PDF modelling of turbulent mixing in stratified fluids

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Abstract :

We present a phenomenological model for the turbulent mixing of stably stratified fluids. The stratification depends on a scalar, wich can be the salinity or the temperature. We describe the temporal evolution of the whole shape of the scalar distribution. We take into account turbulent diffusion, sedimentation and disspation of fluctuations by turbulent cascades. The parametrization of the problem is discussed. We investigate limit "test" cases to understand the effect of each term. As an application, we present our model predictions in the case of the restratification.

Résumé :

Nous proposons un modèle phénoménologique pour le mélange turbulent d'un fluide stratifié. La stratification dépend d'un scalaire, qui peut être la température, ou la concentration en sel du fluide. Le model donne l'évolution temporelle de la forme de la distribution du scalaire. Il prend en compte les effets de diffusion turbulente, de sédimentation et de dissipation des fluctuations du scalaire par cascade turbulente. Nous discutons la paramétrisation du problème. Des cas limites "test" sont étudiés pour comprendre l'effet de chaque terme. Le modèle est ensuite appliqué au phénomène de la restratification

Key-words :

turbulent mixing ; stratified fluid ; pdf modelling

1 Introduction

Mixing of a stably stratified fluid is a common situation in the atmostphere or the ocean, for instance in the upper layer of the oceans. The thikness of the turbulent layer changes according to wind forcing and heating. The understanding of the physical mechanisms involved in this process is an important issue for parametrization in oceanic modelling Canuto *et al*. (2001).

We propose a statistical approach to this problem, leading to an equation for the temporal evolution of the probability to measure a certain density (salt concentration or temperature) at a given height. The system is closed with an equation for the kinetic energy. Three processes are described in this model i) the turbulent diffusion of the different patches of density : this tends to homogeneize the density of the fluid ii) the sedimentation : this tends to drive back fluid particles at their altitude of equilibrium iii) the effect of the turbulent cascade : this cascade drives large scale structures toward small scales; it is much more efficient than molecular diffusivity to dissipate the fluctuations of density and energy at a given scale.

The interest of this phenomenological approach is that we obtain the evolution of the whole scalar probability distribution. It is then possible to model chemical reactions or other effects that depends on local scalar values.

The buoyancy is defined as $b = (\varrho - \varrho_0)/\varrho_0$ where ϱ is the density of the fluid, ϱ_0 is a reference constant density and g is the acceleration of gravity.

2 Model

For the sake of simplicity, we consider an unidimensional problem, independant of the horizontal coordinates (x, y), without mean flow. Let us consider the probability distribution function (PDF) $\rho(b, z, t)$ to measure a certain buoyancy b at a given altitude and time. It is normalized at each altitude : $\int \rho db = 1$. we will use the convention $< \cdot >$ for ensemble average at a given altitude. Our model describes the temporal evolution of this PDF and of the average turbulent kinetic energy density $e(z, t)$.

$$\partial_t \rho = -\partial_z \mathcal{J} + \mathcal{D} \tag{1}$$

$$\partial_t e = \partial_z \left(\beta l e^{1/2} \partial_z e\right) + \langle b\mathcal{J}\rangle - \alpha l^{-1} e^{3/2} + \mathcal{P} \quad \alpha, \beta \in \mathbb{R} \tag{2}$$

The quantity $\mathcal{J}(b, z, t)$ is the flux of probability, which is zero at the bottom and the top of the physical system. The term $\mathcal{D}$ models the dissipation of scalar fluctuations, while preserving the mean : $\int \mathcal{D} b db = 0$. To satisfy the normalization constraint of the PDF, this model must be such that $\int \mathcal{J} db = 0$ and $\int \mathcal{D} db = 0$. We will give a model of both quantities in the next subsections.

In the energy equation (2), there is a production term $\mathcal{P}$ depending on the process of turbulence generation, a turbulent diffusion term, a dissipation term due to Kolmogoroff cascade, and a term $\langle b\mathcal{J}\rangle(z, t) = \int \mathcal{J} b db$ of exchange between potential and kinetic energy.

To close the system, it is necessary to model the caracteristic lengthscale l of the flow at a given height. In the limit of large Richardson numbers, l is nothing but the buoyancy length scale $\propto e^{1/2}/\sqrt{\partial_z \langle b\rangle}$. In the limit of small Richardson number, l is the integral scale of the flow l_f, related to the large scale forcing of the experiment. We use the same form as in Balmforth *et al.* (1998), that satisfies both limits :.

$$l = \frac{l_f e^{1/2}}{e^{1/2} + \delta l_f \sqrt{\partial_z \langle b\rangle}} \tag{3}$$

Turbulent diffusion and sedimentation If they were no sedimentation effect, the flux would be proportional to the vertical gradient of the PDF : $\mathcal{J} = -K_b \partial_z \rho$. This hypothesis holds for times larger than Lagrangian time scales of turbulence (see Batchelor49). The constant of proportionality is the eddy diffusivity, given by the product of the rms turbulent energy with the integral length scale : $K_b \propto l e^{1/2}$. A similar expression has been used to model the turbulent diffusion of energy in equation (2).

In addition tothis diffusive process, we must take into acount a sedimentation mechanism : if fluid parcels keep their buoyancy, they tend to reach a *reference state*. This state is obtained by sorting all fluid paticles and contructing a configuration with the lowest potential energy possible. Thus, the reference state depends only on the global fluctuations ; in this state, the PDF is $\rho(b, z, t) = \delta(b - B(z))$ where $B(z)$ is an increasing function of the altitude. To reach this state, we consider the action of gravity on a fluid particle of buoyancy b at a given altitude. Let us suppose that the fluid particle "sees" a surrounding fluid having the average buoyancy $< b > (z, t)$ at this altitude. Gravity has an opposite effect according to the sign of $b- < b >$. If only this process is active, we can model the flux by $\mathcal{J} = -\tau_b \rho (b - \langle b\rangle)$. This term preserves the mean and the normalization constraint, and is the simplest way to account for the restoring effect of sedimentation. If there is no dissipation of fluctuation, and no external source of energy, the system will tend to the *reference state*. The constant of proportionality has the dimension of a time; we could use dimensional arguments to choose its form as a function of other quantities of the problem, but for the sake of simplicity, we will suppose that τ_b is constant.

Notice that if there is no dissipation of buoyancy flucuation ($\mathcal{D} = 0$), the reference state is preserved through temporal evolution. This comes from the conservation of the quantity $g(b) = \int \rho\,(b,z,t)\,dz$ for all value of buoyancy b. However, buoyancy of fluids particles is not preserved in time in real flows : turbulent cascade tends to smooth out the scalar field at a coarse grained scale. It is the main cause of dissipation of buoyancy flucuations. Qualitatively, it implies that two fluid particle of buoyancy 0 and B in a turbulent area can create new particles of buoyancy $B/2$. This process is described by the term $\mathcal{D}$ which will be modelled in the next subsection. In presence of such dissipation, the reference state evolves with time : it corresponds to an irreversible increase of the minimum total potential energy.

Dissipation of scalar fluctuations In order to describe this mixing process, we consider a simple but realistic way to model dissipation of scalar fluctuations at a coarse grained scale, which has been developped in the context of passive scalar mixing Venaille & Sommeria (2007). The idea is that a probe of width l sees structures coming from larger and larger scales, wich implies the self convolution of the distribution. Let us consider a succession of scalar sheets of width l stired by fluid motion. After the time $\Delta t_{1/2}$ needed for the width of the sheet to be divided by 2, the scalar field filtered at scale l becomes the average of two realisations of the field at the previous time. The probabilities of scalar values in adjacent strips can be assumed independent, as they result of the straining of regions which were initially far apart, at a distance beyond the integral scale of the scalar. Thus the new probability distribution is the self-convolution of the previous one, describing the sum of the independent random variables, followed by a contraction by a factor 2:

$$\rho(b, t + \Delta t_{1/2}) = 2 \int \rho(b', t)\rho(2b - b', t)db' \tag{4}$$

As far as we can choose the reference density ϱ_0 such that b remains always positive, it is convenient to use the Laplace transform of the PDF :

$$\widetilde{\rho}(s) = \int \rho(b)e^{-sb}db \tag{5}$$

The previous expression is then transformed in a product of characteristic functions : $\widetilde{\rho}(2s, t + \Delta t_{1/2}) = [\widetilde{\rho}(s, t)]^2$. Similarly, calling $\Delta t_{\frac{1}{n}}$ the time to divide the thickness of a sheet of fluid by a factor n, the PDF at time $t + \Delta t_{\frac{1}{n}}$ will be an n^{th}-selfconvolution.

$$\widetilde{\rho}(ns, t + \Delta t_{1/n}) = [\widetilde{\rho}(s, t)]^n. \tag{6}$$

In order to get a differential equation in time, we now take $n = 1+\varepsilon$ with $\varepsilon = s(t)dt$ in (6). The meaning of $s(t)$ will become clear later. This yields: $\widetilde{\rho}(s + \varepsilon s, t + dt) = [\widetilde{\rho}(s, t)]^{1+\varepsilon}$. Taking the limit $\varepsilon \longrightarrow 0$, we can express $\rho(s + \varepsilon s, t + dt)$ in terms of partial derivatives with respect to t and s:

$$\partial_t\widetilde{\rho} = s(t)[\widetilde{\rho}\ln\widetilde{\rho} - s\partial_s\widetilde{\rho}] \tag{7}$$

One can check that the normalisation and the mean scalar are conserved throught this process. The variance decreases as

$$\partial_t \left\langle b^2 - \left\langle b^2 \right\rangle \right\rangle = -s(t) \left\langle b^2 - \left\langle b^2 \right\rangle \right\rangle \tag{8}$$

. In the usual Kolmogoroff regime, the dissipation of scalar variance is equal to its cascad flux, independant of the cut-off scale. In that respect, the previous equation allows us to interpret $s(t)$ as the cascade rate. Notice that if we apply this model of to the distribution of velocities, and if we look at the evolution of its variance e, we recover the usual dissipation term in equation **??**.

This approach ignores the fluctuations of $s(t)$, which are known to generate internal intermittence. Nevertheless, equation (10) can be a good model if cascade is of limited extent in wave numbers, or if intermittency is dominated by spatial gradients effects. Another limitation of the model comes from the hypothesis that the scalar field is not spatially correlated at coarse-grained scale l.

final model In summary, we obtain a closed model in terms of the fields $\rho\,(b, z, t)$ and $e\,(z, t)$, and the scale $l(z, t)$.

$$\partial_t \rho = \partial_z \left(\kappa_b l e^{1/2} \partial_z \rho + \tau_b \rho \left(b - \langle b \rangle \right) \right) + \mathcal{D} \tag{9}$$

$$\widetilde{\mathcal{D}} = \epsilon_b l^{-1} e^{1/2} \left(\widetilde{\rho} \ln \widetilde{\rho} - s \partial_s \widetilde{\rho} \right) \tag{10}$$

$$\partial_t e = \partial_z \left(\kappa_e \beta l e^{1/2} \partial_z e \right) + l e^{1/2} \partial_z \langle b \rangle + \tau_b \left\langle b^2 - \langle b \rangle^2 \right\rangle - \epsilon_e l^{-1} e^{3/2} + \mathcal{P} \tag{11}$$

$$l = \frac{l_f e^{1/2}}{e^{1/2} + \gamma l_f \sqrt{\partial_z \langle b \rangle}} \tag{12}$$

where

$$\widetilde{\mathcal{D}}\,(s, z, t) = \int e^{-sb} \mathcal{D} db \tag{13}$$

$$\langle \cdot \rangle \,(z, t) = \int \cdot \rho db \tag{14}$$

This model depends on 6 constants :

- $\kappa_b,\ \kappa_e$ characterize the diffusion coefficient for the scalar and the energy (no dimension).
- ϵ_b, ϵ_e characterize the rate of turbulent cascad of scalar and kinetic energy (they can be fitted from homogeneous turbulence results, and has no dimension).
- τ_b characterizes the effect of sedimentation (dimension of a time).
- γ characterizes the buoyancy length scale (no dimension).

The integral length scale l_f depends on the mechanism of turbulent generation ; it can be obtained by experimental results on velocity correlations in the homogeneous case. Similarly, the production term $\mathcal{P}$ can be modelled on different way according to the experiment.

Notice that taking the ensemble average of the first equation and neglecting the term due to the scalar fluctuations in both equations ($\alpha = \epsilon_b = 0$), we recover the model of Balmforth *et al*. (1998).

3 Application : modelling of restratification

We model the effect of the uniform injection of turbulent energy in a tank filled with a stable linear profile of buoyancy, initially without scalar fluctuations. The parameters are given on the figure. We look for the evolution of this profile once the inection of energy is stopped. This can be realized experimentaly by producing turbulence using a vertical grid. In the homogeneous case, it is a good approximation to consider that l_f is one quarter of the grid mesh. The length unit will be half of the vertical size of the tank, and we choose a time scale such that the buoyancy range is $[0\ 1]$. In this unit system, we choose $l_f = 0.25 * 0.$.

We observe that if there is no scalar fluctuation, there is an initial period of time were the mean profile seems to be mixed, but then the restratification occurs, and as expected, the system tend to its reference state (the initial linear profile without fluctuation). If we take into account dissipation of scalar fluctuations, there is irreversible mixing : the final state is a new reference state, whose potential energy is greater than the one of the initial profile.. .

4 Conclusions

To our knowledge, it exists no other model in the litterature that describe the evolution of the whole scalar probability distribution in the case of turbulent mixing in stratified fluids. Although strong hypothesis were assumed, our phenomenological approach capture the main physical features of mixing, and has the advantage of conceptual simplicity : it takes into account turbulent diffusion, sedimentation, and dissipation of scalar fluctuations. It is in good agreement with experimental results on grid trubulence experiments. This model could be extended to more complex case, as gravity currents, where there is a mean current in addition to the turbulent field.

References

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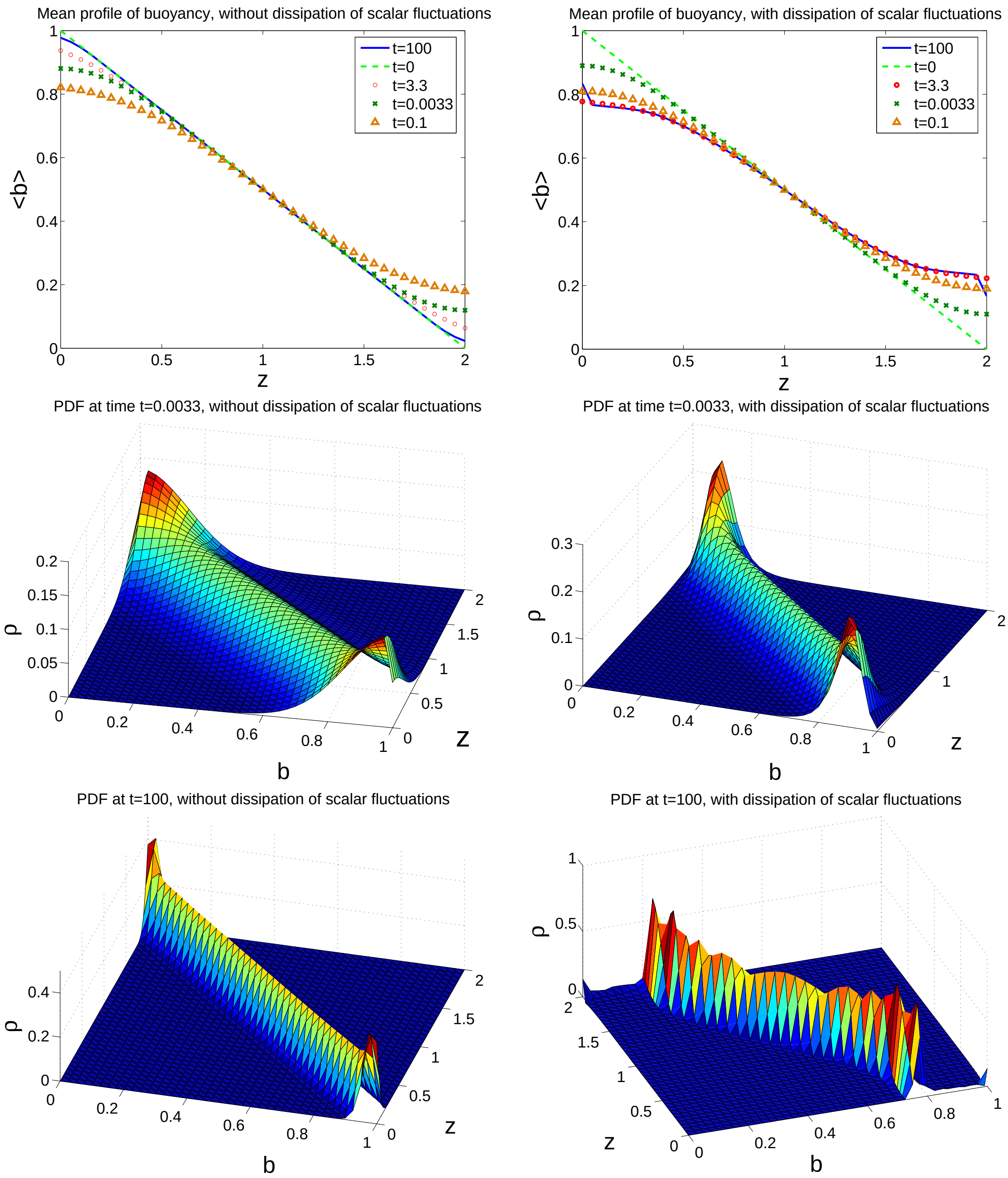

Figure 1: Results of the model with and without the dissipation term of scalar fluctuations $\mathcal{D}$. The initial condition is a linear profile of buoyancy, without fluctuations. Units are given in the text. We take the simple case of an initial injection of energy ($\forall z,\ e(z) = 2600$ at $t = 0$) without production term $\mathcal{P}$ (decaying turbulence). We take $\kappa_b = \kappa_e = 20$, $\tau_s = 1$, and $\epsilon_b = \epsilon_e = 1$. The qualitative behaviour of the system is the same for a wide range of value of those parameters