Sofic approximations and quantitative measure couplings

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Abstract

Measure equivalence was introduced by Gromov as a measured analogue of quasi-isometry. Unlike the latter, measure equivalence does not preserve the large scale geometry of groups and happens to be very flexible in the amenable world. Indeed the Ornstein-Weiss theorem shows that all infinite countable amenable groups are measure equivalent to the group of integers.

To refine this equivalence relation and make it responsive to geometry, Delabie, Koivisto, Le Maître and Tessera introduced a quantitative version of measure equivalence. They also defined a relaxed version of this notion called "quantitative measure subgroup coupling".

In this article we offer to answer the inverse problem of the quantification (find a group admitting a measure subgroup coupling with a prescribed group with prescribed quantification) in the case of the lamplighter group.

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1 Introduction

A recurring theme in group theory is the description of large-scale behaviour of groups and their geometry. A well known example is the study of groups up to quasi-isometry: it describes the large-scale (or "coarse") geometry from the *metric* point of view. A *measure* analogue of quasi-isometry – called measure equivalence – was introduced by Gromov in [GNR93]. Two groups G and H are **measure equivalent** if they both act freely and measure preservingly on a standard measure space (Ω, m), such that each action admits a fundamental domain of finite measure. A first elementary illustration of measure equivalent groups is given by lattices in a common locally compact group. We refer to [Gab10, Fur11] for surveys on the topic.

In some cases, measure equivalence can show remarkable rigidity properties. For instance Furman proved [Fur99] that any countable group which is measure equivalent to a lattice in a simple Lie group G of higher rank, is commensurable (up to finite kernel) to a lattice in G. More recently Guirardel and Horbez [GH21] showed that for $n \geqslant 3$, any countable group that is measure equivalent to $\mathrm{Out}(F_n)$ is virtually isomorphic to $\mathrm{Out}(F_n)$. On the contrary, completely opposite to the aforementioned results, a famous theorem of Ornstein and Weiss [OW80] implies that all amenable groups are measure equivalent. In particular – unlike quasi-isometry – measure equivalence does *not* preserve coarse geometric invariants.

To overcome this issue it is therefore natural to look for some refinements of this equivalence notion. We focus here on the *quantiative* version as introduced by Delabie, Koivisto, Le Maître and Tessera [DKLMT22].

1.1 Quantitative measure couplings

Let G and H be two groups that are measure equivalent over a measure space (Ω, m), and denote by X_G (resp. X_H) the fundamental domains associated to the actions. In this case we have natural actions of G on X_H, and H on X_G, both denoted by "$\cdot$" where for a.e. $x \in X_H$ and all $g \in G$, we define $g \cdot x$ to be the unique element of $H * g * x$ contained in X_H (see also Figure 5, page 12 for an illustration). The action of H on X_G is defined analogously. The corresponding cocycles $\alpha : G \times X_H \to H$ and $\beta : H \times X_G \to G$ are defined by

$$\alpha(g, x) = h \Leftrightarrow h \cdot (g \cdot x) \in X_H \quad \text{and} \quad \beta(h, x) = g \Leftrightarrow g \cdot (h \cdot x) \in X_G. \tag{1.1}$$

When $x \mapsto \alpha(g, x)$ and $x \mapsto \beta(h, x)$ are L^p for all $g \in G$ and $h \in H$, we say that the groups are **L^p-measure equivalent**.

This refinement allowed for example Bader, Furman and Sauer [BFS13] to obtain a new rigidity result: they showed that any group L^1-measure equivalent to a lattice in $SO(n, 1)$ for some $n \geqslant 2$ is virtually a lattice in $SO(n, 1)$. It also lead Bowen to prove, in the appendix of [Aus16], that volume growth was invariant under L^1-measure equivalence. Delabie, Koivisto, Le Maître and Tessera offered in [DKLMT22] to extend this quantification to a family of functions larger than $\{x \mapsto x^p,\ p \in [0, +\infty]\}$. They also defined a relaxed version of measure equivalence called *measure subgroup coupling* and showed the monotonicity of the isoperimetric profile under quantitative measure subgroup couplings. A statement we make precise below.

Quantitative measure subgroup couplings Let $G = \langle S_G \rangle$ and $H = \langle S_H \rangle$ be two finitely generated groups. A **measure subgroup coupling** from G to H is a triple (Ω, m, X_H) such that:

- (Ω, m) is a standard measure space equipped with commuting measure-preserving free actions of G and H;

- the G-action on Ω admits a Borel fundamental domain;
- X_H is a Borel fundamental domain of finite measure for the action of H on Ω.

Such couplings arise naturally from coarse embeddings (see Example 3.2, page 11 and Theorem 3.6, page 13) Remark that when (Ω, m, X_H) is a measure subroup coupling, one can define the corresponding cocycle α as in Equation (1.1). We refer to Section 3.1 for more details on these couplings.

Now let $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ be a non-decreasing map. We say that the measure subgroup coupling (Ω, m, X_H) is **φ-integrable** if the cocycle α is φ-integrable, namely if there exists some constant $c > 0$ such that $x \mapsto \varphi\left(|\alpha(g,x)|_{S_G}\right)$ is integrable on X_H. We refer to Remark 3.5, page 13 for the definition of quantitative measure equivalence.

Monotonicity of the isoperimetric profile Let G be a group generated by a finite set S. If $A \subseteq G$ we will denote y $\partial_S A$ the S-boundary of A, namely the set of elements $g \in A$ for which there exists $s \in S$ such that $as \notin A$. We will also denote by $|g|_S$ the word length of an element $g \in G$. Recall that the **isoperimetric profile** of G is defined as[1]

$$I_G(n) := \sup_{|A| \leqslant n} \frac{|A|}{|\partial_S A|}.$$

Remark that due to the Følner criterion, a group is amenable if and only if its isoperimetric profile is unbounded. Hence we can see the isoperimetric profile as a way to measure the amenability of a group: the faster I_G tends to infinity, the more amenable G is. For example the isoperimetric profile of $\mathbb{Z}$ verifies $I_{\mathbb{Z}}(n) \simeq n$ while the profile of a lamplighter group $G := \mathbb{Z}/m\mathbb{Z} \wr \mathbb{Z}$ verifies $I_G(n) \simeq \log(n)$.

Finally, let $k \in \mathbb{N}$. We say that a measure subgroup coupling (Ω, m, X_H) is at **most** k-**to-one** if for every $x \in X_H$ the map $g \mapsto g^{-1} * (g \cdot x)$ has pre-images of size at most m. Delabie, Koivisto, Le Maître and Tessera showed [DKLMT22, Theorem 4.4] that if

- φ and $t \mapsto t/\varphi(t)$ are two non-decreasing maps from $\mathbb{R}_+^*$ to itself;
- and there exists an at most k-to-one φ-integrable measure subgroup coupling from G to H, for some $k \in \mathbb{N}$;

then $\varphi \circ I_H \preccurlyeq I_G$.

In [BZ21] Brieussel and Zheng managed to construct amenable groups with prescribed isoperimetric profile. Considering the monotonicity of the isoperimetric profile, the result of Brieussel and Zheng thus triggers a new question: instead of trying to quantify the measure equivalence relation between two given groups, can one find a group that is measure equivalent to a prescribed group with a prescribed quantification?

1.2 Background on the inverse problem

The family of amenable groups being quite large, the evaluation of its diversity is at the heart of numerous works. To do so, one can rely for example on geometric quantities – such as volume growth or the isoperimetric profile – or probabilistic data – such as entropy or return probability of random walks. In order to quantify this diversity, several results have been focusing on inverse problems namely, given a prescribed behaviour, does there exist a group having such behaviour?

[1] We chose to adopt the convention of [DKLMT22]. Note that in [BZ21], the isoperimetric profile is defined as $\Lambda_G = 1/I_G$.

In the case of volume growth for example, the question of the existence of a group with intermediate growth has been solved by Grigorchuk [Gri85]. Later on, Bartholdi and Erschler exhibited uncountably many groups with intermediate growth [BE14, BE12]. Another illustration is the striking work of Brieussel and Zheng [BZ21], answering the inverse problem for a large family of quantities, such as the isoperimetric profile, entropy, or equivariant L^p-compression.

The inverse problem of the quantification (that we formalise below) can be seen as a way to quantify the diversity of amenable groups, from the *ergodic* point of view.

If f and g are two non-decreasing real functions, write $f \preccurlyeq g$ if there exists some constant $C > 0$ such that $f(x) = \mathcal{O}\big(g(Cx)\big)$ as x tends to infinity. We write $f \simeq g$ if $f \preccurlyeq g$ and $g \preccurlyeq f$.

Question 1.1 (Inverse Problem)**.** Given a group H and a non-decreasing function φ, does there exist a group G such that:

- there exists a φ-integrable measure subgroup coupling from G to H;
- and this quantification is optimal, that is to say such that any other φ'-integrable coupling verifies $\varphi' \preccurlyeq \varphi$?

Note that the monotonicity of the isoperimetric profile recalled above (see page 3) provides us with an upper bound to the possible optimal integrability. For example when $H = \mathbb{Z}$, we obtain that if there exists a φ-integrable measure sugroup coupling from G to H, then $\varphi \preccurlyeq I_G$. More generally, for any group H the question can be rephrased as follows.

Question 1.2. Given a group H and a non-decreasing function φ, does there exist a group G such that $I_G \sim \varphi \circ I_H$, and such that there exists a φ-integrable measure subgroup coupling from G to H?

Relying on Brieussel-Zheng groups, we tackled these questions in [Esc24] for $H = \mathbb{Z}$. This article studies the case when H is a lamplighter group.

1.3 Main result

The main result of this article is Theorem 1.3 below, which studies the inverse problem for a coupling with a lamplighter group.

To prove it, we rely on the diagonal products of lamplighter groups defined by Brieussel and Zheng [BZ21], providing groups with prescribed isoperimetric profile. Such groups can be defined for profiles of the form $I_G \sim \rho \circ \log$ where ρ belongs to the following family of functions:

$$\mathcal{C} := \left\{ \rho : [1,+\infty) \to [1,+\infty) \;\middle|\; \begin{array}{c} \rho \text{ continuous,} \\ \rho \text{ and } x \mapsto x/\rho(x) \text{ non-decreasing} \end{array} \right\}. \tag{1.2}$$

Theorem 1.3

Let $m \geqslant 2$ be an integer and let $L_m := (\mathbb{Z}/m\mathbb{Z}) \wr \mathbb{Z}$. For all $\rho \in \mathcal{C}$ there exists a group G such that

- $I_G \sim \rho \circ \log$ and,
- there exists an at most 1-to-one measure subgroup coupling from G to L_m which is $\rho^{1-\varepsilon}$-integrable for all $\varepsilon > 0$.

Note that the lamplighter group L_m verifies $I_{L_m} \sim \log$. By monotonicity of the profile (see page 3), if there exists an at most k-to-one φ-integrable measure sugroup coupling from G to L_m, then $\varphi \circ \log \preccurlyeq I_G = \rho \circ \log$. Namely φ has to grow slower than ρ. Therefore, the integrability of the coupling given by Theorem 1.3 is almost optimal.

Structure of the paper and strategy of the proof The next two sections present the tools we use to build our couplings. In Section 2 we introduce the necessary material on Brieussel-Zheng diagonal products ; background on measure couplings are recalled in Section 3. The coupling is then build in Section 4 and relies on the criterion given in Theorem 3.9, page 14.

The strategy is the following: given a quantification $\rho \in \mathcal{C}$ we chose G to be a diagonal product with isoperimetric profile $I_G \sim \rho \circ \log$. The goal is then to verify the conditions of Theorem 3.9, namely to define sofic approximations $(\mathcal{G}_n)_n$ and $(\mathcal{H}_n)_n$ in respectively G and H, then injections $\iota_n : \mathcal{G}_n \to \mathcal{H}_n$ such that ι_n respects the geometry (see Equation (3.1), page 14 for the precise statement).

The appropriate sofic approximations in both G and H are defined in Section 4.1. We then define the injections ι_n between them and show that these injections respect the geometry in Section 4.2.

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2 Background on diagonal products

In order for this article to be self contained, we recall necessary material from [BZ21] concerning the definition of *Brieussel-Zheng diagonal products*: we give the definition of such a group, recall some results concerning the range (see Definition 2.5) of an element. Finally we present in Section 2.3 the tools needed to recover such a diagonal product starting with a prescribed isoperimetric profile.

2.1 Definition of diagonal products

Recall that the wreath product of a group G with $\mathbb{Z}$, denoted by $G \wr \mathbb{Z}$, is defined as

$$G \wr \mathbb{Z} := \oplus_{m \in \mathbb{Z}} G \rtimes \mathbb{Z}.$$

An element of $G \wr \mathbb{Z}$ is a pair (f, t) where f is a map from $\mathbb{Z}$ to G with finite support and t belongs to $\mathbb{Z}$. We refer to f as the **lamp configuration** and t as the **cursor.** Finally we denote by supp(f) the **support** of f which is defined as $\mathrm{supp}(f) := \{x \in \mathbb{Z} \mid f(x) \neq e_G\}$.

General definition Let A and B be two finite groups. Let $(\Gamma_m)_{m \in \mathbb{N}}$ be a sequence of finite groups such that each Γ_m admits a generating set of the form $A_m \cup B_m$ where A_m and B_m are finite subgroups of Γ_m isomorphic respectively to A and B. For $a \in A$ we denote a_m the copy of a in A_m and similarly for B_m.

Finally let $(k_m)_{m\in\mathbb{N}}$ be a sequence of integers such that $k_{m+1} \geqslant 2k_m$ for all m. We define $\Delta_m = \Gamma_m \wr \mathbb{Z}$ and endow it with the generating set

$$S_{\Delta_m} := \left\{(\mathrm{id}, 1)\right\} \cup \left\{\left(a_m\delta_0, 0\right) \mid a_m \in A_m\right\} \cup \left\{\left(b_m\delta_{k_m}, 0\right) \mid b_m \in A_m\right\}.$$

Definition 2.1 ([BZ21])

The **diagonal product** associated to the sequences $(\Gamma_m)_{m\in\mathbb{N}}$ and $(k_m)_{m\in\mathbb{N}}$, is the subgroup Δ of $\left(\prod_m \Gamma_m\right) \wr \mathbb{Z}$ generated by

$$\mathcal{S}_\Delta := \left\{\left(\left(\mathrm{id}\right)_m, 1\right)\right\} \cup \left\{\left((a_m\delta_0)_m, 0\right) \mid a \in A\right\} \cup \left\{\left(\left(b_m\delta_{k_m}\right)_m, 0\right) \mid b \in B\right\}.$$

The group Δ is uniquely determined by the sequences $(\Gamma_m)_{m\in\mathbb{N}}$ and $(k_m)_{m\in\mathbb{N}}$. Let us give an illustration of what an element in such a group looks like. We will denote by $\mathbf{g}$ the sequence $(g_m)_{m\in\mathbb{N}}$.

Example 2.2. We represent in Figure 1 the element $(\mathbf{g}, t)$ of Δ verifying

$$(\mathbf{g}, t) = \left((g_m)_{m\in\mathbb{N}}, t\right) := \left((a_m\delta_0)_m, 0\right)\left(\left(b_m\delta_{k_m}\right)_m, 0\right)(0, 3),$$

when $k_m = 2^m$. The cursor is represented by the blue arrow at the bottom of the figure. The only value of g_0 different from the identity is $g_0(0) = (a_0, b_0)$. Now if $m > 0$ then the only values of g_m different from the identity are $g_m(0) = a_m$ and $g_m(k_m) = b_m$.

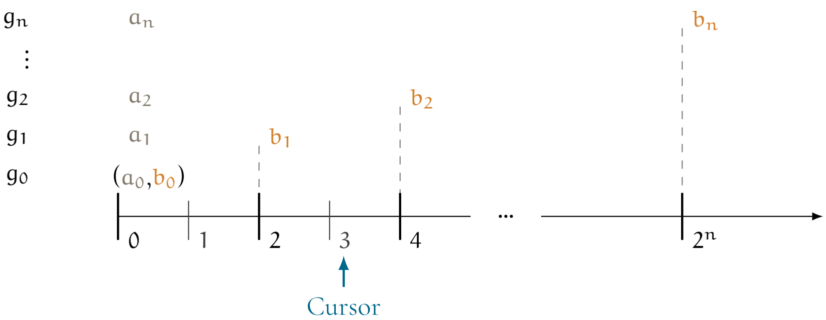


Figure 1: Representation of $(\mathbf{g}, t) = \left((a_m\delta_0)_m, 0\right)\left((b_m\delta_{k_m})_m, 0\right)(0, 3)$ when $k_m = 2^m$.

Relative commutators subgroups For all $m \in \mathbb{N}$ let $\theta_m : \Gamma_m \to \langle\langle[A_m, B_m]\rangle\rangle\backslash\Gamma_m \sim A_m \times B_m$ be the natural projection. Let θ_m^A and θ_m^B denote the composition of θ_m with the projection to A_m and B_m respectively. Now let $m \in \mathbb{N}$ and define $\Gamma'_m := \langle\langle[A_m, B_m]\rangle\rangle$. If (g_m, t) belongs to Δ_m, then there exists a unique $g'_m : \mathbb{Z} \to \Gamma'_m$ such that $g_m(x) = g'_m(x)\theta_m\left(g_m(x)\right)$ for all $x \in \mathbb{Z}$.

Example 2.3. Let $(\mathbf{g}, 3)$ be the element described in Figure 1. Then the only non-trivial value of $\theta_0(g_0)$ is $\theta_0(g_0(0)) = (a_0, b_0)$. If $m > 0$ then the only non trivial values of $\theta_m(g_m)$ are $\theta_m(g_m(0)) = (a_m, \mathbf{e})$ and $\theta_m(g_m(k_m)) = (\mathbf{e}, b_m)$. Finally for all m we have $g'_m = \mathbf{e}$ since there are no commutators appearing in the decomposition of $(\mathbf{g}, 0)$.

Example 2.4. Assume that $k_m = 2^m$ for all $m \in \mathbb{N}^*$, and consider first the element $(\mathbf{f}, 0)$ of Δ defined by $(\mathbf{f}, 0) := (0, -k_1)\big((a_m\delta_0)_m, 0\big)(0, k_1)$. Now define the commutator

$$(\mathbf{g}, 0) = (\mathbf{f}, 0) \cdot \big((b_m\delta_{k_m})_m, 0\big) \cdot (\mathbf{f}, 0)^{-1} \cdot \big((b_m^{-1}\delta_{k_m})_m, 0\big)$$

and let us describe the values taken by $\mathbf{g}$ and the induced maps $\theta_m(g_m)$ and g'_m (see Figure 2 for a representation of $\mathbf{g}$). The only non-trivial commutator appearing in the values taken by $\mathbf{g}$ is $g_1(k_1)$ which is equal to $a_1b_1a_1^{-1}b_1^{-1}$. In other words $g_0 = \mathbf{e}$, thus $\theta_0 = \mathbf{e}$. Moreover when $m = 1$ we have $\theta_1 = \mathbf{e}$ and the only value of $g'_1(x)$ different from $\mathbf{e}$ is $g'_1(k_1) = a_1b_1a_1^{-1}b_1^{-1}$ (on a blue background in Figure 2). Finally if $m > 1$ then g_m trivial thus $\theta_m = \mathbf{e}$ and $g'_m = \mathbf{e}$.

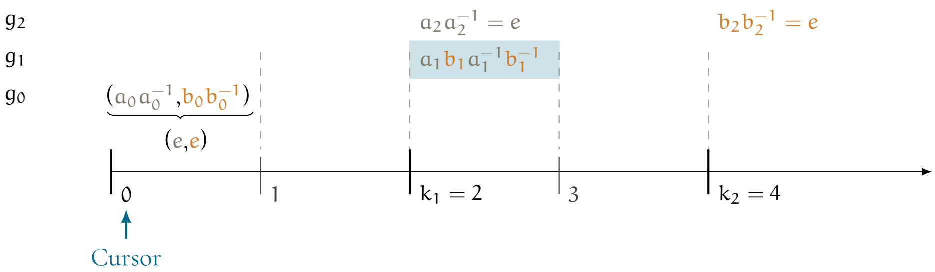


Figure 2: Representation of $(\mathbf{g}, 0)$ defined in Example 2.4

The expanders case In the proof of our main theorem, we will restrict ourselves to a particular familiy of groups $(\Gamma_m)_{m\in\mathbb{N}}$ called **expanders.** Recall that $(\Gamma_m)_{m\in\mathbb{N}}$ is said to be a sequence of **expanders** if the sequence of diameters $(\operatorname{diam}(\Gamma_m))_{m\in\mathbb{N}}$ is unbounded and if there exists $c_0 > 0$ such that for all $m \in \mathbb{N}$ and all $n \leqslant |\Gamma_m|/2$ the isoperimetric profile verifies $I_{\Gamma_m}(n) \leqslant c_0$.

Now, consider a family $(\Gamma_m)_{m\in\mathbb{N}}$ of expanders. Assume that there exists $c > 0$ such that for all $l \geqslant 1$ there exists $\Gamma_{\mathfrak{p}(l)}$ verifying that $\operatorname{diam}\big(\Gamma_{\mathfrak{p}(l)}\big) \simeq l$. We can thus define a "parametrization" by fixing a map $l \mapsto \Gamma_{\mathfrak{p}(l)}$. Consider now two non-decreasing sequences $(k_m)_{m\in\mathbb{N}}$ and $(l_m)_{m\in\mathbb{N}}$ of real numbers greater than 1 and denote by Δ the diagonal product associated to $(\Gamma_{\mathfrak{p}(l_m)})_{m\in\mathbb{N}}$ and $(k_m)_{m\in\mathbb{N}}$. Then Δ is uniquely determined by the data of $(l_m)_{m\in\mathbb{N}}$ and $(k_m)_{m\in\mathbb{N}}$. In what follows, we will abuse notations and denote Γ_m instead of $\Gamma_{\mathfrak{p}(l_m)}$. Moreover we will always make the following assumptions when talking about diagonal products. We refer to [BZ21, Example 2.3] for an explicit example of diagonal product verifying **(H)**.

Hypotheses (H)

1. $q := |A \times B| = 6$;
2. $k_0 = 0$ and $\Gamma_0 = A_0 \times B_0$;
3. $\kappa \geqslant 3$ and $(k_n)_{n\in\mathbb{N}^*}$ is a sub-sequence of $(\kappa^n)_{n\in\mathbb{N}}$;
4. $\lambda \geqslant 2$ and $(l_n)_{n\in\mathbb{N}}$ is a sub-sequence of $(\lambda^n)_{n\in\mathbb{N}}$;
5. $(\Gamma_m)_{m\in\mathbb{N}}$ is a sequence of expanders such that Γ_m is a quotient of $A * B$ and there exists $c_1 > 0$ such that $\operatorname{diam}(\Gamma_m) \leqslant c_1 l_m$ for all $m \in \mathbb{N}$;
6. the natural quotient map $A_m \times B_m \to \langle\langle [A_m, B_m] \rangle\rangle \backslash \Gamma_m$ is an isomorphism, where $\langle\langle [A_m, B_m] \rangle\rangle =: \Gamma'_m$ is the normal closure of $[A_m, B_m]$.

Recall (see [BZ21, page 9]) that in this case there exist $c_1, c_2 > 0$ such that, for all m

$$c_1 l_m - c_2 \leqslant \ln|\Gamma_m| \leqslant c_1 l_m + c_2. \tag{2.1}$$

Finally, we adopt the convention of [BZ21, Notation 2.2] and allow k_m to take the value $+\infty$. In this case Δ_m is the trivial group. In particular when $k_1 = +\infty$ the diagonal product Δ corresponds to the usual lamplighter group $(A \times B) \wr \mathbb{Z}$.

2.2 Range of an element

In this section we recall the notion of *range* of an element $(\mathbf{g}, t)$ in Δ. We denote by $\pi_2 : \Delta \to \mathbb{Z}$ the projection on the second factor, ie. the map that sends $(\mathbf{f}, t) \in \Delta$ to the value of its cursor t.

Definition 2.5

If $w = s_1 \dots s_m$ is a word over S_Δ we define its **range** as

$$\mathrm{range}(w) := \left\{ \pi_2 \left(\prod_{j=1}^{i} s_j \right) \ : \ i = 0, \dots, n \right\}.$$

The range is a finite subinterval of $\mathbb{Z}$. It represents the set of sites visited by the cursor. We refer to Example 2.8 for an illustration.

Definition 2.6

The **range** of an element $(\mathbf{f}, t) \in \Delta$ is defined as the minimal diameter interval obtained as the range of a word over S_Δ representing $(\mathbf{f}, t)$.

In what follows we will consider elements that can be written as a word with range in an interval of the form $[0, n]$, where n belongs to $\mathbb{N}$. Therefore, when there is no ambiguity we will denote $\mathrm{range}(\delta)$ this interval, namely $\mathrm{range}(\delta) = [0, n]$. For all $n \in \mathbb{N}$ we denote by $\mathfrak{l}(n)$ the integer such that $k_{\mathfrak{l}(n)} \leqslant n < k_{\mathfrak{l}(n)+1}$. Let us now recall a useful fact proved in [BZ21].

Claim 2.7 ([BZ21, Fact 2.9])**.** An element $(\mathbf{g}, t) \in \Delta$ is uniquely determined by t, g_0 and the sequence $(g'_m)_{m \leqslant \mathfrak{l}(\mathrm{range}(\mathbf{g},t))}$.

Example 2.8. Let $(\mathbf{g}, 0) \in \Delta$ such that $\mathrm{range}(\mathbf{g}, 0) = [0, 6]$, that is to say: the cursor can only visit sites between 0 and 6. Then the map g_m can "write" elements of A_m only on sites visited by the cursor, that is to say from 0 to 6, and it can write elements of B_m only from k_m to $6 + k_m$. Thus g_0 is supported on $[0, 6]$, since $k_0 = 0$. Moreover, commutators (and hence elements of Γ'_m) can only appear between k_m and 6. Thus $\mathrm{supp}(g'_m) \subseteq [k_m, 6]$ Such a $(\mathbf{g}, 0)$ is represented in Figure 3, page 9 for $k_m = 2^m$.

Finally, the element $(\mathbf{g}, 0)$ is uniquely determined by the data g_0 (that is to say, the values read in the bottom line) and the values of g'_i for $i = 1, 2$ (namely, the value taken in the blue area). Figure 4, page 9 represents the aforementioned characterizing data.

2.3 From the isoperimetric profile to the group

We saw how to define a diagonal product from two sequences $(k_m)_m$ and $(l_m)_m$. In this section we recall the definition given in [BZ21, Appendice B] of a diagonal product from its isoperimetric profile. We conclude on some useful results concerning the metric of these groups.

Definition of Δ Recall that in the particular case of expanders (see Section 2.1) a diagonal product Δ is uniquely determined by the sequences $(k_m)_{m\in\mathbb{N}}$ and $(l_m)_{m\in\mathbb{N}}$ (where l_m corresponds to the diameter of Γ_m). Thus, starting from a prescribed function I, the goal is to define

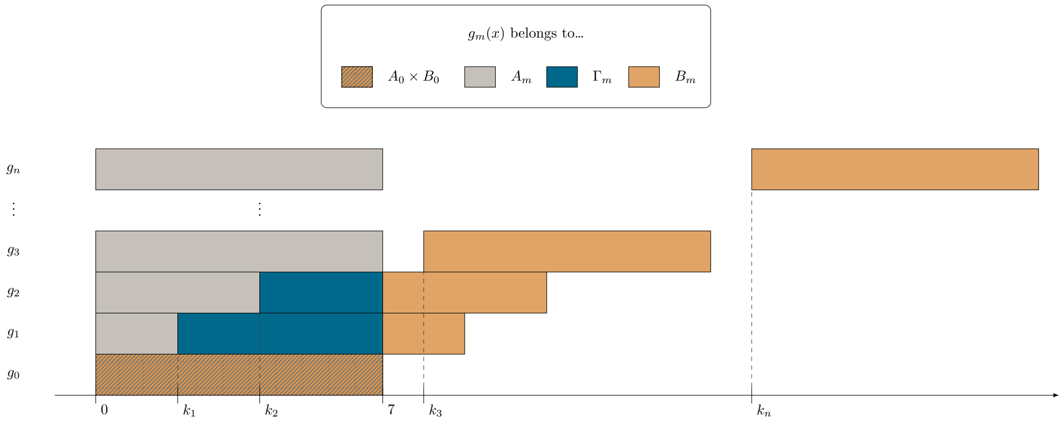


Figure 3: An element of Δ

Recall that $g_m : \mathbb{Z} \to \Gamma_m$. If $m \leqslant \mathfrak{l}(6)$, then $g_m(x)$ belongs to A_m if $x \in [0, k_m - 1]$, it belongs to Γ_m if $x \in [k_m, 6]$ and to B_m if $x \in [7, 6 + k_m]$ and equals $\mathbf{e}$ elsewhere. If $m > \mathfrak{l}(6)$ then $g_m(x)$ belongs to A_m if $x \in [0, 6]$ and to B_m if $x \in [k_m, 6 + k_m]$ and equals $\mathbf{e}$ elsewhere.

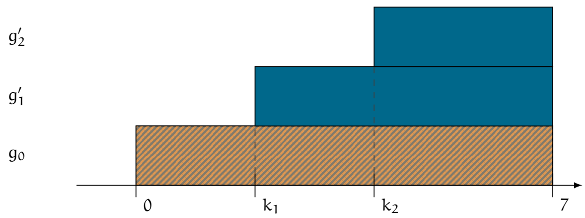


Figure 4: Data needed to characterized $\mathbf{g}$ such that range($\mathbf{g}$) $\subset [0, 6]$ when $k_m = 2^m$

sequences $(k_m)_{m\in\mathbb{N}}$ and $(l_m)_{m\in\mathbb{N}}$ such that the corresponding Δ verifies $I_\Delta \sim I$. Brieussel and Zheng's construction of diagonal products is possible when the profile I is of the form $I \sim \rho \circ \log$ where ρ belongs to the following set $\mathcal{C}$,

$$\mathcal{C} := \left\{ \rho : [1, +\infty) \to [1, +\infty) \;\middle|\; \begin{array}{c} \rho \text{ continue,} \\ \rho \text{ and } x \mapsto x/\rho(x) \text{non-decreasing} \end{array} \right\}.$$

Equivalently this is the set of functions ρ satisfying

$$(\forall x, c \geqslant 1) \qquad \rho(x) \leqslant \rho(cx) \leqslant c\rho(x). \tag{2.2}$$

So let $\rho \in \mathcal{C}$. Combining [BZ21, Proposition B.2 and Theorem 4.6] leads the following proposition (remember that with our convention the isoperimetric profile considered in [BZ21] corresponds to $1/I_\Delta$).

Proposition 2.9

Let $\kappa, \lambda \geqslant 2$. For any $\rho \in \mathcal{C}$ there exists a subsequence $(k_m)_{m\in\mathbb{N}^*}$ of $(\kappa^n)_{n\in\mathbb{N}}$ and a subsequence $(l_m)_{m\in\mathbb{N}}$ of $(\lambda^n)_{n\in\mathbb{N}}$ such that the group Δ defined in Section 2.1 verifies $I_\Delta \sim \rho \circ \log$.

Example 2.10 ([BZ21, Example 4.5]). Let $\alpha > 0$. If $\rho(x) := x^{1/(1+\alpha)}$ then the diagonal product Δ defined by $k_m = \kappa^m$ and $l_m = \kappa^{\alpha m}$ verifies $I_\Delta \sim \rho \circ \log$.

Example 2.11. If $\rho = \log$ then the diagonal product $\mathbb{Z}$ defined by $k_m = \kappa^m$ and $l_m = \kappa^{\kappa^m}$ verifies $I_\Delta \sim \rho \circ \log \circ \log$.

Recall that we allow k_m to take the value $+\infty$ (see below Equation (2.1)).

Example 2.12. If $\rho(x) = x$ then the diagonal product defined by $l_m = 1$ for all m and $k_m = +\infty$ for all $m \geqslant 1$ verifies $\Delta = (A \times B) \wr \mathbb{Z}$ and $I_\Delta \sim \log$.

Technical tools Now let us recall the intermediary functions defined in [BZ21, Appendix B] and some of their properties.

Let $\rho \in \mathcal{C}$. The construction of a group corresponding to the given isoperimetric profile $\rho \circ \log$ is based on the approximation of ρ by a piecewise linear function $\bar{\rho}$. For the quantification of our measure coupling, many of our computations will use $\bar{\rho}$ and some of its properties. We recall below all the needed results, beginning with the definition of $\bar{\rho}$.

Lemma 2.13

Let $\rho \in \mathcal{C}$. Let $(k_m)_m$ and $(l_m)_m$ given by Proposition 2.9 and Δ be the corresponding diagonal product. The function $\bar{\rho}$ defined by

$$\bar{\rho}(x) := \begin{cases} x/l_m & \text{if } x \in [k_m l_m, k_{m+1} l_m], \\ k_{m+1} & \text{if } x \in [k_{m+1} l_m, k_{m+1} l_{m+1}], \end{cases} \tag{2.3}$$

verifies $\bar{\rho} \sim \rho$.

Example 2.14. If $\rho(x) = x$ then $l_m = 1$ for all m and $k_m = +\infty$ for all $m \geqslant 1$.
In this case $\Delta = (A \times B) \wr \mathbb{Z}$.

3 Preliminaries on measure couplings

This section recalls some material from [DKLMT22, Section 2]. We start with general definition on measure equivalence and measure subgroup couplings, then turn to their quantitative version. Finally we present the two couplings building techniques that we use, in order to show our main theorem.

3.1 Measure couplings

A **standard Borel space** $(\Omega, \mathcal{B}(\Omega))$ is a measurable space whose σ-algebra $\mathcal{B}(\Omega)$ consists of the the Borel subsets coming from some Polish (separable and completely metrisable) topology on Ω. A **standard measure space** (Ω, m) is a standard Borel space $(\Omega, \mathcal{B}(\Omega))$ equipped with a non-zero σ-finite measure m.

A **measure-preserving action** of a discrete countable group G on a measured space (Ω, m) is an action of G on Ω such that the map sending (g, x) to $g * x$ is a Borel map and $m(E) = m(g * E)$ for all $E \subseteq \mathcal{B}(X)$ and all $g \in G$. We will say that a measure-preserving action of G on (Ω, m) is **free**, if for almost every $x \in X$ we have $g * x = x$ if and only if $g = \mathbf{e}_G$. A **fundamental domain** for an action of G on (Ω, m) is a Borel subset $X_G \subseteq \Omega$ which intersects almost every G-orbit at exactly one point: in other words, there is a full measure G-invariant Borel set $\Omega^* \subseteq \Omega$ such that for all $x \in \Omega^*$, the intersection $\mathrm{Orb}_G(x) \cap X_G$ is a singleton.

Let G and H be two countable groups. Following the terminology of [DKLMT22], we call a **measure equivalence coupling** from G to H a quadruple (Ω, X_G, X_H, m) such that G and H

both act freely and measure preservingly on the standard measure space (Ω, m), and X_G and X_H are fundamental domains of finite measure for the G- and the H-action on Ω, respectively.

An elementary example of measure equivalence coupling is given by considering a countable group G endowed with the counting measure m, taking G = H and as actions the left and right translations of G on itself. The corresponding coupling is then $(G, \{e_G\}, \{e_G\}, m)$.

When G is an infinite subgroup of H, however, the (left) action by translation of G on H does not admit any fundamental domain of finite measure. This is why Delabie, Koivisto, Le Maître et Tessera introduced the notion of *measure subgroup coupling*, relaxing the condition on the fundamental domain of the G-action.

Definition 3.1 (*[DKLMT22, Def. 2.4]*)

> Let G and H be two countable groups. A **measure subgroup coupling** from G to H is a triple (Ω, X_H, m) such that:
> - (Ω, m) is a standard measure space equipped with commuting measure-preserving free actions of G and H;
> - and the G-action on Ω admits a Borel fundamental domain;
> - and X_H is a Borel fundamental domain of finite measure for the action of H on Ω.

Remark that a measure equivalence coupling (Ω, X_G, X_H, m) from G to H induces two measure subgroup couplings, namely (Ω, X_H, m) from G to H and (Ω, X_G, m) from H to G. Also, as suggested above, when G is a subgroup of H, a measure subgroup coupling grom G to H is given by $(H, \{e_H\}, m)$, where m denotes again the counting measure, and G acts on H by left translation, and H on itself by right translation.

We saw that two lattices in a common locally compact group are measure equivalent. Analogously, another example of measure subgroup coupling is given by considering $\mathcal{G}$ a locally compact group endowed with a Haar measure m, then taking $G \leqslant \mathcal{G}$ to be a discrete group and $H \leqslant \mathcal{G}$ to be a lattice. Letting G and H act respectively by left and right translations on G, and denoting by X_H a fundamental domain for the H-action then produces the triple $(\mathcal{G}, X_H, m)$ which is a measure subgroup coupling from G to H.

Finally, let $k \in \mathbb{N}$. We say that a measure subgroup coupling (Ω, X_H, m) from G to H is at **most** k**-to-one** if for every $x \in X_H$ the map $g \mapsto g^{-1} * (g \cdot x)$ has pre-images of size at most m. Such couplings arise naturally from coarse embeddings.

Example 3.2 (Coarse embeddings and regular maps)**.** A map $f\colon G \to H$ is a **regular map** if it is Lipschitz and verifies $\sup_{h \in H} |f^{-1}(\{h\})| < +\infty$. Coarse embeddings, for example, are regular maps. Recall that a **coarse embedding** is a map $f\colon G \to H$ such that there exists two proper non-decreasing functions $\rho_-, \rho_+ : \mathbb{R}_+ \to \mathbb{R}_+$ verifying for all $g_1, g_2 \in G$,

$$\rho_-\left(d_G\left(g_1, g_2\right)\right) \leqslant d_H\left(f\left(g_1\right), f\left(g_2\right)\right) \leqslant \rho_+\left(d_G\left(g_1, g_2\right)\right).$$

Delabie et al. showed [DKLMT22, Theorem 5.4] that if G is amenable and if there exists a regular map from G to H, then there exists $k \in \mathbb{N}$ such that there is an at most k-to-one measure subgroup coupling from G to H. We will see a more precise version of this statement in Theorem 3.6, page 13, where we add the integrability of the measure subgroup.

Let us now define the quantitative version of such couplings.

3.2 Quantitative couplings

3.2.1 Definitions and examples

Recall that if a finitely generated group G acts on a space Ω and if S_G is a finite generating set of G, one can define the **Schreier graph** associated to this action. It is the graph whose set of vertices is Ω and set of edges is $\{(x, s * x) \mid s \in S_G, x \in \Omega\}$, where "$*$" denotes the action of G on Ω. This graph is endowed with a natural metric d_{S_G} fixing the length of an edge to one. Remark that if S'_G is another generating set of G then there exists $C > 0$ such that for a.e. $x \in \Omega$ and all $g \in G$

$$\frac{1}{C} d_{S_G}(x, g * x) \leqslant d_{S'_G}(x, g * x) \leqslant C d_{S_G}(x, g * x).$$

Finally if (Ω, X_H, m) is a measure subgroup coupling from G to H, we have a natural action of G on X_H (see Figure 5 for an illustration) denoted by "$\cdot$" where for a.e. $x \in X_H$ and all $g \in G$ we define $g \cdot x$ to be the unique element of $H * g * x$ contained in X_H, *viz.*

$$\{g \cdot x\} = \left(H * g * x\right) \cap X_H.$$

Definition 3.3 ([DKLMT22, *Def. 2.20*])

Let $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ be a non-decreasing map. Let G and H be two countable groups and denote by S_H a finite generating set of H. A measure subgroup coupling (Ω, X_H, m) from G to H is said to be **φ-integrable** if for all $g \in G$ there exists $c_g > 0$ such that

$$\int_{X_H} \varphi\left(\frac{1}{c_g} d_{S_H}(g * x, g \cdot x)\right) dm(x) < +\infty.$$

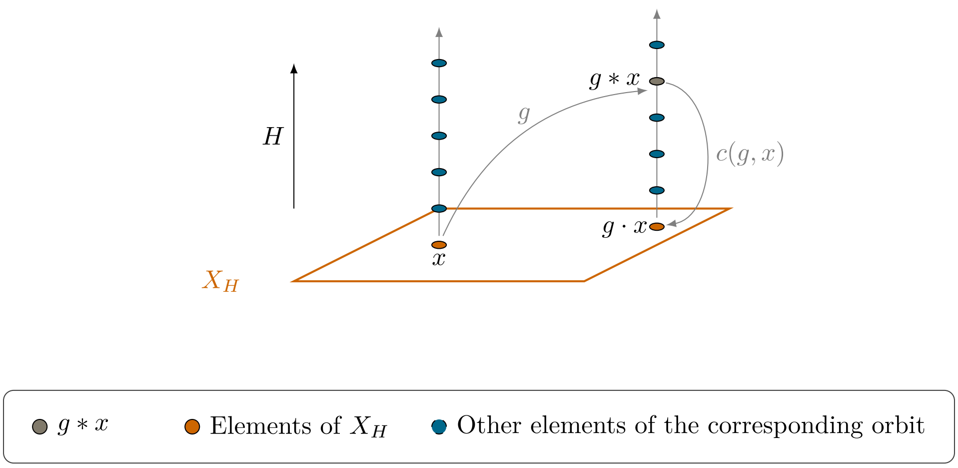


Figure 5: Definition of $g \cdot x$

The constant c_g in the definition is introduced for the integrability to be independent of the choice of generating set S_H.

If $\varphi(x) = x^p$ we will sometimes talk of **L^p-integrability** instead of φ-integrability. In particular, L^0 means that no integrability assumption is made. We will talk about **L^∞-integrability** if $x \mapsto d_{S_H}(g * x, g \cdot x)$ is essentially bounded for all $g \in G$.

Remark 3.4 (Reformulation using cocycles)**.** When (Ω, X_H, m) is a measure subgroup coupling from G to H, the associated **cocycle** $\alpha : G \times X_H \to H$ is defined as being the map that verifies

$\alpha(g,x) * g * x = g \cdot x$, for all $g \in G$ and a.e. $x \in \Omega$. In other words, $\alpha(g,x)$ is the (unique) element of h sending $g * x$ to the fundamental domain X_H.

Now, denote by $|h|_{S_H}$ the length of h in $H = \langle S_H \rangle$. An equivalent manner to formulate the above Definition 3.3 using cocycles, is to replace $d_{S_H}(g * x, g \cdot x)$ in the integral by $|\alpha(g,x)|_{S_H}$.

Remark 3.5 (Quantitative measure equivalence)**.** The authors also defined in [DKLMT22] the quantiative version of measure equivalence. Let (Ω, X_G, X_H, m) be a measure equivalence coupling from G to H, and denote by α and β the corresponding cocycles (see Equation (1.1), page 2). One says that the measure equivalence is (φ,ψ)**-integrable** if α and β are respectively φ-integrable and ψ-integrable. This is equivalent to ask for the measure subgroup coupling (Ω, X_H, m) to be φ-integrable and (Ω, X_G, m) to be ψ-integrable.

Integrable measure subgroup couplings naturally arise from regular maps, and thus, in particular, from coarse embeddings. This is what the following theorem recalls. We refer to Example 3.2, page 11 for the definitions of coarse embeddings and regular maps.

Theorem 3.6 (*[DKLMT22, Theorem 5.4]*)

Let G and H be two finitely generated groups.

- If there exists $k \in \mathbb{N}$ such that there is an at most k-to-one L^∞-integrable measure subgroup coupling from G to H, then G regularly embeds into H.
- If G is amenable and regularly embeds into H, then there exists $k \in \mathbb{N}$ such that there is an at most k-to-one L^∞-integrable measure subgroup coupling from G to H.

3.2.2 Isoperimetric profile

Now recall from the introduction that the **isoperimetric profile** of G is the map I_G defined by $I_G(n) := \sup_{|A| \leqslant n} |A|/|\partial_S A|$. Delabie, Koivisto, Le Maître and Tessera showed the monotonicity of this profile under measure eaquivalence and measure subgroup couplings.

Theorem 3.7 (*[DKLMT22, Theorems 1 and 4.4]*)

Let $\varphi : \mathbb{R}_+^* \to \mathbb{R}_+^*$ be a non-decreasing map such that $t \mapsto t/\varphi(t)$ is also non-decreasing. Let G and H be two finitely generated groups. If either

- there exists a (φ, L^0)-measure equivalence coupling from G to H;
- or there exists an at most k-to-one φ-integrable measure subgroup coupling from G to H, for some $k \in \mathbb{N}$;

then $\varphi \circ I_H \preccurlyeq I_G$.

Assume for example that $H = \mathbb{Z}$. Then its profile verifies $I_{\mathbb{Z}} \sim \mathrm{id}$. The above theorem proves that, if there exists at most k-to-one φ-integrable measure sugroup coupling from a group G to $\mathbb{Z}$, then $\varphi \preccurlyeq I_G$.

3.3 Couplings building techniques

In this section we recall the necessary material from [DKLMT22, CDKT24]. We present the tools needed to build the coupling of Theorem 1.3, page 4, namely sofic approximations and Følner tiling sequences. The necessity of both these tools is discussed in Remark 3.12, page 15.

3.3.1 Sofic approximations

In this paragraph G will be a finitely generated group endowed with a finite generating set S_G, and $(\mathcal{G}_n)_{n\in\mathbb{N}}$ will be a sequence of finite directed graphs, labeled by the elements of S_G.

Let $r > 0$ and denote by $\mathcal{G}_n^{(r)}$ the set of elements $x \in \mathcal{G}_n$ such that $B_{\mathcal{G}_n}(x, r)$ is isomorphic to $B_G(e_G, r)$ seen as directed labeled graphs, *viz.* $\mathcal{G}_n^{(r)} = \left\{x \in \mathcal{G}_n \mid B_{\mathcal{G}_n}(x, r) \simeq B_G(e_G, r)\right\}$. We say that $(\mathcal{G}_n)_{n\in\mathbb{N}}$ is a **sofic approximation** if for every $r > 0$

$$\lim_{n\to\infty} \frac{\left|\mathcal{G}_n^{(r)}\right|}{|\mathcal{G}_n|} = 1.$$

Example 3.8. Any Følner sequence in an amenable group G is a sofic approximation.

In [CDKT24] Carderi, Delabie, Koivisto and Tessera proved a condition for a measure subgroup coupling to be φ-integrable, using sofic approximations.

Theorem 3.9 (*[CDKT24]*)

Let $\varphi\colon \mathbb{R}^+ \to \mathbb{R}^+$ be a non-decreasing map. Let G and H be two finitely generated groups with respective sofic approximations $(\mathcal{G}_n)_n$ and $(\mathcal{H}_n)_n$. Let $\iota_n\colon \mathcal{G}_n \to \mathcal{H}_n$ be an injective map such that, for every $s \in S_G$ there exists $\delta > 0$ such that

$$\lim_{R\to\infty} \sup_n \sum_{r=0}^{R} \varphi(\delta r) \frac{\left|\left\{x \in \mathcal{G}_n^{(1)} \mid d_{\mathcal{H}_n}(\iota_n(x), \iota_n(x \cdot s)) = r\right\}\right|}{|\mathcal{G}_n|} < \infty. \tag{3.1}$$

Then there exists an at most 1-to-one φ-integrable measure subgroup coupling from G to H.

Given two amenable groups G and H, with respective sofic approximations $(\mathcal{G}_n)_n$ and $(\mathcal{H}_n)_n$, defining injections ι_n that verify the above Equation (3.1) is not always straightforward. Our strategy to define such an injection relies on the notion of *Følner tiling sequence*, introduced by Delabie, Koivisto, Le Maître and Tessera [DKLMT22].

3.3.2 Følner tiling sequence: shifts and tiles

Følner tiling sequences are tools to chose the needed sofic approximations, $(\mathcal{G}_n)_n$ and $(\mathcal{H}_n)_n$, and define an injection ι_n from $\mathcal{G}_n$ to $\mathcal{H}_n$, that verifies Equation (3.1).

Definition 3.10 (*[DKLMT22]*)

Let G be an amenable group and $(\Sigma_n)_{n\in\mathbb{N}}$ be a sequence of finite subsets of G. Define by induction the sequence $(T_n)_{n\in\mathbb{N}}$ by $T_0 := \Sigma_0$ and $T_{n+1} := T_n\Sigma_{n+1}$.

We say that $(\Sigma_n)_{n\in\mathbb{N}}$ is a (right) **Følner tiling sequence** if

- $(T_n)_{n\in\mathbb{N}}$ is a (right) Følner sequence, *viz.* $\lim_{n\to\infty} |T_n g \setminus T_n|/|T_n| = 0$ for all $g \in G$;
- $T_{n+1} = \bigsqcup_{\sigma\in\Sigma_{n+1}} \sigma T_n$.

We call Σ_n a set of **shifts** and T_n a **tile**.

Remark that in particular $T_n = \Sigma_n \cdots \Sigma_0$ for all $n \in \mathbb{N}$, and for all $(\mathbf{f}, t) \in T_n$ there exists a unique sequence $(\sigma_i)_{i=0,\dots,n}$ such that

$$(\mathbf{f}, t) = \sigma_n \cdots \sigma_0 \quad \text{and} \quad \sigma_i \in \Sigma_i,\ \forall i \in \{0, \dots, n\}.$$

Example 3.11 ([DKLMT22, Proposition 6.10])**.** Let $G = \mathbb{Z}$ and for all $n \in \mathbb{N}$ let $\Sigma_n := \{0, 2^n\}$. Then $(\Sigma_n)_n$ is a Følner tiling sequence and the corresponding sequence of tiles is the sequence of intervals of the form $T_n := [0, 2^{n+1} - 1]$, for $n \in \mathbb{N}$.

We refer to Section 4.1.1, page 15 for the definition of Følner tiling sequences in diagonal products, and to Section 4.1.2, page 17 for a definition of another tiling in a lamplighter group.

Remark 3.12 (On the choice of tools)**.** In our proof, the criterion given in Theorem 3.9 allows us to build and quantify the coupling. The injections ι_n this criterion requires us to build are obtained using Følner tiling sequences. Indeed, these tiling sequences allow us to define injections ι_n that respect the geometry of the groups, that is to say, that verify Equation (3.1).

On the other hand, relying only on Følner tiling sequences is not possible for us in this case. More precisely, let G and H be two amenable groups with respective Følner tiling sequences $(\Sigma_n)_n$ and $(\Sigma'_n)_n$. In [DKLMT22, Proposition 6.9], the authors give a criterion to show that G and H admit a measure equivalence coupling for which both cocycles are φ-integrable. Their criterion requires that the shifts verify $\Sigma_n = \Sigma'_n$, for all $n \in \mathbb{N}$.

We did not manage to fulfill this strong requirement in our case, namely, when G is a diagonal product and H a lamplighter gorup. This is why, instead, we rely on the criterion given in Theorem 3.9, which allows Σ'_n to contain more elements than Σ_n.

4 Construction of the coupling

We now turn to the proof of Theorem 1.3, page 4. In the following Δ will denote a diagonal product as defined in Section 2 and verifying the hypotheses **(H)** page 7. In particular its isoperimetric profile is of the form $I_\Delta \simeq \rho \circ \log$ for some $\rho \in \mathcal{C}$. To prove Theorem 1.3 we actually show that the diagonal product obtained from the isoperimetric profile $\rho \circ \log$ is the wanted group G. The integrability of the coupling is proved using the criterion of Theorem 3.9.

4.1 Definition of the sofic approximations

The purpose of this section is to define the sofic approximations $(\mathcal{G}_n)_n$ and $(\mathcal{H}_n)_n$ in respectively $G = \Delta$ and $H = (\mathbb{Z}/m\mathbb{Z}) \wr \mathbb{Z}$. We start by exhibiting Følner tiling sequences in both G and H, and then extract appropriate subsequences to work with.

4.1.1 Følner tiling sequences in a diagonal product

In [Esc24] we constructed Følner tiling sequences of diagonal products. In particular, we showed in [Esc24, Lemma 3.10] that the sequence $(T_n)_n$ defined in the following Equation (4.1), can be obtained as a sequence of tiles. We refer to Section 2.2 for details on the range of an element.

Sequence of tiles For all $n \in \mathbb{N}$, let

$$T_n := \left\{ (\mathbf{f}, t) \in \Delta \mid \text{range}(\mathbf{f}, t) \in [0, \kappa^n - 1] \right\}. \tag{4.1}$$

The sequence $(T_n)_n$ is a (right) Følner sequence for Δ [Esc24, Proposition 2.13 and Lemma 3.10]. It verifies $|\partial_{S_\Delta} T_n|/|T_n| \leqslant 2/\kappa^n$.

For all $n \in \mathbb{N}$, let $\mathfrak{L}(n) = \mathfrak{l}(\kappa^n - 1)$, that is to say $\mathfrak{L}(n)$ is the unique integer such that $k_{\mathfrak{L}(n)} \leqslant \kappa^n - 1 < k_{\mathfrak{L}(n)+1}$. For example if $k_n = \kappa^n$, then $\mathfrak{L}(n) = n - 1$. We recall the following useful fact, concerning the growth of the map $\mathfrak{L}$.

Claim 4.1 ([Esc24, Claim 3.8])**.** Let $n \geqslant 0$, then either $\mathfrak{L}(n+1) = \mathfrak{L}(n)$ or $\mathfrak{L}(n+1) = \mathfrak{L}(n) + 1$.

We computed in [Esc24, Lemma 4.2] the value of $|T_n|$ for all $n \in \mathbb{N}$, namely

$$|T_n| = \kappa^n \left(|A||B|\right)^{\kappa^n} \prod_{m=1}^{\mathfrak{L}(n)} \left|\Gamma'_m\right|^{\kappa^n - k_m}, \tag{4.2}$$

and moreover showed [Esc24, Proposition 4.3] that there exist two constants $c_\Delta, C_\Delta > 0$ depending only on Δ, such that for all $n \in \mathbb{N}$

$$c_\Delta \kappa^{n-1} l_{\mathfrak{L}(n)} \leqslant \ln |T_n| \leqslant C_\Delta \kappa^n l_{\mathfrak{L}(n)}. \tag{4.3}$$

We now recall the definition of the corresponding shifts.

Sequence of shifts Let $\Sigma_0 := T_0 = \left\{(\mathbf{f}, t) \in \Delta \,|\, \mathrm{range}(\mathbf{f}, t) \in \{0, 1\}\right\}$. Now for all $n \geqslant 0$ we split Σ_{n+1} in κ parts defined as follows. For all $j \in \{0, \dots, \kappa - 1\}$ we let Σ_n^j be the set of $(\mathbf{g}, j\kappa^n) \in \Delta$ such that the following conditions are verified:

1. $\mathrm{supp}\left(g_0\right) \subseteq \left[0, j\kappa^n - 1\right] \cup \left[(j+1)\kappa^n, \kappa^{n+1} - 1\right]$
2. $\mathrm{supp}\left(g'_m\right) \subseteq \left[k_m, j\kappa^n + k_m - 1\right] \cup \left[(j+1)\kappa^n, \kappa^{n+1} - 1\right]$ for all $m \in [1, \mathfrak{L}(n)]$;
3. If $\mathfrak{L}(n+1) = \mathfrak{L}(n) + 1$ then $\mathrm{supp}\left(g'_{\mathfrak{L}(n+1)}\right) \subseteq \left[k_m, \kappa^{n+1} - 1\right]$.
4. $\mathrm{supp}\left(g'_m\right) = \emptyset$ for all $m \notin [0, \mathfrak{L}(n+1)]$.

We refer to Figure 6, page 16 for a representation of an element in such a part. Finally, we define $\Sigma_{n+1} := \cup_{j=0}^{\kappa-1} \Sigma_{n+1}^j$. This sequence verifies $T_{n+1} = \Sigma_{n+1} T_n$ for all $n \in \mathbb{N}$ [Esc24, Lemma 3.10].

Let $(\mathbf{g}, t)$ be an element of some Σ_{n+1}^j. We represent in Figure 6 the supports and the sets where the maps $g_0, g'_1, \dots, g'_{\mathfrak{L}(n+1)}$ take their values. The light-blue rectangle with dotted outline is in Σ_{n+1}^j if and only if $\mathfrak{L}(n+1) = \mathfrak{L}(n) + 1$.

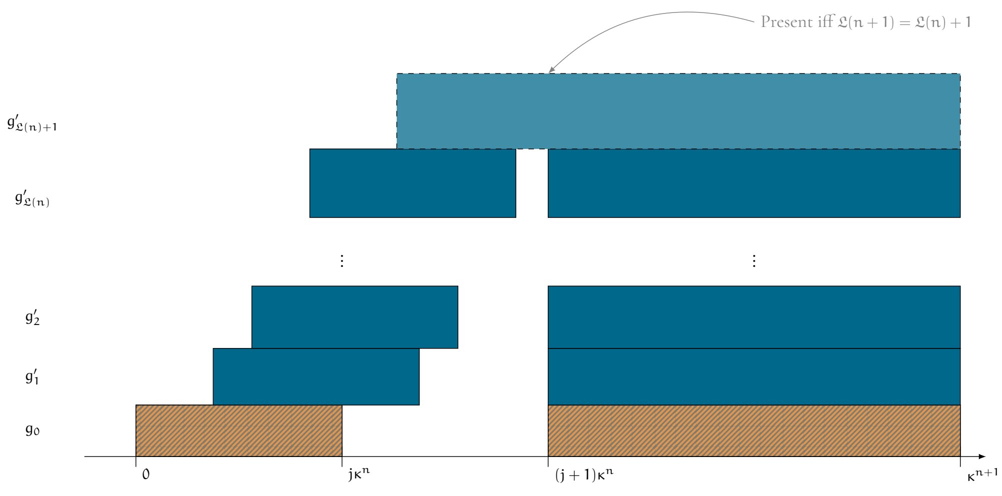


Figure 6: Support and values taken by $(\mathbf{g}, t) \in \Sigma_n^j$

We conclude with some remarks on the cursor of elements of Σ_n. Recall (see page 8) that $\pi_2 : \Delta \to \mathbb{Z}$ denotes the map that sends an element of Δ to its cursor.

Remark 4.2 (Cursor of tiling sequences)**.** First note that by definition of Σ_0 we have $\pi_2(\sigma) = 0$ for all $\sigma \in \Sigma_0$; Now let $n \in \mathbb{N}^*$ and note that if $\sigma \in \Sigma_n$, then $\pi_2(\sigma) = j\kappa^n$, with $j \in \{0, \dots, \kappa - 1\}$.

In particular, let $(\mathbf{f}, t) \in T_n$ and let $(\sigma_i)_{0 \leqslant i \leqslant n}$ denote the (unique) sequence such that $(\mathbf{f}, t) = \sigma_n \cdots \sigma_0$ and $\sigma_i \in \Sigma_i$ for all $i \in \{0, \dots, n\}$. Then $\pi_2(\sigma_0) = 0$. Now decompose t in base κ as $t = \sum_{i=0}^{n-1} t_i \kappa^i$. Then for all $i \in \{0, \dots, n-1\}$, we have $\pi_2(\sigma_{i+1}) = t_i \kappa^i$.

4.1.2 Tiling of the lamplighter group

In the following we endow $H = (\mathbb{Z}/m\mathbb{Z}) \wr \mathbb{Z}$ with the following finite, symmetric generating set: $S_H := \{(\mathbf{e}, \pm 1)\} \cup \left\{(a\mathbf{1}_0, 0) \mid a \in \mathbb{Z}/m\mathbb{Z}\right\}$. Recall that a (right) Følner sequence of H is given by

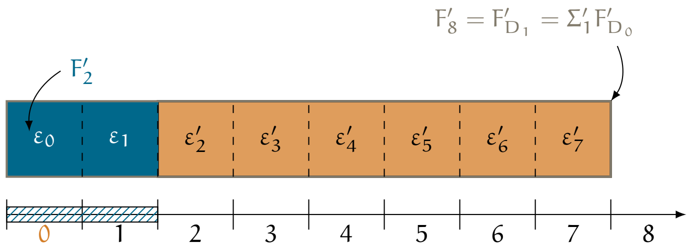
$$F'_n := \left\{\left((\varepsilon_i)_i, t\right) : \ t \in \{0, n-1\}, \ \mathrm{supp}\left((\varepsilon_i)_i\right) \subseteq [0, n-1]\right\}.$$

Our goal here is to extract a subsequence of $(F'_n)_{n\in\mathbb{N}}$ to define a *tiling* for our group H. So let $(d_n)_{n\in\mathbb{N}}$ be a sequence of integers and define $\Sigma'_0 := F'_{d_0}$. Now let $D_n := \prod_{i=0}^{n} d_i$ and consider for all $n \geqslant 0$

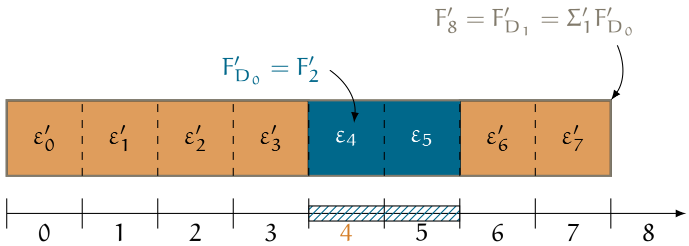
$$\Sigma'_{n+1} := \bigsqcup_{j=0}^{d_{n+1}-1} \left\{\left((\varepsilon_i)_i, jD_n\right) \mid \mathrm{supp}\left((\varepsilon_i)_i\right) \subseteq \left[0, jD_n - 1\right] \cup \left[(j+1)D_n,\ D_{n+1} - 1\right]\right\}. \tag{4.4}$$

These sets will be the shifts of the wanted Følner tiling sequence. This is what Lemma 4.4, page 17 formalises, but first let us give some illustration of this tiling.

Example 4.3. Assume that $d_0 = 2$ and $d_1 = 4$, then $D_0 = 2$ and $D_1 = 8$. Now consider $\left((\varepsilon_i)_i, t\right) \in F'_{D_0}$ and $\left((\varepsilon'_i)_i, jD_0\right) \in \Sigma'_1$. We represent the product of these two elements in Figure 7 for $j = 0$ (Figure 7a) or $j = 2$ (Figure 7b). The dark blue squares correspond to lamp configurations coming from the element in F_{D_0} while the orange ones are coming from the shift. The cursor of this product, namely $t + jD_1$, belongs to the hatched blue rectangle.



(a) Representation of $\left((\varepsilon'_i)_i, 0\right)\left((\varepsilon_i)_i, t\right)$



(b) Representation of $\left((\varepsilon'_i)_i, 2D_1\right)\left((\varepsilon_i)_i, t\right)$

Figure 7: Tiling of the lamplighter

Let us now prove that it actually defines a Følner tiling sequence.

Lemma 4.4

Let $(d_n)_{n\in\mathbb{N}}$ be a sequence of positive integers and for all $n \in \mathbb{N}$, let $D_n = \prod_{i=0}^{n} d_i$. Finally let $(\Sigma'_n)_{n\in\mathbb{N}}$ be as above.

Then $(\Sigma'_n)_{n\in\mathbb{N}}$ is a (right) Følner tiling sequence and $F'_{D_{n+1}} = \Sigma'_{n+1} F'_{D_n}$, for all $n \in \mathbb{N}$.

Proof. Let $(d_n)_{n\in\mathbb{N}}$ be a sequence of positive integers and for all $n \in \mathbb{N}$ let $D_n := \prod_{i=0}^{n} d_n$. Let (Σ'_{n+1}) be defined as in Equation (4.4).

Step 1 Let us prove that $F'_{D_{n+1}} = \Sigma'_{n+1}F'_{D_n}$, for all $n \in \mathbb{N}$.

- We first show that for all $n \in \mathbb{N}$, $\Sigma'_{n+1}F'_{D_n}$ is contained in $F'_{D_{n+1}}$.
  Let $\big((\varepsilon'_i)_i, t\big)$ in F'_{D_n} and $j \in [0, d_{n+1}-1]$ and take $\big((\varepsilon_i)_i, jD_n\big) \in \Sigma'_{n+1}$. Then

$$\big((\varepsilon_i)_i, jD_n\big)\big((\varepsilon'_i)_i, t\big) = \Big(\big(\varepsilon_i + \varepsilon'_{i-jD_n}\big)_i,\ t + jD_n\Big)$$

  By the definition of Σ'_{n+1} given in Equation (4.4) and since $(\varepsilon'_i)_i$ is supported on $[0, D_n-1]$, we have

$$\begin{aligned}\operatorname{supp}\Big(\big(\varepsilon_i + \varepsilon'_{i-jD_n}\big)_{i\in\mathbb{Z}}\Big) &\subseteq \big[0, jD_n - 1\big] \cup \big[(j+1)D_n,\ D_{n+1}-1\big] \cup [jD_n, (j+1)D_n - 1],\\ &= [0, D_{n+1}-1]\,.\end{aligned}$$

  Finally, since $t \leqslant D_n - 1$ and $j \leqslant d_{n+1} - 1$, we get

$$jD_n + t \leqslant (d_{n+1}-1)D_n + D_n - 1 \leqslant D_n d_{n+1} - 1 = D_{n+1} - 1.$$

  Thus $\big((\varepsilon_i)_i, jD_n\big)\big((\varepsilon'_i)_i, t\big)$ belongs to $F'_{D_{n+1}}$.
- Now take $\big((\omega_i)_i, t\big) \in F'_{D_{n+1}}$ and let us show that $\big((\omega_i)_i, t\big)$ belongs to $\Sigma'_{n+1}F'_{D_n}$.
  First, remark that since $t \leqslant D_{n+1} - 1$, and $D_{n+1} = d_{n+1}D_n$, there exists a unique $0 \leqslant j \leqslant d_{n+1} - 1$ such that $jD_n \leqslant t \leqslant (j+1)D_n - 1$. For such a j, let$t' := t - jD_n$ and let $(\varepsilon_i)_i$ and $(\varepsilon'_i)_i$ be such that

$$\varepsilon_i = \begin{cases}\omega_i & \text{if } i \in [0, jD_n - 1] \cup [(j+1)D_n, D_{n+1}-1],\\ \mathbf{e} & \text{else,}\end{cases}$$

$$\varepsilon'_i = \begin{cases}\omega_{i+jD_n} & \text{if } i \in [0, D_n - 1],\\ \mathbf{e} & \text{else.}\end{cases}$$

  Then $\big((\varepsilon_i)_i, jD_n\big) \in \Sigma'_{n+1}$ and $\big((\varepsilon'_i)_i, t'\big) \in F'_{D_n}$ and they verify $\big((\varepsilon_i)_i, jD_n\big)\big((\varepsilon'_i)_i, t'\big) = \big((\omega_i)_i, t\big)$. Hence the equality of the lemma.

Step 2 Let us prove that $(\Sigma'_n)_{n\in\mathbb{N}}$ is a right Følner tiling sequence.

- First, the sequence of tiles $(F'_{D_n})_n$ is a subsequence of the right Følner sequence $(F'_n)_n$ of H. Therefore, it is itself a right Følner sequence.
- We now show that $\sigma F'_{D_n} \cap \sigma' F'_{D_n} = \emptyset$ for all $\sigma \neq \sigma' \in \Sigma'_{n+1}$.
  So take $\big((\varepsilon_i)_i, jD_n\big)$ and $\big((\varepsilon'_i)_i, j'D_n\big)$ in Σ'_{n+1} and take $((\omega_i)_i, t)$ and $((\omega'_i)_i, t')$ in F'_{D_n}. If

$$\big((\varepsilon_i)_i, jD_n\big)((\omega_i)_i, t) = \big((\varepsilon'_i)_i, j'D_n\big)((\omega'_i)_i, t'), \tag{4.5}$$

  then in particular $t + jD_n = t' + j'D_n$. But $t, t' < D_n$ thus the last equality implies $t = t'$ and therefore $j = j'$. In particular $(\varepsilon_i)_i$ and $(\varepsilon'_i)_i$ are supported on the same set, namely $[0, jD_n - 1] \cup [(j+1)D_n, D_{n+1}-1]$. This last set is disjoint from $[jD_n, (j+1)D_n - 1]$ which is the interval where $(\omega_{i-jD_n})_i$ and $(\omega_{i-jD_n})_i$ are supported. Combining this with Equation (4.5) we thus get that $\varepsilon_i = \varepsilon'_i$ for all i. Hence the result. □

We conclude with the following lemma.

Lemma 4.5

Let $(d_n)_{n\in\mathbb{N}}$ be a sequence of positive integers and for all $n \in \mathbb{N}$, let $D_n = \prod_{i=0}^{n} d_i$. Let $(\Sigma'_n)_{n\in\mathbb{N}}$ be as in Equation (4.4), page 17. Then, the sequence of tiles $(F_{D_n})_n$ verifies $\operatorname{diam}\left(F'_n\right) \leqslant 3D_n$ for all $n \in \mathbb{N}$. Moreover, for all $n \in \mathbb{N}$ we have

$$\left|F'_{D_n}\right| = D_n m^{D_n} \quad \text{and} \quad \left|\Sigma'_{n+1}\right| = (d_{n+1}-1)D_n m^{(d_{n+1}-1)D_n}.$$

Proof. By choice of the generating set S_H, the bound on the diameter is immediate.

Let $n \in \mathbb{N}$ and consider $\left((\varepsilon_i)_i, t\right) \in F'_{D_n}$. By definition of F'_{D_n} (see page 17), there are exactly D_n possible values for the cursor t, and m^{D_n} possible values for the sequence $(\varepsilon_i)_i$. Therefore F'_{D_n} contains exactly $D_n m^{D_n}$ elements. Furthermore, since $F'_{D_{n+1}} = \Sigma'_{n+1}F'_{D_n}$, the shift Σ'_{n+1} contains therefore $(d_{n+1}-1)D_n m^{(d_{n+1}-1)D_n}$ elements. ☐

We thus know how to build Følner tiling sequences for H. Now, we have to specify the sequence $(d_n)_{n\in\mathbb{N}}$ such that the obtained tiling will give an appropriate Sofic approximation for our coupling.

4.1.3 Defining the sofic approximations

In the following we denote by $(\Sigma_n)_n$ the Følner tiling sequence of Δ defined in Section 4.1.1, page 15 and by $(T_n)_n$ the corresponding tiling of Δ (see Equation (4.1), page 15). In particular it verifies $T_{n+1} = \Sigma_{n+1}T_n$ for all $n \in \mathbb{N}$.

Given a sequence $(d_n)_n$ of integers we also let $D_n := \prod_{i=0}^{n} d_i$ and denote by $(\Sigma'_n)_n$ the corresponding Folner tiling sequences defined in Equation (4.4). The goal is to define $(d_n)_n$ such that one can embedd Σ_n in Σ'_n for all n.

So for all $n \in \mathbb{N}$, let $\mathcal{G}_n := T_n$, and now let us define inductively the Sofic approximation $\mathcal{H}_n$ of H. First, let $d_0 := \min\{j : |T_0| \leqslant |F'_{d_0}|\}$ and $\mathcal{H}_0 := F'_{d_0}$. By definition of the Følner tiling sequences we have $F'_{d_0} = \Sigma'_{d_0}$ and $\Sigma_0 = T_0$, thus $|\Sigma_0| \leqslant |\Sigma'_0|$ and moreover, by definition of d_0 and Lemma 4.5, page 19, we have

$$(d_0-1)m^{d_0-1} < |\Sigma_0| \leqslant d_0 m^{d_0} \tag{4.6}$$

Then let $n \geqslant 0$ and assume that we have defined the sequence $(d_i)_{0\leqslant i\leqslant n}$ and $\mathcal{H}_n = F'_{D_n}$. Let d_{n+1} be the minimal integer such that the set Σ'_{n+1} defined in Equation (4.4) contains at least $|\Sigma_{n+1}|$ elements, *viz.*

$$(d_{n+1}-1)\, m^{D_n(d_{n+1}-2)} \leqslant |\Sigma_{n+1}| \leqslant d_{n+1} m^{D_n(d_{n+1}-1)}. \tag{4.7}$$

Remark that in particular, one can embed Σ_{n+1} in Σ'_{n+1}. Finally let $\mathcal{H}_{n+1} = F'_{D_{n+1}}$. It defines by induction a sequence $(\mathcal{H}_n)_{n\in\mathbb{N}}$, which is a Sofic approximation of L_m since it is a sub-sequence of a Følner sequence. We refer to table 1, page 20 for a summary of the different objects defined above.

4.2 Quantification

The purpose if this section is to verify the criterion given by Theorem 3.9, page 14.

$\mathcal{G}_0 = \Sigma_0 = T_0$	$\mathcal{G}_{n+1} = T_{n+1} = \Sigma_{n+1} T_n$
$d_0 := \min\{j : \lvert T_0 \rvert \leqslant \lvert F'_{d_0} \rvert\}$	$\mathcal{H}_0 := F'_{d_0} = \Sigma'_0$
d_{n+1} defined by Equation (4.7), page 19 Σ'_{n+1} defined by Equation (4.4), page 17	$D_{n+1} = d_{n+1} D_n = \prod_{i=0}^{n+1} d_i$ $\mathcal{H}_{n+1} = F'_{D_{n+1}} = \Sigma'_{n+1} F'_{n+1}$

Table 1: Definition of the sofic approximations

4.2.1 Definition of the injection

Let us define, for all $n \in \mathbb{N}$, the injection ι_n, embedding $\mathcal{G}_n$ in $\mathcal{H}_n$.

First remark that, by definition of $(\Sigma'_n)_n$, we have $|\Sigma_n| \leqslant |\Sigma'_n|$, for all $n \in \mathbb{N}$ (see Equation (4.7), page 19), and there thus exists an injection ν_n from Σ_n to Σ'_n. From now on, we assume fixed such a sequence of (arbitrarily chosen) injections $(\nu_n)_n$.

Now let $n \in \mathbb{N}$. Since $\left(\Sigma_i\right)_{i \in \mathbb{N}}$ is a Følner tiling sequence, one can write every element of $\mathcal{G}_n$ as a product $\sigma_n \cdots \sigma_0$ where $\sigma_i \in \Sigma_i$ is uniquely determined for all i. Thus we can define without ambiguity the following map ι_n.

Lemma 4.6

Let $n \in \mathbb{N}$. The map defined by

$$\iota_n : \begin{cases} \mathcal{G}_n = T_n & \to \mathcal{H}_n = F'_{D_n}, \\ \sigma_n \cdots \sigma_0 & \mapsto \nu_n(\sigma_n) \cdots \nu_0(\sigma_0). \end{cases}$$

where $\sigma_i \in \Sigma_i$ for all $i \leqslant n$, is an injection from $\mathcal{G}_n$ to $\mathcal{H}_n$.

Proof. Let $n \in \mathbb{N}$. The map ι_n thus defined does not depend on the choice of the decomposition of an element of T_n in a product of shifts. Indeed, by the preceding discussion, there is only one choice possible for the aforementioned decomposition.

Now let $x, x' \in \mathcal{G}_n$. Let $x = \sigma_n \cdots \sigma_0$ and $x' = \sigma'_n \cdots \sigma'_0$ be the (unique) decompositions in product of shifts of x and x'. In particular, for all $i \in \{0, \ldots, n\}$ define σ_i, σ'_i are elements of Σ_i. Then by definition of ι_n we have $\iota_n(x) = \prod_{i=0}^n \nu_i(\sigma_i)$ and $\iota_n(x') = \prod_{i=0}^n \nu_i\left(\sigma'_i\right)$. But $\nu_i(\sigma_i)$ and $\nu_i(\sigma'_i)$ belong to Σ'_i for all i, thus $\prod_{i=0}^n \nu_i(\sigma_i)$ is the decomposition of $\iota_n(x)$ and $\prod_{i=0}^n \nu_i(\sigma_i)$ the decomposition of $\iota_n(x')$ in product of shifts. Since $(\Sigma'_n)_n$ is a Følner tiling shift, this decomposition is unique. Thus if $\iota_n(x) = \iota_n(x')$ then $\nu_i(\sigma_i) = \nu_i(\sigma'_i)$ for all i. Hence $\sigma_i = \sigma'_i$ since ν_i is a bijection for all i, and therefore $x = x'$. Hence the injectivity of ι_n. ☐

In order to verify Equation (3.1), page 14, we now need to estimate the values taken by the distance between $\iota_n\big((\mathbf{f}, t)\big)$ and $\iota_n\big((\mathbf{f}, t) \cdot s\big)$ when $(\mathbf{f}, t)$ belongs to Δ and $s \in S_\Delta$.

4.2.2 Distance

The goal of this subsection is to give an upper bound to the distance between $\iota_n(\mathbf{f}, t)$ and $\iota_n\big((\mathbf{f}, t)s\big)$. We distinguish two cases depending on whether $s = (\mathbf{e}, 1)$ or not. But first let us introduce some notations.

Let $t \in \{0, \ldots, \kappa^n - 1\}$ and let $t = \sum_{i=0}^{n-1} t_i \kappa^i$ be the decomposition in base κ of t. If $(\mathbf{f}, t)$ belongs to $\mathcal{G}_n^{(1)}$ then $t < \kappa^n - 1$ and thus there exists $i \in \{0, \ldots, n-1\}$ such that $t_i < \kappa - 1$. Therefore, we can define

$$i_0(t) := \min\{i \leqslant n \mid t_i < \kappa - 1\}. \tag{4.8}$$

This index corresponds to the one of the coefficient t_i that will absorb the carry when we add one to t. In other words, the decomposition of $t+1$ in base κ is given by $t+1 = (t_{i_0(t)}+1)\kappa^{i_0(t)} + \sum_{i=i_0(t)}^{n-1} t_i\kappa^i$. Finally, let us recall that $\mathcal{G}_n^{(r)}$ is defined as $\mathcal{G}_n^{(r)} = \left\{x \in \mathcal{G}_n \mid B_{\mathcal{G}_n}(x,r) \simeq B_G(e_G, r)\right\}$.

Lemma 4.7

Let $n \in \mathbb{N}$ and $s \in \mathcal{S}_\Delta$. For all $t \in \{0,\dots,\kappa^n-1\}$, let $i_0(t)$ be as in Equation (4.8) Then for all $(\mathbf{f},t) \in \mathcal{G}_n^{(1)}$ we have

$$d\left(\iota_n\big((\mathbf{f},t)\big), \iota_n\big((\mathbf{f},t)\cdot s\big)\right) \leqslant \begin{cases} 3D_0 & \text{if } s \neq (\mathbf{e},1), \\ 3D_{i_0(t)} & \text{if } s = (\mathbf{e},1). \end{cases}$$

The following proof relies on the definition of the range given in Definition 2.5, page 8.

Proof. Recall that $\mathcal{G}_n = T_n$ where T_n is tiled by the sequence of shifts $(\Sigma_i)_{i=0,\dots,n}$. In particular (see Definition 3.10, page 14) $T_n = \Sigma_n\cdots\Sigma_0$ and since $(\mathbf{f},t) \in T_n$, there exists a unique sequence $(\sigma_i)_{0\leqslant i\leqslant n}$ such that $(\mathbf{f},t) = \sigma_n\cdots\sigma_0$ and $\sigma_i \in \Sigma_i$ for all $i \in \{0,\dots,n\}$. Similarly, we denote by $(\sigma'_n)_{0\leqslant i\leqslant n}$ the unique sequence such that $(\mathbf{f},t)s = \sigma'_n\cdots\sigma'_0$ and $\sigma'_i \in \Sigma_i$ for all i.

First case Let $s \in \mathcal{S}_\Delta\backslash\{(\mathbf{e},1)\}$.

- Let us first prove that $\sigma_0 s$ belongs to T_0.
  Since $\sigma_0 \in \Sigma_0 = T_0$ it verifies $\text{range}(\sigma_0) = \{0\}$. Moreover, since s belongs to $\mathcal{S}_\Delta\backslash\{(\mathbf{e},1)\}$ its cursor is equal to 0. Therefore, by definition of the range, we have $\text{range}(\sigma_0 s) = \{0\}$. Hence $\sigma_0 s \in T_0$.
- Let us now show that $\sigma_i = \sigma'_i$ for all $i > 0$.
  Remark that
  $$(\mathbf{f},t)s = \sigma_n\cdots\sigma_0 s = \sigma_n\cdots\sigma_1(\sigma_0 s).$$
  But by the last point $\sigma_0 s$ belongs to Σ_0, thus the above equality gives a decomposition of $(\mathbf{f},t)s$ in a product of shifts. By uniqueness of this decomposition (see below Definition 3.10, page 14) we thus have $\sigma'_0 = \sigma_0\cdot s$ and $\sigma_i = \sigma'_i$ for all $i > 0$.
- Let us now bound by above the distance between $\iota_n\big((\mathbf{f},t)\big)$ and $\iota_n\big((\mathbf{f},t)s\big)$.
  Using the definition of ι_n given by Lemma 4.6, page 20, then the above point, and finally Lemma 4.5, page 19, we obtain that
  $$\begin{aligned} d\left(\iota_n\big((\mathbf{f},t)\big), \iota_n\big((\mathbf{f},t)s\big)\right) &= d\big(\nu_0(\sigma_0), \nu_0(\sigma_0 s)\big), \\ &\leqslant \text{diam}(\mathcal{H}_0) = 3D_0. \end{aligned}$$

Second case Now assume that $s = (\mathbf{e},1)$.

- Let us prove that $\sigma_{i_0(t)}\cdots\sigma_0\cdot(\mathbf{e},1)$ belongs to $T_{i_0(t)}$.
  First recall (see page 8) that $\pi_2 : \Delta \to \mathbb{Z}$ denotes the map that sends an element of Δ to its cursor. Using Remark 4.2, page 16 we get that $\pi_2(\sigma_0) = 0$ and $\pi_2(\sigma_{i+1}) = t_i\kappa^{i-1}$ for all $i \in \{0,\dots,n-1\}$. Therefore
  $$\pi_2\left(\sigma_{i_0(t)}\cdots\sigma_0\right) = \sum_{i=0}^{i_0(t)} t_i\kappa^{i-1}.$$
  But, by definition of $i_0(t)$, we have $t_i = \kappa-1$ for all $i < i_0(t)$ and $t_{i_0(t)} < \kappa-1$, thus

$\pi_2\left(\sigma_{i_0(t)}\cdots\sigma_0\right) < \kappa^{i_0(t)} - 1$. Hence

$$\pi_2\left(\sigma_{i_0(t)}\cdots\sigma_0(\mathbf{e},1)\right) = \pi_2\left(\sigma_{i_0(t)}\cdots\sigma_0\right) + 1 \leqslant \kappa^{i_0(t)} - 1.$$

Since $\sigma_{i_0(t)}\cdots\sigma_0$ belongs to $T_{i_0(t)}$, its range is included in $[0, \kappa^{i_0(t)} - 1]$. Thus, using the above inequality we get

$$\begin{aligned}\operatorname{range}\left(\sigma_{i_0(t)}\cdots\sigma_0(\mathbf{e},1)\right) &\subseteq \operatorname{range}\left(\sigma_{i_0(t)}\cdots\sigma_0\right)\cup\left\{\pi_2\left(\sigma_{i_0(t)}\cdots\sigma_0(\mathbf{e},1)\right)\right\}\\ &\subseteq [0,\kappa^{i_0(t)}-1],\end{aligned}$$

namely $\sigma_{i_0(t)}\cdots\sigma_0(\mathbf{e},1)$ belongs to $T_{i_0(t)}$.

- Let us now show that $\sigma_i = \sigma_i'$ for all $i > i_0(t)$.
  By the above point there exists a sequence $(\bar\sigma_i)_{0\leqslant i\leqslant i_0(t)}$ such that $\sigma_{i_0(t)}\cdots\sigma_0 s = \bar\sigma_{i_0(t)}\cdots\bar\sigma_0$ and $\bar\sigma_i \in \Sigma_i$ for all $i \leqslant i_0(t)$. Therefore

$$\begin{aligned}(\mathbf{f},t)s = \sigma_n\cdots\sigma_0 s &= \sigma_n\cdots\sigma_{i_0(t)+1}\left(\sigma_{i_0(t)}\cdots\sigma_0 s\right),\\ &= \sigma_n\cdots\sigma_{i_0(t)+1}\left(\bar\sigma_{i_0(t)}\cdots\bar\sigma_0\right),\end{aligned}$$

  where $\bar\sigma_i$ belongs to Σ_i for all $i \leqslant i_0(t)$ and σ_i belongs to Σ_i for all $i > i_0(t)$. Hence the above equality gives a decomposition of $(\mathbf{f},t)s$ in a product of shifts. Using once more the uniqueness of this decomposition (see below Definition 3.10, page 14) we thus have $\sigma_i' = \bar\sigma_i$ for all $i \leqslant i_0(t)$ and $\sigma_i = \sigma_i'$ for all $i > i_0(t)$.
- Using the definition of ι_n given by Lemma 4.6, page 20, then the above point, and finally Lemma 4.5, page 19, we obtain that

$$d\left(\iota_n\left((\mathbf{f},t)\right),\iota_n\left((\mathbf{f},t)s\right)\right) \leqslant \operatorname{diam}(\mathcal{H}_{i_0(t)}) = 3D_{i_0(t)}. \qquad \square$$

We conclude with an estimate of the proportion of elements in $\mathcal{G}_n$ verifying $i_0(t) = i$ for some fixed $i \in \{0,\dots,n-1\}$.

Lemma 4.8

> Let $n \in \mathbb{N}$ and $i_0(t)$ be as in Equation (4.8). Then, for all $i \in \{0,\dots,n-1\}$, we have
>
> $$\left|\left\{(\mathbf{f},t)\in\mathcal{G}_n \;:\; i_0(t) = i\right\}\right| = \frac{(\kappa-1)}{\kappa^i}\left|\mathcal{G}_n\right|.$$

Proof. So let $i \in \{0,\dots,n-1\}$. As above, let $t = \sum_{j=0}^{n-1} t_j\kappa^j$ be the decomposition in base κ of t. For a given $j \in \{0,\dots,n-1\}$, the digit t_j can uniformly take κ different values, namely any value between zero and $\kappa - 1$. Thus, the proportion of $(\mathbf{f},t)\in\mathcal{G}_n$ verifying that $t_i < \kappa - 1$ and $t_j = \kappa - 1$ for all $j < i$ is equal to $(\kappa-1)\kappa^{-i}$. Hence the lemma. $\square$

4.2.3 Integrability

We now turn to the proof of the quantification. We prove that the sequences $(\mathcal{G}_n)_n$ and $(\mathcal{H}_n)_n$ defined before verify, for all $\varepsilon > 0$, the condition of Theorem 3.9, page 14 with $\varphi = \rho^{1-\varepsilon}$. First, we need to bound by above the value of D_n.

Lemma 4.9

> There exists $C_\Delta' \geqslant 1$ depending only on Δ, such that $D_n \leqslant C_\Delta'\kappa^n l_{\mathfrak{L}(n)}$, for all $n \in \mathbb{N}$.

Proof. We start by proving some inequalities and then show the upper bound by induction on n.

Useful remarks

1. First, note that since $m \geqslant 2$ we have $\ln(m) \geqslant 1/2$ and therefore $1/\ln(m) \leqslant 2$.
2. By definition of Σ_0, and item 1 of hypotheses **(H)**, page 7, we have $|\Sigma_0| = q = 6$. Therefore $\ln|\Sigma_0| = \ln(q) \geqslant 1$.
3. Let us show that $d_{n+1} \geqslant 2$ for all $n \in \mathbb{N}$.
   Let $n \in \mathbb{N}$ and assume towards a contradiction that $d_{n+1} = 1$. By the right most inequality of Equation (4.7), page 19, this implies $|\Sigma_{n+1}| \leqslant m^{d_0 \cdots d_n \cdot 0} = 1$. But the value of $|T_i|$ is strictly increasing. Since $T_{n+1} = \Sigma_n T_n$ this forces Σ_{n+1} to contain at least 2 elements. Hence the contradiction.
4. Let us now prove that, for all $n \in \mathbb{N}^*$ we have $\ln|\Sigma_n| \leqslant C_\Delta \kappa^n l_{\mathfrak{L}(n)}$.
   Let $n \in \mathbb{N}$. Recall that the sequence of shifts $(\Sigma_i)_i$ verifies $T_{n+1} = \Sigma_{n+1} T_n$. In particular $|\Sigma_{n+1}| = |T_{n+1}|/|T_n|$. Combining this with the upper bound in Equation (4.3), page 16, leads
   $$\ln|\Sigma_{n+1}| = \ln|T_{n+1}| - \ln|T_n| \leqslant \ln|T_{n+1}| \leqslant C_\Delta \kappa^{n+1} l_{\mathfrak{L}(n+1)}.$$

Now let $C'_\Delta := \max\left\{3\ln(q), 6C_\Delta\right\}$. Remark in particular that $C'_\Delta \geqslant 1$ (see the above item 2). We now aim to show that $D_n \leqslant C'_\Delta \kappa^n l_{\mathfrak{L}(n)}$, for all $n \in \mathbb{N}$

Base case Let us first treat the case where $n = 0$.

- Remark that by Equation (4.6), page 19 we have $(d_0 - 1)m^{d_0 - 1} \leqslant |\Sigma_0|$. Therefore, using first the latter inequality, then that $\ln|\Sigma_0| \geqslant 1$ (item 2), then that $1/\ln(m) \leqslant 2$ (item 1), leads
  $$d_0 \leqslant \frac{\ln|\Sigma_0|}{\ln(m)} + 1 \leqslant \ln|\Sigma_0| \left(\frac{1}{\ln(m)} + 1\right) \leqslant 3\ln|\Sigma_0|.$$

Using that $|\Sigma_0| = q$ (item 2) and recalling that $D_0 = d_0$ thus gives $D_0 \leqslant 3\ln|\Sigma_0| = 3\ln(q)$. The wanted inequality then comes noting that $C'_\Delta \geqslant 3\ln(q)$ and that $\kappa^0 l_{\mathfrak{L}(0)} \geqslant \kappa^0 \lambda^0 = 1$.

Induction Now let $n \in \mathbb{N}$ and assume that $D_n \leqslant C'_\Delta \kappa^n l_{\mathfrak{L}(n)}$. Recall that we showed (item 3) that $d_{n+1} \geqslant 2$. We distinguish two cases depending on whether $d_{n+1} = 2$ or $d_{n+1} \geqslant 3$.

- If $d_{n+1} = 2$, then $D_{n+1} = 2D_n$. Using our assumption on D_n, and that $\kappa \geqslant 3$, we obtain
  $$D_{n+1} = 2D_n \leqslant 2 \cdot C'_\Delta \kappa^n l_{\mathfrak{L}(n)} \leqslant C'_\Delta \kappa^{n+1} l_{\mathfrak{L}(n)}$$
  We obtain the wanted inequality by recalling that $\mathfrak{L}(n) \leqslant \mathfrak{L}(n+1)$ (see Claim 4.1, page 15). Since $(l_i)_i$ is a subsequence of a geometric sequence (see item 5 of **(H)**, page 7), in particular it is non-decreasing. Thus $l_{\mathfrak{L}(n)} \leqslant l_{\mathfrak{L}(n+1)}$ and hence we obtain the wanted inequality in this case.
- Assume now that $d_{n+1} \geqslant 3$. Taking the logarithm of the left most inequality of Equation (4.7), page 19 then leads $D_n(d_{n+1} - 2)\ln(m) \leqslant \ln|\Sigma_{n+1}|$. Since $D_{n+1} = d_{n+1} D_n$ we deduce
  $$D_{n+1} \leqslant \frac{\ln|\Sigma_{n+1}|}{\ln(m)} \frac{d_{n+1}}{(d_{n+1} - 2)}.$$
  But since $d_{n+1} \geqslant 3$, we have $d_{n+1}/(d_{n+1} - 2) \leqslant 3$. Recalling that $1/\ln(m) \leqslant 2$ (item 1 of the above useful remarks) thus leads $D_{n+1} \leqslant 6\ln|\Sigma_{n+1}|$. Then applying item 4 of the above remarks, and using that $C'_\Delta \geqslant 6C_\Delta$, implies
  $$D_{n+1} \leqslant 6C_\Delta \kappa^{n+1} l_{\mathfrak{L}(n+1)} \leqslant C'_\Delta \kappa^{n+1} l_{\mathfrak{L}(n+1)}. \quad \square$$

We now turn to the proof of our main result.

Proof of Theorem 1.3. So let $\rho \in \mathcal{C}$, let $\varepsilon > 0$ and define $\varphi_\varepsilon = \rho^{1-\varepsilon}$. Consider Δ verifying the conditions **(H)** page 7 and such that $I_\Delta \sim \rho \circ \log$. Let $G := \Delta$ and $H := (\mathbb{Z}/m\mathbb{Z}) \wr \mathbb{Z}$. Let $(\mathcal{G}_n)_n$ and $(\mathcal{H}_n)_n$ be as defined in Section 4.1, page 15

Consider $s \in S_\Delta$ and fix $R > 0$ and $n \in \mathbb{N}$. We distinguish two cases depending on whether $s = (\mathbf{e}, 1)$ or not

First case Let $s \in \mathcal{S}_\Delta \backslash \{\mathbf{e}, 1\}$. By Lemma 4.7, page 21 the distance between $\iota_n(\mathbf{f}, t)$ and $\iota_n\left((\mathbf{f}, t)s\right)$ is bounded on $\mathcal{G}_n$ by $3D_0$. Therefore

$$\sum_{r=0}^{R} \varphi_\varepsilon(r) \frac{\left|\left\{(\mathbf{f}, t) \in \mathcal{G}_n^{(1)} \mid d_{\mathcal{H}_n}(\iota_n(\mathbf{f}, t), \iota_n\left((\mathbf{f}, t)s\right) = r\right\}\right|}{|\mathcal{G}_n|} \leqslant \varphi_\varepsilon(3D_0) < +\infty.$$

Since this upper bound does not depend on R nor on n, the condition of Equation (3.1), page 14 is verified in this case.

Second case We now assume that $s = (\mathbf{e}, 1)$. By Lemma 4.7, page 21 and Lemma 4.8, page 22, we have

$$\sum_{r=0}^{R} \varphi_\varepsilon(r) \frac{\left|\left\{(\mathbf{f}, t) \in \mathcal{G}_n^{(1)} \mid d_{\mathcal{H}_n}(\iota_n(\mathbf{f}, t), \iota_n\left((\mathbf{f}, t) \cdot s\right) = r\right\}\right|}{|\mathcal{G}_n|} \leqslant \sum_{i=0}^{n} \varphi_\varepsilon(3D_i) \frac{\kappa - 1}{\kappa^i}.$$

But, when $i > 0$, by Lemma 4.9, page 22 there exists a constant $C'_\Delta \geqslant 1$ such that $D_i \leqslant C'_\Delta \kappa^i l_{\mathfrak{L}(i)}$. Therefore, using that ρ is non-decreasing, then Equation (2.2), page 9, we obtain that for all $i > 0$

$$\rho(3D_i) \leqslant \rho\left(3C'_\Delta \kappa^i l_{\mathfrak{L}(i)}\right) \leqslant 3C'_\Delta \rho\left(\kappa^i l_{\mathfrak{L}(i)}\right).$$

Now, recall that the map $\bar{\rho}$ given by Lemma 2.13, page 10 verifies $\bar{\rho} \sim \rho$ and $\bar{\rho}\left(\kappa^i l_{\mathfrak{L}(i)}\right) = \kappa^i$. In other word, there exists some constant $c \geqslant 1$ depending only on Δ such that $\rho\left(\kappa^i l_{\mathfrak{L}(i)}\right) \leqslant c\kappa^i$. Combining these inequalities and recalling that $\varphi_\varepsilon = \rho^{1-\varepsilon}$, we thus get

$$\begin{aligned}\sum_{i=0}^{n} \varphi_\varepsilon\left(3D_i\right) \frac{\kappa-1}{\kappa^i} &\leqslant \rho(3D_0)^{1-\varepsilon}(\kappa - 1) + \sum_{i=1}^{n} \left(3C'_\Delta c\kappa^i\right)^{(1-\varepsilon)} \frac{\kappa-1}{\kappa^i}, \\ &= \rho(3D_0)^{1-\varepsilon}(\kappa - 1) + (3C'_\Delta c)^{(1-\varepsilon)}(\kappa - 1) \sum_{i=1}^{n} \kappa^{-\varepsilon i}.\end{aligned}$$

Finally, since $\kappa \geqslant 3$, we have $\kappa^{-\varepsilon} < 1$ and thus the sequence $(\kappa^{-\varepsilon i})_i$ is summable. Hence the sum $\sum_{i=1}^{n} \kappa^{-\varepsilon i}$ is bounded by above by a constant that does not depend on n nor on R. Therefore Equation (3.1) is verified.

Conclusion on the integrability. By Theorem 3.9, page 14, there exists an at most 1-to-one φ-integrable measure subgroup coupling from G to H. □

Notation index

$\preccurlyeq, \simeq$ See page 4.
$|X|$ Cardinal of the set X.
∂F Boundary of the set F.
Δ See Definition 2.1.
Δ_m See Section 2.1.
$\mathcal{C}$ See Equation (1.2), page 4.
F'_n A Følner sequence of $(\mathbb{Z}/m\mathbb{Z}) \wr \mathbb{Z}$ (see Section 4.1.2, page 17).
$\mathbf{g}$ The sequence of maps $(g_m)_{m\in\mathbb{N}}$.
$\mathbf{g}'$ The sequence of maps $(g'_m)_{m>0}$.
g'_m See Section 2.1.
$\mathcal{G}_n$ Sofic approximation of G.
$\mathcal{G}_n^{(r)}$ The set $\{x \in \mathcal{G}_n \mid B_{\mathcal{G}_n}(x, r) \simeq B_G(e_G, r)\}$.
Γ'_m Normal closure of $[A_m, B_m]$.
$\mathcal{H}_n$ Sofic approximation of H.
$i_0(t)$ Defined by $i_0(t) := \min\{i \leqslant n \mid t_i < \kappa - 1\}$ (see Equation (4.8)).
I_G Isoperimetric profile of G.
ι_n Injection from $\mathcal{G}_n$ to $\mathcal{H}_n$.
L_m Lamplighter group $(\mathbb{Z}/m\mathbb{Z}) \wr \mathbb{Z}$.
$\mathfrak{l}(n)$ Integer such that $k_{\mathfrak{l}(n)} \leqslant n < k_{\mathfrak{l}(n)+1}$.
$\mathfrak{L}(n)$ Integer such that $k_{\mathfrak{L}(n)} \leqslant \kappa^n - 1 < k_{\mathfrak{L}(n)+1}$, *i.e.* $\mathfrak{L}(n) = \mathfrak{l}(\kappa^n - 1)$.
ν_i Injections from Σ_i to Σ'_i (see page 20);
π_2 Map that send an element $(\mathbf{f}, t) \in \Delta$ to its cursor t (see Section 2.2, page 8)
$\bar{\rho}$ The affine approximation of the map ρ (see Lemma 2.13, page 10).
$\mathrm{range}(\mathbf{f}, t)$ See Section 2.2, page 8.
S_G A generating set of the group G.
Σ_n Shifts in Δ (see page 16)
Σ'_n Shifts in the lamplighter group H (see Equation (4.4), page 17)
T_n Tiles in Δ such that $T_{n+1} = \Sigma_{n+1} T_n$ (see Equation (4.1), page 15)
$\theta_m^A(f_m)$ Natural projection of f_m on A_m (see Section 2.1, page 6).
$\theta_m^B(f_m)$ Natural projection of f_m on B_m (see Section 2.1, page 6).

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