Are oceanic rings and jets statistical equilibrium states ?

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Abstract. Oceanic flows have the property to self-organize into large scale coherent structures such as rings (robust vortices) and jets. The concomitant turbulent nature of such flows, and their self-organization property, gives a strong incentive for a statistical mechanics approach. Here we apply equilibrium statistical mechanics theory to a simple ocean model, in order to answer basic questions: What are the shape of the coherent structures ? What is their drift ? When are they unstable ?

1. Introduction

Oceanic flows are organized into large scale coherent structures such as long lived vortex of diameter $\sim 300km$, the so-called rings, or strong and localized eastward jets originating from western boundary currents at mid-latitudes.

These self-organization phenomena involve a huge number of degrees of freedom coupled together via complex non-linear interactions. This situation makes any deterministic approach illusory, if not impossible. It is then appealing to study this problem with a statistical mechanics approach, which reduces the problem of large-scale organization of the flow to the study of states depending on a few key parameters only, such as the energy of the circulation of the flow.

Real oceanic flows are forced and dissipated. However, a natural first step toward a clear theoretical understanding of self-organization phenomena is to consider a initial value problem in the freely evolving case: for a given initial condition, what is the final state organization ? Or more generally: which class of coherent structures can be interpreted as statistical equilibrium states ?

The appropriate tool to tackle this question is equilibrium statistical mechanics, which explains the spontaneous organization of unforced and undissipated two-dimensional and geophysical flows (Miller, 1990; Robert & Sommeria, 1991), (RSM hereafter). From the knowledge of the energy and the global distribution of potential vorticity levels provided by an initial condition, the RSM theory predicts the large scale flow as the most probable outcome of turbulent mixing.

In the framework of a simple model of geophysical turbulent flows (1.5 layer quasi-geostrophic dynamics), it has been shown that the computation of such statistical equilibria can be simplified into a Van der Waals-Cahn-Hilliard variational problem that describes first order phase transitions in thermodynamics (Bouchet, 2001). The existence of this formal analogy between the formation of bubbles and the self-organization of potential vorticity in geophysical flows has been very fruitful in the description of Jovian vortices (Bouchet & Sommeria, 2002). Here we put forward this approach in the oceanic context. More precisely, the following question is addressed: can rings and jets be interpreted as statistical equilibria ? Most results are presented in (Venaille & Bouchet, 2011*a*); we propose in these proceedings a complementary phenomenological interpretation.

2. 1.5 layer quasi-geostrophic turbulent flows

2.1. The model

Most of oceanic coherent structures are surface intensified, with most of their kinetic energy located above the thermocline (around 1000 m depth). In addition, oceanic jets separating front of potential vorticity are characterized by a width of the order of the first baroclinic Rossby radius of deformation $R \sim 50km$. We consider the simplest ocean model that takes into account this vertical structure and this typical horizontal length scale, namely an equivalent barotropic, 1.5 layer quasi-geostrophic model (Pedlosky, 1998):

$$\frac{\partial q}{\partial t} + J(\psi, q) = 0, \quad \text{with } J(\psi, q) = \partial_x \psi \partial_y q - \partial_y \psi \partial_x q \ , \tag{1}$$

$$q = \nabla^2 \psi - \frac{\psi}{R^2} + \beta_c y \ . \tag{2}$$

This model is expressed as the advection by a non-divergent velocity field $\mathbf{v} = (-\partial_y \psi, \ \partial_x \psi)$ of an active tracer, the potential vorticity q, so that the area $\gamma(\sigma)\, d\sigma$ occupied by a given vorticity level σ is a dynamical invariant. The conservation of the global distribution $\gamma(\sigma)$ is equivalent to the conservation of any moment of the potential vorticity $\int_{\mathcal{D}} \mathrm{d}\mathbf{r}\, q^n$, and is related to particle relabelling symmetry (Salmon, 1998). Importantly, the dynamics conserves also the total energy

$$E = \frac{1}{2} \int_{\mathcal{D}} \mathrm{d}\mathbf{r} \ \left[(\nabla \psi)^2 + \frac{\psi^2}{R^2} \right] = -\frac{1}{2} \int_{\mathcal{D}} \mathrm{d}\mathbf{r} \ (q - \beta_c y)\, \psi. \tag{3}$$

2.2. Potential vorticity homogenization with constraints

The turbulent dynamics tend to mix the potential vorticity field, but complete homogenization would break the energy conservation law. In order to satisfy the energy constraint, the flow must present some large scale structure. This large scale organization can be predicted by equilibrium statistical mechanics, assuming ergodicity, since (1) satisfy a Liouville theorem (Miller, 1990; Robert & Sommeria, 1991). The observed flow is then interpreted as the most probable among all the states that satisfy the constraints of the dynamics, see (Bouchet & Venaille, 2011) and references therein for details and discussions of the theory.

When the Rossby radius R is small compared to the domain size, the energy constraint can be satisfied together with complete homogenization of the potential vorticity field by the formation of two sub-domains or "phases" characterized by different values of potential vorticity, see Fig. 1. The value of homogenized potential vorticity in each sub-domain, and the area of these sub-domains depends on the vorticity distribution and the energy of the initial condition. Note that at lowest oder in R, the potential vorticity is proportional to the streamfunction $q \sim \psi/R^2$ and the total energy (4) is dominated by the potential energy term $E \approx R^2 \int \mathrm{d}x\mathrm{d}yq^2$.

The interfaces between these sub-domains are fronts of potential vorticity associated with strong jets of typical width given by the Rossby radius of deformation R, also called "ribbons" (Arbic & Flierl, 2003).

This qualitative ideas has been formalized in the framework of the RSM theory, and it has been rigorously shown that in the limit of small Rossby radius of deformation R, the computation of statistical equilibrium states amounts to solve a Van der Waals-Cahn-Hilliard variational problem that describes phase coexistence in thermodynamics.

Just as in the case of bubbles in usual thermodynamics, the interface between sub-domains "cost" free energy and the system will therefore tend to minimize these interfaces for a given area of the sub-domains.

Statistical mechanics provides therefore a physical explanation for the potential homogenization theory of Rhines & Young (1982), without invoking any dissipation mechanism. It accounts for the irreversible nature of mixing: an overwhelming number of fine-grained microstates are associated with a given coarse grained equilibrium state. The only effect of a weak small scale dissipation process would be to smooth out locally fine-grained fluctuations of potential vorticity, leaving unchanged its coarse-grained structure.

2.3. Energy cascade

The previous results are consistent with the classical phenomenology of cascade processes. Let us write the potential vorticity on the form $q = \zeta - \psi/R^2$ with $\zeta = \Delta\psi$.

At small scales ($r \ll R$), the potential vorticity is dominated by the relative vorticity ($q \approx \zeta$) and the dynamics is essentially the Euler ones, with direct cascade of enstrophy ($Z = \int \mathrm{d}\mathbf{r}\ \zeta^2/2$) and inverse cascade of kinetic energy ($E_c = -\int \mathrm{d}\mathbf{r}\ \psi\zeta/2$).

At large scales ($r \gg R$), the dynamics is the so-called planetary geostrophic model $\partial_\tau \psi + J(\zeta, \psi) = 0$, with $\tau = Rt$ (Larichev & McWilliams, 1991). The role of ζ and ψ are switched with respect to the Euler dynamics. Using simple arguments *a la Kolmogorov*, see e.g. Pierrehumbert *et al.* (1994); Smith *et al.* (2002), one predicts then a direct cascade of kinetic energy ($E_c = -\int \mathrm{d}\mathbf{r}\ \psi\zeta/2$), and an inverse cascade of potential energy ($E_p = \int \mathrm{d}\mathbf{r}\ \psi^2/(2R^2)$).

The previous argument predicts a condensation of the kinetic energy at the Rossby radius scale, and a condensation of potential energy (approximately equal to the total energy) at the domain scale. These two results imply the formation of regions of homogenized potential vorticity at domain scale, separated by strong jets, just as predicted by statistical mechanics arguments.

2.4. Consequences: rings and zonal jets as statistical equilibria

According to the previous subsections, equilibrium states are flow fields presenting regions of homogenized potential vorticity separated by interfaces of minimal length, which are circle or straight lines in two dimensions. In that respect, oceanic rings and jets can be interpreted as statistical equilibrium states in the *f-plane* case, i.e. when there are no planetary vorticity gradients ($\beta_c = 0$). We discuss in the next subsection the important role of such planetary vorticity gradients (this is the *beta plane* case, for which $\beta_c \neq 0$).

3. The role of beta effect

3.1. Oceanic rings drift

Observations show that both cyclonic and anticyclonic rings propagate westward at speed $\beta_c R^2$ (Morrow *et al.*, 2004; Chelton *et al.*, 2007). They also present an additional small meridional drift, poleward for cyclonic rings and equatorward for anti-cyclonic rings.

We consider here the case of a domain invariant by translation in the zonal direction. A drifting ring could not exist as statistical equilibria in a closed domain, since it would be destroyed when arriving on the western boundary.

The westward drift of oceanic rings can be understood as a consequence of the potential vorticity homogenization (see Bouchet & Venaille (2011) for an argument using Galilean invariance and Venaille & Bouchet (2011*a*) for an argument using linear momentum conservation): the flow field must cancel the term $\beta_c y$ in the expression (1) of the potential

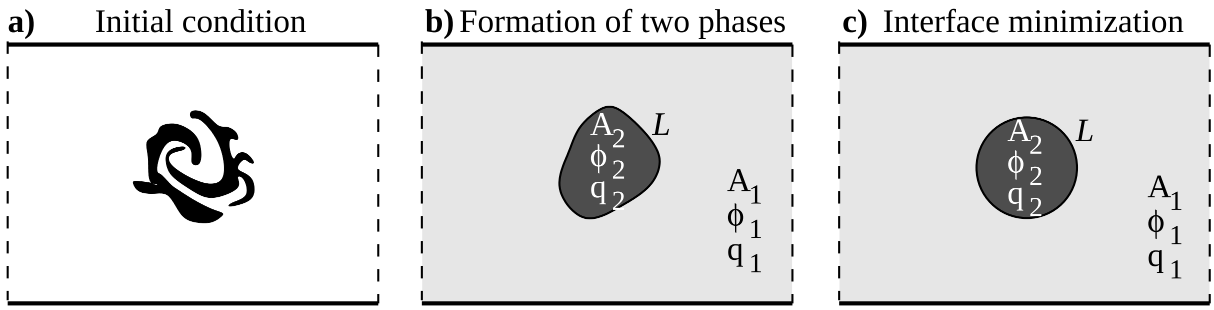

Figure 1. Potential vorticity homogenization with energy constraint. a) Example of an initial condition for the potential vorticity field. Note that this initial condition could as well have many levels of potential vorticity. b) At zeroth order in R, the potential vorticity is homogenized in two sub-domains. These sub-domains are separated by strong jets of typical width R and velocity $U_{jet} = (q_2 - q_1)R$. c) The actual shape of the structure, or equivalently the position of the jets, is obtained by minimizing its perimeter L for a fixed value of the area A_2.

vorticity q, which implies a contribution $\psi_{\beta_c} = R^2\beta_c y$ for the streamfunction in the region where the potential vorticity is homogenized. This corresponds to a westward drift with velocity $R^2\beta_c$.

If the flow actually reaches a local statistical equilibrium, then not only the ring is composed of an homogenized region of potential vorticity, but also the background flow. In Fig. 2 is represented the case of an isolated patch of potential vorticity. Since the background potential vorticity is not homogenized, this state is not a statistical equilibrium state.

The observed asymmetric small meridional drift of cyclonic and anticyclonic rings can be understood as a tendency for the system to reach the statistical equilibrium in such a configuration. One needs for that purpose to consider the conservation of the linear momentum:

$$\mathcal{L} = \int_{\mathcal{D}} \mathrm{d}\mathbf{r}\ qy\ ,$$

which is the dynamical invariant associated with the zonal translational symmetry.

As explained in previous subsection, the background potential vorticity has to be homogenized to reach statistical equilibrium, which leads to a loss of linear momentum that must be compensated by a latitude shift of the ring center. Rings with low potential vorticity values (dark color in Fig. 2) decrease their latitude, while rings with high values of potential vorticity (bright color on Fig. 2) increase their latitude. This corresponds always to a poleward drift for cyclonic structures and an equatorward drift for for anticyclonic structures, just as what is reported from altimetry measurements (Morrow *et al.*, 2004; Chelton *et al.*, 2007).

3.2. Destabilization of mid-basin eastward jets

We now ask whether mid-latitude eastward jets such as the one of Fig. 5 can be interpreted as statistical equilibria in presence of beta effect. Without beta effect, we have shown previously that such states are statistical equilibria because they minimize the interface length between regions of homogenized potential vorticity. However, the reverse configuration, i.e. a flow presenting a westward jets, is equivalent in term of statistical mechanics.

This degeneracy is removed by the beta effect, which acts as a magnetic field. This can be seen by writing the total energy on the form

$$E = -\frac{1}{2}\int_{\mathcal{D}} \mathrm{d}\mathbf{r}\, q\psi + \beta_c \int_{\mathcal{D}} \mathrm{d}\mathbf{r} y\psi. \qquad (4)$$

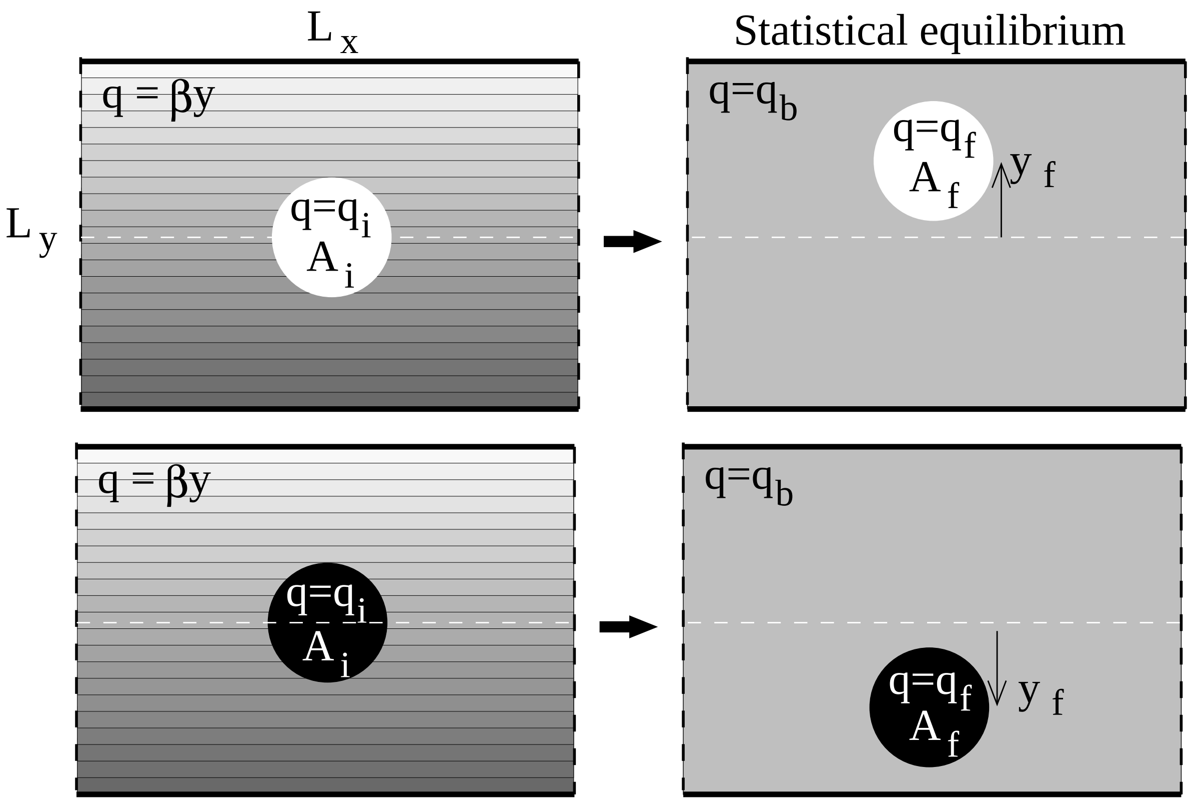

Figure 2. Explanation of the meridional drift of rings, as a tendency to reach the statistical equilibria. On the left, initial conditions, with a white disk of positive potential vorticity in the upper panel, and a black disk of negative potential vorticity in the lower panel. In both cases the disk evolves on an initial beta plane with no background flow. On the right, the corresponding statistical equilibrium.

Because statistical equilibria presenting two regions of homogenized potential vorticity are negative temperature states, which can also be interpreted as maximum energy states, a positive value of β_c favors the configuration where the phase with the higher values of ψ (and then the lower values of q) is in the northern part of the domain. It corresponds to the solution presenting a westward jet. For sufficiently low values of β_c, the eastward jet solution is metastable, and it becomes metastable when β_c exceeds a critical values $\beta_{c,critic}$. Using dimensional analysis, one finds that without beta effect, increasing the length of the interface as in Fig. 4 costs energy with a contribution $\Delta E \sim -U_{jet}^2 R L_x l^2/L_x^2$, where $U_{jet} = |q_2 - q_1|R$ is the jet velocity and l a typical value of the length of the interface perturbation in the meridional direction, with the total interface length given by $\mathcal{L} = \int_0^{L_x} \mathrm{d}x \, \left(1 + (dl/dx)^2\right)^{1/2}$. In presence of beta effect, the same perturbation switches the latitude of negative and potential vorticity around the jet, which increases the energy with a contribution $\Delta E \sim \beta_c U_{jet} R L_x l^2$.

It leads to the necessary condition $\beta_c > \beta_{c,critic} \sim U_{jet}/L_x^2$ for the eastward jet solution to be unstable in term of statistical mechanics. It can actually be shown, using less straightforward considerations, that it is also a sufficient condition for instability. The *destabilizing* effect of increasing values of β_c contrasts with its stabilizing effect in classical criteria for barotropic instability, see e.g. Vallis (2006).

For a fixed value of β_c, eastward jets are local free energy maxima if the domain zonal

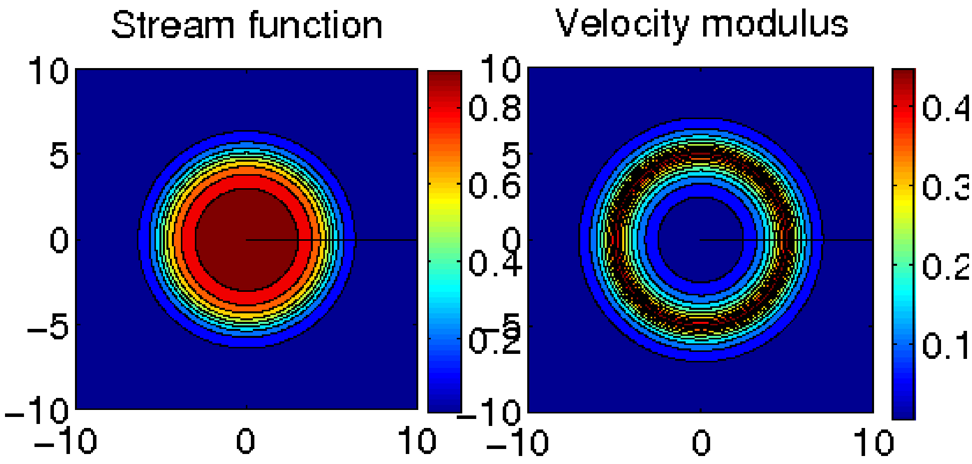

Figure 3. Circular vortex as a statistical equilibrium of the quasi-geophysical model, with $R < L_{ring}$. Although analytical computations are carried in the limit $R \ll L_{ring}$, the results are expected to hold when this scale separation does not exit. It is a circular patch of (homogenized) potential vorticity in a background of homogenized potential vorticity, with two different values. The velocity field (right panel) has a ring structure. The width of the jet surrounding the ring has the order of magnitude of the Rossby radius of deformation R.

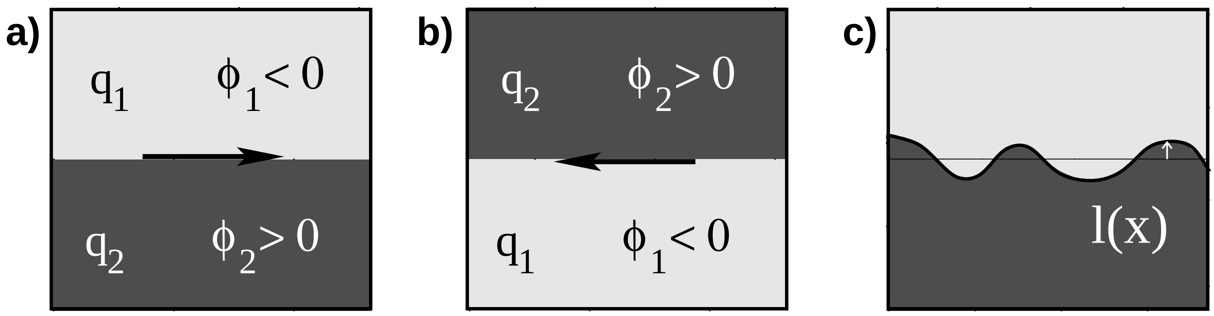

Figure 4. a) Eastward jet configuration b) Westward jet configuration c) Perturbation of the interface for the eastward jet configuration, to determine when this solution is a local equilibrium. Without beta effect, both the eastward and the westward configurations are statistical equilibrium states. With positive beta effect the westward jet becomes the global statistical equilibrium state, and the eastward jet becomes metastable provided that beta is small enough.

extension is smaller than a critical value: $L_x < (U_{jet}/\beta_c)^{1/2}$. The instability is consistent with the fact that strong meanders and pinch off process occurs downstream of oceanic eastward jets. But the marginal nature of this instability is also consistent with the overall robustness of the global structure of the flow, which becomes a statistical equilibrium when the extension of the jet is small enough.

For sufficiently large values of β_c, the solution presenting an eastward jet does not exist, and the equilibrium state is then a Fofonoff solution with $\psi = R^2\beta_c y$ in the interior. More generally, one can show that such Fofonoff modes are global statistical equilibria in a low energy limit Venaille & Bouchet (2011*b*). This happens when the energy of the initial condition is much smaller than the maximum possible energy associated with the same distribution of potential

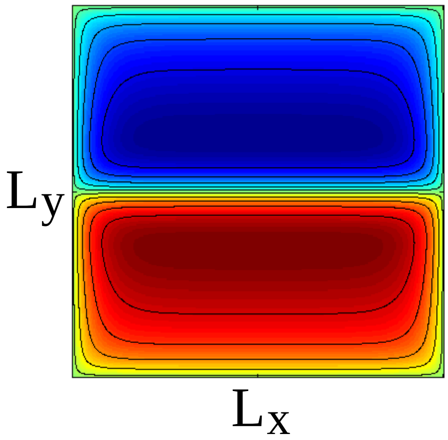

Figure 5. Streamfunction of the solution presenting an eastward jet with beta effect (red: positive values, blue: negative values), associated with case (a) of Fig. 4. The jet width is of order R. The solution with an eastward jet is a statistical equilibrium for $L_x < \pi(U_{jet}/\beta_c)^{1/2}$.

vorticity.

4. Summary

To conclude, the main interest of equilibrium statistical mechanics is to provide a physical explanation and a prediction for the self-organization of large scale oceanic coherent structure, independently of the underlying generation mechanism. It predicts the formation of sub-domains of homogenized potential vorticity, with intense jets (or "ribbons") at their interface. Mesoscale rings can be interpreted as local equilibrium states of the RSM theory. Their shape and their drift can be understood in this framework. Mid-basin eastward jets are found marginally unstable states of the RSM theory, consistently with observations of these jets.

The interest of this approach relies on its generality (it does not depends on a particular flow configuration) and on its ability to describe qualitatively different observed regimes of self-organization, such as rings and zonal jets. The present study was achieved in the framework of a 1.5 layer quasi-geostrophic model, which is too simplistic to describe oceanic eddies quantitatively; however, generalizations and further investigations in the framework of more complex models can be built upon these results.

A caveat of this approach is that forcing and dissipation are not taken into account in the framework of the equilibrium theory: the input of the RSM theory is given by the dynamical invariants. In the case of mesoscale rings, even if the dynamics can be considered close to an equilibrium state, forcing and dissipation play an important role in setting these dynamical invariants. In the case of basin scale jets, their marginal instability suggest that one can not avoid taking into account forcing and dissipation mechanisms to explain these structures.

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