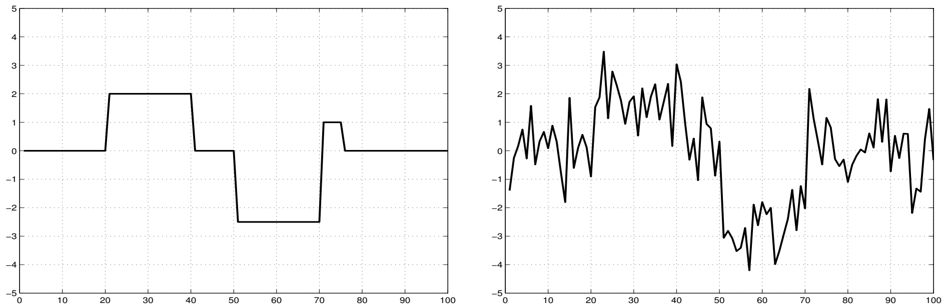
DM optimization: piecewise linear denoising

Let $\overline{x} = (\overline{x}^{(i)})_{1 \le i \le N} \in \mathbb{R}^N$ be a sampled piecewise constant noisy signal. We denote $y = \overline{x} + \varepsilon$ a noisy version of $\overline{x}$ with $\varepsilon \sim \mathcal{N}(0, \sigma^2)$. An illustration of $\overline{x}$ and y is displayed in Figure 1.



Original signal$\overline{x}$ Noisy signal y

FIGURE 1 – Illustration of a piecewise constant signal with $N = 100$ samples degraded with a white Gaussian noise of variance $\sigma^2 = 1$.

The objective of this exercice is to obtain a piecewise constant estimate $\widehat{x}$ which is the closest to the original $\overline{x}$ from data y. A solution consists in minimizing the following objective function :

$$\widehat{x}_\lambda = \arg \min_{x \in \mathbb{R}^N} \frac{1}{2} \|x - y\|_2^2 + \lambda \|Lx\|_1$$

where $(Lx)^{(i)} = x^{(i+1)} - x^{(i)}$ for every $i \in \{1, \dots, N-1\}$, $y \in \mathbb{R}^N$ and $\lambda > 0$. $L \in \mathbb{R}^{(N-1) \times N}$ denotes the finite difference operator.

1. Prove that the dual problem can be written as

$$\widehat{u}_\lambda \in \operatorname{Argmin}_{u \in \mathbb{R}^N} \frac{1}{2} \|y - L^* u\|_2^2 \quad \text{s.t.} \quad \|u\|_\infty \le \lambda,$$

   and that the relation with the dual solution is

$$\widehat{x}_\lambda = y - L^* \widehat{u}_\lambda.$$

2. Derive the closed form expression of the proximity operator of the ℓ_1-norm.

3. Derive the closed form expression of the proximity operator associated with $f(u) = \frac{1}{2}\|y - L^* u\|_2^2$.

4. Derive the closed form expression of the projection associated to the convex set $C = \{u \in \mathbb{R}^N \mid \|u\|_\infty \leq \lambda\}$
5. Detail the forward-backward iterations applied on the dual problem to estimate $\widehat{u}_\lambda$ and deduce $\widehat{x}_\lambda$. What are the convergence guarantees of such iterations ?
6. Detail the Douglas-Rachford iterations applied on the dual problem $\widehat{u}_\lambda$ and deduce $\widehat{x}_\lambda$. What are the convergence guarantees of such iterations ?
7. Detail the Chambolle-Pock primal-dual iterations applied on the primal problem $\widehat{x}_\lambda$. What are the convergence guarantees of such iterations ?
8. Would it be possible to improve the convergence rate of these schemes ? If yes, specify how.