\mathrm{PSL}_n(\mathbb{K})$-orits). We now prove that if $\mathrm{im}(q)$ intersects with a given $\mathrm{PSL}_n(\mathbb{K})$-orbit then it contains the entire orbit.

Lemma 3.1.3

> If X verifies the hypothesis of Theorem 1.2.6, then $\mathrm{PSL}_n(\mathbb{K})$ is included in $\rho(\mathrm{Isom}(X))$. Moreover, if $q(X)$ contains a vertex of a certain type i, then $q(X)$ contains all the vertices of type i.

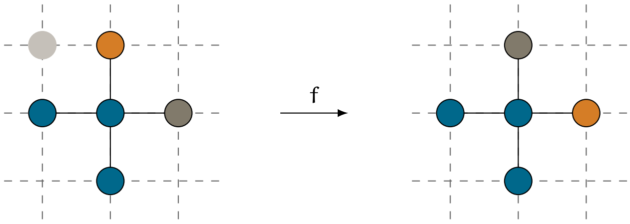


Figure 3.2.: Local isometry that can not coincide with a global one on its entire domain of definition

Proof. Since $\rho(\mathrm{Isom}(X))$ is of finite index in the isometry group of the building $\mathcal{X}$, the same goes for its normal core $\cap_{g \in \mathrm{Isom}(\mathcal{X})} g\rho(\mathrm{Isom}(X))g^{-1}$. Then, by simplicity of $\mathsf{PSL}_n(\mathbb{K})$, the normal core of $\rho(\mathrm{Isom}(X))$ contains $\mathsf{PSL}_n(\mathbb{K})$. Hence the result.

Then, the second part of the lemma follows from the equivariance of q and the transitivity of $\mathsf{PSL}_n(\mathbb{K})$ on vertices of the same type. □

Without loss of generality, we can assume that $\mathrm{im}(q)$ contains type 0 vertices, that is to say $\tau^{-1}(0) \subset \mathrm{im}(q)$. Moreover, using Proposition 1.1.12 we obtain that X is simply connected at some scale $k > 0$.

3.1.3 Hypothesis

The aim of the next two sections is to prove Theorem 1.2.6 for $n \geqslant 4$. For the sake of clarity we recapitulate here the needed assumptions for the proof.

Hypothesis (H)

1. Let X be a k-simply-connected transitive graph;
2. Let Y be a graph R-locally X and k-simply connected;
3. Let $n \geqslant 4$ and $\mathbb{K}$ a non-Archimedean local skew field of characteristic zero and denote by $\mathcal{X}$ the Bruhat-Tits building of $\mathsf{PSL}_n(\mathbb{K})$;
4. Let $\rho : \mathrm{Isom}(X) \to \mathrm{Isom}(\mathcal{X})$ be an injective homomorphism and q be an $\mathrm{Isom}(X)$-equivariant injective quasi-isometry from X to $\mathcal{X}$;
5. Assume that $\rho(\mathrm{Isom}(X))$ is of finite index in $\mathrm{Isom}(\mathcal{X})$ and that $q(X)$ contains $\tau^{-1}(0)$.

Unless otherwise stated we will assume from now on that **(H)** is verified. We can now define our hybrid graph $\mathcal{Y}$.

3.2 DEFINING OUR HYBRID GRAPH

This section is dedicated to the definition of a graph locally the same as $\mathcal{X}$ which we will call $\mathcal{Y}$. Before moving to the detailed definition let us explain the idea of the construction. Recall that the vertices of $\mathcal{X}$ are partitioned into different types (see Section 2.1) denoted by integers in $\{0, \dots, n-1\}$. By Lemma 3.1.3 if $q(X)$ contains a vertex of a certain type then it contains all the vertices of that type. Denote by T the set of types that are not contained in $q(X)$, namely $T = \{0, \ldots, n-1\} \setminus \tau(q(X))$. We have the following partition

$$\mathcal{X} = q(X) \sqcup \left(\sqcup_{i \in T} \tau^{-1}(i)\right). \tag{3.1}$$

Example 3.2.1. Take $\mathbb{K} = \mathbb{Q}_2$ and assume that $\mathrm{im}(q)$ is composed only of type zero vertices. When $n = 2$ we have $T = \{1\}$ and the building is represented in Figure 2.1a. The partition in eq. (3.1) corresponds to the partition of vertices in two different colors.
When $n = 3$, we get $T = \{1, 2\}$. An apartment of $\mathcal{X}$ is represented in Figure 2.2 and the partition of this part of $\mathcal{X}$ corresponds to the partition in three different colors.

Example 3.2.2. Let $n = 4$ and $\mathbb{K} = \mathbb{Q}_2$ and assume that $\mathrm{im}(q)$ contains type zero and type 2 vertices. Then $T = \{1, 3\}$. We will not try to represent $\mathcal{X}$ or an apartment but recall that it is tiled by tetrahedrons. The partition is illustrated on a tetrahedron in Figure 3.3, where $\mathrm{im}(q)$ corresponds to the two blue vertices.

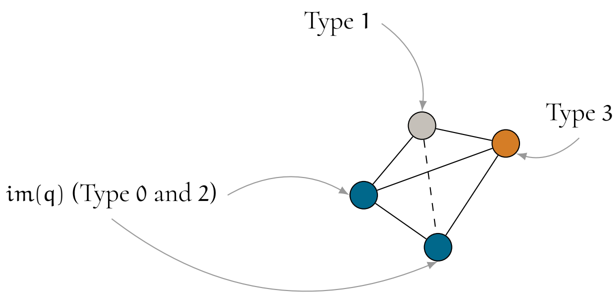


Figure 3.3.: Partition of a simplex

The idea of the construction of $\mathcal{Y}$ is to take the vertices of Y and add to them vertices of the missing types, *ie.* vertices with type in T (see Figure 3.6 for an example). But we want to build this vertices only with informations encoded in $V(Y)$. That is why we introduced the notion of *print* of a vertex in the building (see Definition 2.3.1).

The construction of $\mathcal{Y}$ is organised as follows. Using a well chosen set of isometries from Y to X (see Section 3.2.1), we transfer this print notion to Y, each print in Y corresponding to a vertex of a missing type (see Section 3.2.2). We conclude with the definition of our graph $\mathcal{Y}$.

3.2.1 Atlas of local isometries

To build our graph locally the same as $\mathcal{X}$, we need to restrict ourselves to a particular set of local isometries from Y to X. More precisely, if y_1 and y_2 are close in Y and f_1 (resp. f_2) is an isometry from $B_Y(y_1, R)$ (resp. $B_Y(y_2, R)$) to X, we want the transition map $f_2 f_1^{-1}$ to coincide with an element in $\rho^{-1}\mathrm{PSL}_n(\mathbb{K})$ on a small ball. This is what we formalize here and schematize in Figure 3.4.

In order to avoid any ambiguity regarding the notion of center of a ball, let us precise our definition of ball in a graph. What we mean when we talk of "a ball of radius R" is actually a *pointed ball of radius* R that is to say, a couple $(\mathcal{B}, y)$ such that y is a vertex in Y and $\mathcal{B} = B_Y(y, R)$. We will abuse notation by denoting such a pointed ball $B_Y(y, R)$ (instead of $(B_Y(y, R), y)$). This way, the center of a ball is always well defined.

Definition 3.2.3

Let $\mathfrak{A}$ be a set of isometries from balls of radius R in Y to X, We say that $\mathfrak{A}$ is an **atlas** of local isometries from Y to X if the map that associates to each isometry in $\mathfrak{A}$ the center of its ball of definition is a bijection from $\mathfrak{A}$ to Y. That is to say, we can write

$$\mathfrak{A} := \left\{ f_y \ : \ B_Y(y, R) \to X \mid y \in Y \right\},$$

where the map that associates f_y to y is bijective.
We say that f_y is the **isometry associated to** y in $\mathfrak{A}$.

Let $H_0 := \rho^{-1}\mathrm{PSL}_n(\mathbb{K})$. Now, we show that we can construct an atlas of local isometries from Y to X such that the transition maps between two isometries defined on balls with neighbouring centers coincide with elements of H_0. We will note a path between two vertices v_1 and v_2 as a sequence $(v_1, \dots, v_l)$ of adjacent vertices.

Lemma 3.2.4

Let $r_A > 0$ and let $H_0 := \rho^{-1}\mathrm{PSL}_n(\mathbb{K})$. For R large enough, if Y is R-locally X, then there exists an atlas $\mathfrak{A}$ such that for any two neighbours y and z in Y

$$\exists a \in H_0 \quad f_y \cdot {f_z^{-1}}_{|B(f_z(z), r_A)} = a_{|B(f_z(z), r_A)}. \tag{3.2}$$

Before proving it, let us schematize the framework of this lemma. In Figure 3.4 we represent two isometries f_y and f_z with z neighbour to y. The larger discs correspond to balls of radius R and the smaller ones to balls of radius r_A. The map $f_y f_z^{-1}$ restricted to $B(f_z(z), r_A)$ takes $f_z(z)$ to $f_y(z)$ which is a neighbour of $f_y(y)$ and coincides on this ball with and element in H_0. Let us discuss the idea of the proof. First, for two neighbours

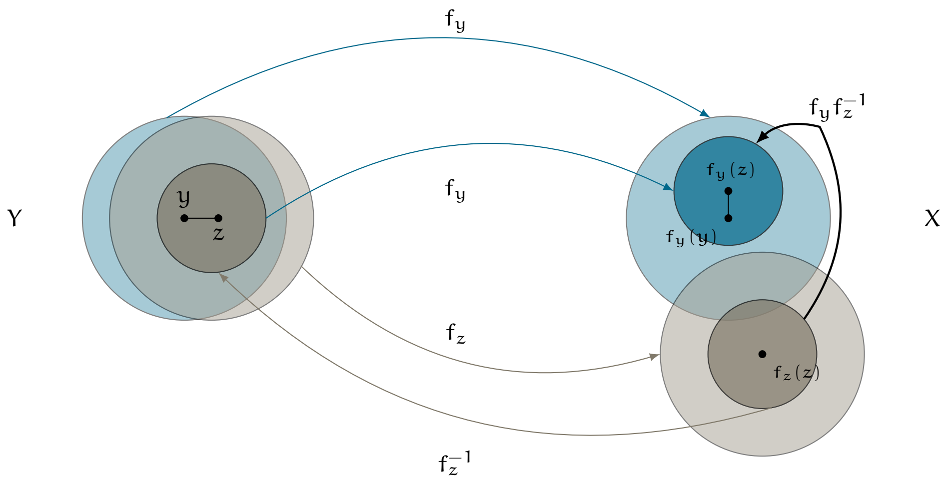


Figure 3.4.: Composition of isometries with neighbouring centers

y and z we use Proposition 3.1.1 to prove that $f_y f_z^{-1}$ coincides on a small ball with an element a in $\mathrm{Isom}(X)$. This isometry corresponds to the “default” of belonging to H_0 we want to correct. Hence, we consider in our atlas the new isometry defined on $B(z, R)$ by af_z. Finally, we extend this construction along paths in Y and prove that the wanted property for $\mathfrak{A}$ does not depend on the choice of path.

Proof of Lemma 3.2.4. Let $r_A > 0$ and let $H_0 := \rho^{-1}\mathrm{PSL}_n(\mathbb{K})$. Now, let $y \in Y$ and f_y be an isometry from $B(y, R)$ to X. Let z be a neighbour of y in Y and $\tilde{f}_z$ be an isometry from $B(z, R)$ to X. Then the map

$$f_y \cdot \tilde{f}_z^{-1} \ : \ B_X\Big(\tilde{f}_z(z), R-1\Big) \to B_X\left(f_y(z), R-1\right)$$

is a well defined local-isometry of X. By Proposition 3.1.1 if R is large enough, there exists a in $\mathrm{Isom}(X)$ such that $f_y \cdot \tilde{f}_z^{-1}$ coincides with a on $B_X\left(\tilde{f}_z(z), r_A + k\right)$, where we recall that k refers to the scale at which Y is simply connected. We will see below why we need to consider such a radius.

Now let $f_z := a\tilde{f}_z$. By definition we have

$$f_z \ : \ \begin{cases} B_Y(z, R) & \to B_X\left(f_y(z), R\right), \\ z & \mapsto a\tilde{f}_z(z) = f_y(z), \end{cases}$$

thus the transition map $f_y f^{-1}{}_z$ is well defined on $B_X(f_z(z), R-1)$. Moreover, by choice of f_z we get that $f_y f^{-1}{}_z$ restricted to $B(f_y(z), r_A + k)$ coincides with the identity and thus belongs to H_0.

Extending this construction along paths in Y we get an atlas $\mathfrak{A}$ of local isometries from Y to X.

Now if $y \in Y$ and f_y is the associated isometry in $\mathfrak{A}$, we want to show that (up to a multiplication by an element in $\mathrm{PSL}_n(\mathbb{K})$) this isometry does not depend on the choice of path. So let $y \in Y$ and $(y_0 = y, y_1, \dots, y_l = y)$ be a loop of length l. Take f_0 to be an isometry from $B_Y(y_0, R)$ to X and using the process detailed above, build a sequence of isometries $f_1, \dots, f_l$ such that f_i is defined on $B_Y(y_i, R)$ and

$$\forall i \in \{1, \dots, l\}\ \exists a_i \in H_0 \mid \left(f_{i-1}f_i^{-1}\right)_{|B(f_i(y_i), r_A+k)} = a_{i|B(f_i(y_i), r_A+k)}.$$

We have to prove that the restrictions to $B(y_0, r_A)$ of f_0 and f_l are equal up to a multiplication by an element in H_0. Since Y is simply connected at scale k, we only have to prove this for loops of length smaller than k. Hence, we assume that $l \leqslant k$.

First, remark that for all $i \in \{0, \dots, l-1\}$

$$\begin{cases} f_{i-1}f_i^{-1} \ : \ B_X\left(f_i(y_i), r_A + k\right) & \to B_X\left(f_{i-1}(y_i), r_A + k\right), \\ f_i f_{i+1}^{-1} \ : \ B_X\left(f_{i+1}(y_{i+1}), r_A + k\right) & \to B_X\left(f_i(y_{i+1}), r_A + k\right). \end{cases}$$

Now since y_i and y_{i+1} are at distance 1, the ball $B_X\left(f_i(y_{i+1}), r_A + k - 1\right)$ is included in $B_X(f_i(y_i), r_A + k)$. Hence the map $\left(f_{i-1}f_i^{-1}\right)\left(f_i f_{i+1}^{-1}\right)$ is well defined and coincides with $a_i a_{i+1}$ on $B_X\left(f_{i+1}(y_{i+1}), r_A + k - 1\right)$.

By induction we get that for all x in $B_X\left(f_{i+1}(y_{i+1}), r_A + k - l + 1\right)$

$$f_0 f_l^{-1}(x) = \left(f_0 f_1^{-1}\right) \cdots \left(f_{l-1}f_l^{-1}\right)(x) = a_1 \cdots a_l(x).$$

Since $\prod_{i=1}^{l} a_i$ belongs to H_0 and l is smaller than k, it implies that f_0 is equal to f_l on $B_Y(y_0, r_A)$ up to multiplication by an element in H_0. □

The atlas is defined such that a transition map between two isometries defined on balls with neighbouring centers belongs to H_0. But in fact, this property is also true when the centers are at a slightly bigger distance.

Lemma 3.2.5

Let $r > 0$ and $\mathfrak{A}$ be an atlas verifiying the conditions of Lemma 3.2.4 with $r_A > 3r$. Let y and z in Y be at distance less than $2r$ and f_y, f_z the associated isometries in $\mathfrak{A}$. Then

$$\exists a \in H_0 \qquad (f_y f_z^{-1})_{|B_Y(z,r)} = a_{|B_Y(z,r)}. \tag{3.3}$$

Proof. Let $r > 0$ and assume $r_A > 3r$. Let $y, z \in Y$ be at distance $l \leqslant 2r$ and let f_y, f_z be two elements of $\mathfrak{A}$ such that

$$f_y \ : \ B_Y(y, R) \to X \quad f_z \ : \ B_Y(z, R) \to X.$$

Take $(y_0 = y, y_1, \dots, y_l = z)$ to be a geodesic between y and z, and for all $i \in \{0, \dots, l\}$, let $f_i \in \mathfrak{A}$ be the isometry associated to y_i. Remark that by definition of an atlas, it implies $f_0 = f_y$ and $f_l = f_z$ and

$$\forall i \in \{0, \dots, l-1\} \quad \exists a_i \in H_0 \quad (f_i f_{i+1}^{-1})_{|B(f_{i+1}(y_{i+1}), r_A)} = a_{i|B(f_{i+1}(y_{i+1}), r_A)}.$$

Now, if $r_A > 3r$ and $l \leqslant 2r$, then $B_Y(z, r)$ is contained in $B_Y(y, r_A)$. Hence the composition of transition maps $(f_0 f_1^{-1}) \cdots (f_{-1} f_r^{-1})$ is well defined on $B_Y(f_l(y_l), r_A - l)$ and verifies on that ball

$$f_0 f_l^{-1} = (f_0 f_1^{-1}) \cdots (f_{l-1} f_l^{-1}) = a_0 \cdots a_{l-1}. \tag{3.4}$$

Hence the result. □

3.2.2 Prints in Y

Using the atlas built above, we can now transfer this print notion to the graph Y. Let $r_{\mathcal{P}} > 0$ and assume that Y is endowed with an atlas of isometries $\mathfrak{A}$ as given by Lemma 3.2.4 with $r_A > 3r_{\mathcal{P}}$. Hence, we have

$$R > r_A > 3r_{\mathcal{P}} > r_{\mathcal{P}}.$$

Definition 3.2.6

Let P be a set of vertices in Y. We say that P is a **print** if there exists y in Y and $f \in \mathfrak{A}$ an isometry from $B_Y(y, R)$ to X such that

- The set P is contained in $B_Y(y, r_{\mathcal{P}})$;
- There exists $x \in \mathcal{X} \backslash \mathrm{im}(q)$ such that $\mathcal{P}(x) = qf(P)$.

Remark 3.2.7. Note that in the definition above we ask that x does *not* belong to $\mathrm{im}(q)$. The definition would also make sense if x belonged to $\mathrm{im}(q)$ but the purpose of these prints is to reconstruct the "missing" vertices, namely vertices that are not in the image of q. Thus to simplify formalism in the next pages, we chose to restrict now the definition to prints of vertices in $\mathcal{X} \backslash \mathrm{im}(q)$.

Example 3.2.8. If $n = 3$ and $p = 2$ there are exactly 3 types of vertices, each represented in Figure 3.5 by a different color. The 1-neighbourhood of a vertex x in $\mathcal{X}$ is then composed of fourteen vertices, represented on the right side of the aforementioned figure (where x is the brown vertex at the center). If $x \in \mathcal{X} \backslash \mathrm{im}(q)$ then seven of these fourteen vertices are in $\mathrm{im}(q)$ (the blue vertices). On the left side of the figure is represented P (the black dots) inside $B(y, r_{\mathcal{P}})$ (the darker disc). The set $qf(P)$ is exactly the set of blue vertices. Hence P is a print.

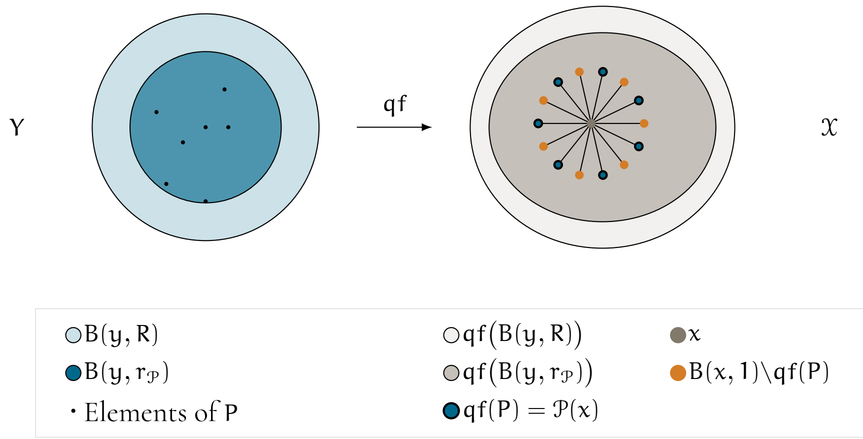


Figure 3.5.: Definition of a print in $\mathcal{Y}$

For now, let's say that P verifying the definition above is a print *associated to* y *and* f. We are going to show that this definition depends neither on y nor f.

Lemma 3.2.9

> Let $y_1, y_2 \in Y$ and f_1, f_2 be the associated isometries in $\mathfrak{A}$. Let P be a print associated to y_1 and f_1. If $P \subset B(y_2, r_{\mathcal{P}})$ then P is a print associated to y_2 and f_2.

Proof. First, remark that since $P \subset B(y_2, r_{\mathcal{P}}) \cap B(y_1, r_{\mathcal{P}})$, then taking any y in P we get

$$d_Y(y_1, y_2) \leqslant d_Y(y_1, y) + d_Y(y, y_2) \leqslant 2r_{\mathcal{P}}.$$

Applying Lemma 3.2.5 with $r = r_{\mathcal{P}}$, we get that there exists $a \in H_0$ such that

$$\left(f_1 f_2^{-1}\right)_{|B_X(f_2(y_2), r_{\mathcal{P}})} = a_{|B_X(f_1(y_2), r_{\mathcal{P}})}.$$

Now let $x \in \mathcal{X}$ be such that $\mathcal{P}(x) = qf_1(P)$. Using the equivariance of q and Lemma 2.3.7, we get

$$qf_2(P) = \rho(a)^{-1} qf_1(P) = \rho(a)^{-1}\mathcal{P}(x) = \mathcal{P}\left(\rho(a)^{-1}(x)\right).$$

Hence P is a print associated to y_2 and f_2. $\square$

This last lemma proves that being a print does not depend on the choice of local isometry.

Remark 3.2.10. In the above proof $\rho(a)^{-1}(x)$ has same type as x since $\rho(a)$ is type preserving. Thus, once we have taken our atlas in $PSL_n(\mathbb{K})$, the type of the source of $qf(P)$ does not depend on the choice of local isometry f.

3.2.3 Definition of $\mathcal{Y}$: a building's replica

The following property defines the graph $\mathcal{Y}$ we will demonstrate to be locally the same as $\mathcal{X}$.

Proposition 3.2.11

Let $r_{\mathcal{P}} > 0$ and $\mathfrak{A}$ be the atlas given by Lemma 3.2.4 for $r_A > 3r_{\mathcal{P}}$. If R is large enough, then the following graph is well defined.
Let $\mathcal{Y}$ be the graph whose vertices are given by

$$V(\mathcal{Y}) := V(Y) \sqcup \{P \ : \ \exists x \in \mathcal{X} \backslash \mathrm{im}(q), \ \mathcal{P}(x) = P\},$$

and edges are given by:

- If $y_1, y_2 \in V(\mathcal{Y})$, then (y_1, y_2) is an edge if there exists z in Y and $f \in \mathfrak{A}$ defined on $B_Y(z, R)$ such that $y_1, y_2 \in B(z, r_{\mathcal{P}})$ and $d_{\mathcal{X}}(qf(y_1), qf(y_2)) = 1$.
- If $y \in V(\mathcal{Y})$ and P is a print, then (y, P) is an edge if there exists z in Y and $f \in \mathfrak{A}$ defined on $B_Y(z, R)$ cointaining y and P and such that $qf(y)$ is at distance 1 from the source of $qf(P)$.
- If P_1 and P_2 are two prints, then (P_1, P_2) is an edge if there exists z in Y and $f \in \mathfrak{A}$ defined on $B_Y(z, R)$ such that $P_1, P_2 \subset B_Y(z, r_{\mathcal{P}})$ and such that the source of $qf(P_1)$ is at distance 1 from the source of $qf(P_2)$.

Before looking at the proof of this property, let us sketch some part of this graph.

Example 3.2.12. If $n = 4$ then $\mathcal{X}$ is composed of vertices of type 0, 1, 2 and 3. Assume that $q(X)$ is composed of vertices of type 0 and 2, then $T = \{1, 3\}$ and we saw the corresponding partition of $\mathcal{X}$ in Example 3.2.2 and Figure 3.3. The appearance of the corresponding $V(\mathcal{Y})$ is represented in Figure 3.6.

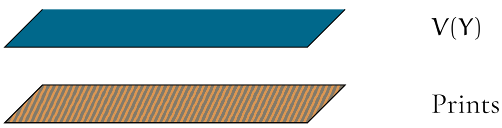


Figure 3.6.: Schematic view of $V(\mathcal{Y})$ in the case of Example 3.2.12

Proof. Let $\mathcal{Y}$ be as in Proposition 3.2.11 and let us show that the definition of the edges does not depend on the choice of f in the atlas.

First, let $y_1, y_2 \in Y$ and $y, z \in Y$ such that y_1 and y_2 belong to $B(y, r_{\mathcal{P}}) \cap B(z, r_{\mathcal{P}})$. Then, take two local maps f_y, f_z in $\mathfrak{A}$ associated to y and z respectively. Then $d(y, z) \leqslant 2r_{\mathcal{P}}$ and by Lemma 3.2.5 there exists $a \in \mathrm{Isom}(X)$ verifying eq. (3.3). Hence, by $\mathrm{Isom}(X)$-equivariance of q we get

$$\begin{aligned} d_{\mathcal{X}}\Big(qf_z(y_1), qf_z(y_2)\Big) &= d_{\mathcal{X}}\Big(\rho(a)qf_z(y_1), \rho(a)qf_z(y_2)\Big) \\ &= d_{\mathcal{X}}\Big(q\left(af_z(y_1)\right), q\left(af_z(y_2)\right)\Big) = d_{\mathcal{X}}\Big(qf_y(y_1), qf_y(y_2)\Big). \end{aligned}$$

Thus $d_{\mathcal{X}}\Big(qf_z(y_1), qf_z(y_2)\Big) = 1$ if and only if $d_{\mathcal{X}}\Big(qf_y(y_1), qf_y(y_2)\Big) = 1$ and the definition of edges between two vertices of Y does not depend on the choice of local isometry.

Now take $y \in Y$ and let $P \subset Y$ be a print. Let z and z' such that y and P are contained in $B(z, r_{\mathcal{P}}) \cap B(z', r_{\mathcal{P}})$ and take f (resp. f') in $\mathfrak{A}$ defined on $B(z, R)$ (resp. $B(z', R)$). Then $d(z, z') \leqslant 2r_{\mathcal{P}}$ and by Lemma 3.2.5 there exists $a \in \mathrm{Isom}(X)$ verifying eq. (3.3). Hence,

$$\begin{aligned} d_{\mathcal{X}}\Big(qf(y), x\Big) &= d_{\mathcal{X}}\Big(\rho(a)qf(y), \rho(a)(x)\Big) \\ &= d_{\mathcal{X}}\Big(q\left(af(y)\right), \rho(a)(x)\Big) = d_{\mathcal{X}}\Big(qf'(y), \rho(a)(x)\Big). \end{aligned}$$

If x is the source of $qf(P)$ then, by Lemma 2.3.7 we get

$$\mathcal{P}(\rho(a)(x)) = \rho(a)\left(\mathcal{P}(x)\right) = \rho(a)qf(P) = qf'(P).$$

Thus, the existence of en edge between y and P in $\mathcal{Y}$ does not depend of the choice of map in $\mathfrak{A}$.

Finally, take $P_1, P_2 \subset Y$ two prints and let z, z' in Y and $f \in \mathfrak{A}$ (resp. f') defined on $B_Y(z, R)$ (resp. $B(z', R)$) such that $P_1, P_2 \subset B_Y(z, r_{\mathcal{P}}) \cap B_Y(z', r_{\mathcal{P}})$. Again $d(z, z') \leqslant 2r_{\mathcal{P}}$ and by Lemma 3.2.5 there exists $a \in \mathrm{Isom}(X)$ verifying eq. (3.3). Hence if x_1 is the source of $qf(P_1)$ and x_2 the source of $qf(P_2)$, then $d(x_1, x_2) = 1$ if and only if $d(\rho(a)(x_1), \rho(a)(x_2)) = 1$. Moreover, by Lemma 2.3.7

$$\forall i = 1,2 \quad \mathcal{P}(\rho(a)(x_i)) = \rho(a)\left(\mathcal{P}(x_1)\right) = \rho(a)qf(P_i) = qf'(P_i).$$

Hence the existence of en edge between P_1 and P_2 in $\mathcal{Y}$ does not depend of the choice of map in the atlas $\mathfrak{A}$. $\square$

3.3 FROM ONE GRAPH TO THE OTHER

In this section we prove the isometry between the graph $\mathcal{Y}$ built and the Bruhat-Tits building and show that it induces an isometry between X and Y.

3.3.1 Isometry with the building

We can now prove that $\mathcal{Y}$ is isometric the Bruhat-Tits building. Recall that r_A is the radius used to define our atlas $\mathfrak{A}$ (see Lemma 3.2.4) and $r_{\mathcal{P}}$ is the radius used to define prints in $\mathcal{Y}$ (see Definition 3.2.6). These constants verify $R > r_A > 3r_{\mathcal{P}} > r_{\mathcal{P}}$.

Lemma 3.3.1

> Let $R_{\mathcal{X}} > 0$. If $r_{\mathcal{P}}$ (and hence R) is large enough, then $\mathcal{Y}$ is $R_{\mathcal{X}}$-locally $\mathcal{X}$.

To prove this lemma, we define explicitely the local isometries on balls of radius $R_{\mathcal{X}}$ and prove that these maps are well defined injections. Then, we compute the minimal value of $r_{\mathcal{P}}$ necessary for these applications to be surjective on balls of radius $R_{\mathcal{X}}$. We conclude by showing that these maps preserve the distance.

Proof. Let $v \in V(\mathcal{Y})$. If $v \in V(Y)$ let $f \in \mathfrak{A}$ be the isometry defined on $B_Y(v, R)$. If v is a print P let y and $f \in \mathfrak{A}$ be such that P is a print associated to y and f. Our goal is to show that the map

$$\phi_f \;:\; \begin{cases} B_{\mathcal{Y}}(v, R_{\mathcal{X}}) & \to \mathcal{X}, \\ z \in Y & \mapsto qf(y), \\ Q & \mapsto x \quad \text{where } \mathcal{P}(x) = qf(Q), \end{cases}$$

is an isometry.

By Proposition 2.3.4, it is a well defined map. Moreover, using the injectivity of q and Proposition 2.3.4 and eq. (3.1) we get that ϕ_f is an injective map.

Now, recall that since q is a quasi-isometry, two elements $q(x_1)$ and $q(x_2)$ joined by an edge in $\mathcal{X}$ might be at distance greater than 1 in X. If we want to prove that ϕ_f is surjective on $B_{\mathcal{X}}(\phi_f(v), R_{\mathcal{X}})$ and preserves the distance, we have to show that there exists a radius $r_{\mathcal{P}}$ allowing us to "reconstruct" all the edges of $B_{\mathcal{X}}(\phi_f(v), R_{\mathcal{X}})$ in $B_{\mathcal{Y}}(v, R_{\mathcal{X}})$. Let $L, \varepsilon > 0$ be such that q is a (L, ε)-quasi-isometry. We distinguish three cases, represented in Figure 3.7.

If $\chi_1, \chi_2 \in \mathrm{im}(q)$, then let $x_1, x_2 \in X$ such that $q(x_i) = \chi_i$. They verify $d_X(x_1, x_2) \leqslant Ld_{\mathcal{X}}(\chi_1, \chi_2) + \varepsilon$. This case is represented in Figure 3.7a.

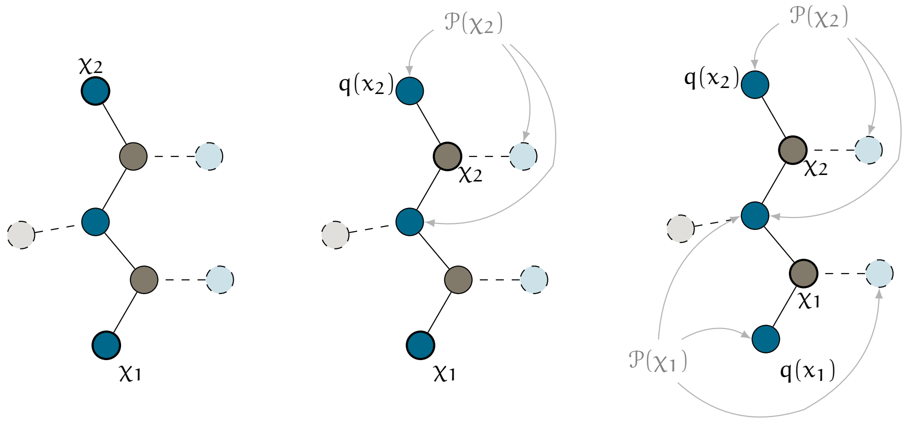
If $\chi_1 \in \mathrm{im}(q)$ and $\chi_2 \notin \mathrm{im}(q)$, let $x_1 = q^{-1}(\chi_1)$. For all $x_2 \in X$ such that $q(x_2) \in \mathcal{P}(\chi_2)$, we have (see Figure 3.7b)

$$d_{\mathcal{X}}(q(x_1), q(x_2)) \leqslant 1 + d_{\mathcal{X}}(\chi_1, \chi_2) \ \Rightarrow d_X(x_1, x_2) \leqslant Ld_{\mathcal{X}}(\chi_1, \chi_2) + L + \varepsilon.$$

If $\chi_1, \chi_2 \notin \mathrm{im}(q)$, let $x_i \in X$ such that $q(x_i) \in \mathcal{P}(\chi_i)$ for $i = 1, 2$. Then (see Figure 3.7b)

$$d_{\mathcal{X}}(q(x_1), q(x_2)) \leqslant 2 + d_{\mathcal{X}}(\chi_1, \chi_2) \ \Rightarrow d_X(x_1, x_2) \leqslant Ld_{\mathcal{X}}(\chi_1, \chi_2) + 2L + \varepsilon.$$

Hence, assume $r_{\mathcal{P}} > LR_{\mathcal{X}} + 2L + \varepsilon$ and let us show that ϕ_f is an isometry.



(a) First case (b) Second case (c) Third case

Figure 3.7.: The three cases ($\mathrm{im}(q)$ is represented by the blue vertices)

Let $\chi \in B_{\mathcal{X}}(\phi_f(v), R_{\mathcal{X}})$, by choice of $r_{\mathcal{P}}$ either $\chi \in \mathrm{im}(q)$ and then there exists $z \in B_Y(y, r_{\mathcal{P}})$ such that $qf(z) = \chi$ or $\chi \notin \mathrm{im}(q)$ and then there exists $P \subset B_Y(y, r_{\mathcal{P}})$ such that $qf(P) = \mathcal{P}(\chi)$. Hence, in both cases $\chi \in \mathrm{im}(\phi_f)$ and thus, ϕ_f is a bijection from $B_{\mathcal{Y}}(v, R_{\mathcal{X}})$ to $B_{\mathcal{X}}(\phi_f(v), R_{\mathcal{X}})$. Now take v_1, v_2 in $B_{\mathcal{Y}}(v, R_{\mathcal{X}})$ at distance l in $\mathcal{Y}$ and let $(w_0 = v_1, w_1, \dots, w_l = v_2)$ be a geodesic in $\mathcal{Y}$. By definition of $\mathcal{Y}$ and choice of $r_{\mathcal{P}}$, for all $i \in \{0, \dots, l-1\}$ if there is an edge between w_i and w_{i+1}, then $d(\phi_f(w_i), \phi_f(w_{i+1})) = 1$. Hence $d_{\mathcal{X}}(\phi_f(v_1), \phi_f(v_2)) \leqslant l$. To get the reversed inequality, take χ_1, χ_2 in $B_{\mathcal{X}}(\phi_f(v), R_{\mathcal{X}})$. Since ϕ_f is bijective there exists $v_0, \dots, v_l$ in $\mathcal{Y}$ such that $(\phi_f(v_0), \dots, \phi_f(v_l))$ is a geodesic between χ_1 and χ_2. Again, by definition of $\mathcal{Y}$ and choice of $r_{\mathcal{P}}$, an edge between $\phi_f(v_i)$ and $\phi_f(v_{i+1})$ gives an edge between v_i and v_{i+1} in $\mathcal{Y}$ and thus $d_{\mathcal{Y}}(v_1, v_2) \leqslant l$.

Hence, if $r_{\mathcal{P}} > LR_{\mathcal{X}} + 2L + \varepsilon$ then ϕ_f is an isometry. $\square$

The LG-rigidity of the building will give us a covering from $\mathcal{X}$ to $\mathcal{Y}$. In order to obtain an *isometry* we need to prove (by Proposition 1.1.11) that $\mathcal{Y}$ is simply connected at the same scale as $\mathcal{X}$.

Lemma 3.3.2

> If $R_{\mathcal{X}}$ (and hence R) is large enough, then $\mathcal{Y}$ is simply connected at scale 3.

We first prove that $\mathcal{Y}$ is quasi-isometric to Y and use it to show that $\mathcal{Y}$ is simply connected at some scale k'. We conclude using the contractibility of the building and the fact that $\mathcal{Y}$ is locally the same as the building. But before looking at the detail of the proof, let us make a remark.

Remark 3.3.3. Let P be a print associated to some $z \in Y$ and $f \in \mathfrak{A}$ and let $y \in P$. If x is the source of $qf(P)$, then $d_{\mathcal{Y}}(P, y) = d_{\mathcal{X}}(x, qf(y)) = 1$.

Proof of Lemma 3.3.2. Let us show that $\mathcal{Y}$ is quasi-isometric to Y. Define $\pi : \mathcal{Y} \to Y$ such that if $y \in V(Y)$ then $\pi(y) = y$ and if P is a print then $\pi(P) = y$ for some $y \in P$ arbitrarily chosen. Let $(v_0, \dots, v_m)$ be a geodesic in $\mathcal{Y}$ and for all $i \in \{0, \dots, m\}$ define $y_i := \pi(v_i)$ and f_i to be the isometry of $\mathfrak{A}$ associated to y_i. Using that q is a (L, ε)-quasi-isometry, we get

$$\begin{aligned} d_Y\left(\pi(v_0), \pi(v_m)\right) = d_Y(y_0, y_m) &\leqslant \sum_{i=0}^{m} d_Y\left(y_i, y_{i+1}\right), \\ &\leqslant \sum_{i=0}^{m} \left[L d_{\mathcal{X}}\left(qf_i(y_i), qf_i(y_{i+1})\right) + \varepsilon\right]. \end{aligned}$$

Now let $i \in \{0, \dots, m\}$. If v_i is a print, denote by x_i the source of $qf(v_i)$ and if v_i belongs to the copy of $V(Y)$ contained in $\mathcal{Y}$ let $x_i := qf_i\pi(v_i)$. Then $d_{\mathcal{Y}}\left(v_i, v_{i+1}\right) = d_{\mathcal{X}}(x_i, x_{i+1})$ for all i. Thus, using remark 3.3.3, we get

$$\begin{aligned} d_{\mathcal{X}}\left(qf_i(y_i), qf_i(y_{i+1})\right) &\leqslant d_{\mathcal{X}}\left(qf_i(y_i), x_i\right) + d_{\mathcal{X}}(x_i, x_{i+1}) + d_{\mathcal{X}}\left(qf_i(y_{i+1}), x_{i+1}\right), \\ &\leqslant 2 + d_{\mathcal{X}}(x_i, x_{i+1}) = 2 + d_{\mathcal{Y}}\left(v_i, v_{i+1}\right). \end{aligned}$$

Since $d_{\mathcal{Y}}\left(v_i, v_{i+1}\right) = 1$, we obtain

$$\begin{aligned} d_Y\left(\pi(v_0), \pi(v_m)\right) = d_Y(y_0, y_m) &\leqslant \sum_{i=0}^{m} \left[L2 + L d_{\mathcal{Y}}\left(v_i, v_{i+1}\right) + \varepsilon\right], \\ &= (3L + \varepsilon)m = (3L + \varepsilon) d_{\mathcal{Y}}\left(v_0, v_m\right). \end{aligned}$$

Now let $v, v' \in \mathcal{Y}$ and let $(\pi(v) = z_0, \dots, \pi(v') = z_l)$ be a geodesic in Y. For all $i \in \{0, \dots, l\}$ take $f'_i \in \mathfrak{A}$ the isometry associated to z_i. Then

$$d_{\mathcal{Y}}(v, v') \leqslant d_{\mathcal{Y}}(v, z_0) + \sum_{i=0}^{l-1} d_{\mathcal{Y}}(z_i, z_{i+1}) + d_{\mathcal{Y}}(z_l, v').$$

But by remark 3.3.3 if v (resp. v') is a print then $d_{\mathcal{Y}}(v, z_0) = 1$ (resp. $d_{\mathcal{Y}}(v', z_l) = 1$). And if v (resp. v') belongs to $V(Y)$ then $v = z_0$ (resp. $v' = z_l$). Thus both $d_{\mathcal{Y}}(v, z_0)$ and $d_{\mathcal{Y}}(v', z_l)$ are always smaller than 1. Hence,

$$\begin{aligned} d_{\mathcal{Y}}(v, v') \leqslant 2 + \sum_{i=0}^{l-1} d_{\mathcal{Y}}(z_i, z_{i+1}) &= 2 + \sum_{i=0}^{l-1} d_{\mathcal{X}}\left(qf'_i(z_i), qf'(z_{i+1})\right), \\ &\leqslant 2 + \sum_{i=0}^{l-1} \left[L d_Y(z_i, z_{i+1}) + \varepsilon\right], \\ &= 2 + (L + \varepsilon)l = 2 + (L + \varepsilon) d_Y(\pi(v), \pi(v')). \end{aligned}$$

Thus π is a quasi-isometry between $\mathcal{Y}$ and Y. Hence Proposition 1.1.12 implies that there exists $k' \in \mathbb{N}^*$ such that $\mathcal{Y}$ is simply-connected at scale k'.

Finally, let ℓ be loop in $\mathcal{Y}$ of length less than k'. If $R_{\mathcal{X}}$ is large enough then ℓ is contained in some ball B in $\mathcal{Y}$. By Lemma 3.3.1 there exists a local isometry ϕ from B to some ball $\mathcal{B}$ in $\mathcal{X}$. But $\phi(\ell)$ is contractible inside its convex hull, by Claim 2.2.2. In particular it is simply-connected. Since $\mathcal{X}$ is 3-simply-connected and if $R_{\mathcal{X}}$ is large enough, the convex hull of $\phi(\ell)$ is contained in the complex obtained by gluing triangles on all the loops of length 3 in $\mathcal{B}$. Which, by local isometry with B, proves the wanted assertion.

□

Thanks to the previous lemma, we can now use the rigidity of the Bruhat-Tits building.

Proposition 3.3.4

If $R_{\mathcal{X}}$ (and hence R) is large enough, then $\mathcal{Y}$ is isometric to $\mathcal{X}$.

Proof. Recall that we have $R > r_A > 3r_{\mathcal{P}} > r_{\mathcal{P}} > 3R_{\mathcal{X}} + 2L + \varepsilon > R_{\mathcal{X}}$.
By Theorem 1.2.1, the building $\mathcal{X}$ is LG-rigid. Moreover, since its isometry group is transitive Proposition 1.1.11 gives us the existence of some radius $R_{sc} > 0$ such that every graph which is 3-simply connected and R_{sc}-locally $\mathcal{X}$ is isometric to $\mathcal{X}$.
By definition of the edges on $\mathcal{Y}$, this graph is simply connected at scale 3. Taking $r_{\mathcal{P}}$ (and hence R) large enough so that $R_{\mathcal{X}} \geqslant R_{sc}$ the preceding paragraph combined with Lemma 3.3.1 give us the existence of an isometry between $\mathcal{X}$ and $\mathcal{Y}$. □

3.3.2 Change of local map, change of global isometry

Let $y \in Y$ and $f_y \in \mathfrak{A}$ be the isometry defined on $B(y, R)$. Let

$$\phi_y \;:\; \begin{cases} B_{\mathcal{Y}}(y, R_{\mathcal{X}}) & \to \mathcal{X} \\ z \in Y & \mapsto qf_y(z) \\ Q & \mapsto x \qquad \text{where } \mathcal{P}(x) = qf_y(Q). \end{cases} \tag{3.5}$$

Lemma 3.3.5

Let y and z be neighbours in Y and $a \in H_0$ such that $f_y f_z^{-1}$ coincide with a on $B_X(f(z), r_A)$. If $R_{\mathcal{X}}$ is large enough, then $\phi_y \phi_z^{-1}$ coincide with $\rho(a)$ on $B_{\mathcal{X}}(\phi_z(z), 2)$.

Proof. Let y and z be neighbours in Y and $a \in H_0$ such that $f_y f_z^{-1}$ coincide with a on $B_X(f(z), r_A)$. If $R_{\mathcal{X}}$ (and hence R) is large enough, then $B_{\mathcal{Y}}(z, 2)$ is contained in $B_{\mathcal{Y}}(y, R_{\mathcal{X}})$. Thus, $\phi_y \phi_z^{-1}$ is well defined on $B_{\mathcal{X}}(\phi_z(z), 2)$.

Let $v \in B_{\mathcal{Y}}(z, 2)$. If $v \in V(Y)$, then

$$\phi_y(v) = qf_y(v) = qaf_z(v) = \rho(a)qf_z(v) = \rho(a)\phi_z(v).$$

If $v = P$ with $P \subset Y$ a print, then

$$\mathcal{P}\big(\phi_y(v)\big) = qf_y(P) = qaf_z(P) = \rho(a)qf_z(P) = \mathcal{P}\big(\rho(a)\phi_z(v)\big),$$

Thus $\phi_y(v) = \rho(a)\phi_z(v)$, since the print determines the vertex. Hence the result. □

Now let $r_{\mathcal{X}} > 0$. If $R_{\mathcal{X}}$ is large enough then, by SLG-rigidity of $\mathcal{X}$ there exists an isometry ι_y from $\mathcal{Y}$ to $\mathcal{X}$ that coincides with ϕ_y on $B(y, r_{\mathcal{X}})$. Thus, the lemma above allows us to work with a set of isometries from $\mathcal{Y}$ to $\mathcal{X}$ that differs only by a multiplication by an element of $\mathsf{PSL}_n(\mathbb{K})$.

Lemma 3.3.6

> If y and z belong to Y and $R_{\mathcal{X}}$ is large enough, then $\iota_y \iota_z^{-1} \in \mathsf{PSL}_n(\mathbb{K})$. Hence for all $y \in Y$, the isometry ι_y sends the copy of $V(Y)$ contained in $\mathcal{Y}$ to $\mathrm{im}(q)$ and sends prints contained in $\mathcal{Y}$ to vertices in $\mathcal{X} \backslash \mathrm{im}(q)$.

Proof. Let y and z be neighbours in Y. Since $\iota_y \iota_z^{-1}$ is an isometry of $\mathcal{X}$ it permutes the $\mathsf{PSL}_n(\mathbb{K})$-orbits. Recall that ι_y coincides with ϕ_y on $B(y, r_{\mathcal{X}})$. Hence, if $r_{\mathcal{X}}$ (and hence R) is large enough, then $B_{\mathcal{Y}}(z, 2)$ is contained in $B_{\mathcal{Y}}(y, r_{\mathcal{X}})$, thus

$$\left(\iota_y \iota_z^{-1}\right)_{|B_{\mathcal{X}}(\iota_z(z),2)} = \phi_y \phi_z^{-1}.$$

But $\phi_y \phi_z^{-1}$ coincides with an element of $\mathsf{PSL}_n(\mathbb{K})$ on $B_{\mathcal{X}}(\phi_z(z), 2)$, by Lemma 3.3.5. Hence $\iota_y \iota_z^{-1}$ restricted to a ball of radius 2 preserves the $\mathsf{PSL}_n(\mathbb{K})$-orbits. Since such a ball contains a vertex of each type, it implies that $\iota_y \iota_z^{-1}$ preserves the $\mathsf{PSL}_n(\mathbb{K})$-orbits and thus belongs to $\mathsf{PSL}_n(\mathbb{K})$.

Now take y and z in Y (not necessarily neighbours), denote by $(y_0 = y, y_1 \dots, y_l = z)$ a geodesic in Y. By the preceding paragraph, there exists a sequence α_1, ..., α_l of elements in $\mathsf{PSL}_n(\mathbb{K})$ such that

$$\forall i \in \{1, \dots, l\} \quad \iota_{y_i} \iota_{y_{i-1}}^{-1} = \alpha_i.$$

Thus, recalling that $z = y_l$ and $y = y_0$, we get $\iota_z = \alpha_l \cdots \alpha_1 \iota_y$. Which proves the first assertion of the lemma.

Let us now prove the second part of the lemma. Let $y \in Y$ and $v \in \mathcal{Y}$. There exists $z \in Y$ such that $v \in B_{\mathcal{Y}}(z, 2)$, and using the paragraph above, there exists $\alpha \in \mathsf{PSL}_n(\mathbb{K})$ such that $\iota_y = \alpha \iota_z$. In particular, since v belongs to $B_{\mathcal{Y}}(z, R_{\mathcal{X}})$,

$$\iota_y(v) = \alpha \iota_z(v) = \alpha \phi_z(v).$$

By definition of ϕ_z, if $v \in V(Y)$ then $\phi_z(v)$ belongs to $\mathrm{im}(q)$ and if $v = P$ with $P \subset Y$ a print, then $\phi_z(v)$ belongs to $\mathcal{X} \backslash \mathrm{im}(q)$. This finish the proof of the lemma. $\square$

Now we have all the tools we need to prove the isometry between Y and X.

3.3.3 Isometry from Y to X

Let κ be the natural injection of Y in $\mathcal{Y}_Z$ and ι an isometry given by Proposition 3.3.4. With the objects constructed so far we get the diagram in Figure 3.8.

The aim of this section is to prove the following result.

Proposition 3.3.7

> For $R_{\mathcal{X}}$ large enough, the graphs Y and X are isometric.

Let us discuss the strategy of the proof. Using the preceding section, we chose an isometry ι from $\mathcal{Y}$ to $\mathcal{X}$ that coincides with a ϕ_y on a small ball. Then, we show that $\kappa \iota q^{-1}$ is

$$\begin{tikzcd}
X \arrow[r, hook, "q"] \arrow[d, dashed, leftrightarrow, "R\text{-loc}"'] & \mathcal{X} \\
Y \arrow[r, hook, "\kappa"'] & \mathcal{Y} \arrow[u, "\iota\text{, isom.}"'] \\
y \arrow[r, mapsto] & \kappa(y)
\end{tikzcd}$$

(a) Maps between graphs

$$\begin{tikzcd}
\mathrm{Isom}(X) \arrow[r, hook, "\rho"] & \mathrm{Isom}(\mathcal{X}) \\
a \arrow[r, mapsto] & \rho(a) \\
H_0 \arrow[r] & \mathrm{PSL}_n(\mathbb{K})
\end{tikzcd}$$

(b) Relations between groups

Figure 3.8.: Relations between the different graphs and groups

locally an isometry, *viz.* there exists a radius r_Y such that $q^{-1}\iota\kappa$ restricted to any ball of radius r_Y preserves the distance. We conclude by showing that it forces $\kappa\iota q^{-1}$ to be an isometry.

Proof of Proposition 3.3.7. By Lemma 3.3.6, for any $y \in Y$ the map $q^{-1}\iota_y\kappa$ is well defined. Now fix $y_0 \in Y$ and consider $\iota := \iota_{y_0}$. We want to prove that $q^{-1}\iota\kappa$ restricted to small balls preserves the distance. Then we will show that it is an isometry from Y to X.

Claim 3.3.8. Let $y \in Y$ and $r_Y \geqslant 1$. If R is large enough, then $q^{-1}\iota\kappa$ restricted to $B_Y(y, r_Y)$ preserves the distance.

Proof of the claim. Let $r_Y \geqslant 1$ and recall that we have $R > r_A > 3r_{\mathcal{P}} > r_{\mathcal{P}} > 3R_{\mathcal{X}} + 2L + \varepsilon > R_{\mathcal{X}} > r_{\mathcal{X}}$. Let $y \in Y$ and recall that L and ε are constants such that q is a (L, ε)-quasi-isometry. If $r_{\mathcal{X}} \geqslant Lr_Y + \varepsilon$ (and hence if R is large enough) then $\kappa(B_Y(y, r_Y))$ is included in $B_{\mathcal{Y}}(y, r_{\mathcal{X}})$. Indeed if $z \in B_Y(y, r_Y)$ then

$$d_{\mathcal{X}}\Big(qf_y(y), qf_y(z)\Big) \leqslant Ld_x\Big(f_y(y), f_y(z)\Big) + \varepsilon = Ld_Y(y, z) + \varepsilon \leqslant Lr_Y + \varepsilon \leqslant r_{\mathcal{X}}.$$

Thus $\phi_y(\kappa(z)) = qf_y(z)$ and

$$d_{\mathcal{Y}}\left(\kappa(y), \kappa(z)\right) = d_{\mathcal{X}}\left(\phi_y(\kappa(y)), \phi_y(\kappa(z))\right) = d_{\mathcal{X}}\Big(qf_y(y), qf_y(z)\Big) \leqslant R_{\mathcal{X}}$$

Now, recall that $H_0 = \rho^{-1}\mathrm{PSL}_n(\mathbb{K})$. Then, by Lemma 3.3.6 there exists $a_y \in H_0$ such that $\iota_y\iota^{-1} = \rho(a_y)$. Hence, using the equivariance of q we get that for all z_1 and z_2 in $B_Y(y, r_Y)$

$$\begin{aligned}
d_X\Big(q^{-1}\iota\kappa(z_1), q^{-1}\iota\kappa(z_1)\Big) &= d_X\Big(a_y q^{-1}\iota\kappa(z_1), a_y q^{-1}\iota\kappa(z_1)\Big) \\
&= d_X\Big(q^{-1}\rho(a_y)\iota\kappa(z_1), q^{-1}\rho(a_y)\iota\kappa(z_1)\Big) \\
&= d_X\Big(q^{-1}\iota_y\kappa(z_1), q^{-1}\iota_y\kappa(z_1)\Big).
\end{aligned}$$

But z_1 and z_2 belong to $B_Y(y, r_Y)$, hence for $i = 1, 2$ we have $\iota_y\kappa(z_i) = qf_y(z_i)$. Thus,

$$\begin{aligned}
d_X\Big(q^{-1}\iota\kappa(z_1), q^{-1}\iota\kappa(z_1)\Big) &= d_X\Big(q^{-1}qf_y(z_1), q^{-1}qf_y(z_2)\Big) \\
&= d_X\Big(f_y(z_1), f_y(z_2)\Big) = d_Y(z_1, z_2).
\end{aligned}$$

Thus $q^{-1}\iota\kappa$ restricted to $B_Y(y, r_Y)$ preserves the distance. $\square$

Let's show that the claim forces $q^{-1}\iota\kappa$ to be an isometry from Y to X. Take $r_Y \geqslant 2$ and let $y, y' \in Y$ and $(y_0 = y, y_1, \dots, y_l = y')$ be a geodesic in Y. Since for all i the vertices y_i and y_{i+1} are adjacent, then Claim 3.3.8 implies that $d_X(q^{-1}\iota\kappa(y_i), q^{-1}\iota\kappa(y_{i+1})) = 1$. Hence

$$d_X\left(q^{-1}\iota\kappa(y), q^{-1}\iota\kappa(y')\right) \leqslant \sum_{i=0}^{l-1} d_X\left(q^{-1}\iota\kappa(y_i), q^{-1}\iota\kappa(y_i)\right) = l.$$

Moreover, if $(x_0 = q^{-1}\iota\kappa(y), x_1, \dots, x_m = q^{-1}\iota\kappa(y'))$ is a geodesic in X, then by bijectivity of $q^{-1}\iota\kappa$ there exists $z_i \in Y$ such that $q^{-1}\iota\kappa(z_i) = x_i$ for all i in $\{1, \dots, m-1\}$. Denote $z_0 = y$ and $z_m = y'$. Since for all i the vertices x_i and x_{i+1} are adjacent, then Claim 3.3.8 implies that $d_X(z_i, z_{i+1}) = d_{\mathfrak{X}}(q^{-1}\iota\kappa(z_i), q^{-1}\iota\kappa(z_{i+1}))$. Thus

$$d_Y(y, y') \leqslant \sum_{i=0}^{m-1} d_Y(z_i, z_{i+1}) = \sum_{i=0}^{m-1} d_X\left(q^{-1}\iota\kappa(z_i), q^{-1}\iota\kappa(z_{i+1})\right) = \sum_{i=0}^{m-1} d_X(x_i, x_{i+1}) = m.$$

□

We conclude by the proof of Theorem 1.2.6.

Proof of Theorem 1.2.6. Let $n \neq 3$ and X verifying the hypothesis of Theorem 1.2.6. If $n = 2$ then $\mathfrak{X}$ is the $(p+1)$-regular tree, thus by Example 1.1.3 if X is quasi-isometric to $\mathfrak{X}$ then X is LG-rigid. If $n \geqslant 4$, let $k \in \mathbb{N}$ such that X is simply connected at scale k. Then by Proposition 3.3.7 for R large enough, any k-simply-connected graph Y being R-locally the same as X is isometric to X. Thus X is LG-rigid. Finally for any $n \neq 3$, since X is assumed transitive it is actually SLG-rigid by Proposition 1.1.14. □

RIGIDITY OF LATTICES

4

> — *Qu'est-ce que tu sais sur les réseaux cocompacts ?*
> — *Les risottos trop compacts ? Il faut les délayer avec du bouillon.*
>
> — **Conversation téléphonique avec F. Caron**

In this chapter we prove Theorem 1.2.5 which we recall hereunder.

Theorem 1.2.5

> Let $n \neq 3$ and $\mathbb{K}$ be a non-Archimedean local skew field of characteristic zero. The torsion-free lattices of $\mathsf{SL}_n(\mathbb{K})$ are SLG-rigid.

Let $n \neq 3$, let $\mathbb{K}$ be a non-Archimedean skew field of characteristic zero and $\Gamma \leqslant \mathsf{SL}_n(\mathbb{K})$ be a lattice without torsion. Denote by (Γ, S) one of its Cayley graphs. Recall that any lattice in $\mathsf{SL}_n(\mathbb{K})$ is uniform (*i.e.* cocompact).

4.1 QUASI-ISOMETRY BETWEEN BUILDING AND LATTICE

To show the theorem, we first check that the lattice is quasi-isometric to the building. Then, using a famous result of Kleiner and Leeb we show that the isometry group of the lattice acts on the building and that the quasi-isometry can be chosen to be equivariant under this action.

Lemma 4.1.1

> Let Λ be a lattice of $\mathsf{SL}_n(\mathbb{K})$. Then Λ is quasi-isometric to $\mathcal{X}$.

Proof. First, recall that any lattice in $\mathsf{SL}_n(\mathbb{K})$ is uniform, *viz.* cocompact (see for example [BQ14]).

Since Λ is a lattice of $\mathsf{SL}_n(\mathbb{K})$, there is a natural action on the Bruhat-Tits building induced by the action of $\mathsf{PSL}_n(\mathbb{K})$. Moreover, since Λ is cocompact and the $\mathsf{PSL}_n(\mathbb{K})$ action has exactly n orbits, the Λ action is also cocompact. Hence by the Svarc-Milnor's lemma Λ is quasi-isometric to X. □

By a result of Kleiner and Leeb [KL97] and Cornulier [Cor18, Theorem 3.B.1] applied to our lattice Γ, this quasi-isometry implies the existence of a homomorphism from $\mathrm{Isom}(\Gamma, S)$ to $\mathrm{Isom}(X)$ and a quasi-isometry from (Γ, S) to X which is $\mathrm{Isom}(\Gamma, S)$-equivariant. Since Γ is assumed to be torsion-free, we can refine the informations about these two applications.

Lemma 4.1.2

> Let Λ be a lattice of $\mathsf{SL}_n(\mathbb{K})$ and T a symmetric generating set. If Λ is torsion-free, then there exists an injective homomorphism
>
> $$\rho \;:\; \mathrm{Isom}(\Lambda, \mathsf{T}) \;\to \mathrm{Isom}(\mathsf{X}),$$
>
> and an injective quasi-isometry which is $\mathrm{Isom}(\Lambda, \mathsf{T})$-equivariant
>
> $$\mathsf{q} \;:\; (\Lambda, \mathsf{T}) \to \mathsf{X}.$$

Proof. Since we assumed that Λ has no torsion element, by Proposition 1.2.7 the isometry group of (Λ, T) contains no non-trivial compact normal subgroup. Hence the morphism ρ given by Kleiner-Leeb's theorem is injective.

Assume that there exist $\lambda_1, \lambda_2 \in \Lambda$ such that $\lambda_1 \neq \lambda_2$ and $\mathsf{q}(\lambda_1) = \mathsf{q}(\lambda_2)$. Then, the equivariance of q implies that

$$\mathsf{q}\left(\left\{\left(\lambda_1\lambda_2^{-1}\right)^n \;:\; n \in \mathbb{N}\right\}\right) = \{\mathsf{q}\,(e)\},$$

which contradicts the fact that q is a quasi-isometry. $\square$

4.2 RELATION BETWEEN THE ISOMETRY GROUPS

To apply Theorem 1.2.6, we still need to check that $\mathrm{Isom}(\Gamma, \mathsf{S})$ is of finite index in $\mathrm{Isom}(\mathcal{X})$. As stated in the lemma below, this is not always the case: the lattice's isometry group can also be discrete. But as we will see in Section 4.3 we will be able to prove the rigidity of the lattice in that case too.

Lemma 4.2.1

> Using the previous notations,
> - Either $\mathrm{Isom}(\Gamma, \mathsf{S})$ is discrete.
> - Or $\mathrm{Isom}(\Gamma, \mathsf{S})$ is of finite index in $\mathrm{Isom}(\mathsf{X})$ and contains $\mathsf{PSL}_n(\mathbb{K})$.

Before proving this lemma, let us recall a useful consequence of a theorem of Benoist and Quint. The original and more general statement can be found in [BQ14, Corollary 4.5].

Proposition 4.2.2 (*Benoist, Quint [BQ14]*)

> Let G be p-adic Lie group and H be a finite covolume closed subgroup of G, with Lie algebra $\mathfrak{h}$. If G has no proper cocompact normal subgroup, then G normalizes $\mathfrak{h}$.

Proof of Lemma 4.2.1. Let $\mathsf{G} = \mathsf{PSL}_n(\mathbb{K})$ and $\mathsf{H} = \mathrm{Isom}(\Gamma, \mathsf{S}) \cap \mathsf{G}$ and note $\mathfrak{h} =: \mathsf{Lie}(\mathsf{H})$ and $\mathfrak{g} := \mathsf{Lie}(\mathsf{G})$ their respective Lie algebras. Since Γ is a lattice in $\mathsf{SL}_n(\mathbb{K})$, we get that $\rho(\Gamma) \cap \mathsf{PSL}_n(\mathbb{K})$ is a lattice in $\mathsf{PSL}_n(\mathbb{K})$. Hence H contains the uniform lattice $\rho(\Gamma) \cap \mathsf{G}$ of G, thus H has finite covolume in $\mathsf{PSL}_n(\mathbb{K})$.
If $\mathbb{K}$ is a non-Archimedean local skew field of characteristic zero then it is an extension of $\mathbb{Q}_p$ for some prime p (see for example [dlST16, Section 1]). In particular G is a p-adic Lie group. Thus the above property applied to G and H implies that G normalises $\mathfrak{h}$, in other words $\mathfrak{h}$ is an ideal of $\mathfrak{g}$. Since $\mathfrak{g}$ is simple, we get that $\mathfrak{h}$ is either trivial or the full

Lie algebra $\mathfrak{g}$. If Isom(Γ, S) isn't discrete, then it is a closed subgroup of Isom(X). Hence H is a closed subgroup of G and its Lie algebra is non-trivial. By the previous point it can only be $\mathfrak{g}$. Hence, it implies that H is an open subgroup of G. Since it is also cocompact, it is necessarily of finite index in G. Thus, we get that $\rho\,(\text{Isom}(\Gamma, S))$ is of finite index in Isom(X).

Let's show that $\mathsf{PSL}_n(\mathbb{K}) \leqslant \rho\,(\text{Isom}\,(\Gamma, S))$. First assume that $\rho(\text{Isom}(\Gamma, S))$ is strictly contained in $\mathsf{PSL}_n(\mathbb{K})$. Since these two groups are of finite index in Isom(X), we get that $\rho\,(\text{Isom}(\Gamma, S))$ is of finite index in $\mathsf{PSL}_n(\mathbb{K})$. But then the core:

$$\bigcap_{g \in \mathsf{PSL}_n} g \cdot \rho\,(\text{Isom}(\Gamma, S)) \cdot g^{-1}$$

of $\rho\,(\text{Isom}(\Gamma, S))$ is itslef of finite index in $\mathsf{PSL}_n(\mathbb{K})$ (and different from $\mathsf{PSL}_n(\mathbb{K})$), which contradicts the simplicity of $\mathsf{PSL}_n(\mathbb{K})$.

Let us now go back to the general case. Assume that $\mathsf{PSL}_n(\mathbb{K})$ is not included in $\rho(\text{Isom}(\Gamma, S))$ and remark that:

$$\mathfrak{h} = \mathsf{Lie}\,(\text{Isom}(X)) = \mathsf{Lie}\,(\mathsf{PSL}_n(\mathbb{K}))\,.$$

In particular $\rho(\text{Isom}(\Gamma, S))$ is "locally" $\mathsf{PSL}_n(\mathbb{K})$ so, up to apply what precedes to an open set centered on e_Γ sufficiently small of $\rho(\text{Isom}(\Gamma, S))$, we obtain a contradiction.
Hence $\mathsf{PSL}_n(\mathbb{K})$ is contained in $\rho(\text{Isom}(\Gamma, S))$. □

4.3 RIGIDITY OF LATTICES

We conclude by the proof of Theorem 1.2.5.

Proof of Theorem 1.2.5. Let $n \neq 3$ and p be a prime. Let Γ be a torsion-free lattice of $\mathsf{PSL}_n(\mathbb{K})$ and S be a symmetric generating part.
If $n = 2$, then $\mathcal{X}$ is the $(p+1)$-regular tree. Since by Lemma 4.1.1, the graph (Γ, S) is quasi-isometric to $\mathcal{X}$, Example 1.1.3 implies that (Γ, S) is LG-rigid.

Assume now that $n > 3$. If Isom(Γ, S) is discrete the LG-rigidity of the lattice is given by Theorem 1.1.15.
If Isom(Γ, S) is non-discrete, then by Lemma 4.2.1 it has finite index in Isom(X) and in this case the hypothesis of Theorem 1.2.6 are satisfied, hence the rigidity of the lattice.

Finally, for all $n \neq 3$ the lattice Γ acts transitively on (Γ, S) thus, by Proposition 1.1.14, it is SLG-rigid. □

CONCLUSION AND OPEN PROBLEMS

5

"

Maintenant nous savons ce que nous ne savons pas comment faire !

— **Buck**
Dans *L'Âge de glace 5 : Les Lois de l'Univers* de
M.Thurmeier et G.T.Chu

Our main result is proved for graphs quasi-isometric to the Bruhat-Tits building of $\mathsf{PSL}_n(\mathbb{K})$ and the key idea of the proof is to use the rigidity of this building to "transfer it" to the graph quasi-isometric thereto. One can ask wether we can generalize this idea to other LG-rigid graphs.

Question 5.0.1. Let $\mathcal{G}$ be quasi-isometric to a LG-rigid graph $\mathcal{H}$, both having cocompact isometry group. If the quasi-isometry is $\mathrm{Isom}(\mathcal{G})$-equivariant, is $\mathcal{G}$ LG-rigid?

Remark that if $\mathcal{H}$ and $\mathcal{G}$ are two Cayley graphs of the same group, we can chose $\mathcal{H}$ to be LG-rigid and $\mathcal{G}$ to be non-rigid (see the discussion below Counter-example 1.1.4 for more details). In that case the hypothesis of the preceding question are satisfied without $\mathcal{G}$ being LG-rigid. Thus, more restrictive hypothesis will be needed to get the rigidity of our graph $\mathcal{G}$.

Our result on lattices is proved for $n \neq 3$; when $n = 3$ we don't know (yet) the answer. Indeed, our proof is based on the rigidity of the Bruhat-Tits building of $\mathsf{PSL}_n(\mathbb{K})$, a result known to be true only for $n \neq 3$. In the $n = 3$ case, a lot of flexibility seems to be allowed (see for example [BP07]) obstructing any local recognizability result. Hence the following question:

Question 5.0.2. Are torsion-free lattices of $\mathsf{SL}_3(\mathbb{K})$ LG-rigid?

Lattices in p-adic Lie groups can be viewed as particular cases of S-arithmetic lattices.

Definition 5.0.3

> Let S be a set of prime.
> We say that Γ is an **S-arithmetic** lattice if it's a lattice in a product of the form $\prod_i G_i$ where G_i is either a real Lie group or a p-adic Lie group for $p \in S$.

Hence, one we can ask what happens in that more general case.

Question 5.0.4. Are torsion-free S-arithmetic lattices LG-rigid?

A result by Bader, Furman and Sauer [BFS20, Theorem B] can be used to deal with irreducible torsion-free S-arithmetic lattices. Indeed, if the product $\prod_i G_i$ contains at least a non-compact real factor, then the aforementioned theorem implies that the isometry group of a Cayley graph of Γ is discrete. Thus, by Theorem 1.1.15 the lattice is LG-rigid. Now, if the product contains a compact real factor then the isometry group of the Cayley graph might not be discrete and in that case, the problem is still open.

When the lattice is reducible, we now know that the projection on the p-adic factors gives LG-rigid lattices. Moreover, if we suppose the real factors to be simple and connected, then a result by de la Salle and Tessera [dlST19] shows that the projection on these factors are also LG-rigid. Hence it remains to understand how to combine these results on the *factors* in order to get a result on the *product*.

Part II

ORBIT AND MEASURE EQUIVALENCE

“

It’s better to fail while striving for something wonderful, challenging, adventurous, and uncertain, than to say, “I don’t want to try because I may not succeed completely.”

— Jimmy Carter

6 AN INTRODUCTION TO MEASURE AND ORBIT EQUIVALENCE

> *Qui a fait l'expérience de penser dans un autre domaine l'emporte toujours sur celui qui ne pense pas du tout ou très peu.*
>
> — **A. Einstein**
> *Comment je vois le monde*

A recurring theme in group theory is the description of large-scale behaviour of groups and their geometry. A well known example is the study of groups up to quasi-isometry: it describes the large-scale (or "coarse") geometry from the *metric* point of view. A *measure* analogue of quasi-isometry was introduced by Gromov in [GNR93] and is called measure equivalence. A first illustration of measure equivalent groups is given by lattices in a common locally compact group. Another impressive example is given by a famous result of Ornstein and Weiss (see Theorem 6.1.7) which implies that all amenable groups are measure equivalent. In particular —unlike quasi-isometry— measure equivalence does *not* preserve coarse geometric invariants.

To overcome this issue it is therefore natural to look for some refinements of this measure equivalence notion. Assume for example that G and H are two finitely generated measure equivalent groups over a probability space (X, μ). Recall that if a finitely generated group K acts on X and S_K is a finite generating set of K, we can define the Schreier graph associated to this action by being the graph whose set of vertices is X and set of edges is $\{(x, s \cdot x) \mid s \in S_K\}$. So let us consider the Schreier graph associated to the action of G (resp. H) on X and equip it with the usual metric d_{S_G} (resp. d_{S_H}), fixing the length of an edge to one. A first way to refine the measure equivalence is to quantify how close the two actions are by studying for all $g \in G$ and $h \in H$ the integrability of the two following maps

$$x \mapsto d_{S_G}(x, h \cdot x) \qquad x \mapsto d_{S_H}(x, g \cdot x).$$

When these two maps are L^p we say that the groups are L^p-measure equivalent (see [BFS13] for more details). This refinement allowed for example Bowen to prove in the appendix of [Aus16] that volume growth was invariant under L^1-measure equivalence. Delabie, Koivisto, Le Maître and Tessera offered in [DKLMT20] to extend this quantification to a family of functions larger than $\{x \mapsto x^p,\ p \in [0, +\infty]\}$ (see Definition 6.1.8). They furthermore showed the monotonicity of the isoperimetric profile under this quantified measure equivalence definition (see Theorem 6.1.16). In [BZ21] Brieussel and Zheng managed to construct amenable groups with prescribed isoperimetric profile. Considering the monotonicity of the isoperimetric profile, the striking result of Brieussel and

Zheng thus triggers a new question: instead of trying to quantify the measure equivalence relation between two given groups, can one find a group that is measure equivalent to a prescribed group with a prescribed quantification?

This is the problem we address in this article. Using Brieussel-Zheng's groups we first exhibit a group that is measure equivalent to $\mathbb{Z}$ with a prescribed quantification (see Theorem 6.3.1). In a second time we construct a measure-subgroup coupling (which is a relaxed version of measure equivalence, see Definition 6.1.1 for a definition) with the lamplighter group $\mathbb{Z}/q\mathbb{Z} \wr \mathbb{Z}$ with prescribed integrability. In both cases we compare the obtained couplings to the constraints given by Theorem 6.1.16 and show that our couplings are close to being optimal in a sense that we precise in Section 6.3. Before looking at these results, we recall some material about quantitative measure equivalence couplings in Section 6.1 and expose tools to build such couplings in Section 6.2.

6.1 QUANTITATIVE MEASURE AND ORBIT EQUIVALENCE

Let us recall some material of [DKLMT20]. A *standard measure space* is a couple (X, μ) where X is a measurable space endowed with a measure μ on the σ-algebra $\mathcal{B}(X)$ given by the Borel σ-algebra of some separable and completely metrizable topology on X. The elements of $\mathcal{B}(X)$ are called *Borel*. We say that (X, μ) is a standard Borel probability space if $\mu(X) = 1$. A *measure-preserving action* of a discrete countable group G on (X, μ) is an action of G on X such that the map $(g, x) \mapsto g \cdot x$ is a Borel map and $\mu(E) = \mu(g \cdot E)$ for all $E \subseteq \mathcal{B}(X)$ and all $g \in G$. We will say that a measure-preserving action of G on (X, μ) is *free* if for almost every $x \in X$ we have $g \cdot x = x$ if and only if $g = e_G$.

We recall below the definitions of measure and orbit equivalence and their quantified version as introduced by Delabie, Koivisto, Le Maître and Tessera. We conclude by studying the relation between isoperimetric profile and measure equivalence.

6.1.1 Measure and orbit equivalence

Let G be a countable group acting on a standard measure space (X, μ). A *fundamental domain* for the action of G on (X, μ) is a Borel $X_G \subseteq X$ which intersects almost every G-orbit at exactly one point. We say that the action is *smooth* if it admits a fundamental domain. Before giving the definition of measure equivalence let us introduce a relaxed version of this notion.

Definition 6.1.1 (*[DKLMT20, Def. 2.4]*)

Let G and H be two countable groups. A *measure subgroup coupling* from G to H is a triple (X, X_H, μ) such that:

- (X, μ) is a standard measure space equipped with commuting smooth free measure-preserving actions of G and H,
- X_H is a fundamental domain of finite measure for the action of H on X.

Example 6.1.2. If $G \leqslant H$ then $(H, \{e_H\}, \mu)$ is a measure subgroup coupling where μ is the counting measure and G (resp. H) acts by left (resp. right) translation on H.

Remark that this definition does not require G to admit a fundamental domain of *finite* measure, in particular this coupling notion is asymmetric. If we add the condition that G must admit a fundamental domain of finite measure we obtain a *measure equivalence coupling.*.

Definition 6.1.3 (*[DKLMT20, Def. 2.3]*)

> Let G and H be two countable groups. A *measure equivalence coupling* from G to H is a quadruple (X, X_G, X_H, μ) such that:
> - (X, μ) is a standard measure space equipped with commuting free actions measure-preserving smooth of G and H,
> - X_G (resp. X_H) is a fundamental domain of finite measure for the action of G (resp H) on X.
>
> We say that G and H are *measure equivalent* if there exists a measure equivalence coupling from G to H.

Example 6.1.4. Let G and H be two lattices in the same locally compact group $\mathcal{G}$ and let $\lambda_{\mathcal{G}}$ be the Haar measure of $\mathcal{G}$. Remark that $\mathcal{G}$ is unimodular since it admits a lattice. Then G (resp. H) acts freely on $\mathcal{G}$ by left (resp. right) translation with fundamental domain X_G (resp. X_H) of finite measure. These two actions commute and preserve the Haar measure thus $(\mathcal{G}, X_G, X_H, \lambda_{\mathcal{G}})$ is a measure equivalence coupling from G to H.

More examples will be given in Section 6.2. Remark that our definition is asymmetric, we talk indeed of a coupling *from* one group *to* another. This asymmetry might be unsettling for now since it is called measure *equivalence* but it will make sense when we introduce the quantification of the coupling (see Definition 6.1.8). Let us now introduce a stronger equivalence relation between groups which comes from ergodic theory.

Definition 6.1.5

> Let G and H be two finitely generated groups. We say that G and H are *orbit equivalent* if there exists a probability space (X, μ) and a measure-preserving free action of G (resp. H) on (X, μ) such that for almost every $x \in X$ we have $G \cdot x = H \cdot x$. We call (X, μ) an *orbit equivalence coupling* from G to H.

We called this equivalence relation stronger than measure equivalence because orbit equivalence *implies* measure equivalence. But the converse is not always true. To ensure that two measure equivalent groups are orbit equivalent we need the two fundamental domains X_G and X_H to be equal. This is what we formalise below.

Proposition 6.1.6

> Two countable groups G and H are orbit equivalent if and only if there exists a measure equivalence coupling (X, X_G, X_H, μ) from G to H such that $X_H = X_G$.

Although this orbit equivalence relation is stronger than measure equivalence, it does not distinguish amenable groups. Indeed by the Ornstein Weiss theorem [OW80, Th. 6] below, all infinite amenable groups are in the same equivalence class.

Theorem 6.1.7 (*[OW80]*)

> All infinite amenable groups are orbit equivalent to $\mathbb{Z}$.

To refine these equivalence relations and "distinguish" amenable groups we introduce quantification of measure and orbit equivalence relations.

6.1.2 Quantification

Recall that if a finitely generated group G acts on a space X and if S_G is a finite generating set of G, we can define the Schreier graph associated to that action as being the graph whose set vertices is X and set of edges is $\{(x, s \cdot x) \mid s \in S_K\}$. This graph is endowed with a natural metric d_{S_G} fixing the length of an edge to one. Remark that if S'_G is another generating set of G then there exists $C > 0$ such that for all $x \in X$ and $g \in G$

$$\frac{1}{C} d_{S_G}(x, g \cdot x) \leqslant d_{S'_G}(x, g \cdot x) \leqslant C d_{S_G}(x, g \cdot x).$$

Finally if (X, X_G, X_H, μ) is a measure equivalence coupling from G to H we have a natural action of G on X_H (seeFigure 6.1 for an illustration) denoted by $\bullet$ were for all $x \in X_H$ and $g \in G$ we define $g \bullet x$ to be the unique element of $H \cdot g \cdot x$ contained in X_H *viz.*

$$\{g \bullet x\} = H \cdot g \cdot x \cap X_H.$$

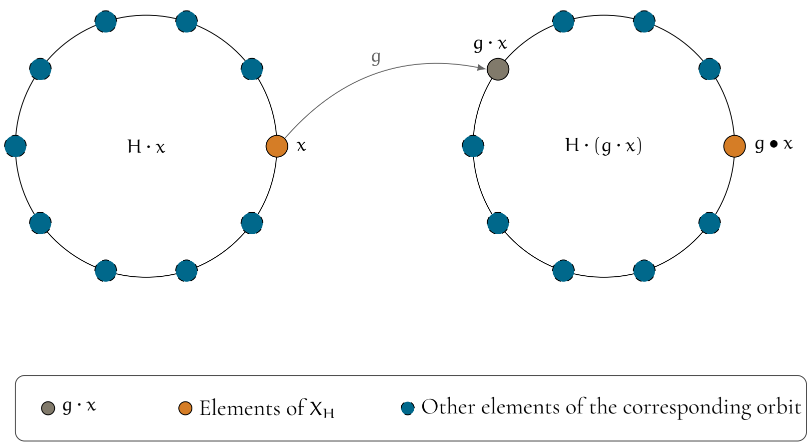


Figure 6.1.: Definition of $g \bullet x$

Definition 6.1.8 (*[DKLMT20, Def. 2.18]*)

> Let $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ be a non-decreasing map. Let G and H be two finitely generated groups and S_H be a generating set of H. We say that a measure subgroup coupling (X, X_H, μ) from G to H is *φ-integrable* if for all $s \in S_G$ there exists $c_s > 0$ such that
>
> $$\int_{X_H} \varphi\left(\frac{1}{c_s} d_{S_H}(s \cdot x, s \bullet x)\right) d\mu(x) < +\infty.$$
>
> We say that the coupling is *L^∞-integrable* if the map $x \mapsto d_{S_H}(s \cdot x, s \bullet x)$ is essentially bounded.

We introduce the constant c_s in the definition for the φ-integrability to be independent of the choice of generating set S_H. If no integrability assumption is made we will say that the coupling is L^0-integrable. Finally if $\varphi(x) = x^p$ we will sometimes talk of L^p-integrability instead of φ-integrability. Finally, note that every L^∞ measure subgroup coupling is φ-integrable for any increasing map $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$.

Definition 6.1.9 (*[DKLMT20, Def. 2.18]*)

> We say that a measure equivalence coupling (X, X_G, X_H, μ) from G to H is (φ, ψ)-*integrable* if the coupling (X, X_H, μ) from G to H is φ-integrable and the coupling (X, X_G, μ) from H to G is ψ-integrable. We say that an orbit equivalence coupling is (φ, ψ)-*integrable* if it is (φ, ψ)-integrable as a measure equivalence coupling.

When $\varphi = \psi$ we will say that the coupling is φ-integrable instead of (φ, φ)-integrable.

Example 6.1.10. Delabie et al. [DKLMT20] showed that there exists an orbit equivalence coupling between $\mathbb{Z}^4$ and the Heisenberg group $\mathrm{Heis}(\mathbb{Z})$ that is L^p-integrable for all $p < 1$.

Example 6.1.11. Let $k \in \mathbb{N}^*$ and $BS(1, k) := \langle a, b \mid a^{-1}ba = b^k \rangle$. The same authors showed [DKLMT20, Th. 8.1] that their exists an orbit equivalence coupling from L_q to $BS(1, k)$ that is $(L^\infty, \exp)$-integrable.

We will see more examples in Section 6.2 where we develop tools to *build* couplings. But before adressing these constructions, a natural question to ask is whether there exists obstructions for finding φ-integrable couplings between two amenable groups, for a given function φ. A first answer —and thus a first obstruction— is given by the isoperimetric profile.

6.1.3 Isoperimetric profile

As stated in the inroduction the measure equivalence notion introduced by Gromov does not preserve the coarse geometric invariants. But the quantified measure equivalence defined above allowed Delabie et al. [DKLMT20] to get a relation between the isoperimetric profiles of two measure equivalent groups which we describe below.

Let $C > 0$. If f and g are two real functions we denote $f \preccurlyeq_C g$ (or $f \preccurlyeq g$ for short) if $f(x) = \mathcal{O}\big(g(Cx)\big)$ as x tends to infinity. We write $f \simeq_C g$ (or $f \simeq g$ for short) if $f \preccurlyeq_C g$ and $g \preccurlyeq_C f$.

Definition 6.1.12

> Let G be a finitely generated group and S a finite generating set. The *isoperimetric profile* of G is defined as
> $$I_G(n) := \sup_{|A| \leqslant n} \frac{|A|}{|\partial A|}.$$

We chose to adopt the convention of [DKLMT20]. Note that in [BZ21], the isoperimetric profile is defined as $\Lambda_G = 1/I_G$. Remark that due to Følner criterion, a group is amenable if and only if its isoperimetric profile is unbounded. Hence we can see the isoperimetric profile as a way to measure the amenability of a group: the faster I_G tends to infinity, the more amenable G is.

Example 6.1.13. The isoperimetric profile of $\mathbb{Z}$ verifies $I_{\mathbb{Z}}(x) \simeq x$.

A famous result of Erschler [Ers03] gives the two following examples.

Example 6.1.14. Let $q \geqslant 2$ and $d \geqslant 1$. If $G := \mathbb{Z}/q\mathbb{Z} \wr \mathbb{Z}^d$ then $I_G(x) \simeq (\log(x))^{1/d}$.

Example 6.1.15. If $G := \mathbb{Z} \wr \mathbb{Z}$ then $I_G(n) \simeq \log(n)/\log \circ \log(n)$.

The behaviour of the isoperimetric profile under measure equivalence coupling is given by the following theorem.

Theorem 6.1.16 (*[DKLMT20, Th.1]*)

Let G and G' be two finitely generated groups admitting a (φ, L^0)-integrable measure equivalence coupling. If φ and $t/\varphi(t)$ are increasing then

$$\varphi \circ I_{G'} \preccurlyeq I_G.$$

This theorem provides an obstruction for finding φ-integrable couplings with certain functions φ between two amenable groups. For example we can deduce from the preceding examples that there is no L^1 measure equivalence coupling from $\mathbb{Z} \wr \mathbb{Z}$ to $\mathbb{Z}/2\mathbb{Z} \wr \mathbb{Z}$.

On the other hand given a group G and a function φ one can ask whether there exists a group G' which is (φ, L^0)-measure equivalent to G. This is the question we try to answer in this paper for $G = \mathbb{Z}$ or $G = \mathbb{Z}/q\mathbb{Z} \wr \mathbb{Z}$. Our construction is based on the powerful machinerie developped by Brieussel and Zheng in [BZ21] to construct amenable groups with a prescribed isoperimetric profile (see Chapter 7 for a definition of these groups and appendix A.1 for the technical construction from the isoperimetric profile). This machinerie builds the wanted *group* G', we now have to construct the *coupling*.

6.2 BUILDING COUPLINGS

To obtain and quantify couplings we will use two different constructions which we recall below. The first one is based on *Følner tiling sequences* described in [DKLMT20, Section 6] and the second one using *Sofic approximations* is described in [DKT21].

6.2.1 Følner tiling sequences

A first way of building an orbit equivalence between two groups is to build *Følner tiling sequences*. These sequences are Følner sequences that are defined recursively: the term of rank $(n+1)$ is composed of a finite number of translates of the n-th term of the sequence.

Definition 6.2.1

Let G be an amenable group and $(\Sigma_n)_{n\in\mathbb{N}}$ be a sequence of finite subsets of G. Define by induction the sequence $(T_n)_{n\in\mathbb{N}}$ by $T_0 := \Sigma_0$ and $T_{n+1} := T_n\Sigma_{n+1}$. We say that $(\Sigma_n)_{n\in\mathbb{N}}$ is a (left) *Følner tiling sequence* if

- $(T_n)_{n\in\mathbb{N}}$ is a left Følner sequence, *viz.*

$$(\forall g \in G) \qquad \lim_{n\to\infty} \frac{|gT_n \setminus T_n|}{|T_n|} = 0,$$

- $T_{n+1} = \bigsqcup_{\sigma\in\Sigma_{n+1}} \sigma T_n$.

We call Σ_n the set of *shifts* and $(T_n)_{n\in\mathbb{N}}$ the *tiles*.

Let finally S be a generating part of G. We say that $(\Sigma_n)_{n\in\mathbb{N}}$ is a (R_n, ε_n)-Folner tiling sequence if for all n we have

$$\operatorname{diam}(T_n) \leqslant R_n, \qquad |sT_n \setminus T_n| \leqslant \varepsilon_n |T_n| \quad (\forall s \in S).$$

Delabie et al. showed in [DKLMT20] the two following examples.

Example 6.2.2. If $G = \mathbb{Z}$ the sequence defined by $\Sigma_{n+1} := \{0, 2^n\}$ is a $(2^n, 2^{1-n})$-Følner tiling sequence and the sequence (T_n) thus defined verifies $T_n = [0, 2^n - 1]$. We represent the tiling of T_{n+1} by T_n in Figure 6.2.

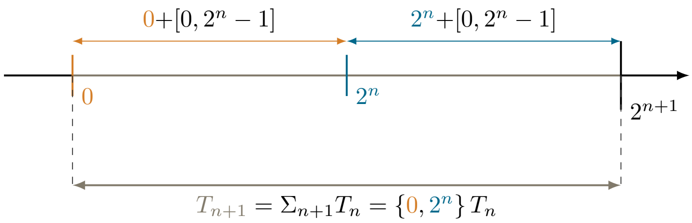


Figure 6.2.: A tiling for $\mathbb{Z}$.

Example 6.2.3. If $G = (\mathbb{Z}/2\mathbb{Z}) \wr \mathbb{Z}$ then the sequence $(\Sigma_n)_{n\in\mathbb{N}}$ defined by

$$\begin{cases} \Sigma_0 & := \left\{(f, 0) \in G \mid \operatorname{supp}(f) \subseteq \{0, 1\}\right\}, \\ \Sigma_{n+1} & := \left\{(f, 0) \in G \mid \operatorname{supp}(f) \subseteq [2^n, 2^{n+1} - 1]\right\} \\ & \quad \cup \left\{(f, 2^n) \in G \mid \operatorname{supp}(f) \subseteq [0, 2^n - 1]\right\}, \end{cases}$$

is a right $(3 \cdot 2^n, 2^{-n})$-Følner tiling sequence. Moreover the tiling $(T_n)_{n\in\mathbb{N}}$ thus defined verifies $T_n = \left\{(f, m) \in G \mid \operatorname{supp}(f) \subseteq [0, 2^n - 1],\ m \in [0, 2^n - 1]\right\}$.

Delabie et al. [DKLMT20] gave a condition for two amenable groups admitting both a Følner tiling sequence to be orbit equivalent. Indeed if G admits a Følner tiling sequence $(\Sigma_n)_{n\in\mathbb{N}}$ then we can define $X := \prod_{n\in\mathbb{N}} \Sigma_n$ and endow it with an action of G. Up to measure zero, two elements of X will be in the same orbit under that action if and only if they differ from a finite number of indices. The equivalence relation thus induced is called the *cofinite equivalence relation*. Now if G' admit a Følner tiling sequence $(\Sigma'_n)_{n\in\mathbb{N}}$ verifying $|\Sigma_n| = |\Sigma'_n|$ for all integer n, then there exists a natural bijection between X and $X' := \prod_{n\in\mathbb{N}} \Sigma'_n$ which preserves the cofinite equivalence relation. That is to say G and G' are orbit equivalent. Furthermore they showed that if we know the diameter and the ratio of elements in the boundary of each tile then we can deduce the integrability of the coupling. This is what the following proposition sums up.

Theorem 6.2.4 (*[DKLMT20, Prop. 6.6]*)

Let G and G' be two discrete amenable groups and let $(\Sigma_n)_n$ be an (ε_n, R_n)-Følner tiling sequence for G and $(\Sigma'_n)_n$ be an (ε'_n, R'_n)-Følner tiling sequence for G'.

If $|\Sigma_n| = |\Sigma'_n|$, then the groups are orbit equivalent.

Moreover if $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ is a non-decreasing function such that the sequence $(\varphi(2R'_n)(\varepsilon_{n-1} - \varepsilon_n))_{n\in\mathbb{N}}$ is summable, then the coupling from G to G' is (φ, L^0)-integrable.

Using this tiling technique and the above theorem, Delabie et al. [DKLMT20] showed Example 6.1.10 stated before and the two following examples.

Example 6.2.5. For all n and m there exists an orbit equivalence coupling from $\mathbb{Z}^m$ to $\mathbb{Z}^n$ which is $(\varphi_\varepsilon, \psi_\varepsilon)$-integrable for every $\varepsilon > 0$ where

$$\varphi_\varepsilon(x) = \frac{x^{n/m}}{\log(x)^{1+\varepsilon}} \quad \psi_\varepsilon(x) = \frac{x^{m/n}}{\log(x)^{1+\varepsilon}}.$$

Remark that in particular if $n < m$ then for all $p < n/m$ there exists a $(L^p, L^{1/p})$-orbit equivalence coupling from $\mathbb{Z}^m$ to $\mathbb{Z}^n$.

Example 6.2.6. Let $m \geqslant 2$. There exists an orbit equivalence coupling between $\mathbb{Z}$ and $\mathbb{Z}m\mathbb{Z} \wr \mathbb{Z}$ that is $(\exp, \varphi_\varepsilon)$-integrable for all $\varepsilon > 0$ where

$$\varphi_\varepsilon(x) = \frac{\log(x)}{\log(\log(x))^{1+\varepsilon}}.$$

Given two finitely generated groups G and H, Theorem 6.1.16 gives an upper bound for the integrability of a coupling from G to H. As we will see in Chapter 10 Følner tiling sequences do not always induce couplings with a "good" integrability: the map φ quantifying the coupling obtained with Følner tiling sequence can grow much slower than the one suggested by Theorem 6.1.16. Moreover, given two amenable groups it is not always easy (or even possible) to find Følner tiling sequences verifying $|\Sigma_n| = |\Sigma'_n|$. Hence this tiling technique does have its limitations and we thus need other tools to build couplings.

6.2.2 Sofic approximations

Another technique to build measure equivalence couplings is given by Sofic approximations. In this paragraph G will be a finitely generated group endowed with a finite generating set S_G and $(\mathcal{G}_n)_{n\in\mathbb{N}}$ will be a sequence of finite, directed, labeled graphs. Let $r > 0$ and denote by $\mathcal{G}_n^{(r)}$ the set of elements $x \in \mathcal{G}_n$ such that $B_{\mathcal{G}_n}(x, r)$ is isomorphic to $B_G(e_G, r)$ seen as directed labeled graphs, *viz.* $\mathcal{G}_n^{(r)} = \left\{x \in \mathcal{G}_n \mid B_{\mathcal{G}_n}(x, r) \simeq B_G(e_G, r)\right\}$.

Definition 6.2.7

We say that $(\mathcal{G}_n)_{n\in\mathbb{N}}$ is a *Sofic approximation* if for every $r > 0$

$$\lim_{n\to\infty} \frac{\left|\mathcal{G}_n^{(r)}\right|}{|\mathcal{G}_n|} = 1.$$

Example 6.2.8. Any Følner sequence in an amenable group G is a Sofic approximation.

In [DKT21] Delabie, Koivisto and Tessera proved a condition for a measure subgroup to be φ-integrable using Sofic approximations.

Theorem 6.2.9

Let G and H be two finitely generated groups with Sofic approximations $(\mathcal{G}_n)_n$ and $(\mathcal{H}_n)_n$ and let $\iota_n \colon \mathcal{G}_n \to \mathcal{H}_n$ be an injective map. Let $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ be a non-decreasing map. If for every $s \in S_G$ there exists $\delta > 0$ such that

$$\lim_{R\to\infty} \sup_n \sum_{r=0}^{R} \varphi(\delta r) \frac{\left|\left\{x \in \mathcal{G}_n^{(s)} \mid d_{\mathcal{H}_n}(\iota_n(x), \iota_n(x.s)) = r\right\}\right|}{|\mathcal{G}_n|} < +\infty \tag{6.1}$$

then there exists a φ-integrable measure subgroup coupling from G to H.

We will use this theorem to obtain a measure subgroup with the Lamplighter with prescribed integrability (see Theorem 6.3.3). But before that, let us state its analogue for measure and orbit *equivalence* couplings.

Theorem 6.2.10 (*[DKT21]*)

Let $\varphi, \psi \colon \mathbb{R}^+ \to \mathbb{R}^+$ be two non-decreasing maps. Let G and H be two finitely generated groups with Sofic approximations $(\mathcal{G}_n)_n$ and $(\mathcal{H}_n)_n$. Let $\iota_n \colon \mathcal{G}_n \to \mathcal{H}_n$ and $\iota'_n : \mathcal{H}_n \to \mathcal{G}_n$ be two injective maps that satisfy the following:

1. There exists $C > 0$ such that the image ι_n is C-dense for all $n \in \mathbb{N}$;
2. For every $s \in S_G$ there exists $\delta > 0$ such that

$$\lim_{R\to\infty} \sup_n \sum_{r=0}^{R} \varphi(\delta r) \frac{\left|\left\{x \in \mathcal{G}_n^{(s)} \mid d_{\mathcal{H}_n}(\iota_n(x), \iota_n(x.s)) = r\right\}\right|}{|\mathcal{G}_n|} < \infty;$$

3. For every $h \in H$ there exists $\delta > 0$ such that the following limit is finite

$$\lim_{R\to\infty} \sup_n \sum_{r=0}^{R} \psi(\delta r) \frac{\left|\left\{y \in \iota'_n(\mathcal{G}_n) \cap \mathcal{H}_n^{(h)} \mid \operatorname{diam}\left(\iota_n'^{-1}(y) \cup \iota_n'^{-1}(y \cdot h)\right) = r\right\}\right|}{|\mathcal{G}_n|}.$$

Then there exists a (φ, ψ)-integrable measure equivalence coupling from G to H. Moreover if the maps ι_n are bijective then there is an (φ, ψ)-integrable orbit equivalence coupling from G to H.

Given two finitely generated amenable groups, we now have tools to build and quantify couplings between them. But we can also address the problem the other way round.

6.3 BUILDING PRESCRIBED EQUIVALENCES

Instead of looking for a quantification for a given coupling from one group to another, one can ask if given a group G and a non-decreasing function φ we can find a group H and a measure (or orbit) equivalence coupling from G to H that is (φ, L^0)-integrable. As we saw before, given G and φ as above, Theorem 6.1.16 gives obstructions to find such a group H: it provides a bound on the growth of the isoperimetric profile of H.

Considering the work of Brieussel and Zheng [BZ21] giving an engineering to build groups with prescribed isoperimetric profile, we exhibit groups orbit equivalent to $\mathbb{Z}$ with prescribed integrability. In a second result we build a measure subgroup coupling with the Lamplighter group with prescribed integrability. We discuss the case of the measure *equivalence* coupling with the Lamplighter group below Theorem 6.3.3.

6.3.1 Main results

In this part we show the two following theorems and their three corollaries.

Theorem 6.3.1

For all non-decreasing function $\rho : [1, +\infty[\to [1, +\infty[$ such that $\rho(1) = 1$ and $x/\rho(x)$ is non-decreasing, there exists a group G such that

- $I_G \simeq \rho \circ \log$;
- there exists an orbit equivalence coupling from G to $\mathbb{Z}$ that is $(\varphi_\varepsilon, \exp\circ\rho)$-integrable for all $\varepsilon > 0$, where $\varphi_\varepsilon(x) := \rho \circ \log(x)/(\log\circ\rho\circ\log(x))^{1+\varepsilon}$.

Let us discuss the optimality of this result. Consider a (φ, L^0)-integrable orbit equivalence coupling from some group G to $\mathbb{Z}$. By Theorem 6.1.16 it verifies $\varphi \circ I_{\mathbb{Z}} \preccurlyeq I_G$. In particular since $I_{\mathbb{Z}}(x) \simeq x$, we can not have a better integrability than $\varphi(x) \simeq I_G$. Since $I_\Delta \simeq \rho \circ \log$ our above theorem is optimal up to a logarithmic error.

Let us now consider the possible generalisations of this result to other groups than the group of integers. To do so we can use the *composition* of couplings described in Appendix B. We also refer to Figure B.1 in this last appendix wich sums up the our main results and illustrates the discussion to come.

Given the above theorem, once we have a measure subgroup coupling from $\mathbb{Z}$ to a group H we can compose the two couplings to obtain a measure subgroup coupling from G to H. If the growth of the isoperimetric profile of H is close from the one of $\mathbb{Z}$, the integrability of the obtained coupling will be close to the optimal one given by Theorem 6.1.16. It is for example the case when $H = \mathbb{Z}^d$.

Corollary 6.3.2

Let $d \in \mathbb{N}^*$. For all non-decreasing function $\rho : [1, +\infty[\to [1, +\infty[$ such that $\rho(1) = 1$ and $x/\rho(x)$ is non-decreasing, there exists a group G such that

- $I_G \simeq \rho \circ \log$;
- there exists an orbit equivalence coupling from G to $\mathbb{Z}^d$ that is $(\varphi_\varepsilon, L^0)$-integrable for all $\varepsilon > 0$, where $\varphi_\varepsilon(x) := \rho \circ \log(x)/(\log\circ\rho\circ\log(x))^{1+\varepsilon}$.

Now if the growth of I_H is quite slower than the one of $I_{\mathbb{Z}}$ (it is for example the case when H is the lamplighter group) the coupling obtained by composition will not have the optimal integrability: by using $\mathbb{Z}$ as an intermediary to build our coupling we lose information about the geometry of G and H and thus lose precision in the integrability. To obtain a coupling with a finer quantification it is thus necessary to construct a coupling without making a "detour" by $\mathbb{Z}$. It is what our second main result offers to do in the case of the lamplighter.

We denote by L_q the lamplighter group with q lamp configurations, that is to say $L_q = \mathbb{Z}/q\mathbb{Z} \wr \mathbb{Z}$.

Theorem 6.3.3

For all $\varepsilon > 0$ and all $\alpha > 0$ there exists a group G such that

- $I_G(x) \simeq (\log(x))^{1/(1+\alpha)}$;
- if we define $\varphi_\varepsilon(x) := x^{\frac{1}{1+\alpha+\varepsilon}}$ then there exists a φ_ε-integrable measure subgroup coupling from G to L_q.

Let us discuss the conclusion of this theorem. First remark that it implies the existence of a measure *subgroup* coupling. Indeed our demonstration is based on the construction of Sofic approximations $(\mathcal{G}_n)_{n\in\mathbb{N}}$ of Δ and $(\mathcal{H}_n)_{n\in\mathbb{N}}$ of L_q, but the construction we make allows us to obtain an injective map between $\mathcal{G}_n$ and $\mathcal{H}_n$, not an injective and C-*dense* one. The hypothesis of Theorem 6.2.10 implying the existence of a measure *equivalence* coupling are thus not verified. However, we can apply Theorem 6.2.9 to obtain a measure *subgroup*. Nonetheless, let us mention that even though the result is not present in this manuscript, we worked on the construction of Sofic approximations and injective C-dense maps between them in order to obtain a measure equivalence coupling between Δ and L_q. The (quite technical) proof could not be written before rendering this thesis.

Second, note that the integrability is almost optimal. Indeed, consider a (φ, L^0)-integrable orbit equivalence coupling from some group G to L_q. By Theorem 6.1.16 it verifies $\varphi \circ I_{L_q} \preccurlyeq I_{L_q}$. In particular since $I_{L_q}(x) \simeq \log(x)$ we can not have a better integrability than $\varphi \simeq I_G \circ \exp$. Since $I_\Delta(x) \simeq \log(x)^{1/(1+\alpha)}$, our above theorem is thus optimal up to a small error.

Finally, let us remark that the above theorem applies to a map ρ of the form $\rho(x) = x^{1/(1+\alpha)}$ while Theorem 6.3.1 worked for a larger family of functions ρ. As we will see in Section 9.2, the construction we make require us to estimate the ratio $l_{\mathfrak{L}(p(n+1))}/l_{\mathfrak{L}(pn)}$ in order to get the integrability of the coupling. But in a general case we have no control on that last ratio: there exists diagonal product such that the sequence $(l_m)_{m\in\mathbb{N}}$ grows as fast as possible. That is why we restrict ourselves to the case where $l_m = \kappa^{\alpha m}$ for some $\kappa \geqslant 2$ and $\alpha > 0$ and thus only obtain diagonal product Δ with isoperimetric profile $I_\Delta(n) \simeq \log(x)^{1/(1+\alpha)}$.

We can deduce from the above theorem two corollaries. First define $H := \mathbb{Z}^2 \rtimes_A \mathbb{Z}$ where A is the matrix

$$A := \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}.$$

Using once more the composition of couplings we will deduce from Theorem 6.3.3 the following result.

Corollary 6.3.4

For all $\varepsilon > 0$ and all $\alpha > 0$ there exists a group G such that

- $I_G(x) \simeq \left(\log(x)\right)^{1/(1+\alpha)}$;
- if we define $\varphi_\varepsilon(x) := x^{\frac{1}{1+\alpha+\varepsilon}}$ then there exists a measure subgroup coupling from G to H that is φ_ε-integrable.

Second consider $k \in \mathbb{N}^*$ and the Baumslag-Solitar group defined by

$$\mathrm{BS}(1,k) = \left\langle a, b \mid a^{-1}ba = b^k \right\rangle.$$

Using Example 6.1.11 and Theorem 6.3.3 we will also show this last corollary.

Corollary 6.3.5

For all $\varepsilon > 0$ and all $\alpha > 0$ there exists a group G such that

- $I_G(x) \simeq \left(\log(x)\right)^{1/(1+\alpha)}$;
- if we define $\varphi_\varepsilon(x) := x^{\frac{1}{1+\alpha+\varepsilon}}$ then there exists a measure subgroup coupling from G to $\mathrm{BS}(1,k)$ that is φ_ε-integrable.

6.3.2 Idea of the proofs

Let us discuss the idea of the above theorems proofs.

The proof of Theorem 6.3.1 is based on Følner tiling sequences and on the criterion given by Theorem 6.2.4. What we actually show is that a Brieussel-Zheng's diagonal product Δ admits a coupling with $\mathbb{Z}$ satisfying Theorem 6.3.1. Hence, we first compute a Følner tiling sequence $(T_n)_n$ for Δ and estimate the value of the tiles diameter and the value of $|\partial T_n|/|T_n|$ in Section 8.1. In Section 8.2 we build a tiling of $\mathbb{Z}$ and compute $(R'_n)_n$ and $(\varepsilon'_n)_n$ such that the tiling gives a (R'_n, ε'_n)-Følner tiling sequence. As we will see, the values of $(R'_n)_n$ and $(\varepsilon'_n)_n$ depend on the cardinality of the tiles $(T_n)_n$ of Δ. Thus —in order to use the criterion given by Theorem 6.2.4— we compute bounds to $\ln|T_n|$. We conclude by showing the integrability of the orbit equivalence coupling obtained and use it to prove that Δ thus considered satisfies Theorem 6.3.1.

The main tools of the proof of Theorem 6.3.3 are Sofic approximations and Theorem 6.2.9. Analogous to the proof of the theorem with $\mathbb{Z}$, we prove that G satisfying Theorem 6.3.3 can be taken to be a Brieussel-Zheng's diagonal product Δ. We thus start by defining Sofic approximations $(\mathcal{G}_n)_{n\in\mathbb{N}}$ of Δ and $(\mathcal{H}_n)_{n\in\mathbb{N}}$ of $\mathbb{Z}$ (see Section 9.1) starting from the following observation: given two Følner tiling sequences $(\Sigma_n)_n$ of Δ and $(\Sigma''_n)_n$ of L_q such that $|\Sigma_n| \leqslant |\Sigma''_n|$ for all n, one can define an embedding ν_n from Σ_n to Σ''_n. Then denote $T_n = \prod_{i=0}^{n} \Sigma_i$ and $T''_n := \prod_{i=0}^{n} \Sigma''_i$ and let $x = \sigma_n \cdots \sigma_0$ such that $\sigma_i \in \Sigma_i$ for all i. We can define without ambiguity a map ι_n from T_n to T''_n such that $\iota_n(x) = \prod_{i=0}^{n} \nu_i(\sigma_i)$. Since $(\Sigma_n)_n$ and $(\Sigma''_n)_n$ are Følner tiling sequences this map is injective. We are thus going to take sequences of *tiles* as Sofic approximations. Moreover, in order to obtain a coupling with the wanted integrability we need the embedding ι_n to preserve the geometry of the groups. As we will see in Section 9.1 we will have to consider *subsequences* of the tiles sequences for the hypothesis of Theorem 6.2.9 to be verified. Once our Sofic approximations are defined we compute some useful bounds on $\ln|\Sigma_n|$, show the integrability of the coupling thus obtained and use it to prove that Δ thus considered satisfies Theorem 6.3.3.

6.3.3 Structure of Part ii

We first describe Brieussel-Zheng's diagonal product and recall all necessary material in Chapter 7. Theorem 6.3.1 is shown in Chapter 8 using Følner tiling sequences. Chapter 9 is devoted to the proof of Theorem 6.3.3 which involves Sofic approximations. We first define the desired Sofic approximations of Δ and L_q, compute some useful inequalities regarding these two sequences and show the integrability of the measure subgroup coupling using Theorem 6.2.9. More details on the proofs of Theorems 6.3.1 and 6.3.3 will be given in the related chapters. We finally conclude this part with Chapter 10 were we discuss unachieved proofs, work in progress and open problems.

7 DIAGONAL PRODUCTS OF LAMPLIGHTER GROUPS

“

Avec un escalier prévu pour la montée on réussit souvent à monter plus bas qu'on ne serait descendu avec un escalier prévu pour la descente.

— Proverbe Shadok

We recall here necessary material from [BZ21] concerning the definition of *Brieussel-Zheng's diagonal products*. We give the definition of such a group, recall and prove some results concerning the range (see Definition 7.2.1) of an element in such a group and conclude by identifying a Følner sequence.

7.1 DEFINITION OF DIAGONAL PRODUCTS

Recall that the wreath product of a group G with $\mathbb{Z}$ denoted $G \wr \mathbb{Z}$ is defined as $G \wr \mathbb{Z} := \oplus_{m\in\mathbb{Z}} G \rtimes \mathbb{Z}$. An element of $G \wr \mathbb{Z}$ is a pair (f, t) where f is a map from $\mathbb{Z}$ to G with finite support and t belongs to $\mathbb{Z}$. We refer to f as the *lamp configuration* and t as the *cursor*.

7.1.1 General definition

Let A and B be two finite groups. Let $(\Gamma_m)_{m\in\mathbb{N}}$ be a sequence of finite groups such that each Γ_m admits a generating set of the form $A_m \cup B_m$ where A_m and B_m are finite subgroups of Γ_m isomorphic respectively to A and B. For $a \in A$ we denote a_m the copy of a in A_m and similarly for B_m. Finally let $(k_m)_{m\in\mathbb{N}}$ be a sequence of integers such that $k_{m+1} \geqslant 2k_m$ for all m. We define $\Delta_m = \Gamma_m \wr \mathbb{Z}$ and endow it with the generating set

$$S_{\Delta_m} := \Big\{(\mathrm{id}, +1)\Big\} \cup \Big\{(a_m\delta_0, 0) \mid a_m \in A_m\Big\} \cup \Big\{(b_m\delta_{k_m}, 0) \mid b_m \in A_m\Big\}.$$

Definition 7.1.1

The *Brieussel-Zheng's diagonal product* associated to $(\Gamma_m)_{m\in\mathbb{N}}$ and $(k_m)_{m\in\mathbb{N}}$ is the subgroup Δ of $(\prod_m \Gamma_m) \wr \mathbb{Z}$ generated by

$$S_\Delta := \Big\{\big((\mathrm{id})_m, +1\big)\Big\} \cup \Big\{\big((a_m\delta_0)_m, 0\big) \mid a \in A\Big\} \cup \Big\{\big((b_m\delta_{k_m})_m, 0\big) \mid b \in B\Big\}.$$

The group Δ is uniquely determined by the sequences $(\Gamma_m)_{m\in\mathbb{N}}$ and $(k_m)_{m\in\mathbb{N}}$. Let us give an illustration of what an element in such a groups looks like.

Example 7.1.2. We represent in Figure 7.1 the element $((g_m)_{m\in\mathbb{N}}, t)$ of Δ verifying

$$((g_m)_{m\in\mathbb{N}}, t) = \big((a_m\delta_0)_m, 0\big)\big((b_m\delta_{k_m})_m, 0\big)(0,3),$$

when $k_m = 2^m$. The cursor is represented by the blue arrow at the bottom of the figure. The only value of g_0 different from the identity is $g_0(0) = (a_0, b_0)$. Now if $m > 0$ then the only values of g_m different from the identity are $g_m(0) = a_m$ and $g_m(k_m) = b_m$.

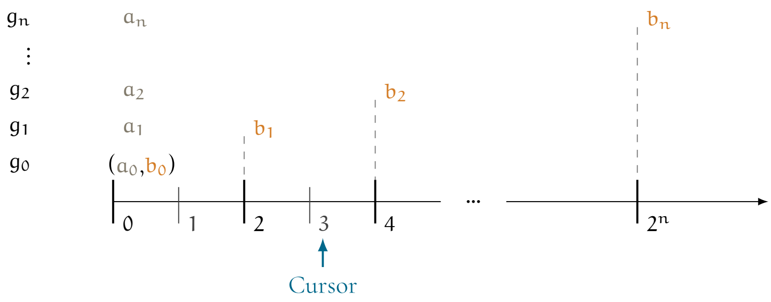


Figure 7.1.: Representation of $((g_m)_m, t) = ((a_m\delta_0), 0)((b_m\delta_{k_m}), 0)(0,3)$ when $k_m = 2^m$.

7.1.2 The expanders case

In this part we will restrict ourselves to a particular familiy of groups $(\Gamma_m)_{m\in\mathbb{N}}$ called *expanders*. Recall that $(\Gamma_m)_{m\in\mathbb{N}}$ is said to be a sequence of *expanders* if the sequence of diameters $(\operatorname{diam}(\Gamma_m))_{m\in\mathbb{N}}$ is unbounded and if there exists $c_0 > 0$ such that for all $m \in \mathbb{N}$ and all $n \leqslant |\Gamma_m|/2$ the isoperimetric profile verifies $I_{\Gamma_m}(n) \leqslant c_0$.

Consider a family $(\Gamma_n)_{n\in\mathbb{N}}$ of expanders for which there exists $c > 0$ such that for all $l \geqslant 1$ there exists $\Gamma_{p(l)}$ such that $\operatorname{diam}(\Gamma_{p(l)}) \simeq_c l$ (for an example see Example 7.1.3). We can thus define a "parametrization" by fixing a map $l \mapsto \Gamma_{p(l)}$. Consider now two non-decreasing sequences $(k_m)_{m\in\mathbb{N}}$ and $(l_m)_{m\in\mathbb{N}}$ of real numbers greater than 1 and denote Δ the diagonal product associated to $(\Gamma_{p(l_m)})_{m\in\mathbb{N}}$ and $(k_m)_{m\in\mathbb{N}}$. Then Δ is uniquely determined by the data of $(l_m)_{m\in\mathbb{N}}$ and $(k_m)_{m\in\mathbb{N}}$. In what follows, we will abuse notations and denote Γ_m instead of $\Gamma_{p(l_m)}$. Moreover we will always make the following assumptions when talking about diagonal products.

Hypothesis (H)

- $(k_m)_m$ and $(l_m)_m$ are sub-sequences of geometric sequences.
- $k_{m+1} \geqslant 2k_m$ for all $m \in \mathbb{N}$;
- $(\Gamma_m)_{m\in\mathbb{N}}$ is a sequence of expanders such that Γ_m is a quotient of $A * B$ and $\operatorname{diam}(\Gamma_m) \simeq l_m$;
- $k_0 = 0$ and $\Gamma_0 = A_0 \times B_0$;
- $\Gamma_m/\langle\langle[A_m, B_m]\rangle\rangle \simeq A_m \times B_m$ where $\langle\langle[A_m, B_m]\rangle\rangle$ denotes the normal closure of $[A_m, B_m]$.

We recall below an example of such expanders $(\Gamma_m)_{m\in\mathbb{N}}$ and refer to [BZ21, Example 2.3] for more details.

Example 7.1.3. Assume $(k_m)_{m\in\mathbb{N}}$ and $(l_m)_{m\in\mathbb{N}}$ given and consider the sequence $(\Lambda_n)_{n\in\mathbb{N}}$ defined by $\Lambda_n := \mathrm{SL}_3\left(\mathbb{F}_p[X]/(X^n-1)\right)$ and let

$$A := \left\langle \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & X & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right\rangle \simeq \mathbb{F}_p^2 \quad B := \left\langle \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \right\rangle \simeq \mathbb{Z}/3\mathbb{Z}.$$

Now we need the last assertion of **(H)** to be verified, so define Γ_n to be the diagonal product of Λ_n with $A\times B$, that is to say Γ_n is the subgroup of $\Lambda_n \times A \times B$ generated by

$$\Big\{\big(a_n, (a, e)\big) \mid a \in A\Big\} \cup \Big\{\big(b_n, (e, b)\big) \mid b \in B\Big\}.$$

Finally, denote by $\Gamma_m := \Gamma_{p(l_m)}$, then $(\Gamma_m)_{m\in\mathbb{N}}$ verifies **(H)**.

Recall (see [BZ21, page 9]) that in this case there exist $c_1, c_2 > 0$ such that, for all m

$$c_1 l_m - c_2 \leqslant \ln|\Gamma_m| \leqslant c_1 l_m + c_2. \tag{7.1}$$

7.1.3 Relative commutators subgroups

For all $m \in \mathbb{N}$ let $\theta_m : \Gamma_m \to \Gamma_m/\langle\langle[A_m, B_m]\rangle\rangle \simeq A_m \times B_m$ be the natural projection. Let θ_m^A and θ_m^B denote the composition of θ_m with the projection to A_m and B_m respectively. Now let $m \in \mathbb{N}$ and define $\Gamma'_m := \langle\langle[A_m, B_m]\rangle\rangle$. If (g_m, t) belongs to Δ_m then there exists a unique $g'_m : \mathbb{Z} \to \Gamma'_m$ such that $g_m = \theta_m(g_m)g'_m$.

Example 7.1.4. Let $(\mathbf{g}, 3)$ be the element described in Section 7.1.1. Then the only non-trivial value of $\theta_0(g_0)$ is $\theta_0(g_0(0)) = (a_0, b_0)$. If $m > 0$ then the only non trivial values of $\theta_m(g_m)$ are $\theta_m(g_m(0)) = (a_m, e)$ and $\theta_m(g_m(k_m)) = (e, b_m)$. Finally for all m we have $g'_m = \mathrm{id}$ since there are no commutators appearing in the decomposition of $(\mathbf{g}, 0)$.

Example 7.1.5. Assume that $k_m = 2^m$ and consider first the element $(f, 0)$ of Δ defined by $(f, 0) := (0, k_1)\big((a_m\delta_0)_m, 0\big)(0, -k_1)$. Now define the commutator

$$(\mathbf{g}, 0) = (f, 0) \cdot \big((b_m\delta_{k_m})_m, 0\big) \cdot (f, 0)^{-1} \cdot \big((b_m^{-1}\delta_{k_m})_m, 0\big)$$

and let us describe the values taken by $\mathbf{g}$ and the induced maps $\theta_m(g_m)$ and g'_m (see Figure 7.2 for a representation of $\mathbf{g}$). The only non-trivial commutator appearing in the values taken by $\mathbf{g}$ is $g_1(k_1)$ which is equal to $a_1b_1a_1^{-1}b_1^{-1}$. In other words g_0 is the indentity, thus $\theta_0 = \mathrm{id}$. Moreover when $m = 1$ we have $\theta_1 = \mathrm{id}$ and the only value of $g'_1(x)$ different from e is $g'_1(k_1) = a_1b_1a_1^{-1}b_1^{-1}$ (on a blue background in Figure 7.2). Finally if $m > 1$ then g_m is the identity thus $\theta_m = \mathrm{id}$ and $g'_m = \mathrm{id}$.

Let us study the behaviour of this decomposition under product of lamp configurations.

Claim 7.1.6. If $g_m, f_m : \mathbb{Z} \to \Gamma_m$ then $(g_m f_m)' = \big(\theta_m(f_m)\big)^{-1} g'_m \theta_m(f_m) f'_m$.

Proof. Since $g_m = \theta_m(g_m)g'_m$ and $f_m = \theta_m(f_m)f'_m$ we can write

$$(g_m f_m)' = \theta_m(g_m)g'_m \cdot \theta_m(f_m)f'_m = \theta_m(g_m)\theta_m(f_m) \cdot \Big(\theta_m(f_m)\Big)^{-1} g'_m\theta_m(f_m)f'_m.$$

But $\theta_m(g_m)\theta_m(f_m)$ takes values in $A_m \times B_m$ and Γ'_m is a normal subgroup thus the map $\big(\theta_m(f_m)\big)^{-1} g'_m\theta_m(f_m)$ takes values in Γ'_m. Hence the claim. ☐

Finally let $\mathbf{g} := (g_m)_{m\in\mathbb{N}}$. Combining Lemma 2.7 and Fact 2.9 of [BZ21], we get the following result.

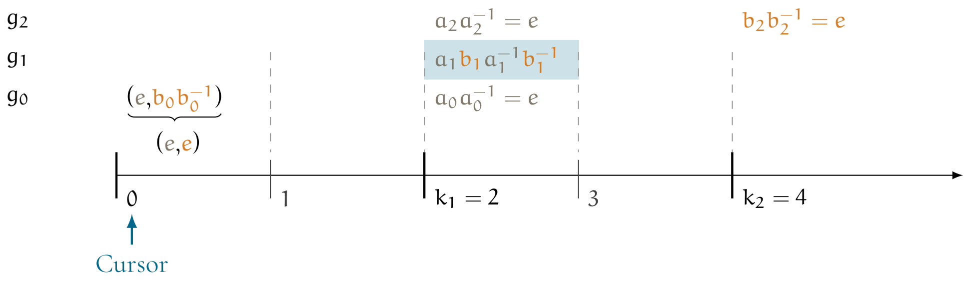


Figure 7.2.: Representation of $(\mathbf{g}, 0)$ defined in Example 7.1.5

Lemma 7.1.7

> Let $(\mathbf{g}, t) \in \Delta$. For all $m \in \mathbb{N}$ and $x \in \mathbb{Z}$
>
> $$g_m(x) = \theta^A_m(g_0(x))\theta^B_m(g_0(x - k_m))g'_m(x).$$
>
> In particular the sequence $\mathbf{g} = (g_m)_{m\in\mathbb{N}}$ is uniquely determined by g_0 and $(g'_m)_{m\in\mathbb{N}}$.

In the next section we are going to prove that we actually need only a *finite* number of elements of the sequence $(g'_m)_{m\in\mathbb{N}}$ to characterize $\mathbf{g}$.

7.2 RANGE AND SUPPORT

In this section we introduce the notion of *range* of an element $(\mathbf{g}, t)$ in Δ and link it to the supports of the lamp configurations $(g_m)_{m\in\mathbb{N}}$.

7.2.1 Range

We denote by $\pi_2 : \Delta \to \mathbb{Z}$ the projection on the second factor and for all $n \in \mathbb{N}$ denote by $l(n)$ the integer such that $k_{l(n)} \leqslant n < k_{l(n)+1}$.

Definition 7.2.1

> If $w = s_1 \dots s_m$ is a word over S_Δ we define its *range* as
>
> $$\mathrm{range}(w) := \left\{ \pi_2\left(\prod_{j=1}^{i} s_j\right) \mid i = 1, \dots, n \right\}.$$

The range is a finite subinterval of $\mathbb{Z}$. It represents the set of sites visited by the cursor.

Definition 7.2.2

> The *range* of an element $\delta \in \Delta$ is defined as the minimal diameter interval obtained as the range of a word over S_Δ representing δ.

When there is no ambiguity we will sometimes denote range(δ) the diameter of this interval.

Example 7.2.3. Let $(\mathbf{g}, 0) \in \Delta$ such that $\mathrm{range}(\mathbf{g}, 0) = [0, 6]$, that is to say: the cursor can only visit sites between 0 and 6. Then the map g_m can "write" elements of A_m only on sites visited by the cursor, that is to say from 0 to 6, and we can write elements of B_m only from k_m to $6+k_m$. Thus g_0 is supported on $[0,6]$, since $k_0=0$. Moreover, commutators (and hence elements of Γ'_m) can only appear between k_m and 6, thus $\mathrm{supp}(g'_m) \subseteq [k_m, 6]$.

Such a $(\mathbf{g}, 0)$ is represented in Figure 7.3 for $k_m = 2^m$.

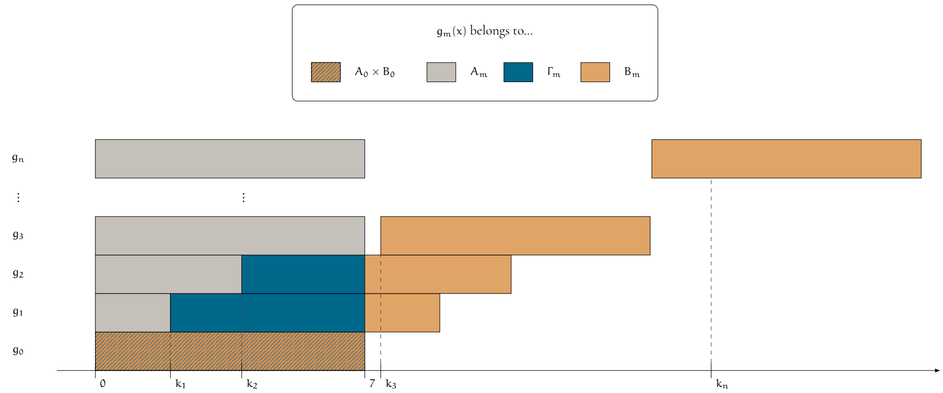


Figure 7.3.: An element of Δ

The element g_m of $\mathbf{g}$ is the function $g_m : \mathbb{Z} \to \Gamma_m$. If $m \leqslant \iota(6)$, then $g_m(x)$ belongs to A_m if $x \in [0, k_m - 1]$, it belongs to Γ_m if $x \in [k_m, 6]$ and to B_m if $x \in [7, 6+k_m]$ and equals e elsewhere. If $m > \iota(6)$ then $g_m(x)$ belongs to A_m if $x \in [0,6]$ and to B_m if $x \in [k_m, 6+k_m]$ and equals e elsewhere.

Let us now recall a useful fact proved in [BZ21]. In order to emphasize the behaviour of Γ_m we also recall its proof.

Claim 7.2.4 ([BZ21, Fact 2.9]). An element $(\mathbf{g}, t) \in \Delta$ is uniquely determined by t, g_0 and the sequence $(g'_m)_{m \leqslant \iota(\mathrm{range}(\mathbf{g},t))}$.

To illustrate this proof we refer to Figure 7.3 which represents the support and values taken by $\mathbf{g}$ and to Figure 7.4 which pictures the corresponding characterizing data given by the claim.

Proof. If $m > \iota(\mathrm{range}(\mathbf{g}))$, then $k_m > \mathrm{range}(\mathbf{g}, t)$ which means that the map g_m takes values in $A_m \cup B_m$, in particular $g'_m \equiv e$. Thus, using Lemma 7.1.7 we get $g_m(x) = \theta_m^A(g_0(x))\theta_m^B(g_0(x-k_m))$. Hence g_m is uniquely determined by g_0.

Finally, if $m \leqslant \iota(\mathrm{range}(\mathbf{g}))$ then g_m is uniquely determined by g_0 and $(g'_m)_{m \in \mathbb{N}}$, by Lemma 7.1.7. Hence the result. □

Example 7.2.5. Consider again $(\mathbf{g}, 0) \in \Delta$ such that $\mathrm{range}(\mathbf{g}, 0) = [0,6]$, which was illustrated in Figure 7.3. Since $k_3 = 8 > 6$, the element $(\mathbf{g}, 0)$ is uniquely determined by the data g_0 (that is to say, the values read in the bottom line) and the values of g'_i for $i = 1, 2$ (namely, the value taken in the blue area). Figure 7.4 represents the aforementioned characterizing data.

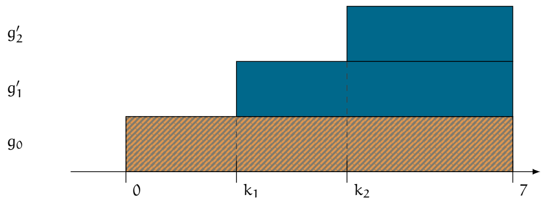


Figure 7.4.: Data needed to characterized $\mathbf{g}$ such that range($\mathbf{g}$) $\subset [0,6]$ when $k_m = 2^m$

7.2.2 Relation between range and support

Recall that for all $m \in \mathbb{N}$ we can write

$$g_m(x) = \theta_m^A(g_0(x))\theta_m^B(g_0(x-k_m)){g'}_m(x).$$

To work with the Følner sequence we compute in Section 7.3 and deduce a Følner *tiling* sequence from it, we will need to link the range of $(\mathbf{g}, t)$ in Δ with the support of g_0 and the sequence of supports of $({g'}_m)_{m\in\mathbb{N}}$. This is what the following lemma formalises.

Lemma 7.2.6

Let $n \in \mathbb{N}$ and take $(\mathbf{g}, t) \in \Delta$. Then range$(\mathbf{g}, t)$ is included in $[0, n]$ if and only if

$$\begin{cases} t \in [0,n] & \\ \operatorname{supp}(g_0) \subset [0,n] & \\ \operatorname{supp}({g'}_m) \subseteq [k_m, n] & \forall 1 \leqslant m \leqslant l(n) \\ {g'}_m \equiv e & \forall m > l(n). \end{cases}$$

Proof. Let $n \in \mathbb{N}$ and let $(\mathbf{g}, t) = \prod_{i=0}^{l} s_i$ be a decomposition in a product of elements of S_Δ of minimal length.

First assume that range$(\mathbf{g}, t) \subseteq [0, n]$, that is to say: the cursor can only visit sites between 0 and n. Let $m \in \mathbb{N}$, then by definition of S_Δ an element s_i can "write" elements of A_m only between 0 and n, and it can write elements of B_m only between k_m and $n+k_m$. Thus g_0 is supported on $[0, n]$, since $k_0 = 0$. And commutators can only appear between k_m and n, hence $\operatorname{supp}(g'_m) \subseteq [k_m, n]$. In particular if $k_m > n$ then $g'_m \equiv e$. Finally we obtain that t belongs to $[0, n]$ by noting that $t = \pi_2\left(\prod_{j=1}^{l} s_j\right)$.

Now let us prove the other way round. Take $m \in [1, l(n)]$, then $k_m \leqslant n$ and $\operatorname{supp}(g'_m)$ is contained in $[k_m, n]$. Now let $x \in \operatorname{supp}(g'_m) \subseteq [k_m, n]$. Since ${\Gamma'}_m \subseteq \Gamma_m$ which is generated by $A_m \times B_m$ we can decompose g'_m as

$$g'_m(x) = \prod_{i=1}^{|g'_m(x)|} a_i b_i.$$

Let $i \in \left[1, |g'_m(x)|\right]$. If the cursor is at some $u \in [0, n]$ then, to write a_i at the position x it needs to visit sites from u to x. Since x belongs to $[k_m, n]$, the cursor stays in $[0, n]$ all along. Now, to write b_i it will need to go from x to $x - k_m$. But since $x \in [k_m, n]$, it will only visit sites between $[0, x]$ and will therefore always stay in $[0, n]$. Hence, to write $\prod_{i=1}^{|g'_m(x)|} a_i b_i$ at the position $g'_m(x)$ the cursor will only visit values between $[0, n]$.

Finally, for all x the cursor needs only to visit position x in order to write $g_0(x)$. Since $\mathrm{supp}(g_0)$ is contained in $[0, n]$ then the cursor needs only to visit sites between 0 and n.

Combining what precedes with Lemma 7.1.7 we get that the cursor needs only to visit cites between $[0, n]$ to write $(g_m)_{m \in \mathbb{N}}$. Hence the lemma.

□

7.3 FØLNER SEQUENCES

In this section we describe a Følner sequence $(F_n)_{n\in\mathbb{N}}$ for Δ. Recall that $l(n)$ denotes the integer such that $k_{l(n)} \leqslant n < k_{l(n)+1}$.

Proposition 7.3.1

> The following sequence is a Følner sequence of Δ
>
> $$F_n := \left\{(f, t) \mid \quad \mathrm{range}\,(f, t) \subseteq \{0, \dots, n-1\}\right\}.$$

Proof. Let $n \in \mathbb{N}$ and $\delta \in F_n$ and let $s_1, \dots, s_l \in S_\Delta$ such that $\delta = s_1 \cdots s_l$. Now take $s_{l+1} \in S_\Delta$. If $s_{l+1} = \big((a_m \delta_0), 0\big)$ for some $a \in A$ or if $s_{l+1} = \big((b_m \delta_{k_m}), 0\big)$ for some $b \in B$ then

$$\mathrm{range}(\delta s_{l+1}) = \left\{\pi_2 \left(\prod_{j=1}^{i} s_j\right) \mid i = 1, \dots, l+1\right\} = \mathrm{range}(\delta),$$

since the cursor of s_{l+1} equals 0. Thus $\delta s_{l+1} \in F_n$. Finally denote by $[x, y]$ the range of δ, then using the same formula as above we get

$$\begin{aligned} \mathrm{range}(\delta \cdot (\mathrm{id}, 1)) &\subseteq [x, y+1] \text{if } t < y, \\ \mathrm{range}(\delta \cdot (\mathrm{id}, 1)) &\subseteq [x, y] \text{if } t = y. \end{aligned}$$

Hence for all $t < n-1$ we have $\mathrm{range}(\delta \cdot (\mathrm{id}, 1)) \subseteq [x, y+1] \subseteq [0, n-1]$. Now if $t = n-1$ then the cursor of $\delta(\mathrm{id}, 1)$ visits the site n, thus $\mathrm{range}(\delta \cdot (\mathrm{id}, 1))$ is not included in $[0, n-1]$ and therefore $\delta(\mathrm{id}, 1)$ does not belong to F_n. A similar argument shows that $\delta(0, -1)$ belongs to F_n if and only if $t \neq 0$. Hence $\partial F_n = \{(f, t) \in F_n \ : \ t = 0, n\}$ and thus

$$\frac{|\partial F_n|}{|F_n|} = \frac{2}{n} \underset{n \to \infty}{\longrightarrow} 0.$$

□

Since we will need an estimation of the diameter of F_n in our proofs to come, let us conclude this section by computing an upper bound for that diameter.

Lemma 7.3.2

> There exists $C_{\mathrm{diam}} > 0$ depending only on Δ such that $\mathrm{diam}\,(F_n) \leqslant C_{\mathrm{diam}} n l_{l(n-1)}$ for all $n \in \mathbb{N}$. In particular if $k_m = \kappa^m$ and $l_m = \kappa^{\alpha m}$ for some $\kappa \geqslant 2$ and $\alpha > 0$, then $\mathrm{diam}\,(F_n) \leqslant C_{\mathrm{diam}} n^{1+\alpha}$.

To show this result, we use Proposition A.2.2.

Proof. Let $n \in \mathbb{N}$ and $(f,t) \in F_n$. First, take $m \leqslant \iota(n-1)$ and let us bound E_m by above. Recall that $I_j^m = [jk_m/2, (j+1)k_m/2 - 1]$. Since (f,t) belongs to F_n its range is included in $[0, n-1]$, thus

$$\begin{aligned}
\left|\left\{j \in \mathbb{Z} : \mathrm{range}(f_m, t) \cap I_j^m \neq \emptyset\right\}\right| &\leqslant \left|\{j \in \mathbb{Z} :\ [0, n-1] \cap [jk_m/2, (j+1)k_m/2 - 1] \neq \emptyset\}\right|, \\
&\leqslant \left|\{j \in \mathbb{Z} :\ jk_m/2 \leqslant n-1 \text{ and } (j+1)k_m/2 \geqslant 1\}\right|, \\
&\leqslant \frac{2(n-2)}{k_m} + 1.
\end{aligned}$$

Moreover remark that $|f_m(x)|_{\Gamma_m} \leqslant \mathrm{diam}\,(\Gamma_m) = l_m$ for all x, thus

$$\begin{aligned}
E_m(f_m) &= k_m \sum_{j : \mathrm{range}(f_m, t) \cap I_j^m \neq \emptyset} \max_{x \in I_j^m} \left(|f_m(x)|_{\Gamma_m} - 1\right)_+, \\
&\leqslant k_m \sum_{j : \mathrm{range}(f_m, t) \cap I_j^m \neq \emptyset} l_m, \\
&\leqslant k_m l_m \left(\frac{2(n-2)}{k_m} + 1\right) = l_m(2(n-2) + k_m).
\end{aligned}$$

Thus, applying the second part of Proposition A.2.2 we get

$$\begin{aligned}
|(f_m, t)|_{\Delta_m} &\leqslant 9\left(\mathrm{range}(f_m, t) + E_m(f_m)\right), \\
&\leqslant 9\left(n + l_m\left(2(n-2) + k_m\right)\right).
\end{aligned}$$

But if $m \leqslant \iota(n-1)$ then $k_m \leqslant n-1 \leqslant n$ thus we can bound $|(f_m,t)|_{\Delta_m}$ by above by $9n(3l_m + 1)$. Now remark that $\iota\left(\mathrm{range}(f,t)\right) \leqslant \iota(n-1)$. Thus, using the preceding inequality and the first part of Proposition A.2.2, we get

$$\begin{aligned}
|(f,t)|_{\Delta} &\leqslant 500 \sum_{m=0}^{\iota(\mathrm{range}(f,t))} |(f_m, t)|_{\Delta_m}, \\
&\leqslant 500 \sum_{m=0}^{\iota(n-1)} 9n\,(3l_m + 1), \\
&\leqslant 4500n \sum_{m=0}^{\iota(n-1)} (3l_m + 1)
\end{aligned}$$

Thus $\mathrm{diam}\,(F_n) \leqslant 4500n \sum_{m=0}^{\iota(n-1)} (3l_m + 1)$. Finally, since l_m is a subsequence of a geometric sequence, there exists $C_r > 0$ such that $\sum_{m=0}^{\iota(n-1)} (3l_m + 1) \leqslant C_r l_{\iota(n-1)}$. Hence the first part of the lemma.

Now let $\kappa \geqslant 2$ and $\alpha > 0$ and assume that $k_m = \kappa^m$ and $l_m = \kappa^{\alpha m}$. Then $\mathrm{diam}\,(F_n) \leqslant C_{\mathrm{diam}} n \kappa^{\alpha \iota(n-1)}$. But by definition of $\iota(n-1)$ we have $\kappa^{\iota(n-1)} \leqslant n-1$ thus $\kappa^{\alpha\iota(n-1)} \leqslant n^{\alpha}$. Hence the second part. $\square$

ORBIT EQUIVALENCE COUPLING WITH THE GROUP OF INTEGERS

8

> *Actually it's very simple, but simple things are always the hardest to explain*
>
> — **David Eddings**
> *Belgariade, Book 3 : Magician's Gambit*

Our aim in this chapter is to show Theorem 6.3.1 which we recall below.

Theorem 6.3.1

For all non-decreasing function $\rho : [1, +\infty[\to [1, +\infty[$ such that $\rho(1) = 1$ and $x/\rho(x)$ is non-decreasing, there exists a group G such that

- $I_G \simeq \rho \circ \log$;
- there exists an orbit equivalence coupling from G to $\mathbb{Z}$ that is $(\varphi_\varepsilon, \exp\circ\rho)$-integrable for all $\varepsilon > 0$, where $\varphi_\varepsilon(x) := \rho \circ \log(x)/(\log\circ\rho \circ \log(x))^{1+\varepsilon}$.

What we actually show is that the group Δ built in appendix A.1 is the wanted group G. To do so we first exhibit a Følner tiling sequence for Δ in Section 8.1. Then in Section 8.2 we define an appropriate Følner tiling sequence for $\mathbb{Z}$, compute some useful inequalities for the quantification and show the theorem using Theorem 6.2.4.

8.1 TILING OF Δ

Let us define a Følner tiling sequence for our group Δ. Our goal is to obtain a tiling verifying $T_n = F_{\kappa^n}$. After defining the shifts sets Σ_n we prove that the sequence $(\Sigma_n)_{n\in\mathbb{N}}$ is actually a Følner tiling sequence. Finally we precise this last statement by computing $(R_n)_{n\in\mathbb{N}}$ and $(\varepsilon_n)_{n\in\mathbb{N}}$ such that $(\Sigma_n)_{n\in\mathbb{N}}$ is a (R_n, ε_n)-Følner tiling sequence (see Definition 6.2.1).

8.1.1 Definition of the shifts

For any $n \in \mathbb{N}$, let $\mathfrak{L}(n) = \mathfrak{l}(\kappa^n - 1)$, that is to say $\mathfrak{L}(n)$ is the integer such that $k_{\mathfrak{L}(n)} \leqslant \kappa^n - 1 < k_{\mathfrak{L}(n)+1}$.

Example 8.1.1. If $k_n := \kappa^n$ for all $n \in \mathbb{N}$ then $\mathfrak{L}(n) = n - 1$.

Before defining our sequence $(\Sigma_n)_{n\in\mathbb{N}}$, let us show some practical results on $\mathfrak{L}$. First remark that since $(k_n)_{n\in\mathbb{N}}$ is a subsequence of $(\kappa^n)_{n\in\mathbb{N}}$, it verifies $k_n \geqslant \kappa^n$ for all $n \in \mathbb{N}$. Thus $\mathfrak{L}(n) \leqslant n$ and

$$k_{\mathfrak{L}(n)} < \kappa^n \leqslant k_{\mathfrak{L}(n)+1}.$$

Claim 8.1.2. Let $n \geqslant 0$, then either $\mathfrak{L}(n+1) = \mathfrak{L}(n)$ or $\mathfrak{L}(n+1) = \mathfrak{L}(n)+1$. Moreover in this second case $k_{\mathfrak{L}(n+1)} = \kappa^n$.

Proof. Recall that by definition $\mathfrak{L}(m) = \max\{i \in \mathbb{N} \mid k_i \leqslant \kappa^m - 1\}$ for all $m \in \mathbb{N}$.

Let $n \in \mathbb{N}$, then $\mathfrak{L}(n+1) \geqslant \mathfrak{L}(n)$. Moreover if $k_{\mathfrak{L}(n)+1} \geqslant \kappa^{n+1}$ then $\mathfrak{L}(n+1) < \mathfrak{L}(n)+1$. That is to say $\mathfrak{L}(n+1) \leqslant \mathfrak{L}(n)$ and thus $\mathfrak{L}(n+1) = \mathfrak{L}(n)$.

On the contrary, if $k_{\mathfrak{L}(n)+1} < \kappa^{n+1}$ then $\mathfrak{L}(n+1) \geqslant \mathfrak{L}(n)+1$. But, by definition of $\mathfrak{L}(n)$ it verifies $k_{\mathfrak{L}(n)+1} \geqslant \kappa^n$ and by construction of $(k_m)_{m\in\mathbb{N}}$ we also have $k_{\mathfrak{L}(n)+2} \geqslant \kappa k_{\mathfrak{L}(n)+1}$ thus $k_{\mathfrak{L}(n)+2} \geqslant \kappa^{n+1}$. Hence $\mathfrak{L}(n+1) < \mathfrak{L}(n)+2$ and the first assertion.

Finally if $\mathfrak{L}(n+1) = \mathfrak{L}(n)+1$ then by definition of $\mathfrak{L}$

$$k_{\mathfrak{L}(n)} < \kappa^n \leqslant k_{\mathfrak{L}(n)+1} = k_{\mathfrak{L}(n+1)} \leqslant \kappa^{n+1} - 1.$$

But $(k_m)_{m\in\mathbb{N}}$ is a subsequence of κ^m thus the above inequality implies $k_{\mathfrak{L}(n+1)} = \kappa^n$. $\square$

Now, let us define the shifts sets. First let $\Sigma_0 := F_0$, then if $n \geqslant 0$ we distinguish two cases depending on whether $\mathfrak{L}(n+1) = \mathfrak{L}(n)$ or $\mathfrak{L}(n+1) = \mathfrak{L}(n)+1$ and in both cases we split the set of shifts Σ_{n+1} in κ parts.

If $\mathfrak{L}(n+1) = \mathfrak{L}(n)$, let for all $j \in \{0, \dots, \kappa-1\}$

$$\Sigma^j_{n+1} := \left\{ (g, j\kappa^n) \in \Delta \;\middle|\; \begin{array}{l} \quad\operatorname{supp}(g_0) \subseteq \left[0, j\kappa^n - 1\right] \cup \left[(j+1)\kappa^n, \kappa^{n+1} - 1\right], \\ \forall m \in [1, \mathfrak{L}(n)] \\ \quad\operatorname{supp}(g'_m) \subseteq \left[k_m, j\kappa^n + k_m - 1\right] \cup \left[(j+1)\kappa^n, \kappa^{n+1} - 1\right], \\ \forall m \notin [0, \mathfrak{L}(n)] \\ \quad\operatorname{supp}(g'_m) = \emptyset. \end{array} \right\}.$$

Now if $\mathfrak{L}(n+1) = \mathfrak{L}(n)+1$ we add the condition that $g'_{\mathfrak{L}(n)+1}$ has support contained in $[k_{\mathfrak{L}(n+1)}, \kappa^{n+1} - 1]$, namely

$$\Sigma^j_{n+1} := \left\{ (g, j\kappa^n) \in \Delta \;\middle|\; \begin{array}{l} \quad\operatorname{supp}(g_0) \subseteq \left[0, j\kappa^n - 1\right] \cup \left[(j+1)\kappa^n, \kappa^{n+1} - 1\right] \\ \forall m \in [1, \mathfrak{L}(n)] \\ \quad\operatorname{supp}(g'_m) \subseteq \left[k_m, j\kappa^n + k_m - 1\right] \cup \left[(j+1)\kappa^n, \kappa^{n+1} - 1\right], \\ \operatorname{supp}\left(g'_{\mathfrak{L}(n)+1}\right) \subseteq \left[k_{\mathfrak{L}(n)+1}, \kappa^{n+1} - 1\right], \\ \forall m \notin [0, \mathfrak{L}(n+1)] \operatorname{supp}(g'_m) = \emptyset. \end{array} \right\}.$$

Finally, in both cases we define $\Sigma_{n+1} := \cup_{j=0}^{\kappa-1} \Sigma^j_{n+1}$.

Let (g, t) be an element of some Σ^j_{n+1}. We represent in Figure 8.1 the supports and the sets where the maps $g_0, {g'}_1, \dots, {g'}_{\mathfrak{L}(n+1)}$ take their values. The light-blue rectangle with dotted outline is in Σ^j_{n+1} if and only if $\mathfrak{L}(n+1) = \mathfrak{L}(n)+1$.

Now that we have the shifts sequence, let us turn to the definition of the tiles.

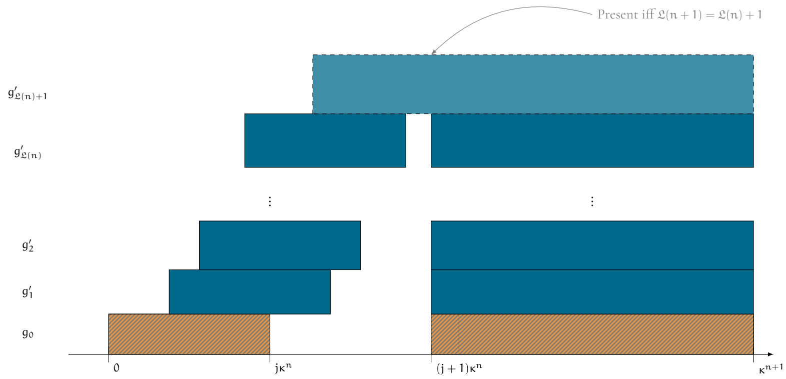


Figure 8.1.: Support and values taken by $(\mathbf{g}, t) \in \Sigma_n^j$

8.1.2 Tiling

Recall that $(F_n)_{n \in \mathbb{N}}$ denotes the Følner sequence of Δ defined in Proposition 7.3.1. The aim of this section is to show the theorem below.

Theorem 8.1.3

> The sequence $(\Sigma_n)_{n \in \mathbb{N}}$ defined in Section 8.1.1 is a Følner tiling sequence of Δ.

Before showing that the sequence of tiles $(T_n)_{n \in \mathbb{N}}$ thus induced verifies indeed the conditions of Definition 6.2.1, let us show the following lemma.

Lemma 8.1.4

> The sequence $(T_n)_{n \in \mathbb{N}}$ defined by $T_0 := F_0$ and $T_{n+1} := \Sigma_{n+1} T_n$ for all $n > 0$ verifies
>
> $$(\forall n \in \mathbb{N}) \quad T_n = F_{\kappa^n}.$$

Let us discuss the idea of the proof. We proceed by induction and use a double inclusion argument to prove the induction step. To show that $\Sigma_{n+1} T_n$ is included in $F_{\kappa^{n+1}}$ we rely on Lemma 7.2.6, that is to say we verify that every element of $\Sigma_{n+1} T_n$ has range included in $[0, \kappa^{n+1} - 1]$. For the reversed inclusion we consider an element $(\mathbf{h}, t)$ of $F_{\kappa^{n+1}}$ and explicit the elements $(\mathbf{g}, j\kappa^n)$ of Σ_{n+1} and $(\mathbf{f}, t')$ of T_n such that $(\mathbf{h}, t) = (\mathbf{g}, j\kappa^n)(\mathbf{f}, t')$. To show this we have to split $\operatorname{supp}(\mathbf{h})$ into smaller subintervals and show that last equality on each subintervals separately. Indeed for each $m > 1$ the supports of g_m and f_m partly overlap (see Figure 8.2 for an illustration). We thus have to consider differently subintervals where the two supports overlap from the ones where f_m or g_m equals $\mathbf{e}$.

Mind the involved maps here: we study the values of g_m and f_m instead of the "derived" functions g'_m, f'_m usually considered.

Proof of the lemma. The assertion is true for T_0. Now let $n \leqslant 0$ and assume that $T_n = F_{\kappa^n}$. We show the induction step by double inclusion.

FIRST INCLUSION

Let us show that $\Sigma_{n+1} T_n \subseteq F_{\kappa^{n+1}}$. Recall that Σ_{n+1} can be decomposed as $\Sigma_{n+1} = \cup_{j=0}^{\kappa-1} \Sigma_{n+1}^j$.

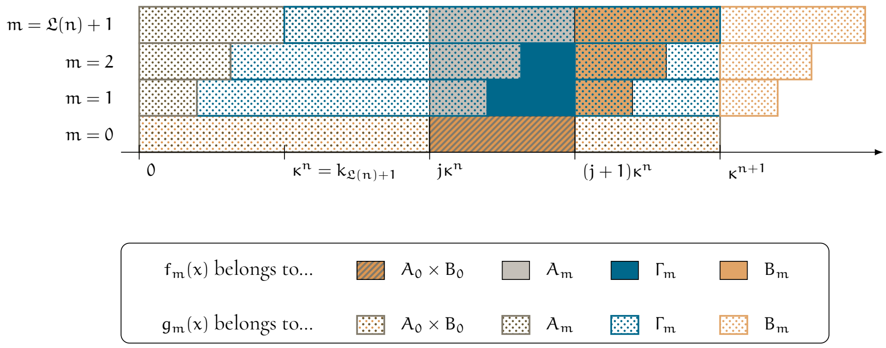


Figure 8.2.: Supports overlap

Let $(\mathbf{f}, t) \in T_n$ and $j \in \{0, \dots, \kappa - 1\}$. Take $(\mathbf{g}, j\kappa^n) \in \Sigma^j_{n+1}$, then the following product

$$(\mathbf{g}, j\kappa^n)\,(\mathbf{f}, t) = \left(\left(g_m f_m\left(\cdot - j\kappa^n\right)\right)_m, t + j\kappa^n\right)$$

verifies $t + j\kappa^n \in \left[j\kappa^n, \kappa^n - 1 + (\kappa - 1)\kappa^n\right] \subset \left[0, \kappa^{n+1} - 1\right]$ and

$$g_0(x) f_0(x - j\kappa^n) = \begin{cases} g_0(x) & \text{if } x \in \left[0, j\kappa^n\right] \cup \left[(j+1)\kappa^n, \kappa^{n+1} - 1\right] \\ f_0(x - j\kappa^n) & \text{if } x \in \left[j\kappa^n, (j+1)\kappa^n - 1\right] \\ 0 & \text{else.} \end{cases}$$

Thus $\operatorname{supp}(g_0 f_0(\cdot - j\kappa^n)) \subseteq \left[0, \kappa^{n+1} - 1\right]$.

Moreover, for all $m \in \{1, \dots, \mathfrak{L}(n)\}$

$$\operatorname{supp}(g'_m) \subset \left[k_m, j\kappa^n + k_m - 1\right] \cup \left[(j+1)\kappa^n, \kappa^{n+1} - 1\right]$$
$$\operatorname{supp}\left(f'_m\left(\cdot - j\kappa^n\right)\right) \subseteq [j\kappa^n + k_m, (j+1)\kappa^n - 1],$$

hence by Claim 7.1.6 the support of $(g_m f_m(\cdot - j\kappa^m))'$ is contained in $[k_m, \kappa^{n+1} - 1]$.

Now if $\mathfrak{L}(n+1) = \mathfrak{L}(n) + 1$ let us denote $m = \mathfrak{L}(n) + 1$. In that case $f'_m \equiv \mathbf{e}$ since $m > \mathfrak{L}(n)$. Thus $(g_m f_m(\cdot - j\kappa^m))' = g'_n$ whose support is contained in $[k_{\mathfrak{L}(n)+1}, \kappa n + 1 - 1]$.

Finally $(g_m f_m(\cdot - j\kappa^m))' \equiv 0$ for all $m \notin [0, \mathfrak{L}(n+1)]$. Hence by Lemma 7.2.6 the product $(\mathbf{g}, j\kappa^n)\,(\mathbf{f}, t)$ has range included in $\left[0, \kappa^{n+1} - 1\right]$ and thus belongs to $F_{\kappa^{n+1}}$.

Second Inclusion

Let us show that $F_{\kappa^{n+1}}$ is contained in $\Sigma_{n+1} T_n$. Take $(\mathbf{h}, t) \in F_{\kappa^{n+1}}$. We want to define $(\mathbf{f}, t') \in T_n$ and $(\mathbf{g}, j\kappa^n) \in \Sigma_{n+1}$ such that $(\mathbf{g}, j\kappa^n)\,(\mathbf{f}, t') = (\mathbf{h}, t)$.

First remark that $t < \kappa^{n+1}$, since $(\mathbf{h}, t) \in F_{\kappa^{n+1}}$. Thus there exists $t_0, \dots, t_n$ in $[0, \kappa - 1]$ such that $t = \sum_{i=0}^{n} t_i \kappa^i$. Let $j = t_n$ and $t' = \sum_{i=0}^{n-1} t_i \kappa^i$. Then j does belong to $[0, \kappa - 1]$ and t' to $[0, \kappa^n - 1]$. We now have to define $\mathbf{f}$ and $\mathbf{g}$ such that

$$\left(\left(g_m f_m\left(\cdot - j\kappa^n\right)\right), t' + j\kappa^n\right) = (\mathbf{h}, t).$$

Recall that the supports of g_m and f_m overlap. One can refer to Figure 8.2 for an illustration of such overlaps. Let

$$f_0(x) := \begin{cases} h_0(x + j\kappa^n) & \text{if } x \in [0, \kappa^n - 1], \\ \mathbf{e} & \text{else,} \end{cases}$$

$$g_0(x) := \begin{cases} h_0(x) & \text{if } x \in [0, j\kappa^n - 1] \cup [(j+1)\kappa^n, \kappa^{n+1} - 1], \\ \mathbf{e} & \text{else.} \end{cases}$$

One can verify immediately that $g_0 f_0(\cdot - j\kappa^n) = h_0$. Then take $m \in [1, \mathfrak{L}(n)]$ and let

$$f'_m(x) := \begin{cases} h'_m(x + j\kappa^n) & \text{if } x \in [k_m, \kappa^n - 1], \\ e & \text{else,} \end{cases}$$

$$g'_m(x) := \begin{cases} h'_m(x) & \text{if } x \in [0, j\kappa^n - 1] \\ & \cup[(j+1)\kappa^n + k_m, \kappa^{n+1} - 1], \\ \theta_0^A(h_0(x))\, h'_m(x) \theta_0^A(h_0(x))^{-1} & \text{if } x \in [j\kappa^n, j\kappa^n + k_m - 1], \\ \theta_0^B(h_0(x - k_m))\, h'_m(x) \theta_0^B(h_0(x - k_m))^{-1} & \text{if } x \in \big[(j+1)\kappa^n, \\ & \qquad (j+1)\kappa^n + k_m - 1\big], \\ e & \text{else.} \end{cases}$$

Note that to define g'_m on $[j\kappa^n, j\kappa^n + k_m - 1]$ and $[(j+1)\kappa^n, (j+1)\kappa^n + k_m - 1]$ we need to conjugate h'_m because these two intervals are included in the support of f_m (see Figure 8.2).

Now if $\mathfrak{L}(n+1) = \mathfrak{L}(n) + 1$ then $k_{\mathfrak{L}(n+1)} \geqslant \kappa^n$ and in that case define

$$g'_m(x) := \begin{cases} h'_m(x) & \text{if } x \in [\kappa^n, j\kappa^n - 1] \\ & \quad \cup [(j+2)\kappa^n, \kappa^{n+1} - 1], \\ \theta_0^A(h_0(x))\, h'_m(x) \theta_0^A(h_0(x))^{-1} & \text{if } x \in [j\kappa^n, (j+1)\kappa^n - 1], \\ \theta_0^B(h_0(x - k_m))\, h'_m(x) \theta_0^B(h_0(x - k_m))^{-1} & \text{if } x \in [(j+1)\kappa^n, (j+2)\kappa^n - 1], \\ e & \text{else.} \end{cases}$$

Finally let $f'_{\mathfrak{L}(n+1)} \equiv e$ and if $m > \mathfrak{L}(n+1)$ let $g'_m \equiv e \equiv f'_m$.

With the above definitions $\mathbf{f}$ and $\mathbf{g}$ are uniquely defined. Moreover, by definition $((g_m)_m, j\kappa^n)$ belongs to Σ^j_{n+1} and by Lemma 7.2.6 we have range$(\mathbf{f}, t) \subseteq [0, \kappa^n - 1]$ thus $(\mathbf{f}, t')$ belongs to T_n.
Now, using Lemma 7.1.7 we verify that $g_m f_m(\cdot - j\kappa^n) = h_m$ thus $(\mathbf{h}, t) \in \Sigma_{n+1} T_n$.

Hence, combining the first and second inclusion we get $F_{\kappa^{n+1}} = T_n$. □

We now know that $(T_n)_{n\in\mathbb{N}}$ is a Følner sequence. To prove Theorem 8.1.3 we have to show that $(\Sigma_n)_{n\in\mathbb{N}}$ a Følner *tiling* sequence. We thus only have to verify that the set of shifts $(\Sigma_i)_{i\leqslant n}$ tiles T_n.

Proof of Theorem 8.1.3. The sequence $(T_n)_{n\in\mathbb{N}}$ is a Følner sequence, by the last lemma. Thus we only have to show that for all $\sigma \neq \tilde\sigma \in \Sigma_{n+1}$, $\sigma T_n \cap \tilde\sigma T_n = \emptyset$. So let us denote by $(\mathbf{h}, t)$ an element of $\sigma T_n \cap \tilde\sigma T_n$. We distinguish two cases.

First if $\sigma \in \Sigma^j_{n+1}$ and $\tilde\sigma \in \Sigma^i_{n+1}$ for some $i \neq j$, then the cursor of σ is equal to $j\kappa^n$ and the one of $\tilde\sigma$ to $i\kappa^n$. Thus

$$(\mathbf{h}, t) \in \sigma T_n \Rightarrow t \in [j\kappa^n, (j+1)\kappa^n - 1],$$
$$(\mathbf{h}, t) \in \tilde\sigma T_n \Rightarrow t \in [i\kappa^n, (i+1)\kappa^n - 1].$$

But since $i \neq j$ these two intervals are disjoint, thus $\sigma T_n \cap \tilde\sigma T_n = \emptyset$.

Now fix $j \in \{0, \dots, \kappa - 1\}$ and take $\sigma, \tilde\sigma \in \Sigma^j_{n+1}$. Let $\sigma := (\mathbf{g}, j\kappa^n)$ and $\tilde\sigma := (\tilde{\mathbf{g}}, j\kappa^n)$. Assume that there exists $(\mathbf{f}, t), (\tilde{\mathbf{f}}, \tilde t) \in T_n$ such that $(\mathbf{g}, j\kappa^n)(\mathbf{f}, t) = (\tilde{\mathbf{g}}, j\kappa^n)(\tilde{\mathbf{f}}, \tilde t)$. Then

$$\forall m \in \mathbb{N} \quad g_m f_m(\cdot - j\kappa^n) = \tilde g_m \tilde f_m(\cdot - j\kappa^n). \tag{8.1}$$

First remark that

$$\begin{aligned}
\sigma, \tilde{\sigma} \in \Sigma_{n+1}^{j} \quad &\Longrightarrow \quad \operatorname{supp}(g_0), \operatorname{supp}(\tilde{g}_0) \subseteq [0, j\kappa^n - 1] \cup [(j+1)\kappa^n, \kappa^{n+1} - 1] \\
(f,t), (\tilde{f}, \tilde{t}) \in T_n \quad &\Longrightarrow \quad \operatorname{supp}(f_0(\cdot - j\kappa^n)), \operatorname{supp}(\tilde{f}_0(\cdot - j\kappa^n)) \subseteq [j\kappa^n, (j+1)\kappa^n - 1].
\end{aligned}$$

In other word the support of g_0 (resp $\tilde{g}_0$) is disjoint from the one of $f_0(\cdot - j\kappa^n)$ (resp $\tilde{f}_0(\cdot - j\kappa^n)$). Combining this with eq. (8.1) we obtain that $g_0 = \tilde{g}_0$ and $f_0 = \tilde{f}_0$.

Now let $m > 0$ and let us show that $g_m = \tilde{g}_m$. Due to supports overlap (see Figure 8.2) we need to decompose $[0, \kappa^{n+1} - 1]$ in five subintervals, namely

$$\begin{aligned}
\left[0, \kappa^{n+1} - 1\right] = &\left[0, j\kappa^n - 1\right] \sqcup \left[j\kappa^n, j\kappa^n + k_m - 1\right] \sqcup \left[j\kappa^n + k_m, (j+1)\kappa^n - 1\right], \\
&\sqcup \left[(j+1)\kappa^n, (j+1)\kappa^n + k_m - 1\right] \sqcup \left[(j+1)\kappa^n + k_m, \kappa^{n+1} - 1\right].
\end{aligned}$$

If $x \leqslant j\kappa^n - 1$ or $x \geqslant (j+1)\kappa^n + k_m$, then $f_m(x - j\kappa^n) = e = \tilde{f}_m(x - j\kappa^n)$ and thus $g_m(x) = \tilde{g}_m(x)$ by eq. (8.1).

If $x \in [j\kappa^n, j\kappa^n + k_m - 1]$ then using Lemma 7.1.7 and the fact that on that subinterval $f_0 = \tilde{f}_0$, we get

$$f_m(x - j\kappa^n) = \theta_0^A\left(f_0\left(x - j\kappa^n\right)\right) = \theta_0^A\left(\tilde{f}_0\left(x - j\kappa^n\right)\right) = \tilde{f}_m(x - j\kappa^n).$$

Hence by eq. (8.1) we get $g_m(x) = \tilde{g}_m(x)$.

If x belongs to $[j\kappa^n + k_m, (j+1)\kappa^n - 1]$ then $g_m(x) = \tilde{g}_m(x) = e$ and thus eq. (8.1) implies that $f_m(x - j\kappa^n) = \tilde{f}_m(x - j\kappa^n)$, that is to say f_m and $\tilde{f}_m$ coïncide on $[k_m, \kappa^n - 1]$.

Finally if $x \in [(j+1)\kappa^n, (j+1)\kappa^n + k_m - 1]$ then using Lemma 7.1.7 and the fact that $f_0 = \tilde{f}_0$ on that subinterval, we get

$$f_m(x - j\kappa^n) = \theta_0^B\left(f_0\left(x - j\kappa^n - k_m\right)\right) = \theta_0^B\left(\tilde{f}_0\left(x - j\kappa^n - k_m\right)\right) = \tilde{f}_m(x).$$

Hence by eq. (8.1), we have $g_m(x) = \tilde{g}_m(x)$.

Thus $\mathbf{g} = \tilde{\mathbf{g}}$ and then $\sigma = \tilde{\sigma}$. Which concludes the proof of the theorem. □

We showed that $(\Sigma_n)_{n\in\mathbb{N}}$ is a Følner tiling sequence. Let us now compute the sequences $(R_n)_{n\in\mathbb{N}}$ and $(\varepsilon_n)_{n\in\mathbb{N}}$ such that it is a (R_n, ε_n)-Følner tiling sequence.

8.1.3 Quantification

In order to build our orbit equivalence with $\mathbb{Z}$ we need to estimate the diameter of the tiles and the value of $|\partial T_n|/|T_n|$. Recall that we gave an estimation of the diameter of F_m in Lemma 7.3.2.

Lemma 8.1.5

The sequence $(\Sigma_n)_{n\in\mathbb{N}}$ defined in Section 8.1.1 is a (R_n, ε_n)-Følner tiling sequence for

$$R_n = C_R \kappa^n l_{\mathfrak{L}(n)} \qquad \varepsilon_n = \frac{2}{\kappa^n},$$

for some strictly positive constant C_R.

Prof of Lemma 8.1.5. First remark that by the proof of Proposition 7.3.1 we have

$$\varepsilon_n = \frac{|\partial T_n|}{|T_n|} = \frac{|\partial F_{\kappa^n}|}{|F_{\kappa^n}|} = \frac{2}{\kappa^n}.$$

Now by Lemma 7.3.2 we have $\operatorname{diam}(T_n) = \operatorname{diam}(F_{\kappa^n}) \leqslant \kappa^n l_{\mathfrak{L}(n)}$. □

Example 8.1.6. Let $\alpha > 0$. If $k_n := \kappa^n$ and $l_n = \kappa^{\alpha n}$ for all $n \in \mathbb{N}$, then $\mathfrak{L}(n) = n - 1$ and thus $R_n = C_R \kappa^{(1+\alpha)n}$.

We can now use our Følner tiling sequence to build and quantify an orbit equivalence coupling with $\mathbb{Z}$.

8.2 COUPLING

Our aim in this section is to show Theorem 6.3.1. The main tool is Theorem 6.2.4. We thus start by defining an appropriate Følner tiling sequence for $\mathbb{Z}$ in Section 8.2.1 and give useful bounds for $\ln|T_n|$ in Section 8.2.2. Finally we prove in Section 8.2.3 that the orbit equivalence coupling induced by the two built Følner tiling sequences is $(\psi_\varepsilon, \exp\circ\rho)$-integrable. We conclude this chapter by showing that the orbit equivalence coupling from Δ to $\mathbb{Z}$ thus obtained is satisfies Theorem 6.3.1.

8.2.1 Tiling sequence for Z

We will denote by $(\Sigma'_n)_{n\in\mathbb{N}}$ a Følner tiling sequence of $\mathbb{Z}$ and by T'_n the corresponding tiles.

In order to use Theorem 6.2.4 to get an orbit equivalence coupling between $\mathbb{Z}$ and Δ we need Σ_{n+1} and Σ'_{n+1} to have the same number of elements. We thus define

$$\begin{cases} & \Sigma'_0 = [0, |T_0| - 1] \\ \forall n \in \mathbb{N} & \Sigma'_{n+1} := \{0, |T_n|, 2|T_n|, \ldots, (|\Sigma_{n+1}| - 1)\,|T_n|\}. \end{cases} \tag{8.2}$$

It induces a sequence $(T'_n)_{n\in\mathbb{N}}$ defined by $T'_0 = \Sigma'_0$ and $T'_{n+1} = \Sigma'_{n+1}T'_n$ for all $n \geqslant 0$. We are going to prove that $(\Sigma'_n)_{n\in\mathbb{N}}$ is a Følner tiling sequence for $\mathbb{Z}$. We represent in Figure 8.3 the construction of T'_{n+1} from Σ'_{n+1} and T'_n for $|\Sigma_{n+1}| = 3$.

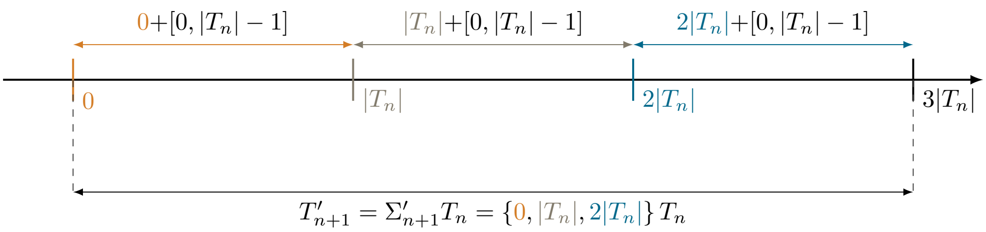


Figure 8.3.: Example of tiling of T'_{n+1} with T'_n for $\Sigma'_{n+1} = \{0, |T_n|, 2|T_n|\}$

.

Proposition 8.2.1

The sequence $(\Sigma'_n)_{n\in\mathbb{N}}$ defined by eq. (8.2) is a (R'_n, ε'_n)-Følner tiling sequence for $\mathbb{Z}$ with

$$R'_n = |T_n| \qquad \varepsilon'_n = \frac{2}{|T_n|}.$$

Moreover the induced sequence $(T'_n)_{n\in\mathbb{N}}$ verifies $T'_n = [0, |T_n| - 1]$ for all $n \in \mathbb{N}$.

Proof. Let $(\Sigma'_n)_{n\in\mathbb{N}}$ be as defined by eq. (8.2) and recall that the induced tiling $(T'_n)_{n\in\mathbb{N}}$ is the sequence defined by $T'_0 := \Sigma'_0$ and $T'_{n+1} = \Sigma'_{n+1}T'_n$ for all $n \in \mathbb{N}$. One can easily prove that for all $n \geqslant 0$

$$T'_n = [0, |T_n| - 1]. \tag{8.3}$$

It is now immediate to check that $\operatorname{diam}(T'_n) = |T_n|$ and $|\partial T'_n|/|T'_n| = 2/|T_n|$. Furthermore note that if $\sigma, \sigma' \in \Sigma'_{n+1}$ such that $\sigma \neq \sigma'$ then $|\sigma - \sigma'| \geqslant |T_n| = \operatorname{diam}(T'_n)$. Thus for such σ and σ' we get $\sigma T'_n \cap \sigma' T_n = \emptyset$. Therefore $(\Sigma_n)_{n\in\mathbb{N}}$ is a Følner tiling sequence and the proposition follows from the above quantifications on T_n. □

By the first part of Theorem 6.2.4 we now know that there exists an orbit equivalence coupling between $\mathbb{Z}$ and Δ. To quantify the integrability of this coupling we need to precise and bound the values of R'_n and ε'_n.

8.2.2 Useful inequalities

The integrability of the coupling between $\mathbb{Z}$ and Δ depends on (R_n, ϵ_n) and (R'_n, ε'_n) but by the last proposition, that last couple depends on the value of the cardinality of the tiles $(T_n)_{n\in\mathbb{N}}$. The aim of this subsection is to give estimations of $|T_n|$ involving only terms of $(k_m)_{m\in\mathbb{N}}$ and $(l_m)_{m\in\mathbb{N}}$. First let us precise the value of $|T_n|$.

Lemma 8.2.2

> The sequence $(T_n)_n$ defined in Theorem 8.1.3 verifies
>
> $$|T_n| = \kappa^n \left(|A||B|\right)^{\kappa^n} \prod_{m=1}^{\mathfrak{L}(n)} \left|\Gamma'_m\right|^{\kappa^n - k_m}.$$

Proof. Recall that $T_n = F_{\kappa^n} = \{(f,t) \mid \operatorname{range}(f,t) \subseteq \{0, \dots, \kappa^n - 1\}\}$ for all $n \in \mathbb{N}$. Let $n \in \mathbb{N}$ and take $(f,t) \in T_n$, then there are exactly κ^n values of t possible. Moreover f is uniquely determined by f_0 and $f'_1, \dots, f'_{\mathfrak{L}(n)}$ (see Lemma 7.1.7). But f_0 is supported on $[0, \kappa^n - 1]$ which is set of cardinal κ^n so there are exactly $\left(|A||B|\right)^{\kappa^n}$ possible values for f_0. Moreover if $m > 0$ then remark that f'_m is supported on $[k_m, \kappa^n - 1]$ which has $\kappa^n - k_m$ elements so there are exactly $\left|\Gamma'_m\right|^{\kappa^n - k_m}$ possible values for f'_m. Thus the number of elements in T_n is

$$\kappa^n \left(|A||B|\right)^{\kappa^n} \prod_{m=1}^{\mathfrak{L}(n)} \left|\Gamma'_m\right|^{\kappa^n - k_m}.$$

□

Now let us bound $|T_n|$ such that the bounds depend only on $(\kappa^m)_{m\in\mathbb{N}}$ and $(l_m)_{m\in\mathbb{N}}$.

Proposition 8.2.3

> There exists two constants $C_2, C_3 > 0$ such that for all $n \in \mathbb{N}$,
>
> $$C_2 \kappa^{n-1} l_{\mathfrak{L}(n)} \leqslant \ln|T_n| \leqslant C_3 \kappa^n l_{\mathfrak{L}(n)}.$$

Remember that $|T_n|$ is given by Lemma 8.2.2. Before showing the above proposition let us give an estimation of the right factor of the expression of $|T_n|$.

Lemma 8.2.4

There exists two constants $C_1, C_2 > 0$ such that for all $n \in \mathbb{N}$,

$$C_2 \kappa^{n-1} l_{\mathfrak{L}(n)} \leqslant \ln\left(\prod_{m=1}^{\mathfrak{L}(n)} \left|\Gamma'_m\right|^{\kappa^n - k_m}\right) \leqslant C_1 \kappa^n l_{\mathfrak{L}(n)}.$$

Proof. Recall that by eq. (7.1) there exists $c_1, c_2 > 0$ such that, for all m

$$c_1 l_m - c_2 \leqslant \ln|\Gamma_m| \leqslant c_1 l_m + c_2.$$

Since $\Gamma'_m \leqslant \Gamma_m$ we thus have

$$\begin{aligned}
\ln\left(\prod_{m=1}^{\mathfrak{L}(n)} \left|\Gamma'_m\right|^{\kappa^n - k_m}\right) &\leqslant \sum_{m=1}^{\mathfrak{L}(n)} (\kappa^n - k_m) \ln|\Gamma_m|, \\
&\leqslant \sum_{m=1}^{\mathfrak{L}(n)} (\kappa^n - k_m)(c_1 l_m + c_2).
\end{aligned}$$

But we can bound $\kappa^n - k_m$ from above by κ^n and since $(l_m)_{m\in\mathbb{N}}$ is a subsequence of a sequence having geometric growth, the sum $\sum_{m=1}^{\mathfrak{L}(n)} (c_1 l_m + c_2)$ is bounded from above by its last term up to a multiplicative constant. That is to say: there exists $C_1 > 0$ such that

$$\ln\left(\prod_{m=1}^{\mathfrak{L}(n)} \left|\Gamma'_m\right|^{\kappa^n - k_m}\right) \leqslant C_1 \kappa^n l_{\mathfrak{L}(n)}.$$

Hence the upper bound. Now, using that $[\Gamma_m : \Gamma'_m] = |A||B|$ we have

$$\ln\left(\prod_{m=1}^{\mathfrak{L}(n)} \left|\Gamma'_m\right|^{\kappa^n - k_m}\right) = \sum_{m=1}^{\mathfrak{L}(n)} (\kappa^n - k_m) \ln|\Gamma'_m| = \sum_{m=1}^{\mathfrak{L}(n)} (\kappa^n - k_m) \ln\left(\frac{|\Gamma_m|}{|A||B|}\right).$$

Bounding the sum from below by its last term and using eq. (7.1), we get

$$\begin{aligned}
\ln\left(\prod_{m=1}^{\mathfrak{L}(n)} \left|\Gamma'_m\right|^{\kappa^n - k_m}\right) &\geqslant (\kappa^n - k_{\mathfrak{L}(n)}) \ln\left(\frac{|\Gamma_{\mathfrak{L}(n)}|}{|A||B|}\right), \\
&\geqslant (\kappa^n - k_{\mathfrak{L}(n)}) \left(c_1 l_{\mathfrak{L}(n)} - c_2 - \ln(|A||B|)\right), \\
&\geqslant C_2(\kappa^n - k_{\mathfrak{L}(n)}) l_{\mathfrak{L}(n)},
\end{aligned}$$

for some $C_2 > 0$. We get the wanted inequality by noting that $\kappa^n - k_{\mathfrak{L}(n)} \geqslant \kappa^{n-1}$. □

Proof of Proposition 8.2.3. By Lemma 8.2.2 and Lemma 8.2.4

$$\begin{aligned}
\ln|T_n| &= \ln\left(\kappa^n (|A||B|)^{\kappa^n} \prod_{m=1}^{\mathfrak{L}(n)} \left|\Gamma'_m\right|^{\kappa^n - k_m}\right) \\
&\leqslant \ln(\kappa^n) + \kappa^n \ln(|A||B|) + C_1 \kappa^n l_{\mathfrak{L}(n)}.
\end{aligned}$$

Thus, there exists $C_3 > 0$ such that $\ln|T_n| \leqslant C_3 \kappa^n l_{\mathfrak{L}(n)}$.

The minoration comes imediately from Lemma 8.2.4. □

Equipped with these bounds on $|T_n|$ we can now show the wanted integrability for the coupling.

8.2.3 Integrability and proof of Theorem 6.3.1

Let us now prove Theorem 6.3.1. So recall that $\rho : [1, +\infty[\to [1, +\infty[$ is a non-decreasing function such that $\rho(1) = 1$ and $x/\rho(x)$ is non-decreasing and that Δ is the Brieussel-Zheng's diagonal product associated to ρ (see appendix A.1 for a defintion of Δ from the map ρ). We will show that Δ is the group satisfying Theorem 6.3.1, but first let us quantify the integrability of the orbit equivalence coupling with $\mathbb{Z}$ induced by the Følner tiling sequences we built.

Theorem 8.2.5

> Let $\rho : [1, +\infty[\to [1, +\infty[$ be a non-decreasing function such that $\rho(1) = 1$ and $x/\rho(x)$ is non-decreasing and take Δ to be the Brieussel-Zheng's diagonal product defined from ρ. Let $\varepsilon > 0$ and $\Psi := \exp \circ \rho$ and let
>
> $$\varphi_\varepsilon(x) := \frac{\rho \ln(x)}{\left(\ln \rho \ln(x)\right)^{1+\varepsilon}}.$$
>
> There exists an orbit equivalence coupling from Δ to $\mathbb{Z}$ that is $(\varphi_\varepsilon, \Psi)$-integrable.

Let us discuss the strategy of the proof. The demonstration is based on Theorem 6.2.4, thus we first prove that $(\Psi(2R_n)\varepsilon'_{n-1})_n$ is summable and then that $(\varphi_\varepsilon(2R'_n)\varepsilon_{n-1})_n$ is. In both cases we use Proposition 8.2.3 to get majorations. So far, we have the following quantifications.

$$R_n = C_R \kappa^n l_{\mathfrak{L}(n)} \qquad R'_n = |T_n| - 1$$

$$\varepsilon_n = \frac{2}{\kappa^n} \qquad \varepsilon'_n = \frac{2}{|T_n|}.$$

$$|T_n| = \kappa^n \left(|A||B|\right)^{\kappa^n} \prod_{m=1}^{\mathfrak{L}(n)} \left|\Gamma'_m\right|^{\kappa^n - k_m}.$$

Proof of Theorem 8.2.5. Let $\rho : [1, +\infty[\to [1, +\infty[$ be a non-decreasing function such that $\rho(1) = 1$ and $x/\rho(x)$ is non-decreasing and take Δ to be the Brieussel-Zheng's diagonal product defined from ρ as described in appendix A.1.

To begin, let us recall some preliminary results about ρ. Remember that $\rho \simeq \tilde{\rho}$ where $\tilde{\rho}$ is defined below eq. (A.2). By definition of $\mathfrak{L}(n)$ we have $k_{\mathfrak{L}(n)} l_{\mathfrak{L}(n)} \leqslant \kappa^n l_{\mathfrak{L}(n)} \leqslant k_{\mathfrak{L}(n)+1} l_{\mathfrak{L}(n)}$, thus by eq. (A.2)

$$\tilde{\rho}(\kappa^n l_{\mathfrak{L}(n)}) = \kappa^n. \tag{8.4}$$

Now let us show that the coupling from $\mathbb{Z}$ to Δ is Ψ-integrable. To do so we prove that $\left(\Psi(2R_n)\varepsilon'_{n-1}\right)$ is summable. First note that by Proposition 8.2.3 we have the following lower bound on $|T_{n-1}|$

$$|T_{n-1}| \geqslant \exp\left(C_2 \kappa^{n-2} l_{\mathfrak{L}(n-1)}\right). \tag{8.5}$$

Moreover recall that $R_n = C_R \kappa^n l_{\mathfrak{L}(n)}$ and $\varepsilon'_{n-1} = 2/|T_{n-1}|$ thus by the inequality above

$$\begin{aligned}\Psi(2R_n)\varepsilon'_{n-1} &= \exp\left[\rho(2C_R\kappa^n l_{\mathfrak{L}(n)})\right]\frac{2}{|T_{n-1}|},\\ &\leqslant 2\exp\left[\rho\left(2C_R\kappa^n l_{\mathfrak{L}(n)}\right) - C_2\kappa^{n-2}l_{\mathfrak{L}(n-1)}\right].\end{aligned}$$

But remember that $\rho \simeq \tilde{\rho}$. Thus using eqs. (8.4) and (A.1) we get

$$\rho\left(2C_R\kappa^n l_{\mathfrak{L}(n)}\right) \simeq \tilde{\rho}\left(2C_R\kappa^n l_{\mathfrak{L}(n)}\right) \leqslant 2C_R\tilde{\rho}\left(\kappa^n l_{\mathfrak{L}(n)}\right) = 2C_R\kappa^n. \tag{8.6}$$

Combining the above result with the previous inequality, we get

$$\begin{aligned}\Psi(2R_n)\varepsilon'_{n-1} &\preccurlyeq 2\exp\left[2C_R\kappa^n - C_2\kappa^{n-2}l_{\mathfrak{L}(n-1)}\right],\\ &= 2\exp\left[\kappa^{n-2}\left(2C_R\kappa^2 - C_2 l_{\mathfrak{L}(n-1)}\right)\right],\end{aligned}$$

which is summable. Indeed $l_{\mathfrak{L}(n)}$ tends to infinity and thus $\left(2C_R\kappa^2 - C_2 l_{\mathfrak{L}(n-1)}\right) < 0$ for n large enough. Hence by Theorem 6.2.4 the orbit equivalence from $\mathbb{Z}$ to Δ si Ψ-integrable.

Now, let us show that for all $\varepsilon > 0$ the coupling from Δ to $\mathbb{Z}$ is φ_ε-integrable. Based on Theorem 6.2.4 we only have to prove that $\varphi_\varepsilon(2R'_n)\varepsilon_{n-1}$ is summable. Recall that $R'_n = |T_n|$ and $\varepsilon_{n-1} = 2/\kappa^{n-2}$ and remark that by both the lower and upper bounds given in Proposition 8.2.3 we have

$$\varphi_\varepsilon(2R'_n)\varepsilon_{n-1} = \frac{2\rho\big(\ln(2|T_n|)\big)}{\Big(\ln\rho\ln 2|T_n|\Big)^{1+\varepsilon}\kappa^{n-1}} \leqslant \frac{2\rho\big(2C_3\kappa^n l_{\mathfrak{L}(n)}\big)}{\Big(\ln\rho\left(2C_2\kappa^{n-1}l_{\mathfrak{L}(n)}\right)\Big)^{1+\varepsilon}\kappa^{n-1}}.$$

Let us give a lower bound for $\rho\left(2C_2\kappa^{n-1}l_{\mathfrak{L}(n)}\right)$. Recall that $\rho \simeq \tilde{\rho}$ furthemore if $2C_2 \geqslant 1$ then by eq. (8.4) and since $\tilde{\rho}$ is non-decreasing

$$\begin{aligned}\kappa^{n-1} = \tilde{\rho}\left(\kappa^{n-1}l_{\mathfrak{L}(n)}\right) &\leqslant \tilde{\rho}\left(2C_2\kappa^{n-1}l_{\mathfrak{L}(n)}\right),\\ &\simeq \rho\left(2C_2\kappa^{n-1}l_{\mathfrak{L}(n)}\right).\end{aligned}$$

Now if $2C_2 < 1$ using Claim A.1.4 with $c' = 2C_2$ and $x' = \kappa^{n-1}l_{\mathfrak{L}(n)}$ we get (for n large enough)

$$2C_2\kappa^{n-1} = 2C_2\tilde{\rho}(\kappa^{n-1}l_{\mathfrak{L}(n)}) \leqslant \tilde{\rho}(2C_2\kappa^{n-1}l_{\mathfrak{L}(n)}) \simeq \rho\left(2C_2\kappa^{n-1}l_{\mathfrak{L}(n)}\right)$$

Hence, in both cases $\kappa^{n-1} \preccurlyeq \rho(2C_2\kappa^{n-1}l_{\mathfrak{L}(n)})$. Finally replacing C_R by C_3 in eq. (8.6) we can show that $\rho\left(2C_3\kappa^n l_{\mathfrak{L}(n)}\right) \preccurlyeq \kappa^n$. Thus, combining the two preceding results we obtain

$$\begin{aligned}\varphi_\varepsilon(R'_n)\varepsilon_{n-1} &\leqslant \frac{2\rho\big(C_3\kappa^n l_{\mathfrak{L}(n)}\big)}{\Big(\ln\rho\left(C_2\kappa^{n-1}l_{\mathfrak{L}(n)}\right)\Big)^{1+\varepsilon}\kappa^{n-1}}\\ &\preccurlyeq \frac{\kappa^n}{\Big(\ln\left(\kappa^{n-1}\right)\Big)^{1+\varepsilon}\kappa^{n-1}} = \frac{\kappa}{\left((n-1)\ln(\kappa)\right)^{1+\varepsilon}},\end{aligned}$$

which is a summable sequence. Hence by Theorem 6.2.4 the orbit equivalence coupling from Δ to $\mathbb{Z}$ si φ_ε-integrable. □

Remark 8.2.6. This result is stated in the general case, that is to say for an abstract ρ. Nonetheless, for some particular functions ρ the quantification can be improved. For example the case where $k_n = 2^n$ and $l_n = 2^{\alpha n}$ corresponds to $\rho(x) \simeq x^{1/(1+\alpha)}$. In that case $\mathfrak{L}(n) = n-1$ and we can show that the coupling from $\mathbb{Z}$ to Δ is exp-integrable (instead of $\exp\circ\rho$-integrable). Indeed, let $c_\varphi < C_2/(C_R 2^{3+\alpha})$ and $\Psi(x) := \exp(c_\varphi x)$, then by eq. (8.5)

$$\begin{aligned}\Psi(2R_n)\varepsilon'_{n-1} &= \exp\left[c_\varphi 2C_R k_n l_{n-1}\right]\frac{2}{|T_{n-1}|}\\ &\leqslant \exp\left[c_\varphi 2C_R 2^n 2^{\alpha(n-1)} - C_2 2^{n-2}2^{\alpha(n-2)}\right]2\\ &= 2\exp\left[2^{n-2}2^{\alpha(n-2)}\left(c_\varphi C_R 2^{3+\alpha} - C_2\right)\right].\end{aligned}$$

Which is summable by choice of c_φ.

Remark 8.2.7. We can verify that the integrability obtained for the coupling from Δ to $\mathbb{Z}$ is "almost" optimal. Indeed if the coupling from Δ to $\mathbb{Z}$ is φ-integrable, then by Theorem 6.1.16 we have

$$\varphi\circ I_{\mathbb{Z}} \preccurlyeq I_\Delta$$

where we recall that $I_{\mathbb{Z}}(n) \simeq n$ and $I_\Delta(n) \simeq \rho\circ\ln(n)$. Thus using the inequality above, we get $\varphi(n) \preccurlyeq \rho\circ\ln(n)$. Hence the quantification of Theorem 8.2.5 is optimal up to a logarithmic factor.

It is now easy to prove our first main theorem.

Proof of Theorem 6.3.1. Let $\rho : [1,+\infty[\to [1,+\infty[$ be a non-decreasing function such that $\rho(1) = 1$ and $x/\rho(x)$ is non-decreasing. Let Δ be the group defined in Proposition A.1.1. By the aforementioned proposition it verifies $I_\Delta \simeq \rho\circ\log$. Moreover by Theorem 8.2.5 there exists an orbit equivalence coupling from Δ and $\mathbb{Z}$ that is $(\varphi_\varepsilon, \exp\circ\rho)$-integrable for all $\varepsilon > 0$. □

Finally let us show Corollary 6.3.2 concerning the coupling with $\mathbb{Z}^d$.

Proof. Let $\rho : [1,+\infty[\to [1,+\infty[$ be a non-decreasing function such that $\rho(1) = 1$ and $x/\rho(x)$ is non-decreasing. Let Δ be the group defined in Proposition A.1.1, in particular it verifies $I_\Delta \simeq \rho\circ\log$.

Let $d \geqslant 1$ and recall (see Example 6.2.5) that for all $p < 1/d$ there exists a $(L^p, L^{1/p})$-integrable orbit equivalence coupling from $\mathbb{Z}$ to $\mathbb{Z}^d$. Hence, using the composition of couplings described in appendix B we can deduce from Theorem 9.3.1 and Proposition B.1.3 that there exists a $(\varphi_\varepsilon(\cdot^p), L^0)$-integrable orbit equivalence coupling from Δ to $\mathbb{Z}^d$. But $p < 1$ thus by Claim A.1.4

$$p\rho\circ\log \preccurlyeq \rho(p\log) = \rho\circ\log(\cdot^p) \preccurlyeq \rho\circ\log.$$

Thus $\varphi_\varepsilon(\cdot^p) \simeq \varphi_\varepsilon$. Hence the corollary. □

MEASURE SUBGROUP COUPLING WITH THE LAMPLIGHTER GROUP 9

> *Happiness can be found, even in the darkest of times,*
> *if one only remembers to turn on the light.*
>
> — **J.K. Rowling**
> *Harry Potter and the Prisoner of Azkaban*

Let Δ be as defined in Section 7.1 and let $q = |A \times B|$. Denote by L_q the wreath product $(\mathbb{Z}/q\mathbb{Z}) \wr \mathbb{Z}$ and endow it with the following generating part:

$$S_{L_q} := \left\{ \left(x\delta_0, 0 \right) \mid x \in \mathbb{Z}/q\mathbb{Z} \right\} \cup \{(e_{A\times B}, \pm 1)\}.$$

The aim of this section is to show Theorem 6.3.3 which we recall below.

Theorem 6.3.3

For all $\varepsilon > 0$ and all $\alpha > 0$ there exists a group G such that
- $I_G(x) \simeq \left(\log(x)\right)^{1/(1+\alpha)}$;
- if we define $\varphi_\varepsilon(x) := x^{\frac{1}{1+\alpha+\varepsilon}}$ then there exists a φ_ε-integrable measure subgroup coupling from G to L_q.

Using Sofic approximations, we are going to prove that the Brieussel-Zheng's diagonal product Δ verifying $k_m = \kappa^m$ and $l_m = \kappa^{\alpha m}$ is the wanted group G. Drawing inspiration from the Følner tiling technique described in Definition 6.2.1, we define tilings of Δ and L_q in Section 9.1. However —unlike the result obtained in the previous chapter— we can not obtain Følner tiling sequences having same number of elements at each step. That is why we work in the framework of *Sofic approximations* which only requires that we can *embed* one approximation into the other. Section 9.2 is devoted to the proof of inequalities useful for the quantification. The latter is proved in Section 9.3 as well as Theorem 6.3.3.

9.1 CONSTRUCTION OF THE SOFIC APPROXIMATIONS

In this section we define a Sofic approximation $(\mathcal{G}_n)_{n\in\mathbb{N}}$ of Δ and a Sofic approximation $(\mathcal{H}_n)_{n\in\mathbb{N}}$ of L_q. Recall that $(T_n)_{n\in\mathbb{N}}$ denotes the tiling of Δ defined in Lemma 8.1.4 and let us describe the idea of the construction. We want $(\mathcal{G}_n)_{n\in\mathbb{N}}$ to be a subsequence of $(T_n)_{n\in\mathbb{N}}$ and $(\mathcal{H}_n)_{n\in\mathbb{N}}$ to be a subsequence of some Følner sequence of L_q such that $\mathcal{G}_n$ embedds in $\mathcal{H}_n$ for all n. Since we want these Sofic approximations to verify eq. (6.1) with

$\varphi(x) = x^{1/(1+\alpha+\varepsilon)}$, we also need the embedding to respect the geometry of the groups. To do so, we draw inspiration from the tiling techniques developped in the preceding sections. Indeed, we define $(\mathcal{H}_n)_{n\in\mathbb{N}}$ such that $\mathcal{H}_{n+1}$ is tiled by $\mathcal{H}_n$ for all n, which means that there should exist a finite subset Σ''_{n+1} such that $\mathcal{H}_{n+1} = \bigsqcup_{\sigma\in\Sigma''_{n+1}} \sigma\mathcal{H}_n$.

As we will see in the next paragraphs, we will take $(\mathcal{G}_n)_{n\in\mathbb{N}}$ such that $\mathcal{G}_n = T_{pn}$ for some large enough $p > 0$ to be determined later. The need to take such a subsequence of $(T_n)_{n\in\mathbb{N}}$ comes from the following discussion. Assume first that $\mathcal{G}_n = T_n$ and that it embeds in some sequence $\mathcal{H}_n$ as described above for all n. Then these sequences verify $|\mathcal{H}_n| > |\mathcal{G}_n|$ and due to the tiling process, the quotient of these two cardinalities is multiplied at each step. More precisely, since we want $\mathcal{H}_n$ to be tiled by $\mathcal{H}_{n-1}$ *and* $\mathcal{H}_n$ to embed in $\mathcal{G}_n$ we thus need Σ''_n to verify $|\Sigma_n| \leqslant |\Sigma''_n|$. We consequently "approach" Σ_n by Σ''_n (in a sense that will be precised by eq. (9.3)) and when taking the product $\Sigma''_0 \cdots \Sigma''_n$ we hence multiply the errors. Since $\mathcal{H}_n$ is the product of $\Sigma''_0, \dots, \Sigma''_n$, the error is thus cumulative. Hence the size of $\mathcal{H}_n$ grows faster than $\mathcal{G}_n$, and actually *too fast*. Indeed denote by ι_n the embedding from $\mathcal{G}_n$ to $\mathcal{H}_n$ and take $x \in \mathcal{G}_n$ and $s \in S_\Delta$; to obtain the wanted integrability we will need to control the distance between $\iota_n(x)$ and $\iota_n(xs)$ in $\mathcal{H}_n$ But if the size —and thus the diameter— of $\mathcal{H}_n$ grows too fast we will not be able to have good enough control over the distance between $\iota_n(x)$ and $\iota_n(xs)$. To avoid this pitfall we define $(\mathcal{G}_n)_{n\in\mathbb{N}}$ to be a subsequence $(T_{pn})_{n\in\mathbb{N}}$ of $(T_n)_{n\in\mathbb{N}}$. The speed at which errors of approximation of $\mathcal{G}_n$ by $\mathcal{H}_n$ accumulate is slower. Consequently $|\mathcal{H}_n|$ grows slowly and thus if p is large enough we will obtain the wanted control on the aforementioned distance. We refer to page 94 for the concrete use of this parameter p in the proof of Theorem 9.3.1.

Before defining these Sofic approximations, let us state a useful fact about tilings of the lamplighter group.

9.1.1 Tiling of the Lamplighter group

Recall that a Følner sequence of L_q is given by

$$F''_n := \left\{ \Big((\varepsilon_i)_i, t \Big) \ : \ t \in \{0, n-1\}, \operatorname{supp}\Big((\varepsilon_i)_i \Big) \subseteq [0, n-1] \right\}.$$

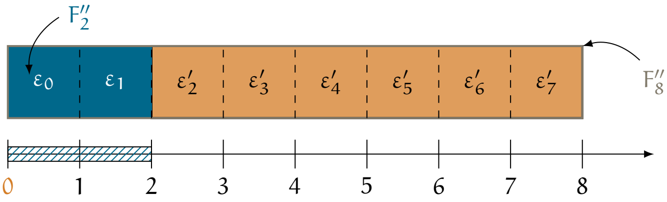
Our goal here is to extract a subsequence of $(F''_n)_{n\in\mathbb{N}}$ to define a Følner *tiling* sequence for our group L_q. So let $(d_n)_{n\in\mathbb{N}}$ be a sequence of integers and define $\Sigma''_0 := F''_{d_0}$. Now let $D_n := \prod_{i=0}^n d_i$ and consider for all $n \geqslant 0$

$$\begin{aligned} \Sigma''_{n+1} := \bigsqcup_{j=0}^{d_{n+1}-1} \Big\{ \big((\varepsilon_i), jD_n \big) \mid \operatorname{supp}\big((\varepsilon_i)_i \big) \subseteq & \big[0, jD_n - 1\big] \\ & \cup \big[(j+1)D_n,\ D_{n+1} - 1\big] \Big\}. \end{aligned} \tag{9.1}$$

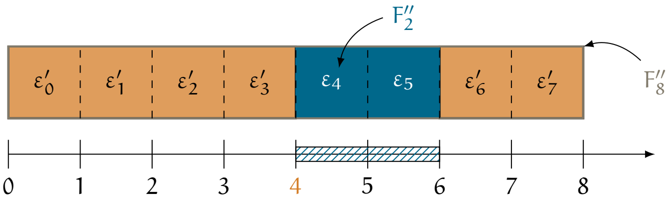
These sets will be the shifts of the wanted Følner tiling sequence. This is what Lemma 9.1.2 formalises, but first let us give some illustration of this tiling.

Example 9.1.1. Assume $d_0 = 2$ and $d_1 = 4$ then $D_0 = 2$ and $D_1 = 8$. Let $\big((\varepsilon_i)_i, t\big) \in F''_{D_0}$ and $\big((\varepsilon'_i)_i, jD_0\big) \in \Sigma''_1$. We represent the product of these two elements in Figure 9.1 for $j = 0$ (Figure 9.1a) or $j = 2$ (Figure 9.1b). The dark blue squares correspond to lamp configurations coming from the element in F_{D_0} while the orange ones are coming from the shift. The cursor of that product, namely $t + jD_1$, belongs to the hatched blue rectangle.

Let us now prove that it actually defines a Følner tiling sequence.



(a) Representation of $((\varepsilon_i')_i, 0)((\varepsilon_i)_i, t)$



(b) Representation of $((\varepsilon_i')_i, 2D_1)((\varepsilon_i)_i, t)$

Figure 9.1.: Tiling of the lamplighter

Lemma 9.1.2

> Let $(d_n)_{n\in\mathbb{N}}$ be a sequence of positive integers and define Σ''_{n+1} as above. Then $(\Sigma''_n)_{n\in\mathbb{N}}$ is a Følner tiling sequence and $F''_{D_{n+1}} = \Sigma''_{n+1}F''_{D_n}$.

Proof. The strategy of the proof is the same as the one of Lemma 8.1.4.

Let $(d_n)_{n\in\mathbb{N}}$ be a sequence of positive integers and define Σ''_{n+1} as in eq. (9.1). Recall that $D_n = \prod_{i=0}^{n} d_n$ for all $n \in \mathbb{N}$. First let us show that $\Sigma''_{n+1}F''_{D_n}$ is contained in $F''_{D_{n+1}}$. Let $((\varepsilon_i')_i, t)$ in F''_{D_n} and $j \in [0, d_{n+1}-1]$ and take $((\varepsilon_i)_i, jD_n) \in \Sigma''_{n+1}$. Then

$$((\varepsilon_i)_i, jD_n)((\varepsilon_i')_i, t) = \left((\varepsilon_i + \varepsilon'_{i-jD_n})_i,\ t + jD_n\right)$$

By eq. (9.1) and since $(\varepsilon_i')_i$ is supported on $[0, D_n - 1]$, we have

$$\begin{aligned}\operatorname{supp}\left((\varepsilon_i + \varepsilon'_{i-jD_n})_{i\in\mathbb{Z}}\right) &\subseteq \left[0, jD_n - 1\right] \cup \left[(j+1)D_n,\ D_{n+1} - 1\right] \cup [jD_n, (j+1)D_n - 1],\\ &= \left[0, D_{n+1} - 1\right].\end{aligned}$$

Finally, since $t \leqslant D_n - 1$ and $j \leqslant d_{n+1} - 1$, we get

$$jD_n + t \leqslant (d_{n+1} - 1)D_n + D_n - 1 \leqslant D_n d_{n+1} - 1 = D_{n+1} - 1.$$

Thus $((\varepsilon_i)_i, jD_n)((\varepsilon_i')_i, t)$ belongs to $F''_{D_{n+1}}$.

Now take $((\omega_i)_i, t) \in F''_{D_{n+1}}$ and let us show that $((\omega_i)_i, t)$ belongs to $\Sigma''_{n+1}F''_{D_n}$. First, remark that since $t \leqslant D_{n+1} - 1$ there exists a unique $0 \leqslant j \leqslant d_{n+1} - 1$ such that $0 \leqslant t - jD_n \leqslant D_n - 1$. For such a j let $t' := t - jD_n$ and let $(\varepsilon_i)_i$ and $(\varepsilon_i')_i$ such that

$$\varepsilon_i = \begin{cases} \omega_i & \text{if } i \in [0, jD_n - 1] \cup [(j+1)D_n, D_{n+1} - 1],\\ e & \text{else,}\end{cases}$$

$$\varepsilon_i' = \begin{cases} \omega_{i+jD_n} & \text{if } i \in [0, D_n - 1],\\ e & \text{else.}\end{cases}$$

Then $((\varepsilon_i)_i, jD_n)((\varepsilon'_i)_i, t') = ((\omega_i)_i, t)$. Hence the equality of the lemma. To prove that $(\Sigma''_n)_{n\in\mathbb{N}}$ is indeed a Følner tiling sequence we now only have to show that $\sigma F''_{D_n} \cap \sigma' F''_{D_n} = \emptyset$ for all $\sigma \neq \sigma' \in \Sigma''_{n+1}$. So take $((\varepsilon_i)_i, jD_n)$ and $((\varepsilon'_i)_i, j'D_n)$ in Σ''_{n+1} and take $((\omega_i)_i, t)$, and $((\omega'_i)_i, t')$ in F''_{D_n}. If

$$((\varepsilon_i)_i, jD_n)((\omega_i)_i, t) = ((\varepsilon_i)_i, jD_n)((\omega'_i)_i, t) \tag{9.2}$$

then in particular $t + jD_n = t' + j'D_n$. But $t, t' < D_n$ thus the last equality implies $t = t'$ and $j = j'$. In particular $(\varepsilon_i)_i$ and $(\varepsilon'_i)_i$ are supported on the same set, namely $[0, jD_n - 1] \cup [(j+1)D_n, D_{n+1} - 1]$. That last set is disjoint from $[jD_n, (j+1)D_n - 1]$ which is the interval where $((\omega_{i-jD_n})_i)$ and $(\omega_{i-jD_n})_i$ are supported. Combining this with eq. (9.2) we thus get that $\varepsilon_i = \varepsilon'_i$ for all i. Hence the result. □

We thus know how to build Følner tiling sequences for L_q. Now we have to specify the sequence $(d_n)_{n\in\mathbb{N}}$ such that the obtained tiling will give an appropriate Sofic approximation for our coupling.

9.1.2 Sofic approximations

Let $p \in \mathbb{N}$. Let $\mathcal{G}_n := T_{pn}$ where $(T_n)_n$ is the tiling of Δ defined in Lemma 8.1.4. Recall that the sequence of shifts $(\Sigma_n)_{n\in\mathbb{N}}$ is defined such that $T_{n+1} = \Sigma_{n+1}T_n$. In particular $T_{p(n+1)} = \Sigma_{p(n+1)}\Sigma_{pn+p-1}\cdots\Sigma_{pn+1}T_{pn}$. Let $\bar{\Sigma}_0 := T_0$ and

$$\bar{\Sigma}_{n+1} := \Sigma_{p(n+1)}\Sigma_{pn+p-1}\cdots\Sigma_{pn+1}.$$

By the above discussion, we thus have $\mathcal{G}_{n+1} = \bar{\Sigma}_{n+1}\mathcal{G}_n$. Now, let us define inductively the Sofic approximation $\mathcal{H}_n$ of L_q. First, let $\mathcal{H}_0 := F''_1$ and define $d_0 = 1$. Then let $n \geqslant 1$ and assume $\mathcal{H}_n := F''_{D_n}$ defined. Let d_{n+1} be the minimal integer such that $F''_{D_n d_{n+1}}$ contains at least $\left|\bar{\Sigma}_{n+1}\right|$ translates of F''_{D_n}. In other words d_{n+1} is the minimal integer such that the set Σ''_{n+1} defined in eq. (9.1) contains at least $\left|\bar{\Sigma}_{n+1}\right|$ elements, *viz.*

$$(d_{n+1} - 1)\, q^{d_0\cdots d_n(d_{n+1}-2)} \leqslant \left|\bar{\Sigma}_{n+1}\right| \leqslant d_{n+1} q^{d_0\cdots d_n(d_{n+1}-1)}. \tag{9.3}$$

Remark that in particular, one can embed $\bar{\Sigma}_n$ in Σ''_n. Finally let $\mathcal{H}_{n+1} = F''_{D_{n+1}}$. It defines by induction a sequence $(\mathcal{H}_n)_{n\in\mathbb{N}}$, which is a Sofic approximation of L_q since it is a subsequence of a Følner sequence.

9.1.3 Injection between Sofic approximations

Let us now define the embedding from $\mathcal{G}_n$ to $\mathcal{H}_n$. First remark that there exists a natural bijection ι_0 between $\mathcal{G}_0$ and $\mathcal{H}_0$ that maps $(f, 0) \in \mathcal{G}_0$ to the element $((\varepsilon_i)_i, 0)$ of $\mathcal{H}_0$ where $\varepsilon_0 = f_0(0)$ and $\varepsilon_i = e$ if $i \neq 0$. Now let $n \geqslant 1$ and denote by ν_n an embedding of $\bar{\Sigma}_n$ in Σ''_n arbitrarily chosen. Since $(\bar{\Sigma}_n)_{n\in\mathbb{N}}$ is a Følner tiling sequence, one can write every element of $\mathcal{G}_n$ as a product $\sigma_n \cdots \sigma_0$ where $\sigma_i \in \bar{\Sigma}_i$ is uniquely determined for all i. Thus we can define without ambiguity the following injection.

Lemma 9.1.3

Let $n \in \mathbb{N}$. The map defined by

$$\iota_n : \begin{cases} \mathcal{G}_n = T_{pn} & \to \mathcal{H}_n, \\ \prod_{i=0}^n \sigma_i & \mapsto \prod_{i=0}^n \nu_i(\sigma_i), \end{cases}$$

is a well defined injection from $\mathcal{G}_n$ to $\mathcal{H}_n$.

Proof. Let $n \in \mathbb{N}$. By the preceding discussion, this map is well defined. Now let $x, x' \in \mathcal{G}_n$. For all $i \in \{0, \dots, n\}$ define σ_i (resp. σ'_i) to be the element in $\bar{\Sigma}_i$ such that $x = \sigma_n \cdots \sigma_0$ (resp. $x' = \sigma'_n \cdots \sigma'_0$). Then by definition of ι_n we have $\iota_n(x) = \prod_{i=0}^{n} \nu_i(\sigma_i)$ and $\iota_n(x') = \prod_{i=0}^{n} \nu_i(\sigma'_i)$. But $\nu_i(\sigma_i)$ and $\nu_i(\sigma'_i)$ belong to $\bar{\Sigma}_i$ for all i, thus $\prod_{i=0}^{n} \nu_i(\sigma_i)$ is the decomposition of $\iota_n(x)$ and $\prod_{i=0}^{n} \nu_i(\sigma_i)$ the decomposition of $\iota_n(x')$) in product of shifts. Since $(\Sigma''_n)_n$ is a Følner tiling sequence, that decomposition is unique thus if $\iota_n(x) = \iota_n(x')$ then $\nu_i(\sigma_i) = \nu_i(\sigma'_i)$ for all i. Thus $\sigma_i = \sigma'_i$ since ν_i is a bijection for all i, and therefore $x = x'$. Hence the injectivity of ι_n. $\square$

9.2 USEFUL INEQUALITIES

In order to quantify the relation between Δ and L_q we will need some bounds on $\ln\left|\bar{\Sigma}_n\right|$. The first lemma bounds $\ln\left|\bar{\Sigma}_{n+1}\right|$ by above and the second one bounds $\ln\left|\bar{\Sigma}_n\right|$ by below, both in the general case (that is to say for abitrary sequences $(k_m)_{m\in\mathbb{N}}$ and $(l_m)_{m\in\mathbb{N}}$). Corollary 9.2.3 gives a relation between $\ln\left|\bar{\Sigma}_{n+1}\right|$ and $\ln\left|\bar{\Sigma}_n\right|$ in the particular case when $k_m = \kappa^m$ and $l_m = \kappa^{\alpha m}$ for some $\alpha > 0$.

First remark that, by the definition of $\bar{\Sigma}_{n+1}$ and Σ_{n+1}

$$\left|\bar{\Sigma}_{n+1}\right| = \kappa^p q^{\kappa^{p(n+1)}-\kappa^{pn}} \prod_{i=1}^{\mathfrak{L}(pn)} |\Gamma'_i|^{\kappa^{p(n+1)}-\kappa^{pn}} \prod_{i=\mathfrak{L}(pn)+1}^{\mathfrak{L}(p(n+1))} |\Gamma'_i|^{\kappa^{p(n+1)}-k_i}. \tag{9.4}$$

We can now show the wanted bounds on $\ln\left|\bar{\Sigma}_{n+1}\right|$ and $\ln\left|\bar{\Sigma}_n\right|$.

Lemma 9.2.1

There exists a constant $\hat{C}_1 > 0$ such that, for all $n \in \mathbb{N}$

$$\ln\left|\bar{\Sigma}_{n+1}\right| \leqslant \hat{C}_1 \kappa^{p(n+1)} l_{\mathfrak{L}(p(n+1))}.$$

To show this lemma we first bound the two products on the right of eq. (9.4). Recall that there exist $c_1, c_2 > 0$ such that $\ln|\Gamma'_i|$ is bounded above by $c_1 l_i + c_2$ for all i. Moreover, since $(l_m)_m$ is a subsequence of some $(u_n)_n$ having geometric growth, a sum of of terms of $(l_m)_{m\in\mathbb{N}}$ is bounded by its last term up to a multiplicative constant. We use this to bound by above (up to a multiplicative constant) $\ln|\bar{\Sigma}_{n+1}|$ by the highest term of l_m appearing in eq. (9.4) times the highest possible power.

Proof. First remark that $\kappa^{p(n+1)} - k_i \leqslant \kappa^{p(n+1)} - \kappa^{pn}$ for all $i \in [\mathfrak{L}(pn)+1, \mathfrak{L}(p(n+1))]$, thus

$$\ln\left[\prod_{i=1}^{\mathfrak{L}(pn)} |\Gamma'_i|^{\kappa^{p(n+1)}-\kappa^{pn}} \prod_{i=\mathfrak{L}(pn)+1}^{\mathfrak{L}(p(n+1))} |\Gamma'_i|^{\kappa^{p(n+1)}-k_i}\right] \leqslant \left(\kappa^{p(n+1)} - \kappa^{pn}\right) \sum_{i=1}^{\mathfrak{L}(p(n+1))} \ln|\Gamma'_i|.$$

But recall that by eq. (7.1) we have $\ln|\Gamma'_i| \leqslant c_1 l_m + c_2$ and that $(l_m)_{m\in\mathbb{N}}$ is a subsequence of a geometric sequence thus a sum of its terms can be bounded by above by the last term up to a positive constant. Thus

$$\sum_{i=1}^{\mathfrak{L}(p(n+1))} \ln|\Gamma'_i| \leqslant \sum_{i=1}^{\mathfrak{L}(p(n+1))} (c_1 l_i + c_2) \leqslant C_0 l_{\mathfrak{L}(p(n+1))},$$

for some $C_0 > 0$ depending only on Δ. Thus using eq. (9.4) we get,

$$\begin{aligned}
&\ln\left|\bar{\Sigma}_{n+1}\right| \\
&= \ln(\kappa^p) + \ln\left[q^{\kappa^{p(n+1)}-\kappa^{pn}} \prod_{i=1}^{\mathfrak{L}(pn)} |\Gamma_i'|^{\kappa^{p(n+1)}-\kappa^{pn}} \prod_{i=\mathfrak{L}(pn)+1}^{\mathfrak{L}(p(n+1))} |\Gamma_i'|^{\kappa^{p(n+1)}-k_i}\right] \\
&\leqslant \ln(\kappa^p) + \left(\kappa^{p(n+1)} - \kappa^{pn}\right)\left[C_0 l_{\mathfrak{L}(p(n+1))} + \ln(q)\right].
\end{aligned}$$

But $\ln(q)$ is constant and $\ln(\kappa^p) \leqslant \left(\kappa^{p(n+1)} - \kappa^{pn}\right)$ thus there exists some constant $\hat{C}_1 > 0$ such that $\ln\left|\bar{\Sigma}_{n+1}\right| \leqslant \hat{C}_1\left(\kappa^{p(n+1)} - \kappa^{pn}\right) l_{\mathfrak{L}(p(n+1))}$. We get the lemma by (roughly) bounding $\left(\kappa^{p(n+1)} - \kappa^{pn}\right)$ by above by $\kappa^{p(n+1)}$. ☐

Now for the lower bound on $\ln\left|\bar{\Sigma}_n\right|$. Recall that C_2 is the positive constant given in Lemma 8.2.4.

Lemma 9.2.2

For all $n \in \mathbb{N}$,

$$\ln\left|\bar{\Sigma}_n\right| \geqslant C_2 \kappa^{pn-1} l_{\mathfrak{L}(pn)}$$

where $C_2 > 0$ is a constant.

Proof. Bounding $\ln\left|\bar{\Sigma}_n\right|$ by below by the last factor appearing in eq. (9.4) and using the fact that $k_{\mathfrak{L}(pn)} \leqslant \kappa^{pn-1}$ we get

$$\begin{aligned}
\ln\left|\bar{\Sigma}_n\right| &\geqslant \left(\kappa^{pn} - k_{\mathfrak{L}(pn)}\right) \ln|\Gamma'_{\mathfrak{L}(pn)}| \\
&\geqslant \left(\kappa^{pn} - \kappa^{pn-1}\right) \ln|\Gamma'_{\mathfrak{L}(pn)}|,
\end{aligned}$$

We conclude as in the proof of Lemma 8.2.4 and by bounding $\kappa^{pn} - \kappa^{pn-1}$ by below by κ^{pn-1}. ☐

Now let $\rho(x) = x^{1/(1+\alpha)}$ for some $\alpha > 0$ and consider the diagonal product Δ associated to such a map ρ (see appendix A.1). In that case $k_m = \kappa^m$ and $l_m = \kappa^{\alpha m}$ for all m and we can deduce from the two preceding lemmas the following corollary.

Corollary 9.2.3

Let $\alpha > 0$ and assume that $k_m = \kappa^m$ and $l_m = \kappa^{\alpha m}$ for all $m \in \mathbb{N}$. Then there exists some constant $\hat{C}_2 > 0$ such that for for all $n \in \mathbb{N}$

$$\ln\left|\bar{\Sigma}_{n+1}\right| \leqslant \hat{C}_2 \kappa^{p(\alpha+1)} \ln\left|\bar{\Sigma}_n\right|$$

Proof. Let $\alpha > 0$ and assume that $k_m = \kappa^m$ and $l_m = \kappa^{\alpha m}$ for all m. Remark that in that case $\mathfrak{L}(pj) = pj - 1$ thus $l_{\mathfrak{L}(pj)} = \kappa^{\alpha(pj-1)}$ for all j. Using Lemmas 9.2.1 and 9.2.2 we get,

$$\begin{aligned}
\frac{\ln\left|\bar{\Sigma}_{n+1}\right|}{\ln\left|\bar{\Sigma}_n\right|} &\leqslant \frac{\hat{C}_1 \kappa^{p(n+1)} \kappa^{\alpha(p(n+1)-1)}}{C_2 \kappa^{pn-1} \kappa^{\alpha(pn-1)}}, \\
&= \frac{\hat{C}_1}{C_2} \kappa^{p+1} \kappa^{\alpha(p(n+1)-1-pn+1)}, \\
&= \frac{\hat{C}_1}{C_2} \kappa^{p+1} \kappa^{\alpha p}.
\end{aligned}$$

Let $\hat{C}_2 := \hat{C}_1 \kappa / C_2$, then $\ln\left|\bar{\Sigma}_{n+1}\right| \leqslant \hat{C}_2 \kappa^{p(1+\alpha)} \ln\left|\bar{\Sigma}_n\right|$. Hence the lemma. ☐

9.3 QUANTIFICATION

Let us now prove Theorem 6.3.3. First let $\alpha > 0$ and consider the diagonal product Δ defined by $k_m = \kappa^m$ and $l_m = \kappa^{\alpha m}$. For such a group we start by quantifying the integrability of the measure subgroup coupling with L_q obtained using Sofic approximations.

Theorem 9.3.1

Let $\alpha > 0$ and Δ be the diagonal product defined by $k_m = \kappa^m$ and $l_m = \kappa^{\alpha m}$. Then for all $\varepsilon > 0$ there exists a measure subgroup coupling from Δ to L_q that is φ_ε-integrable where

$$\varphi_\varepsilon(x) := x^{\frac{1}{1+\alpha+\varepsilon}}.$$

Using Theorem 6.2.9, we will show that there exists $p \in \mathbb{N}$ such that the sequences $\mathcal{G}_n$ and $\mathcal{H}_n$ defined in Section 9.1.2 verify for every $s \in S_\Delta$

$$\lim_{R\to\infty} \sup_n \sum_{r=0}^{R} \varphi_\varepsilon(r) \frac{\left|\left\{x \in \mathcal{G}_n^{(1)} \mid d_{\mathcal{H}_n}(\iota_n(x), \iota_n(x\cdot s)) = r\right\}\right|}{|\mathcal{G}_n|} < \infty. \tag{9.5}$$

Let us discuss the strategy of the proof. Take $(\mathbf{f}, \mathbf{t}) \in \mathcal{G}_n$. We first study the distance between $\iota_n(\mathbf{f}, \mathbf{t})$ and $\iota_n((\mathbf{f}, \mathbf{t}) \cdot s)$ for some generator s and we distinguish two cases, depending on whether $s = (\mathbf{e}, \pm 1)$ (see Claim 9.3.3) or not (see Claim 9.3.2). In the first case, the bound on the distance obtained involves terms of $(D_i)_{i\in\mathbb{N}}$. Thus we compute an upper bound to D_i (see Claim 9.3.4). Finally we show that eq. (9.5) is verified in both cases.

Proof of Theorem 9.3.1. Let $\varepsilon > 0$, let $p \in \mathbb{N}$ and $\mathcal{G}_n$ and $\mathcal{H}_n$ as defined in Section 9.1.2. Let ι_n be the map defined in Section 9.1.3. Let $(\mathbf{f}, \mathbf{t}) \in \mathcal{G}_n$ and $s \in S_\Delta$ such that $(\mathbf{f}, \mathbf{t}) \cdot s \in \mathcal{G}_n$. Recall that for all $i = 0, \dots, n$ there exists a unique $\left(\mathbf{g}^i, t_i\kappa^{p(i-1)}\right) \in \bar{\Sigma}_i$ such that

$$(\mathbf{f}, \mathbf{t}) = \left(\mathbf{g}^n, t_n\kappa^{p(n-1)}\right) \cdots \left(\mathbf{g}^0, 0\right). \tag{9.6}$$

Let us study the value of $\iota_n\Big((\mathbf{f}, \mathbf{t}) \cdot s\Big)$ when $s \in S_\Delta$.

Claim 9.3.2. If $s = ((a_m\delta_0)_m, 0)$ for some $a \in A$ or if $s = ((b_m\delta_{k_m})_m, 0)$ for some $b \in B$ then $d\left(\iota_n((\mathbf{f}, \mathbf{t})), \iota_n((\mathbf{f}, \mathbf{t}) \cdot s)\right) = 1$

Proof of the Claim. First assume that $s = ((a_m\delta_0)_m, 0)$ for some $a \in A$. Let $(\mathbf{h}, 0)$ be such that

$$(\forall m \in \mathbb{N}) \quad h_m(x) = \begin{cases} g_m^0(x) & \text{if } x \neq 0, \\ g_m^0(0)a_m & \text{if } x = 0. \end{cases}$$

By definition it verifies $(\mathbf{h}, 0) = (\mathbf{g}^0, 0)\big((a\delta_0)_m, 0\big)$ and its range is the same as the one of $(\mathbf{g}^0, 0)$. Thus $(\mathbf{h}, 0)$ belongs to $\bar{\Sigma}_0$. Hence the decomposition

$$(\mathbf{f}, \mathbf{t}) \cdot s = \left[\prod_{i=1}^{n} (\mathbf{g}^i, t_i\kappa^{p(i-1)})\right] (\mathbf{h}, 0)\,,$$

gives the unique decomposition of $(\mathbf{f}, \mathbf{t}) \cdot s$ in product of shifts. Hence by definition of ι_n

$$\iota_n\big((\mathbf{f}, \mathbf{t}) \cdot s\big) = \prod_{i=1}^{n} \nu_i\left(\mathbf{g}^i, t_i\kappa^{p(i-1)}\right) \cdot \iota_0\left(\mathbf{h}, 0\right).$$

Thus, using the definition of ι_0

$$\begin{aligned} d_{L_q}\Big(\iota_n\big((\mathbf{f},t)\big),\ \iota_n\big((\mathbf{f},t)\cdot s\big)\Big) &= d_{L_q}\Big(\iota_0\big(\mathbf{g}^0,0\big),\ \iota_0\big(\mathbf{h},0\big)\Big) \\ &= d_{L_q}\Big(\big(g_0^0(0)\delta_0,0\big),\ \big(h_0(0)\delta_0,0\big)\Big) = 1. \end{aligned}$$

If $s = ((b_m\delta_{k_m})_m, 0)$ for some $b \in B$, a similar argument shows that

$$d_{L_q}\Big(\iota_n\big((\mathbf{f},t)\big), \iota_n\big((\mathbf{f},t)\cdot s\big)\Big) = d_{L_q}\Big(\iota_0\big(\mathbf{g}^0,0\big), \iota_0\big((\mathbf{g}^0,0)\cdot s\big)\Big) = 1.$$

Hence the claim. $\square$

Let us study the case when $s = (\mathbf{e}, 1)$. Remark that by eq. (9.6) the decomposition of t in base κ^p is given by the sequence $(t_i\kappa^{p(i-1)})_i$ of the cursors of elements of $\bar{\Sigma}_i$, that is to say $t = \sum_{i=0}^{n} t_i\kappa^{p(i-1)}$. Now, denote by $i_0(t)$ the integer such that

$$(\forall i < i_0(t))\ t_i = \kappa^{p(i-1)} - 1 \quad \text{and} \quad t_{i_0(t)} < \kappa^{p(i_0(t)-1)} - 1. \tag{9.7}$$

That is to say $i_0(t)$ is the digit that will "absorb" the carry when we add one to t decomposed in base κ^p. Finally, let us recall that $\mathcal{G}_n^{(r)}$ is defined as

$$\mathcal{G}_n^{(r)} = \left\{x \in \mathcal{G}_n \mid B_{\mathcal{G}_n}(x,r) \simeq B_G(e_G,r)\right\}.$$

Claim 9.3.3. If $s = (\mathbf{e}, \pm 1)$ and $(\mathbf{f},t) \in \mathcal{G}_n^{(1)}$, then $d\left(\iota_n\big((\mathbf{f},t)\big), \iota_n\big((\mathbf{f},t)\cdot s\big)\right) \leqslant 3D_{i_0(t)}$. Moreover, to a given $i_0 \leqslant n$, the proportion of elements in $\mathcal{G}_n$ verifying eq. (9.7) is $(\kappa^p - 1)/\kappa^{p i_0}$.

When there is no ambiguity we will sometimes abuse notations and denote i_0 instead of $i_0(t)$. Now, let us discuss the strategy of the proof. The action of s adds one to the cursor and thus changes the the decomposition in base κ^p up to t_{i_0} (the coefficient that will "absorb" the carry). Using this we show that the action of s on $(\mathbf{f},t)$ changes only the (i_0+1)-right-most factors in the decomposition given by eq. (9.6). Thus $(\mathbf{f},t)$ and $(\mathbf{f},t+1)$ differ at most from an element in $F_{D_{i_0}}$. Hence the distance between these two elements is bounded by the diameter of the aforementioned Følner set.

Proof of the claim. Assume that $s = (\mathbf{e},1)$ and $(\mathbf{f},t) \in \mathcal{G}_n^{(1)}$ and let $i_0 = i_0(t)$ as defined by eq. (9.7). Remark that we have in particular $\sum_{i=0}^{i_0} t_i\kappa^{p(i-1)} < \kappa^{p(i_0-1)} - 1$. Now consider the element of $T_{p i_0}$ defined by

$$(\mathbf{p},t') := \prod_{i=0}^{i_0}\left(\mathbf{g}^i, t_i\kappa^{p(i-1)}\right).$$

Then, since $(\mathbf{p},t')\cdot s = (\mathbf{p},t'+1)$ and since by the above discussion $t'+1 < \kappa^{p(i_0-1)}$, we obtain that $(\mathbf{p},t')\cdot s$ belongs to $T_{p i_0}$. Thus for all $i = 0,\dots,i_0$ there exists a unique $(\mathbf{h}^i, x_i) \in \bar{\Sigma}_i$ such that

$$(\mathbf{p},t')\cdot s = \prod_{i=0}^{i_0}(\mathbf{h}^i, x_i).$$

But by eq. (9.6)

$$\left(\prod_{i=i_0+1}^{n}\left(\mathbf{g}^i, t_i\kappa^{p(i-1)}\right)\right)^{-1}\cdot(\mathbf{f},t)\cdot s = \prod_{i=0}^{i_0}\left(\mathbf{g}^i, t_i\kappa^{p(i-1)}\right)\cdot s = \prod_{i=0}^{i_0}\left(\mathbf{h}^i, x_i\right)$$

Thus, the equality $(\mathbf{f},t)\cdot s = \prod_{i=i_0+1}^{n}\left(\mathbf{g}^i, t_i\kappa^{p(i-1)}\right)\cdot\prod_{i=0}^{i_0}\left(\mathbf{h}^i, x_i\right)$ gives the decomposition of $(\mathbf{f},t)s$ in product of shifts. Hence,

$$\begin{aligned} d_{L_q}\left(\iota_n(\mathbf{f},t),\, \iota_n(\mathbf{f},t)\cdot s\right) &= d_{L_q}\left(\prod_{i=0}^{i_0}\nu_i\left(\mathbf{h}^i, x_i\right),\, \prod_{i=0}^{i_0}\nu_i\left(\mathbf{g}^i, t_i\kappa^{p(i-1)}\right)\right) \\ &= d_{L_q}\left(\iota_{i_0}\left((\mathbf{p},t')\cdot s\right),\, \iota_{i_0}(\mathbf{p},t')\right), \\ &\leqslant \operatorname{diam}\left(F''_{D_{i_0}}\right). \end{aligned}$$

The diameter of $F''_{D_{i_0}}$ is at most $3D_{i_0}$, hence the first part of the claim. Let us now show the second assertion. The cursor of an element in $\bar{\Sigma}_i$ can uniformly take κ^p different values. Thus the proportion of $(\mathbf{g}^i, t_i\kappa^{p(i-1)})$ in $\bar{\Sigma}_i$ verifying $t_i = \kappa^p - 1$ is $1/\kappa^p$, whereas the proportion verifying $t_i < \kappa^p - 1$ is $(\kappa^p-1)/\kappa^p$. Hence the ratio elements $(\mathbf{f},t) \in \mathcal{G}_n$ such that the decomposition given in eq. (9.6) verifies eq. (9.7) is $(\kappa^p-1)/\kappa^{i_0 p}$.

Finally, an analogous reasoning shows the result for $s = (\mathbf{e}, -1)$. □

To prove eq. (9.5) is verified we now need an estimation of D_n.

Claim 9.3.4. There exists $C_D > 0$ depending only on Δ such that $D_n \leqslant \left(C_D\kappa^{p(1+\alpha)}\right)^n$ for all $n \in \mathbb{N}$.

Proof. To prove the claim, we want to establish an upper bound to d_{n+1}. By the left inequality of eq. (9.3) we get

$$\begin{aligned} &(d_{n+1}-1)\, q^{D_n(d_{n+1}-2)} \leqslant \left|\bar{\Sigma}_{n+1}\right| \\ \Rightarrow &\ln(d_{n+1}-1) + D_n(d_{n+1}-2)\ln(q) \leqslant \ln\left|\bar{\Sigma}_{n+1}\right|, \\ \Rightarrow &d_{n+1} - 2 \leqslant \frac{\ln\left|\bar{\Sigma}_{n+1}\right|}{D_n\ln(q)}. \end{aligned}$$

Using Corollary 9.2.3 and eq. (9.3) applied to $\bar{\Sigma}_n$, we deduce

$$\begin{aligned} d_{n+1} - 2 &\leqslant \hat{C}_2\kappa^{p(1+\alpha)}\frac{\ln\left|\bar{\Sigma}_n\right|}{D_n\ln(q)}, \\ &\leqslant \hat{C}_2\kappa^{p(1+\alpha)}\frac{D_{n-1}(d_n-1)\ln(q) + \ln(d_n)}{D_n\ln(q)}, \\ &\leqslant \hat{C}_2\kappa^{p(1+\alpha)}\left(\frac{d_n-1}{d_n} + \frac{\ln(d_n)}{D_n\ln(q)}\right). \end{aligned}$$

But $(d_n-1)/d_n < 1$ and $\ln(d_n)/(D_n\ln(q)) < 1$, thus $d_{n+1} \leqslant 2\hat{C}_2\kappa^{p(1+\alpha)} + 2$ which we can roughly bound by above by $(2\hat{C}_2+1)\kappa^{p(1+\alpha)}$. Let us define $C_D := 2\hat{C}_2 + 1$ then C_D depends only Δ and we get $D_n \leqslant \left(C_D\kappa^{p(1+\alpha)}\right)^n$. Hence the claim. □

Finally, let us show that eq. (9.5) is verified for some well chosen p. Let $n, p \in \mathbb{N}$ and $R > 0$. If $s = ((a\delta_0), 0)$ or $s = ((b\delta_{k_m}), 0)$ then by Claim 9.3.2

$$\sum_{r=0}^{R}\varphi_\varepsilon(r)\frac{\left|\left\{x \in \mathcal{G}_n^{(s)} \mid d_{\mathcal{H}_n}(\iota_n(x), \iota_n(x\cdot s)) = r\right\}\right|}{|\mathcal{G}_n|} = \varphi_\varepsilon(1)$$

which does not depend on R nor on n, thus eq. (9.5) is verified. Now assume $s = (e, \pm 1)$. By Claim 9.3.3 and the upper bound on D_n given by Claim 9.3.4, we get

$$\begin{aligned}
&\sum_{r=0}^{R} \varphi_\varepsilon(r) \frac{\left|\left\{x \in \mathcal{G}_n^{(s)} \mid d_{\mathcal{H}_n}(\iota_n(x), \iota_n(x \cdot s)) = r\right\}\right|}{|\mathcal{G}_n|} \\
\leqslant &\sum_{i\in\mathbb{N}} \varphi_\varepsilon(3D_i)\frac{\kappa^p - 1}{\kappa^{ip}}, \\
\leqslant &\sum_{i\in\mathbb{N}} \varphi_\varepsilon\left(3C_D^i \kappa^{pi(1+\alpha)}\right)\frac{\kappa^p - 1}{\kappa^{ip}}, \\
= &\sum_{i\in\mathbb{N}} 3^{1/(1+\alpha+\varepsilon)}\left(C_D\kappa^{p(1+\alpha)}\right)^{\frac{i}{1+\alpha+\varepsilon}}\frac{\kappa^p - 1}{\kappa^{ip}}, \\
= &(\kappa^p - 1)3^{1/(1+\alpha+\varepsilon)}\sum_{i\in\mathbb{N}}\left(\frac{(C_D\kappa^{p(1+\alpha)})^{1/(1+\alpha+\varepsilon)}}{\kappa^p}\right)^i.
\end{aligned}$$

But since $(1+\alpha)/(1+\alpha+\varepsilon) - 1 < 0$, there exists $p > 0$ such that

$$C_D^{1/(1+\alpha+\varepsilon)}\kappa^{p\left((1+\alpha)/(1+\alpha+\varepsilon)-1\right)} < 1.$$

For such an integer p we get $(\kappa^p - 1)\sum_{i\in\mathbb{N}}\left((C\kappa^{p(1+\alpha)})^{1/(1+\alpha+\varepsilon)}/\kappa^p\right)^i < +\infty$. Since that term does not depend on R nor n we thus get that eq. (9.5) is verified. Hence the theorem. □

Remark 9.3.5. We can verify that the result above concerning the coupling from Δ to L_q is "almost" optimal. Indeed if the coupling from Δ to L_q is φ-integrable, then by Theorem 6.1.16 we have

$$\varphi \circ I_{L_q} \preccurlyeq I_\Delta$$

where we recall that $I_{L_q}(n) \simeq \log(n)$ and $I_\Delta(n) \simeq \log(n)^{1/(1+\alpha)}$. Thus using the inequality above, we get $\varphi \circ \log(n) \preccurlyeq \log(n)^{1/(1+\alpha)}$. Hence the optimal integrability suggested by this theorem corresponds to $\varphi(x) := x^{1/(1+\alpha)}$. Thus the quantification of Theorem 9.3.1 is almost optimal.

We can now prove Theorem 6.3.3

Proof of Theorem 6.3.3. Let $\alpha > 0$ and Δ be the diagonal product defined by the two sequences $k_m = \kappa^m$ and $l_m = \kappa^{\alpha m}$. Then by appendix A.1 the isoperimetric profile of Δ verifies $I_\Delta(n) \simeq \log(n)^{1/(1+\alpha)}$. Moreover by Theorem 9.3.1 there exists a measure subgroup coupling from Δ to L_q that is φ_ε-integrable for all $\varepsilon > 0$. Hence the theorem. □

Let us conclude by the demonstration of Corollary 6.3.4.

Proof. Let $\alpha > 0$ and $\varepsilon > 0$ and define $\varphi_\varepsilon(x) := x^{\frac{1}{1+\alpha+\varepsilon}}$. By Theorem 6.3.3 there exists a group G such that $I_G(n) \simeq (\log(n))^{1/(1+\alpha)}$ and such that there exists a φ_ε-integrable measure subgroup coupling from G to L_q.

Now let $H := \mathbb{Z}^2 \rtimes_A \mathbb{Z}$ where A is the matrix

$$A := \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}.$$

By [DKT21, Th. 6.1] there exists a measure equivalence coupling from L_q to H that is $(L^\infty, \exp)$-integrable. It is thus $(\psi, \exp)$-integrable for all increasing map $\psi : \mathbb{R}_+ \to \mathbb{R}_+$.

In particular if $\psi = \mathrm{id}$ we can compose the couplings and obtain by Proposition B.1.3 an orbit equivalence coupling from G to H that is $(\varphi_\varepsilon, \mathrm{L}^0)$-integrable. Hence the corollary. $\square$

We show Corollary 6.3.5 similarly, by using Example 6.1.11 instead of [DKT21, Th. 6.1] in the above proof.

10 CONCLUSION, UNFINISHED WORK AND OPEN PROBLEMS

> *Past the mud and the rain I will slowly stand*
> *Yet still a long way to go on the route we planned.*
>
> — **Alpine Universe**
> The Empire of Winds, from *The Empire of Winds (O.S.T)*

We conclude this part on some open problems. This is also the occasion to discuss possible strategies to tackle them and current work in progress.

THE LAMPLIGHTER CASE

Our result concerning the Lamplighter group (Theorem 6.3.3) gives only a measure *subgroup* coupling. It is thus natural to ask the following question.

Question 10.0.1. If $\alpha > 0$ et $\varphi(x) = x^{1/(1+\alpha)}$ does there exist a group G with isoperimetric profile $I_G \simeq \varphi \circ \log(x)$ and such that there exists a φ-integrable measure *equivalence* coupling from G to L_q?

As discussed in the introduction, we worked on this question and we hope to be able to prove the existence of a φ_ε-integrable measure equivalence coupling (for φ_ε as defined in Theorem 6.3.3). However time did not allow us to add this result in this mansucript, since the proof was only achieved during spring 2021.

Another limitation discussed in the introduction was the fact that we needed an estimation of the growth rate of $(l_m)_{m\in\mathbb{N}}$ in order to get the integrability (see discussion below Theorem 6.3.3). We thus had to restrict ourselves to the family of functions of the form $x \mapsto x^{1/(1+\alpha)}$. Nonetheless one can also ask what happens for a larger family of functions.

Question 10.0.2. Let $\rho : [1, +\infty[\to [1, +\infty[$ be a non-decreasing function such that $\rho(1) = 1$ and $x/\rho(x)$ is non-decreasing. Does there exist G such that $I_G \simeq \rho \circ \log$ and a (ρ, L^0)-integrable measure equivalence coupling from Δ to L_q?

FURTHER DIRECTIONS

These are possible refinements of our second main result, but we can also look at problems concerning other kind of groups. For example instead of considering $H := \mathbb{Z}^2 \rtimes_A \mathbb{Z}$ as in Corollary 6.3.4, we can look at the problem involving any polycyclic groups.

Question 10.0.3. Can we build measure or orbit equivalence coupling from some group G to a polycyclic group, with prescribed integrability?

Another possible direction is to consider couplings between two Brieussel-Zheng's diagonal products.

Question 10.0.4. Let $\rho, \tilde{\rho} : [1, +\infty[\to [1, +\infty[$ be non-decreasing functions such that $\tilde{\rho}(1) = \rho(1) = 1$ and $x/\rho(x)$ and $x/\tilde{\rho}(x)$ are non-decreasing. Denote by ρ^{-1} a quasi-inverse for ρ. Given two Brieussel-Zheng's diagonal products Δ and $\tilde{\Delta}$ with respective isoperimetric profiles $I_\Delta \simeq \rho \circ \log$ and $I_{\tilde{\Delta}} \simeq \tilde{\rho} \circ \log(n)$, can we build a measure equivalence coupling from Δ to $\tilde{\Delta}$ that is $(\tilde{\rho} \circ \rho^{-1}, L^0)$-integrable?

We actually worked on this problem but obtained couplings with "bad" integrability. Indeed —as in the Lamplighter case— our constructions draw inspiration from the tiling technique, which does have the kindness to provide us with a coupling *but* with terrible integrability. We adress and detail this issue in the next paragraph.

COUPLINGS BUILDING TECHNIQUES

We exposed and used two different techniques to build couplings between groups: Følner tiling sequences and Sofic approximations. As we saw in the Lamplighter case, the tiling technique —though inspiring— is not always usable to get orbit or measure equivalence couplings. Indeed the condition that the two sequences must have at each step the same cardinality is very restrictive. Furthemore this technique can give couplings with a very "bad" integrability. Indeed we tried to use these tilings to answer Question 10.0.4 but the integrability of the coupling thus obtained was not as good as the optimal one, namely $(L^{(1+\tilde{\alpha})/(1+\alpha)}, L^0)$. This is what we describe below.

Let $\alpha > \tilde{\alpha} > 0$ and take Δ (resp. $\tilde{\Delta}$) to be the diagonal product defined with $k_m = \kappa^m$ and $l_m = \kappa^{\alpha m}$ (resp. $\tilde{k}_m = k_m$ and $\tilde{l}_m = \kappa^{\tilde{\alpha} m}$). Consider the Følner tiling sequence $(\Sigma_n)_{n\in\mathbb{N}}$ of Δ defined in Chapter 8 and assume that we have overcome this cardinality issue. That is to say, assume that we can build a Følner tiling sequences $(\tilde{\Sigma}_n)_{n\in\mathbb{N}}$ for $\tilde{\Delta}$ such that $|\Sigma_n| = |\tilde{\Sigma}_n|$ for all $n \in \mathbb{N}$. We can show that the sequence of tiles $(\tilde{T}_n)_{n\in\mathbb{N}}$ verifies

$$\operatorname{diam}\left(\tilde{T}_n\right) =: \tilde{R}_n \preccurlyeq \kappa^{m(1+\alpha)}.$$

Moreover by Lemma 8.1.5 the tiles in Δ verify $\varepsilon_n = 2/\kappa^n$. Hence to compute the integrability of the coupling we have to find a map φ such that $(\varphi(\tilde{R}_n)\varepsilon_{n-1})_{n\in\mathbb{N}}$ is summable (see Theorem 6.2.4). For example if $\varphi = x^{1/(1+\alpha+\varepsilon)}$ for some $\varepsilon > 0$, then the sequence is summable and thus the coupling is (φ, L^0)-integrable. But Theorem 6.1.16 suggests the optimal integrability of the coupling from Δ to $\tilde{\Delta}$ could be as good as $\varphi(x) = x^{(1+\alpha)/(1+\tilde{\alpha})}$. Hence the integrability of the coupling obtained with the Følner tiling sequences is not as good as the (theoretical) optimal one. Thus the motivation to define and use other couplings building techniques is not just due to the fact that it might be hard to find Følner tiling sequences with same number of elements: it is also necessary because Følner tiling sequences do not always provide couplings with good integrability.

Part III

APPENDIX

“

— Percival
Theme from *The Witcher 3: The Wild Hunt OST*

A THE TECHNICAL CONSTRUCTION OF DIAGONAL PRODUCT

In this section we recall the definition given in [BZ21, Appendice B] of a Brieussel-Zheng's group from its isoperimetric profile and some useful results concerning the metric of these groups.

A.1 FROM THE ISOPERIMETRIC PROFILE TO THE GROUP

The aim of this section is to define —given a function— a group with this asymptotic behavior. We also expose some tools used by Brieussel and Zheng to define such a group, that will be useful for our proofs of orbit equivalence integrability.

A.1.1 Definition of Δ

Recall that a Brieussel-Zheng's group Δ is uniquely determined by the sequences $(\Gamma_m)_{m\in\mathbb{N}}$ and $(k_m)_{m\in\mathbb{N}}$. Moreover, in the particular case of expanders (see Section 7.1.2), the group Δ is uniquely determined by the sequences $(k_m)_{m\in\mathbb{N}}$ and $(l_m)_{m\in\mathbb{N}}$ (where l_m corresponds to the diameter of Γ_m). Thus, starting from a prescribed function ρ, we will define sequences $(k_m)_{m\in\mathbb{N}}$ and $(l_m)_{m\in\mathbb{N}}$ such that the corresponding Δ verifies $I_\Delta \simeq \rho\circ\log$.

First, let

$$\mathcal{C} := \left\{ \zeta : [1,+\infty) \to [1,+\infty) \;\middle|\; \begin{array}{c} \zeta \text{ continue}, \quad \zeta(1)=1 \\ \zeta \text{ and } x\mapsto x/\zeta(x) \text{non-decreasing} \end{array} \right\}.$$

Equivalently this is the set of functions ζ satisfying $\zeta(1)=1$ and

$$(\forall x, c \geqslant 1) \qquad \zeta(x) \leqslant \zeta(cx) \leqslant c\zeta(x). \tag{A.1}$$

So let $\rho\in\mathcal{C}$ and define f such that $\rho(x)=x/f(x)$. Remark that in particular f belongs to $\mathcal{C}$. Combining [BZ21, Proposition B.2 and Theorem 4.6] we can show the following result (remember that with our convention the isoperimetric profile considered in [BZ21] corresponds to $1/I_\Delta$).

Proposition A.1.1

> Let $\kappa,\lambda \geqslant 2$. For any $f\in\mathcal{C}$ there exists a subsequence $(k_m)_{m\in\mathbb{N}}$ of $(\kappa^n)_{n\in\mathbb{N}}$ and a subsequence $(l_m)_{m\in\mathbb{N}}$ of $(\lambda^n)_{n\in\mathbb{N}}$ such that the group Δ defined in Section 7.1.2 verifies $I_\Delta(x)\simeq\rho\circ\log$.

Example A.1.2 ([BZ21, Example 4.5]). Let $\alpha>0$. If $\rho(x):=x^{1/(1+\alpha)}$ then the diagonal product Δ defined by $k_m=\kappa^m$ and $l_m=\kappa^{\alpha m}$ verifies $I_\Delta\simeq\rho\circ\log$.

A.1.2 Technical tools

Now let us recall the intermediary functions defined in [BZ21, Appendix B] and some of their properties.

Let $\rho \in \mathcal{C}$ and let f such that $\rho(x) = x/f(x)$. The construction of a group corresponding to the given isoperimetric profile $\rho \circ \log$, is based on the approximation of f by a piecewise linear function $\tilde{f}$. For the quantification of orbit equivalence, many of our computations will use this function $\tilde{f}$ and some of its properties. We recall below all the needed results, beginning with the definition of $\tilde{f}$.

Lemma A.1.3

Let (k_m) and (l_m) given by Proposition A.1.1 above. The function $\tilde{f}$ defined by

$$\tilde{f}(x) := \begin{cases} l_m & \text{if } x \in [k_m l_m, k_{m+1} l_m], \\ \frac{x}{k_{m+1}} & \text{if } x \in [k_{m+1} l_m, k_{m+1} l_{m+1}], \end{cases} \tag{A.2}$$

verifies $\tilde{f} \simeq f$.

We will denote by $\tilde{\rho}$ the map $x \mapsto x/\tilde{f}(x)$. Remark that both $\tilde{f}$ and $\tilde{\rho}$ belong to $\mathcal{C}$. In particular they verify eq. (A.1), which is only true when c and x are greater than 1. When $c < 1$ we get the following inequality.

Claim A.1.4. If $0 < c' < 1$ and $x' \geqslant 1/c'$ then $c'\tilde{\rho}(x') \leqslant \tilde{\rho}(c'x')$.

Proof. If $0 < c' < 1$ then $1/c' > 1$, thus we can apply eq. (A.1) with $c = 1/c'$ and $x = c'x$ which gives us

$$\tilde{\rho}(x') = \tilde{\rho}\left(\frac{1}{c'}c'x'\right) = \tilde{\rho}(cx) \leqslant c\rho(x) = \frac{1}{c'}\tilde{\rho}(c'x').$$

Thus $c'\rho(x') \leqslant \tilde{\rho}(c'x')$. □

A.2 KNOWN RESULTS ON THE METRIC

We recall here some useful material about the metric of Δ and refer to [BZ21, Section 2.2] for more details. First, let $(x)_+ := \max\{x, 0\}$.

Definition A.2.1

For $j \in \mathbb{Z}$ and $m \in \mathbb{N}$ let $I_j^m := [jk_m/2, (j+1)k_m/2 - 1]$. We define the *essential contribution* of $f_m : \mathbb{Z} \to \Gamma_m$ by

$$E_m(f_m) := k_m \sum_{j:\mathrm{range}(f_m,t)\cap I_j^m \neq \emptyset} \max_{x\in I_j^m} \left(|f_m(x)|_{\Gamma_m} - 1\right)_+ .$$

The following proposition sums up [BZ21, Lemma 2.13, Proposition 2.14]. Recall that $\iota(n)$ denotes the integer such that $k_{\iota(n)+1} > n$ and $k_{\iota(n)} \leqslant n$.

Proposition A.2.2

For any $\delta = (f, t) \in \Delta$ we have

$$|(f,t)|_\Delta \leqslant 500 \sum_{m=0}^{\iota(\mathrm{range}(\delta))} |(f_m, t)|_{\Delta_m},$$

$$|(f_m,t)|_{\Delta_m} \leqslant 9\left(\mathrm{range}(f_m, t) + E_m(f_m)\right).$$

COMPOSITION OF COUPLINGS

B

We recall in this chapter some material of [DKLMT20, Sections 2.3 and 2.5] concerning the composition of couplings. We conclude by a figure summing up the different couplings mentioned in this manuscript and their integrability.

B.1 COMPOSITION AND INTEGRABILITY

B.1.1 Definition of the composition

Let us first recall the definition of the composition of two couplings given in [DKLMT20, Section 2.3]. Please note that we state it for measure subgroup and equivalence couplings since these are the only two notions of coupling we consider in this manuscript, but in [DKLMT20] the authors define the composition in a more general case.

Let Γ, Λ and Σ be three countable groups and let $(X_1, X_{1,\Lambda}, \mu_1)$ be a subgroup coupling from Γ to Λ and $(X_2, X_{2,\Sigma}, \mu_2)$ be a subgroup coupling from Λ to Σ. We define the *composition* of these two couplings to be the subgroup coupling $(X_3, X_{3,\Sigma}, \mu_3)$ obtained as follows: the space of the coupling is defined by $X_3 := (X_1 \times X_2)/\Lambda$ where Λ acts diagonally on $(X_1 \times X_2, \mu_1 \otimes \mu_2)$. This space X_3 is equipped with the measure μ_3 obtained by identifying X_3 with a Λ-fundamental domain. Denote by π_{X_3} the map which takes (almost) every x in $X_1 \times X_2$ to the unique element of $\Lambda \cdot x$ which belongs to the aforementioned fundamental domain and define $X_{3,\Sigma} := \pi_{X_3}(X_{1,\Lambda} \times X_{2,\Sigma})$. Finally equip X_3 with the induced Γ and Λ actions and denote by $X_{1,\Gamma}$ (resp. $X_{2,\Lambda}$) the fundamental domain for the Γ (resp. Λ) action on X_1 (resp. X_2), then $X_{3,\Gamma} := \pi_{X_3}(X_{1,\Gamma} \times X_{2,\Lambda})$ is a fundamental domain for the action of Γ on X_3.

This new coupling verifies the following property.

Proposition B.1.1 (*[DKLMT20, Prop.2.9]*)

Let Γ, Λ and Σ be three countable groups.

- Let $(X_1, X_{1,\Lambda}, \mu_1)$ be a subgroup coupling from Γ to Λ and $(X_2, X_{2,\Sigma}, \mu_2)$ be a subgroup coupling from Λ to Σ. The composition of these two couplings is a subgroup coupling from Γ to Σ.
- If both couplings are measure equivalence couplings, then their composition is also a measure equivalence coupling.

Let us now study the behaviour of integrability under composition.

B.1.2 Integrability

Now that we know how to compose couplings, we need to quantify the integrability of the obtained composition. Such a tool is provided by [DKLMT20, Prop. 2.26] which we recall below.

Proposition B.1.2

> Let $\varphi, \psi : \mathbb{R}^+ \to \mathbb{R}^+$ be two non-decreasing subadditive maps with φ moreover concave and let Γ, Λ and Σ be three finitely generated groups. Let $(X_1, X_{1,\Lambda}, \mu_1)$ be a φ-integrable measure subgroup coupling from Γ to Λ and let $(X_2, X_{2,\Sigma}, \mu_2)$ be a ψ-integrable measure subgroup coupling from Λ to Σ. Then the composition of these two couplings is a $\varphi \circ \psi$-integrable measure subgroup coupling from Γ to Σ.

Using the above proposition and Proposition B.1.1 we can obtain a similar result for the quantification of measure equivalence coupling.

Now recall that two orbit equivalent groups are measure equivalent. In particular every orbit equivalence coupling induces an measure equivalence coupling (see [DKLMT20, Rk. 2.34]). Thus combining Proposition B.1.1 with [DKLMT20, Prop. 2.42] we obtain the following result.

Proposition B.1.3 (*[DKLMT20, Prop. 2.9 and 2.42]*)

> If (X_1, μ_1) (resp. (X_2, μ_2)) is a (φ, L^0)-integrable (resp. (ψ, L^0)-integrable) orbit equivalence coupling from Γ to Λ (resp. Λ to Σ), the composition of the induced measure equivalence couplings gives a $(\phi \circ \psi, L^0)$-integrable orbit equivalence coupling from Γ to Σ.

See Corollary 6.3.2 and the discussion below Question 10.0.3 for examples of quantified composition of couplings.

B.2 OVERVIEW

Figure B.1 sums up the known results on the integrability of couplings between the different groups appearing in this manuscript.

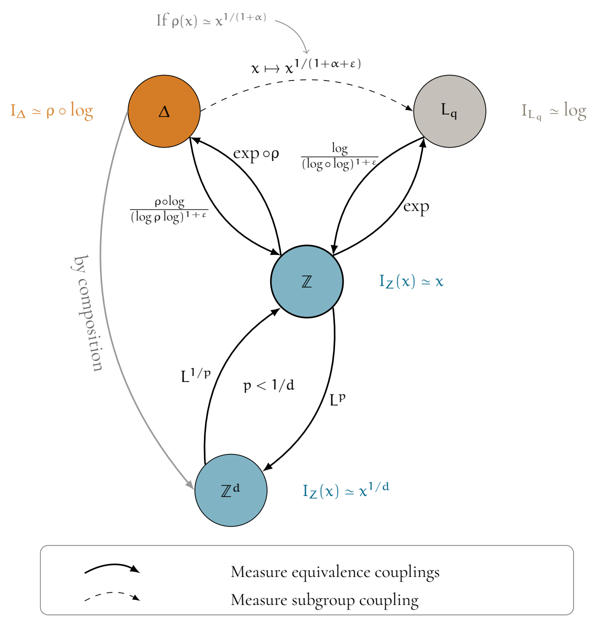


Figure B.1.: Overview of the mentioned couplings

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INDEX

NOTATIONS INDEX

PART 1

$\mathfrak{A}$ Atlas of isometries from Y to X.
$\mathcal{A}$ An apartment in $\mathcal{X}$.
(Γ, S) Cayley graph of Γ with respect to the generating part S.
H_0 The group $\rho^{-1}(\mathrm{PSL}_n(\mathbb{K}))$.
Isom$(\mathcal{G})$ Isometry group of $\mathcal{G}$.
ι_y Isometry from $\mathcal{Y}$ to $\mathcal{X}$ based at y (see page 42).
κ Natural injection of Y in $\mathcal{Y}$ (see Section 3.3.3).
$[L]$ Class modulo homothety of the lattice L.
$\mathcal{P}(x)$ The print of the vertex x (see Definition 2.3.1).
P A print in Y (see Definition 3.2.6).
ϕ_y Local isometry from $\mathcal{Y}$ to $\mathcal{X}$ based at y (see eq. (3.5)).
q Quasi-isometry between X and $\mathcal{X}$.
R Radius such that Y is R-locally the same as X.
ρ Injective homomorphism from Isom(X) to Isom$(\mathcal{X})$.
r_A See Lemma 3.2.4.
$r_{\mathcal{P}}$ Radius considered to define prints (see Definition 3.2.6).
$R_{\mathcal{X}}$ Radius such that $\mathcal{Y}$ is $R_{\mathcal{X}}$-locally $\mathcal{X}$.
$r_{\mathcal{X}}$ Radius such that ι_y coincide with ϕ_y on $B_{\mathcal{Y}}(y, r_{\mathcal{X}})$ (see page 42).
r_Y See Claim 3.3.8.
$\tau(x)$ The type of the vertex x, where x belongs to the Bruhat-Tits building of $\mathrm{PSL}_n(\mathbb{K})$.
$\mathcal{X}$ The Bruhat-Tits building of $\mathrm{PSL}_n(\mathbb{K})$.
$\mathcal{Y}$ Hybrid graph built to be locally the same as the building (see Section 3.2.3).
$(y_1, \dots, y_l)$ A path of adjacent vertices y_1, y_2, ..., y_l.

PART 2

$\preccurlyeq$, $\simeq$ See above Definition 6.1.12.
$|X|$ Cardinal of the set X.
∂F Boundary of the set F.
d_n Integer such that $\bar{\Sigma}_n$ ambedds in Σ''_n (see eq. (9.3))
D_n Equals to the product $\prod_{i=0}^{n} d_i$.
Δ See Definition 7.1.1.
Δ_m See Section 7.1.
F_n Følner sequence of Δ.
F''_n Følner sequence of L_q.
g The sequence of maps $(g_m)_{m\in\mathbb{N}}$.
g'_m See Section 7.1.3.
$\mathcal{G}_n$ Sofic approximation of Δ.

Γ'_m Normal closure of $[A_m, B_m]$.

$\mathcal{H}_n$ Sofic approximatin of L_q.

I_G Isoperimetric profile of G.

ι_n Injection from Σ_n to Σ''_n (see page 88).

L_q Lamplighter group $\mathbb{Z}/q\mathbb{Z} \wr \mathbb{Z}$.

ν_i An injection from $\bar{\Sigma}_i$ to Σ''_i.

R_n Diameter of T_n.

R'_n Diameter of T'_n.

S_G A generating part of the group G.

Σ_n Følner tiling sequence (of Δ).

$\bar{\Sigma}_n$ Følner tiling sequence defined by $\bar{\Sigma}_n = \prod_{i=p(n-1)-1}^{pn} \Sigma_n$.

Σ'_n Følner tiling sequence of $\mathbb{Z}$.

Σ''_n Følner tiling sequence of L_q.

T_n Tile of Δ defined by $T_n = \prod_{i=0}^{n} \Sigma_i$

T'_n Tile of $\mathbb{Z}$ defined by $T'_n = \prod_{i=0}^{n} \Sigma'_i$

$\theta_m^A(f_m)$ Natural projection of f_m on A_m (see Section 7.1.3).

$\theta_m^B(f_m)$ Natural projection of f_m on B_m (see Section 7.1.3).

COLOPHON

Local and Asymptotic Geometry of Groups, Amandine Escalier, September 8, 2021. This thesis was realised under the supervision of Romain Tessera and Jérémie Brieussel. This document was typeset using a custom version of *classicthesis* developed by André Miede. All figures have been encoded by the author using Ti*k*Z. The partition figuring in page 103 has been encoded using *lilypond* by Théo Laurent.