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# Chapter 1
# Vector Spaces

This chapter reviews several important facts about different aspects of vectors spaces that will be used throughout the book. More comprehensive and detailed accounts of these subjects can be found in advanced linear algebra books.

## 1.1 Definition

A *vector space* $\mathbb{E}$ over $\mathbb{R}$ (or a "real vector space") is a set of elements called *vectors* such that the following holds.

(A) For any two vectors $\mathbf{x}, \mathbf{y} \in \mathbb{E}$, there corresponds a vector $\mathbf{x}+\mathbf{y}$, called the *sum* of $\mathbf{x}$ and $\mathbf{y}$, satisfying the following properties:

1. $\mathbf{x} + \mathbf{y} = \mathbf{y} + \mathbf{x}$ for any $\mathbf{x}, \mathbf{y} \in \mathbb{E}$.
2. $\mathbf{x} + (\mathbf{y} + \mathbf{z}) = (\mathbf{x} + \mathbf{y}) + \mathbf{z}$ for any $\mathbf{x}, \mathbf{y}, \mathbf{z} \in \mathbb{E}$.
3. There exists in $\mathbb{E}$ a unique vector $\mathbf{0}$ (called the *zeros vector*) such that $\mathbf{x} + \mathbf{0} = \mathbf{x}$ for any $\mathbf{x}$.
4. For any $\mathbf{x} \in \mathbb{E}$, there exists a vector $-\mathbf{x} \in \mathbb{E}$ such that $\mathbf{x} + (-\mathbf{x}) = \mathbf{0}$.

(B) For any real number (also called *scalar*) $\alpha \in \mathbb{R}$ and $\mathbf{x} \in \mathbb{E}$, there corresponds a vector $\alpha\mathbf{x}$ called the *scalar multiplication* of $\alpha$ and $\mathbf{x}$ satisfying the following properties:

1. $\alpha(\beta\mathbf{x}) = (\alpha\beta)\mathbf{x}$ for any $\alpha, \beta \in \mathbb{R}, \mathbf{x} \in \mathbb{E}$.
2. $1\mathbf{x} = \mathbf{x}$ for any $\mathbf{x} \in \mathbb{E}$.

(C) The two operations (summation, scalar multiplication) satisfy the following properties:

1. $\alpha(\mathbf{x} + \mathbf{y}) = \alpha\mathbf{x} + \alpha\mathbf{y}$ for any $\alpha \in \mathbb{R}, \mathbf{x}, \mathbf{y} \in \mathbb{E}$.
2. $(\alpha + \beta)\mathbf{x} = \alpha\mathbf{x} + \beta\mathbf{x}$ for any $\alpha, \beta \in \mathbb{R}, \mathbf{x} \in \mathbb{E}$.

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## 1.2 Dimension

A set of vectors $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ in a vector space $\mathbb{E}$ is called *linearly independent* or just *independent* if the linear system

$$\sum_{i=1}^{n} \alpha_i \mathbf{v}_i = \mathbf{0}$$

implies that $\alpha_1 = \alpha_2 = \cdots = \alpha_n = 0$. In other words, there does not exist a nontrivial linear combination of vectors that is equal to the zeros vector. A set of vectors $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ is said to *span* $\mathbb{E}$ if for any $\mathbf{x} \in \mathbb{E}$, there exist $\beta_1, \beta_2, \ldots, \beta_n \in \mathbb{R}$ such that

$$\mathbf{x} = \sum_{i=1}^{n} \beta_i \mathbf{v}_i.$$

A *basis* of a vector space $\mathbb{E}$ is an independent set of vectors that spans $\mathbb{E}$. It is well known that the number of vectors in all the bases of a vector space $\mathbb{E}$ is the same; this number is called the *dimension* of the space and is denoted by $\dim(\mathbb{E})$. In this book we will discuss only vector spaces with a finite dimension, namely, *finite-dimensional vector spaces*.

## 1.3 Norms

A *norm* $\|\cdot\|$ on a vector space $\mathbb{E}$ is a function $\|\cdot\| : \mathbb{E} \to \mathbb{R}$ satisfying the following properties:

1. **(nonnegativity)** $\|\mathbf{x}\| \geq 0$ for any $\mathbf{x} \in \mathbb{E}$ and $\|\mathbf{x}\| = 0$ if and only if $\mathbf{x} = \mathbf{0}$.
2. **(positive homogeneity)** $\|\lambda \mathbf{x}\| = |\lambda| \cdot \|\mathbf{x}\|$ for any $\mathbf{x} \in \mathbb{E}$ and $\lambda \in \mathbb{R}$.
3. **(triangle inequality)** $\|\mathbf{x} + \mathbf{y}\| \leq \|\mathbf{x}\| + \|\mathbf{y}\|$ for any $\mathbf{x}, \mathbf{y} \in \mathbb{E}$.

We will sometimes denote the norm of a space $\mathbb{E}$ by $\|\cdot\|_{\mathbb{E}}$ to emphasize the identity of the space and to distinguish it from other norms. The *open ball* with center $\mathbf{c} \in \mathbb{E}$ and radius $r > 0$ is denoted by $B(\mathbf{c}, r)$ and defined by

$$B(\mathbf{c}, r) = \{\mathbf{x} \in \mathbb{E} : \|\mathbf{x} - \mathbf{c}\| < r\}.$$

The *closed ball* with center $\mathbf{c} \in \mathbb{E}$ and radius $r > 0$ is denoted by $B[\mathbf{c}, r]$ and defined by

$$B[\mathbf{c}, r] = \{\mathbf{x} \in \mathbb{E} : \|\mathbf{x} - \mathbf{c}\| \leq r\}.$$

We will sometimes use the notation $B_{\|\cdot\|}[\mathbf{c}, r]$ or $B_{\|\cdot\|}(\mathbf{c}, r)$ to identify the specific norm that is being used.

## 1.4 Inner Products

An *inner product* of a real vector space $\mathbb{E}$ is a function that associates to each pair of vectors $\mathbf{x}, \mathbf{y}$ a real number, which is denoted by $\langle \mathbf{x}, \mathbf{y} \rangle$ and satisfies the following properties:

1. **(commutativity)** $\langle \mathbf{x}, \mathbf{y} \rangle = \langle \mathbf{y}, \mathbf{x} \rangle$ for any $\mathbf{x}, \mathbf{y} \in \mathbb{E}$.

2. **(linearity)** $\langle \alpha_1 \mathbf{x}_1 + \alpha_2 \mathbf{x}_2, \mathbf{y} \rangle = \alpha_1 \langle \mathbf{x}_1, \mathbf{y} \rangle + \alpha_2 \langle \mathbf{x}_2, \mathbf{y} \rangle$ for any $\alpha_1, \alpha_2 \in \mathbb{R}$ and $\mathbf{x}_1, \mathbf{x}_2, \mathbf{y} \in \mathbb{E}$.

3. **(positive definiteness)** $\langle \mathbf{x}, \mathbf{x} \rangle \geq 0$ for any $\mathbf{x} \in \mathbb{E}$ and $\langle \mathbf{x}, \mathbf{x} \rangle = 0$ if and only if $\mathbf{x} = \mathbf{0}$.

A vector space endowed with an inner product is also called an *inner product space*. At this point we would like to make the following important note:

**Underlying Spaces:** In this book the underlying vector spaces, usually denoted by $\mathbb{V}$ or $\mathbb{E}$, are always finite dimensional real inner product spaces with endowed inner product $\langle \cdot, \cdot \rangle$ and endowed norm $\| \cdot \|$.

## 1.5 Affine Sets and Convex Sets

Given a real vector space $\mathbb{E}$, a set $S \subseteq \mathbb{E}$ is called *affine* if for any $\mathbf{x}, \mathbf{y} \in S$ and $\lambda \in \mathbb{R}$, the inclusion $\lambda \mathbf{x} + (1 - \lambda)\mathbf{y} \in S$ holds. For a set $S \subseteq \mathbb{E}$, the *affine hull* of $S$, denoted by $\mathrm{aff}(S)$, is the intersection of all affine sets containing $S$. Clearly, $\mathrm{aff}(S)$ is by itself an affine set, and it is the smallest affine set containing $S$ (w.r.t. inclusion). A *hyperplane* is a subset of $\mathbb{E}$ given by

$$H_{\mathbf{a},b} = \{\mathbf{x} \in \mathbb{E} : \langle \mathbf{a}, \mathbf{x} \rangle = b\},$$

where $\mathbf{a} \in \mathbb{E}$ and $b \in \mathbb{R}$. It is an easy exercise to show that hyperplanes are affine sets.

A set $C \subseteq \mathbb{E}$ is called *convex* if for any $\mathbf{x}, \mathbf{y} \in C$ and $\lambda \in [0, 1]$ it holds that $\lambda \mathbf{x} + (1 - \lambda)\mathbf{y} \in C$. Evidently, affine sets are always convex. Open and closed balls are always convex regardless of the choice of norm. For given $\mathbf{x}, \mathbf{y} \in \mathbb{E}$, the *closed line segment* between $\mathbf{x}$ and $\mathbf{y}$ is a subset of $\mathbb{E}$ denoted by $[\mathbf{x}, \mathbf{y}]$ and defined as

$$[\mathbf{x}, \mathbf{y}] = \{\alpha \mathbf{x} + (1 - \alpha)\mathbf{y} : \alpha \in [0, 1]\}.$$

The *open line segment* $(\mathbf{x}, \mathbf{y})$ is similarly defined as

$$(\mathbf{x}, \mathbf{y}) = \{\alpha \mathbf{x} + (1 - \alpha)\mathbf{y} : \alpha \in (0, 1)\}$$

when $\mathbf{x} \neq \mathbf{y}$ and is the empty set $\emptyset$ when $\mathbf{x} = \mathbf{y}$. Closed and open line segments are convex sets. Another example of convex sets are *half-spaces*, which are sets of the form

$$H^-_{\mathbf{a},b} = \{\mathbf{x} \in \mathbb{E} : \langle \mathbf{a}, \mathbf{x} \rangle \leq b\},$$

where $\mathbf{a} \in \mathbb{E}$ and $b \in \mathbb{R}$.

## 1.6 Euclidean Spaces

A finite dimensional real vector space equipped with an inner product $\langle \cdot, \cdot \rangle$ is called a *Euclidean space* if it is endowed with the norm $\|\mathbf{x}\| = \sqrt{\langle \mathbf{x}, \mathbf{x} \rangle}$, which is referred to as the *Euclidean norm*.

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## 1.7 The Space $\mathbb{R}^n$

The vector space $\mathbb{R}^n$ ($n$ being a positive integer) is the set of $n$-dimensional column vectors with real components endowed with the component-wise addition operator,

$$\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} + \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix} = \begin{pmatrix} x_1 + y_1 \\ x_2 + y_2 \\ \vdots \\ x_n + y_n \end{pmatrix},$$

and the scalar-vector product,

$$\lambda \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} \lambda x_1 \\ \lambda x_2 \\ \vdots \\ \lambda x_n \end{pmatrix},$$

where in the above $x_1, x_2, \ldots, x_n, \lambda$ are real numbers. We will denote the standard basis of $\mathbb{R}^n$ by $\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n$, where $\mathbf{e}_i$ is the $n$-length column vector whose $i$th component is one while all the others are zeros. The column vectors of all ones and all zeros will be denoted by $\mathbf{e}$ and $\mathbf{0}$, respectively, where the length of the vectors will be clear from the context.

By far the most used inner product in $\mathbb{R}^n$ is the *dot product* defined by

$$\langle \mathbf{x}, \mathbf{y} \rangle = \sum_{i=1}^{n} x_i y_i.$$

**Inner Product in $\mathbb{R}^n$:** In this book, unless otherwise stated, the endowed inner product in $\mathbb{R}^n$ is the dot product.

Of course, the dot product is not the only possible inner product that can be defined over $\mathbb{R}^n$. Another useful option is the $\mathbf{Q}$-inner product, which is defined as

$$\langle \mathbf{x}, \mathbf{y} \rangle_{\mathbf{Q}} = \mathbf{x}^T \mathbf{Q} \mathbf{y},$$

where $\mathbf{Q}$ is a positive definite $n \times n$ matrix. Obviously, the $\mathbf{Q}$-inner product amounts to the dot product when $\mathbf{Q} = \mathbf{I}$. If $\mathbb{R}^n$ is endowed with the dot product, then the associated Euclidean norm is the $l_2$-norm

$$\|\mathbf{x}\|_2 = \sqrt{\langle \mathbf{x}, \mathbf{x} \rangle} = \sqrt{\sum_{i=1}^{n} x_i^2}.$$

If $\mathbb{R}^n$ is endowed with the $\mathbf{Q}$-inner product, then the associated Euclidean norm is the $\mathbf{Q}$-*norm*

$$\|\mathbf{x}\|_{\mathbf{Q}} = \sqrt{\mathbf{x}^T \mathbf{Q} \mathbf{x}}.$$

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For a given $p \geq 1$, the $l_p$-norm on $\mathbb{R}^n$ is given by the formula

$$\|\mathbf{x}\|_p = \sqrt[p]{\sum_{i=1}^{n} |x_i|^p}.$$

The $l_\infty$-norm on $\mathbb{R}^n$ is defined by

$$\|\mathbf{x}\|_\infty = \max_{i=1,2,\ldots,n} |x_i|.$$

### 1.7.1 Subsets of $\mathbb{R}^n$

The *nonnegative orthant* is the subset of $\mathbb{R}^n$ consisting of all vectors in $\mathbb{R}^n$ with nonnegative components and is denoted by $\mathbb{R}^n_+$:

$$\mathbb{R}^n_+ = \left\{(x_1, x_2, \ldots, x_n)^T : x_1, x_2, \ldots, x_n \geq 0\right\}.$$

Similarly, the *positive orthant* consists of all the vectors in $\mathbb{R}^n$ with positive components and is denoted by $\mathbb{R}^n_{++}$:

$$\mathbb{R}^n_{++} = \left\{(x_1, x_2, \ldots, x_n)^T : x_1, x_2, \ldots, x_n > 0\right\}.$$

The *unit simplex*, denoted by $\Delta_n$, is the subset of $\mathbb{R}^n$ comprising all nonnegative vectors whose components sum up to one:

$$\Delta_n = \left\{\mathbf{x} \in \mathbb{R}^n : \mathbf{x} \geq \mathbf{0}, \mathbf{e}^T\mathbf{x} = 1\right\}.$$

Given two vectors $\boldsymbol{\ell}, \mathbf{u} \in \mathbb{R}^n$ that satisfy $\boldsymbol{\ell} \leq \mathbf{u}$, the *box* with lower bounds $\boldsymbol{\ell}$ and upper bounds $\mathbf{u}$ is denoted by $\text{Box}[\boldsymbol{\ell}, \mathbf{u}]$ and defined as

$$\text{Box}[\boldsymbol{\ell}, \mathbf{u}] = \{\mathbf{x} \in \mathbb{R}^n : \boldsymbol{\ell} \leq \mathbf{x} \leq \mathbf{u}\}.$$

Thus, for example, $\text{Box}[-\mathbf{e}, \mathbf{e}] = [-1, 1]^n$.

### 1.7.2 Operations on Vectors in $\mathbb{R}^n$

There are several operations on vectors in $\mathbb{R}^n$ that will be frequently used in the book. For a given vector $\mathbf{x} \in \mathbb{R}^n$, the vector $[\mathbf{x}]_+$ is the *nonnegative part* of $\mathbf{x}$ defined by $[\mathbf{x}]_+ = (\max\{x_i, 0\})_{i=1}^n$. For a given $\mathbf{x} \in \mathbb{R}^n$, the vector $|\mathbf{x}|$ is the vector of component-wise absolute values $(|x_i|)_{i=1}^n$, and the vector $\text{sgn}(\mathbf{x})$ is defined as

$$\text{sgn}(\mathbf{x})_i = \begin{cases} 1, & x_i \geq 0, \\ -1, & x_i < 0. \end{cases}$$

For two vectors $\mathbf{a}, \mathbf{b} \in \mathbb{R}^n$, their *Hadamard product*, denoted by $\mathbf{a} \odot \mathbf{b}$, is the vector comprising the component-wise products: $\mathbf{a} \odot \mathbf{b} = (a_i b_i)_{i=1}^n$.

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## 1.8 The Space $\mathbb{R}^{m\times n}$

The set of all real-valued $m \times n$ matrices is denoted by $\mathbb{R}^{m\times n}$. This is a vector space with the component-wise addition as the summation operation and the component-wise scalar multiplication as the "scalar-vector multiplication" operation. The *dot product* in $\mathbb{R}^{m\times n}$ is defined by

$$\langle \mathbf{A}, \mathbf{B} \rangle = \mathrm{Tr}(\mathbf{A}^T\mathbf{B}) = \sum_{i=1}^{m}\sum_{j=1}^{n} A_{ij}B_{ij}, \quad \mathbf{A}, \mathbf{B} \in \mathbb{R}^{m\times n}.$$

The space $\mathbb{R}^{m\times n}$ is sometimes associated with the space $\mathbb{R}^{mn}$ in the sense that each matrix in $\mathbb{R}^{m\times n}$ corresponds to the $mn$-length vector constructed by stacking the columns of the matrix. Unless otherwise stated, we will assume that the inner product in $\mathbb{R}^{m\times n}$ is the dot product.

**Inner Product in $\mathbb{R}^{m\times n}$:** In this book, unless otherwise stated, the endowed inner product in $\mathbb{R}^{m\times n}$ is the dot product.

### 1.8.1 Subsets of $\mathbb{R}^{n\times n}$

The set of all $n \times n$ symmetric matrices is denoted by $\mathbb{S}^n$:

$$\mathbb{S}^n = \left\{ \mathbf{A} \in \mathbb{R}^{n\times n} : \mathbf{A} = \mathbf{A}^T \right\}.$$

Note that $\mathbb{S}^n$ is also a vector space with the same summation and scalar multiplication operations as in $\mathbb{R}^{n\times n}$. The inner product in $\mathbb{S}^n$, unless otherwise stated, is the dot product.
The set of all $n \times n$ positive semidefinite matrices is denoted by $\mathbb{S}^n_+$:

$$\mathbb{S}^n_+ = \left\{ \mathbf{A} \in \mathbb{R}^{n\times n} : \mathbf{A} \succeq \mathbf{0} \right\}.$$

The set of all $n \times n$ positive definite matrices is denoted by $\mathbb{S}^n_{++}$:

$$\mathbb{S}^n_{++} = \left\{ \mathbf{A} \in \mathbb{R}^{n\times n} : \mathbf{A} \succ \mathbf{0} \right\}.$$

Obviously, the inclusion $\mathbb{S}^n_{++} \subseteq \mathbb{S}^n_+ \subseteq \mathbb{S}^n$ holds. Similarly, $\mathbb{S}^n_-$ is the set of all $n \times n$ negative semidefinite matrices, and $\mathbb{S}^n_{--}$ is the set of all $n \times n$ negative definite matrices:

$$\mathbb{S}^n_- = \left\{ \mathbf{A} \in \mathbb{R}^{n\times n} : \mathbf{A} \preceq \mathbf{0} \right\},$$
$$\mathbb{S}^n_{--} = \left\{ \mathbf{A} \in \mathbb{R}^{n\times n} : \mathbf{A} \prec \mathbf{0} \right\}.$$

The set of all $n \times n$ orthogonal matrices is denoted by $\mathbb{O}^n$:

$$\mathbb{O}^n = \left\{ \mathbf{A} \in \mathbb{R}^{n\times n} : \mathbf{A}\mathbf{A}^T = \mathbf{A}^T\mathbf{A} = \mathbf{I} \right\}.$$

### 1.8.2 Norms in $\mathbb{R}^{m\times n}$

If $\mathbb{R}^{m\times n}$ is endowed with the dot product, then the corresponding Euclidean norm is the *Frobenius norm* defined by

$$\|\mathbf{A}\|_F = \sqrt{\mathrm{Tr}(\mathbf{A}^T\mathbf{A})} = \sqrt{\sum_{i=1}^{m}\sum_{j=1}^{n} A_{ij}^2}, \quad \mathbf{A} \in \mathbb{R}^{m\times n}.$$

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Many examples of matrix norms are generated by using the concept of induced norms, which we now describe. Given a matrix $\mathbf{A} \in \mathbb{R}^{m\times n}$ and two norms $\|\cdot\|_a$ and $\|\cdot\|_b$ on $\mathbb{R}^n$ and $\mathbb{R}^m$, respectively, the *induced matrix norm* $\|\mathbf{A}\|_{a,b}$ is defined by

$$\|\mathbf{A}\|_{a,b} = \max_{\mathbf{x}}\{\|\mathbf{A}\mathbf{x}\|_b : \|\mathbf{x}\|_a \le 1\}.$$

It can be easily shown that the above definition implies that for any $\mathbf{x} \in \mathbb{R}^n$, the inequality

$$\|\mathbf{A}\mathbf{x}\|_b \le \|\mathbf{A}\|_{a,b}\|\mathbf{x}\|_a$$

holds. We refer to the matrix norm $\|\cdot\|_{a,b}$ as the $(a,b)$-norm. When $a = b$, we will simply refer to it as an $a$-norm and omit one of the subscripts in its notation, that is, use the notation $\|\cdot\|_a$ instead of $\|\cdot\|_{a,a}$.

**Example 1.1 (spectral norm).** If $\|\cdot\|_a = \|\cdot\|_b = \|\cdot\|_2$, then the induced norm of a matrix $\mathbf{A} \in \mathbb{R}^{m\times n}$ is the maximum singular value of $\mathbf{A}$:

$$\|\mathbf{A}\|_2 = \|\mathbf{A}\|_{2,2} = \sqrt{\lambda_{\max}(\mathbf{A}^T\mathbf{A})} \equiv \sigma_{\max}(\mathbf{A}). \quad \blacksquare$$

**Example 1.2 (1-norm).** When $\|\cdot\|_a = \|\cdot\|_b = \|\cdot\|_1$, the induced matrix norm of a matrix $\mathbf{A} \in \mathbb{R}^{m\times n}$ is given by

$$\|\mathbf{A}\|_1 = \max_{j=1,2,\ldots,n} \sum_{i=1}^{m} |A_{i,j}|.$$

This norm is also called the *maximum absolute column sum norm.* $\blacksquare$

**Example 1.3 ($\infty$-norm).** When $\|\cdot\|_a = \|\cdot\|_b = \|\cdot\|_\infty$, the induced matrix norm of a matrix $\mathbf{A} \in \mathbb{R}^{m\times n}$ is given by

$$\|\mathbf{A}\|_\infty = \max_{i=1,2,\ldots,m} \sum_{j=1}^{n} |A_{i,j}|.$$

This norm is also called the *maximum absolute row sum norm.* $\blacksquare$

## 1.9 Cartesian Product of Vector Spaces

Given $m$ vector spaces $\mathbb{E}_1, \mathbb{E}_2, \ldots, \mathbb{E}_m$ equipped with inner products $\langle\cdot,\cdot\rangle_{\mathbb{E}_i}$, their Cartesian product $\mathbb{E}_1 \times \mathbb{E}_2 \times \cdots \times \mathbb{E}_m$ is the vector space of all $m$-tuples $(\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_m)$ equipped with the component-wise addition between vectors:

$$(\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_m) + (\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_m) = (\mathbf{v}_1 + \mathbf{w}_1, \mathbf{v}_2 + \mathbf{w}_2, \ldots, \mathbf{v}_m + \mathbf{w}_m)$$

and the scalar-vector multiplication operation given by

$$\alpha(\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_m) = (\alpha\mathbf{v}_1, \alpha\mathbf{v}_2, \ldots, \alpha\mathbf{v}_m).$$

The inner product in the Cartesian product space is defined as

$$\langle(\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_m), (\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_m)\rangle_{\mathbb{E}_1\times\mathbb{E}_2\times\cdots\times\mathbb{E}_m} = \sum_{i=1}^{m} \langle\mathbf{v}_i, \mathbf{w}_i\rangle_{\mathbb{E}_i}. \tag{1.1}$$

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The space $\mathbb{R}\times\mathbb{R}$, for example, consists of all two-dimensional row vectors, so in that respect it is different than $\mathbb{R}^2$, which comprises all two-dimensional *column* vectors. However, with only a slight abuse of notation, we will occasionally refer to $\mathbb{R}\times\mathbb{R}$ as $\mathbb{R}^2$.

Suppose that $\mathbb{E}_1, \mathbb{E}_2, \ldots, \mathbb{E}_m$ are vector spaces with endowed norms $\|\cdot\|_{\mathbb{E}_1}, \|\cdot\|_{\mathbb{E}_2}, \ldots, \|\cdot\|_{\mathbb{E}_m}$, respectively. There are many ways to define a norm on the Cartesian product space $\mathbb{E}_1 \times \mathbb{E}_2 \times \cdots \times \mathbb{E}_m$. For example, for any $p \geq 1$, we can define the composite $l_p$-norm as

$$\|(\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_m)\| = \sqrt[p]{\sum_{i=1}^{m} \|\mathbf{u}_i\|_{\mathbb{E}_i}^p}.$$

Another norm is a composite weighted $l_2$-norm:

$$\|(\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_m)\| = \sqrt{\sum_{i=1}^{m} \omega_i \|\mathbf{u}_i\|_{\mathbb{E}_i}^2},$$

where $\omega_1, \omega_2, \ldots, \omega_m$ are given positive real numbers.

We will use the convention that if $\mathbb{E}_1, \mathbb{E}_2, \ldots, \mathbb{E}_m$ are Euclidean spaces, then $\mathbb{E}_1 \times \mathbb{E}_2 \times \cdots \times \mathbb{E}_m$ is also a Euclidean space, and consequently, by the definition (1.1) of the inner product in product spaces,

$$\|(\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_m)\|_{\mathbb{E}_1\times\mathbb{E}_2\times\cdots\times\mathbb{E}_m} = \sqrt{\sum_{i=1}^{m} \|\mathbf{u}_i\|_{\mathbb{E}_i}^2}.$$

## 1.10 Linear Transformations

Given two vector spaces $\mathbb{E}$ and $\mathbb{V}$, a function $\mathcal{A} : \mathbb{E} \to \mathbb{V}$ is called a *linear transformation* if the following property holds for any $\mathbf{x}, \mathbf{y} \in \mathbb{E}$ and $\alpha, \beta \in \mathbb{R}$:

$$\mathcal{A}(\alpha\mathbf{x} + \beta\mathbf{y}) = \alpha\mathcal{A}(\mathbf{x}) + \beta\mathcal{A}(\mathbf{y}).$$

All linear transformations from $\mathbb{R}^n$ to $\mathbb{R}^m$ have the form

$$\mathcal{A}(\mathbf{x}) = \mathbf{A}\mathbf{x}$$

for some matrix $\mathbf{A} \in \mathbb{R}^{m\times n}$. All linear transformations from $\mathbb{R}^{m\times n}$ to $\mathbb{R}^k$ have the form

$$\mathcal{A}(\mathbf{X}) = \begin{pmatrix} \mathrm{Tr}(\mathbf{A}_1^T\mathbf{X}) \\ \mathrm{Tr}(\mathbf{A}_2^T\mathbf{X}) \\ \vdots \\ \mathrm{Tr}(\mathbf{A}_k^T\mathbf{X}) \end{pmatrix}$$

for some $\mathbf{A}_1, \mathbf{A}_2, \ldots, \mathbf{A}_k \in \mathbb{R}^{m\times n}$. The *identity transformation*, denoted by $\mathcal{I}$, is defined by the relation $\mathcal{I}(\mathbf{x}) = \mathbf{x}$ for all $\mathbf{x} \in \mathbb{E}$.