Additive Logistic Regression
a Statistical View of Boosting

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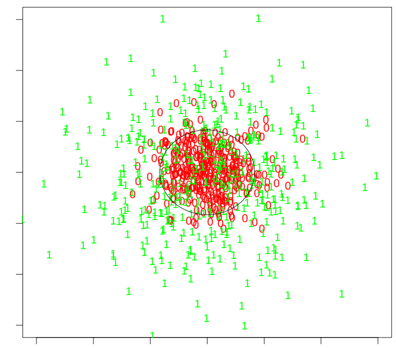
Classification Problem



Data $(X, Y) \in R^p \times \{0, 1\}$.

X is predictor, feature; Y is class label, response.

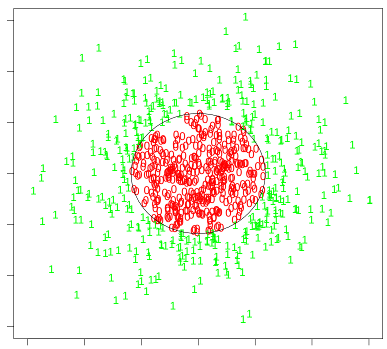
(X, Y) have joint probability distribution $\mathcal{D}$.

Goal: Based on N training pairs $(X_i,\ Y_i)$ drawn from $\mathcal{D}$ produce a classifier $\hat{C}(X) \in \{0, 1\}$

Goal: choose $\hat{C}$ to have low generalization error

$$\begin{aligned} R(\hat{C}) &= P_{\mathcal{D}}(\hat{C}(X) \neq Y) \\ &= E_{\mathcal{D}}[1_{(\hat{C}(X) \neq Y)}] \end{aligned}$$

Deterministic Concepts



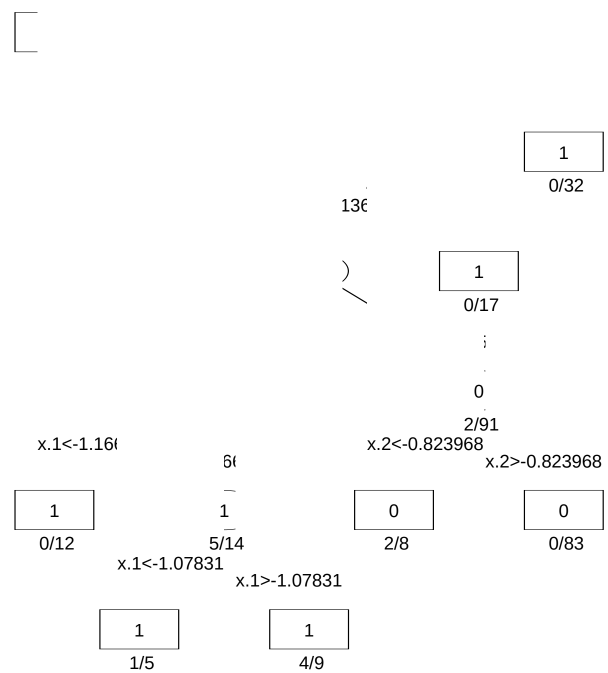
$X \in R^p$ has distribution $\mathcal{D}$.

$C(X)$ is deterministic function $\in$ concept class.

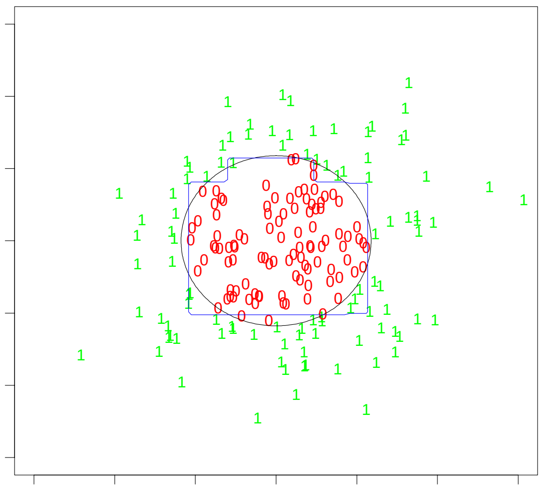
Goal: Based on N training pairs $(X_i,\ Y_i = C(X_i))$ drawn from $\mathcal{D}$ produce a classifier $\hat{C}(X) \in \{0, 1\}$

Goal: choose $\hat{C}$ to have low generalization error

$$\begin{aligned} R(\hat{C}) &= P_{\mathcal{D}}(\hat{C}(X) \neq C(X)) \\ &= E_{\mathcal{D}}[1_{(\hat{C}(X) \neq C(X))}] \end{aligned}$$



Decision Boundary: Tree



When the nested spheres are in R^{10}, CART^{TM} produces a rather noisy and inaccurate rule $\hat{C}(X)$, with error rates around 40%.

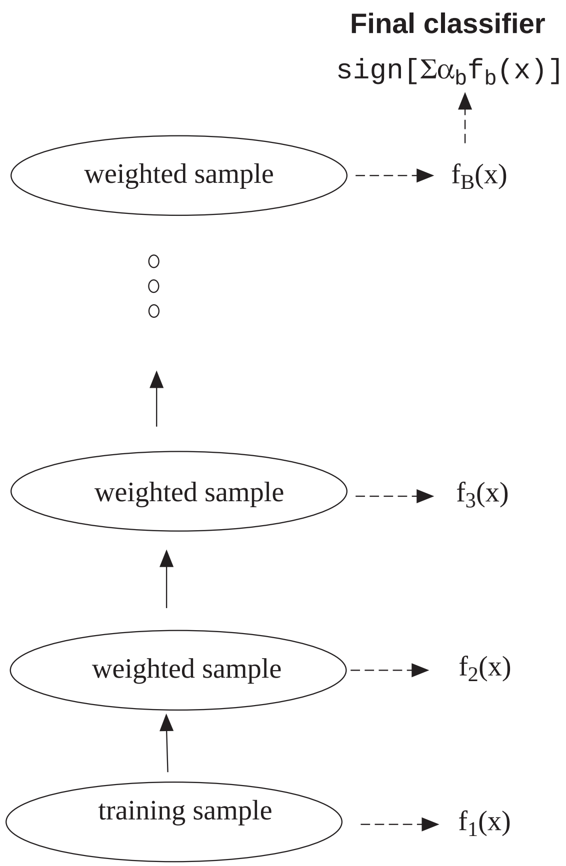
Bagging and Boosting

Classification trees can be simple, but often produce noisy (bushy) or weak (stunted) classifiers.

- Bagging (Breiman, 1996): Fit many large trees to bootstrap-resampled versions of the training data, and classify by majority vote.
- Boosting (Freund & Shapire, 1996): Fit many large or small trees to reweighted versions of the training data. Classify by weighted majority vote.

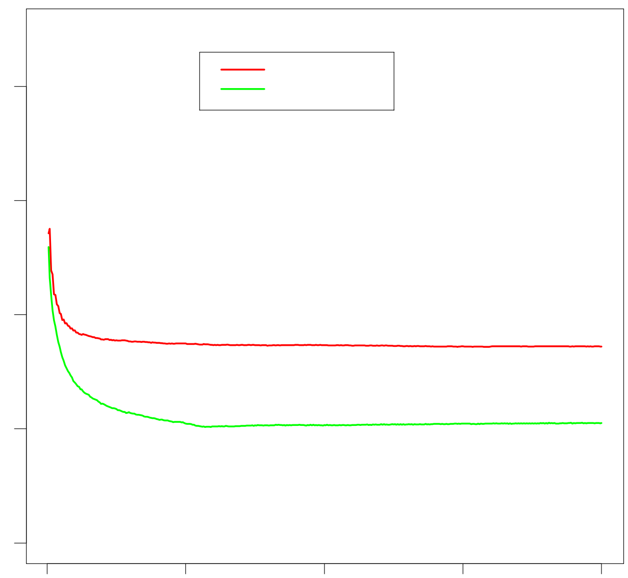
In general Boosting > Bagging > Single Tree.

"AdaBoost $\cdots$ best off-the-shelf classifier in the world" — Leo Breiman, NIPS workshop, 1996.



The weighting in boosting can be achieved by weighted importance sampling.

Bagging and Boosting

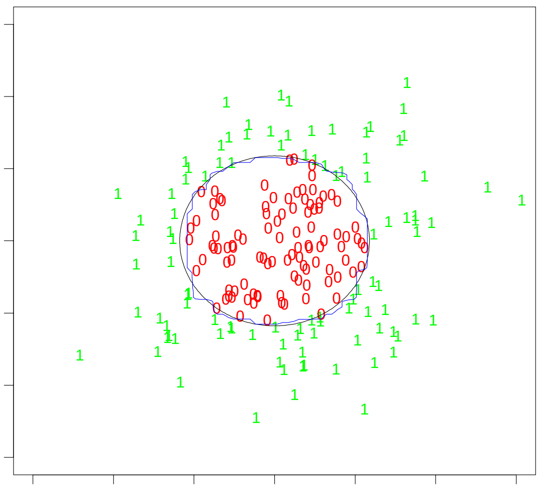


2000 points from Nested Spheres in R^{10}; Bayes error rate is 0%.

Trees are grown Best First without pruning.

Leftmost iteration is a single tree.

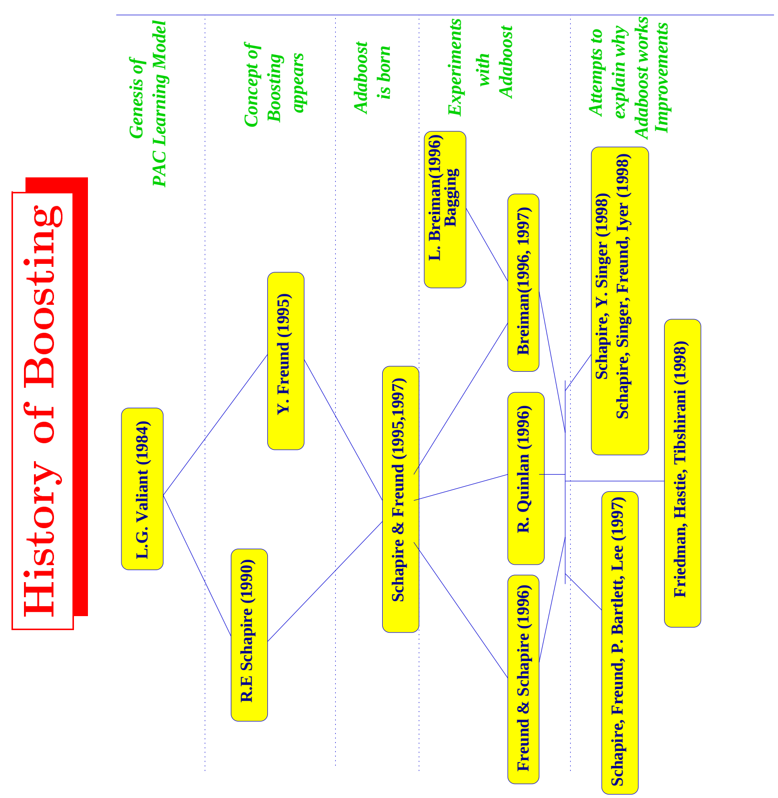
Decision Boundary: Boosting



Bagging and Boosting average many trees, and produce smoother decision boundaries.

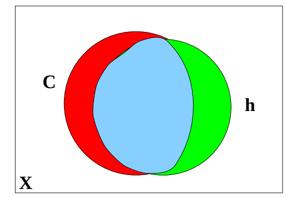
AdaBoost (Freund & Schapire, 1996)

1. Start with weights $w_i = 1/N \ \forall i = 1, \ldots, N$. $y_i \in \{-1, 1\}$.
2. Repeat for $m = 1, 2, \ldots, M$:
   (a) Estimate the weak learner $f_m(x) \in \{-1, 1\}$ from the training data with weights w_i.
   (b) Compute $e_m = E_w[1(y \neq f_m(x)]$, $c_m = \log((1 - e_m)/e_m)$.
   (c) Set $w_i \leftarrow w_i \exp[c_m \cdot 1(y_i \neq f_m(x_i)]$, $i = 1, 2, \ldots N$, and renormalize so that $\sum_i w_i = 1$.
3. Output the majority weight classifier $C(x) = \text{sign}[\sum_{m=1}^{M} c_m f_m(x)]$.



PAC Learning Model

$X \sim \mathcal{D}$: Instance Space
$C : X \mapsto \{0, 1\}$ Concept $\in \mathcal{C}$
$h : X \mapsto \{0, 1\}$ Hypothesis $\in \mathcal{H}$
error$(h) = P_{\mathcal{D}}[C(X) \neq h(X)]$



Definition: Consider a concept class $\mathcal{C}$ defined over a set X of length N. L is a learner (algorithm) using hypothesis space $\mathcal{H}$. $\mathcal{C}$ is PAC learn-able by $\mathcal{L}$ using $\mathcal{H}$ if for all $C \in \mathcal{C}$, all distributions $\mathcal{D}$ over X and all $\epsilon, \delta \in (0, \frac{1}{2})$, learner L will, with $Pr \geq (1 - \delta)$, output an $h \in \mathcal{H}$ s.t. error$_D(h) \leq \epsilon$ in time polynomial in $1/\epsilon, 1/\delta, N$ and size$(\mathcal{C})$.

Such an L is called a strong Learner.

Boosting a Weak Learner

Weak learner L produces an h with error rate $\beta = (\frac{1}{2} - \varepsilon) < \frac{1}{2}$, with $Pr \geq (1 - \delta)$ for any $\mathcal{D}$.

L has access to continuous stream of training data and a class oracle.

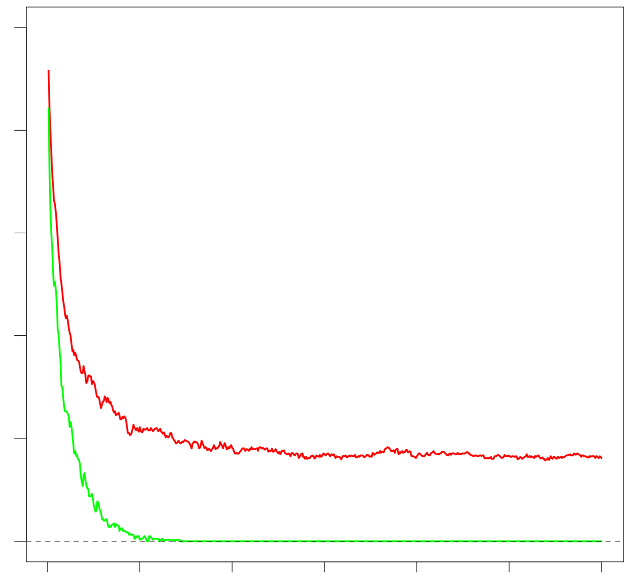
1. L learns h_1 on first N training points.
2. L randomly filters the next batch of training points, extracting $N/2$ points correctly classified by h_1, $N/2$ incorrectly classified, and produces h_2.
3. L builds a third training set of N points for which h_1 and h_2 disagree, and produces h_3.
4. L outputs $h = \mathit{Majority\ Vote}(h_1, h_2, h_3)$

THEOREM (Schapire, 1990): "The Strength of Weak Learnability"

$$\text{error}_D(h) \leq 3\beta^2 - 2\beta^3 < \beta$$

Boosting & Training Error

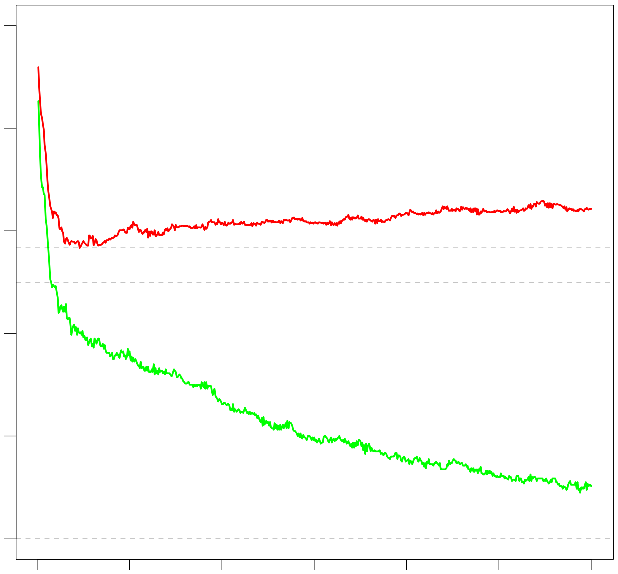
Nested spheres in R^{10} — Bayes error is 0%.



Boosting drives the training error to zero. Further iterations continue to improve test error in many examples.

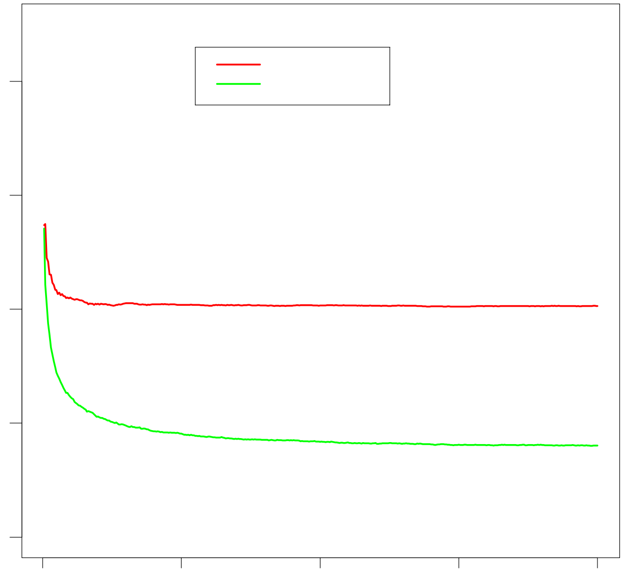
Boosting Noisy Problems

Nested Gaussians in R^{10} — Bayes error is 25%.



Here the test error does increase, but quite slowly.

Bagging and Boosting Smaller Trees

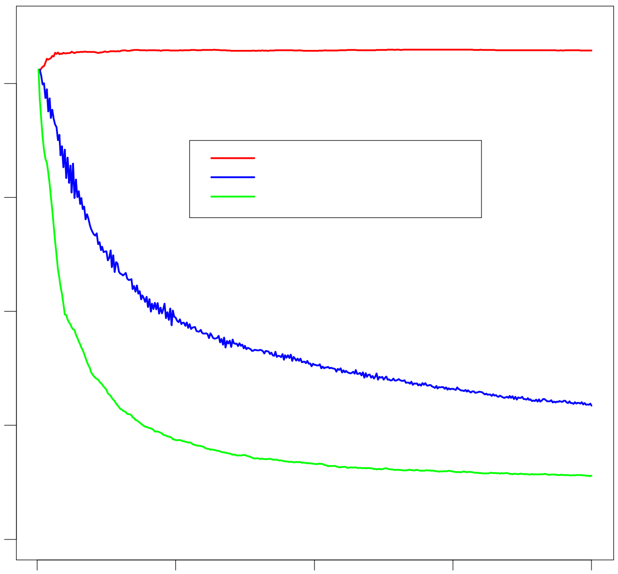


2000 points in R^{10}; Bayes error rate is 0%.

Each tree has 10 terminal nodes, grown best-first.

Bagging-Boosting gap is wider

Bagging and Boosting Stumps



2000 points in R^{10}; Bayes error rate is 0%.

Each tree has 2 terminal nodes, grown best-first.

Bagging fails — boosting does best ever!

Prediction games

Results of Freund and Schapire (1996) and Breiman (1998).

Idea:

- Start with fixed learners $f_1(x), f_2(x) \ldots f_M(x)$.
- Play a two person game: player 1 picks observation weights w_i; player 2 picks learner weights c_m (i.e. he will use learner $f_m(x)$ with probability c_m).
- Player 1 tries to make the prediction problem as hard as possible, while Player 2 does the best he can on the weighted problem. We judge difficulty by the approximate "margin", a smooth version of misclassification loss.

- Result: this is a zero-sum game, and the minimax theorem gives the best strategy for each player. Furthermore, AdaBoost converges to this optimal strategy!
- However: link with statistical properties of actual AdaBoost is tenuous:
  1. why minimize hardest weighted problem (makes test error smaller?);
  2. actual AdaBoost does not use a random choice of learner,
  3. actual AdaBoost finds the learners $f_m(x)$.

Stage-wise Additive Modeling

Boosting builds an additive model $F(x) = \sum_{m=1}^{M} f_m(x)$ and then $C(x) = \text{sign}[F(x)]$.

We do things like that in statistics!

- GAMs: $F(x) = \sum_j f_j(x_j)$
- Basis expansions: $F(x) = \sum_{m=1}^{M} \theta_m h_m(x)$

Traditionally each of the terms $f_m(x)$ is different in nature, and they are fit jointly (i.e. least squares, maximum likelihood)

With Boosting, each term is equivalent in nature, and they are fit in a stagewise fashion.

Simple example: stagewise least-squares? Fix the past $M-1$ functions, and update the Mth using a tree:

$$\min_{f_M \in Tree(x)} E(Y - \sum_{m=1}^{M-1} f_m(x) - f_M(x))^2$$

Boosting and Additive Models

- Discrete AdaBoost builds an additive model

$$F(x) = \sum_{m=1}^{M} c_m f_m(x)$$

by stage-wise optimization of

$$J(F(x)) = Ee^{-y(F(x))}$$

- Given an imperfect $F_{M-1}(x)$, the updates in Discrete AdaBoost correspond to a Newton step towards minimizing

$$J(F_{M-1}(x) + c_M f_M(x)) = Ee^{-y(F_{M-1}(x)+c_M f_M(x))}$$

over $f_M(x) \in \{-1, 1\}$, with step length c_M.

- $Ee^{-y(F(x))}$ is minimized at

$$F(x) = \frac{1}{2} \log \frac{P(y = 1|x)}{P(y = -1|x)}$$

Hence Adaboost is fitting an additive logistic regression model.

Details

f_M:

$$f_M(x) = \arg \min_{g(x) \in \{-1,1\}} E_w(y - g(x))^2$$

with weights $w(x, y) = e^{-yF_{M-1}(x)}$.

c_M:

$$\begin{aligned} c_M &= \arg\min_c E_w e^{-cyf_M(x)} \\ &= \frac{1}{2} \log \frac{1-e}{e} \end{aligned}$$

with $e = E_w[1_{[y \neq f_M(x)]}]$.

- Empirical version: at each stage, $f_M(x)$ is estimated by the classification at the terminal nodes of a tree, grown to appropriately weighted versions of the training data.

- At the Mth stage of the Discrete AdaBoost iterations, the weights are such that f_{M-1} has weighted training error $= 50\%$.

Real AdaBoost

1. Start with weights $w_i = 1/N$, $i = 1, 2, \ldots, N$. $y_i \in \{-1, 1\}$.
2. Repeat for $m = 1, 2, \ldots, M$:
   (a) Fit the class probability estimate $p_m(x) = \hat{P}_w(y = 1|x) \in [0, 1]$ using weights w_i on the training data.
   (b) Set $f_m(x) \leftarrow \frac{1}{2} \log \frac{p_m(x)}{1 - p_m(x)} \in R$.
   (c) Set $w_i \leftarrow w_i \exp[-y_i f_m(x_i)]$, $i = 1, 2, \ldots N$, and renormalize so that $\sum_i w_i = 1$.
3. Output the classifier $\text{sign}[\sum_{m=1}^{M} f_m(x)]$

Real AdaBoost

- Real AdaBoost also builds an additive logistic regression model:

$$\log \frac{P(y = 1|x)}{P(y = -1|x)} = \sum_{m=1}^{M} f_m(x)$$

  by stage-wise optimization of $J(F(x)) = Ee^{-y(F(x))}$.

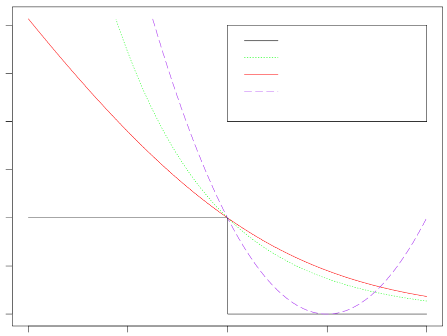
- Given an imperfect $F_{M-1}(x)$, Real AdaBoost minimizes $J(F_{M-1}(x) + f_M(x)) = Ee^{-y(F_{M-1}(x) + f_M(x))}$ over $f_M(x) \in R$, with solution

$$f_M(x) = \frac{1}{2} \log \frac{P_w(y = 1)}{P_w(y = -1)}$$

  where the weights $w(x, y) = e^{-yF_{M-1}(x)}$.

- Empirical version: at each stage, $P_w(\cdot|x)$ is estimated by averages at the terminal nodes of a tree, grown to appropriately weighted versions of the training data.

Why $J(F(x)) = Ee^{-yF(x)}$?



- $e^{-yF(x)}$ is a monotone, smooth upper bound on misclassification loss at x.
- $J(F)$ is an expected χ statistic at its minimum, and equivalent to the binomial log-likelihood to second order.
- Stage-wise binomial maximum-likelihood estimation of additive models based on trees works as least as well.

Stagewise Maximum Likelihood

Consider the model

$$F_M(x) = \log \frac{P(y^* = 1|x)}{P(y^* = 0|x)} = \sum_{m=1}^{M} f_m(x)$$

or

$$P(y^* = 1|x) = p(x) = \frac{e^{F(x)}}{1 + e^{F(x)}}$$

The binomial log-likelihood is

$$\begin{aligned} \ell(F(x)) &= E[y^* \log(p(x)) + (1 - y^*) \log(1 - p(x))] \\ &= E[y^* F(x) - \log(1 + e^F(x)) \end{aligned}$$

Stagewise Maximum Likelihood: Given an imperfect $F_{M-1}(x)$, maximize $\ell(F_{M-1}(x) + f_M(x))$ over $f_M(x) \in R$

The LogitBoost algorithm takes a single Newton-Step at each stage.

LogitBoost

1. Start with weights $w_i = 1/N \;\; i = 1, 2, \ldots, N$, $F(x) = 0$ and probability estimates $p(x_i) = \frac{1}{2}$.
2. Repeat for $m = 1, 2, \ldots, M$:
   (a) Compute the working response and weights

   $$z_i = \frac{y_i^* - p(x_i)}{p(x_i)(1 - p(x_i))}$$
   $$w_i = p(x_i)(1 - p(x_i))$$

   (b) Fit the function $f_m(x)$ by a weighted least-squares regression of z_i to x_i using weights w_i. i.e. a weighted tree

   (c) Update $F(x) \leftarrow F(x) + f_m(x)$ and $p(x)$
3. Output the classifier $\text{sign}[F(x)] = \text{sign}[\sum_{m=1}^{M} f_m(x)]$

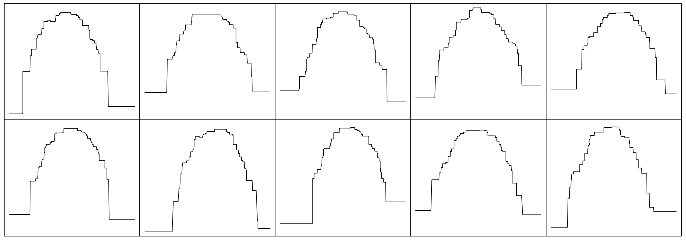
We also have a natural generalization of LogitBoost for multiple classes.

Additive Logistic Trees

Best-first tree growing allows us to limit the size of each tree, and hence the interaction order.

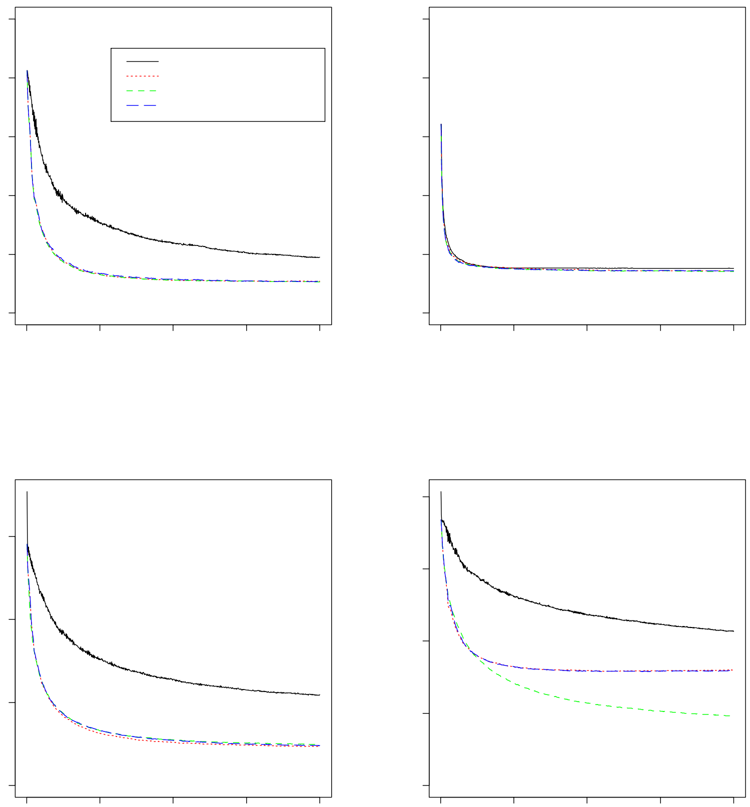
By collecting terms, we get

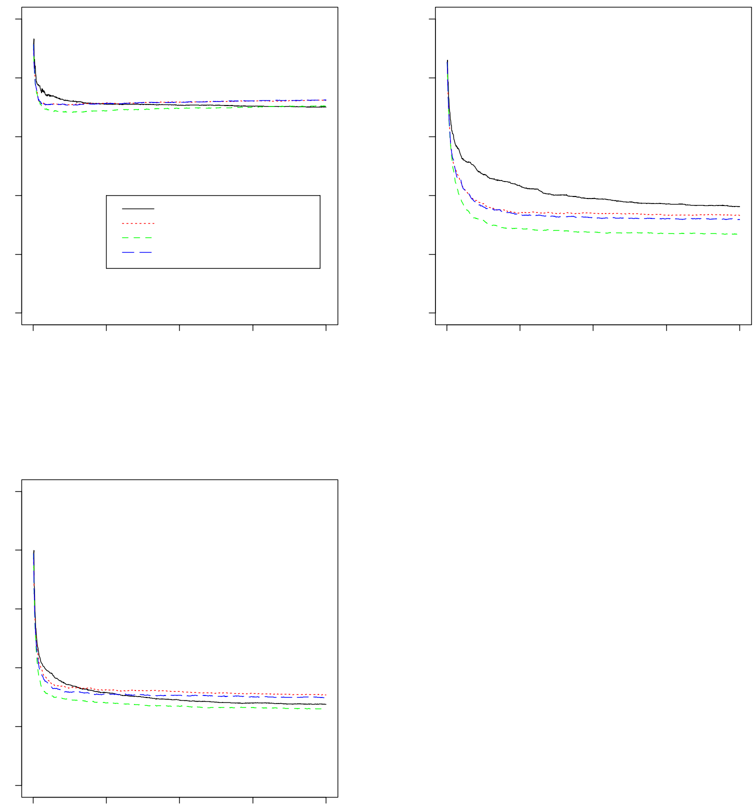
$$F(x) = \sum_j f_j(x_j) + \sum_{j,k} f_{jk}(x_j, x_k) + \sum_{j,k,l} f_{jkl}(x_j, x_k, x_l) + \ldots$$



Coordinate functions for Additive Stumps Model.

Boosting uses stage-wise optimization, as opposed to joint optimization (full least squares, backfitting,$\cdots$).





Large Real Example: Satimage

Method	Terminal Nodes	Iterations 20	50	100	200
Satimage	CART error = .148				
LogitBoost	2	.140	.120	.112	.102
Real AdaBoost	2	.148	.126	.117	.119
Gentle AdaBoost	2	.148	.129	.119	.119
Discrete AdaBoost	2	.174	.156	.140	.128
LogitBoost	8	.096	.095	.092	.088
Real AdaBoost	8	.105	.102	.092	.091
Gentle AdaBoost	8	.106	.103	.095	.089
Discrete AdaBoost	8	.122	.107	.100	.099

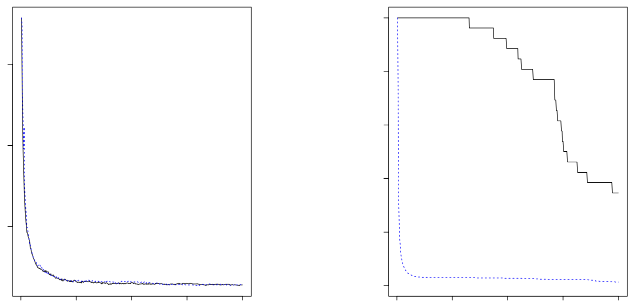
4435 training, 2000 test, 36 features, 6 classes

Large Real Example: Letter

Method	Terminal Nodes	Iterations 20	50	100	200	Fraction
Letter	CART error = .124					
LogitBoost	2	.250	.182	.159	.145	.06
Real AdaBoost	2	.244	.181	.160	.150	.12
Gentle AdaBoost	2	.246	.187	.157	.145	.14
Discrete AdaBoost	2	.310	.226	.196	.185	.18
LogitBoost	8	.075	.047	.036	.033	.03
Real AdaBoost	8	.068	.041	.033	.032	.03
Gentle AdaBoost	8	.068	.040	.030	.028	.03
Discrete AdaBoost	8	.080	.045	.035	.029	.03

16000 training, 4000 test, 16 features, 26 classes

Weight Trimming

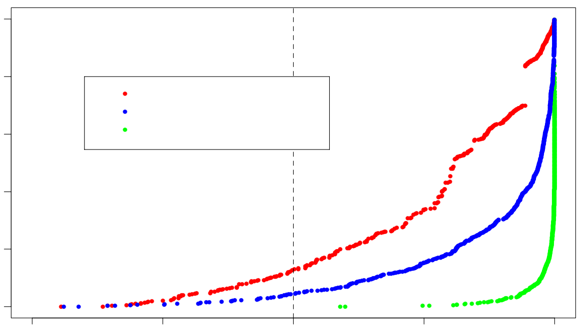


- At each iteration, observations with $w_i < t(\beta)$ are not used for training. $t(\beta)$ is βth quantile of weight distribution, and $\beta \in [0.01, 0.1]$. Works better for LogitBoost:
  - LogitBoost has weights $w_i = p_i(1 - p_i)$ which are large near the decision boundary.
  - AdaBoost has weights $w_i = e^{-y_i F_M(x_i)}$ (recall $y_i \in \{-1, 1\}$). Large for misclassified points.
- For multiple-class procedures, if the class-k logit $F_{mk} > 15 + \log(N)$, training stops for that class.

Summary and Closing Comments

- The introduction of Boosting by Schapire, Freund, and colleagues has brought us an exciting and important set of new ideas.
- Boosting fits additive logistic models, where each component (base learner) is simple. The complexity needed for the base learner depends on the target function.
- Little connection between weighted boosting and bagging; boosting is primarily a bias reduction procedure, while the goal of bagging is variance reduction.
- The distinction becomes blurred when weighting is achieved (in boosting) by importance sampling.

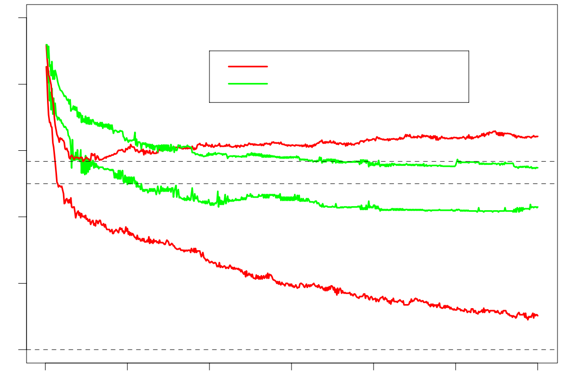
Margins



$$\text{margin}(X) = M(X) = 2\hat{P}_{C(X)} - 1$$

Freund & Schapire (1997): Boosting generalizes because it pushes the training margins well above zero, while keeping the VC dimension under control (also Vapnik, 1996). With $Pr \geq (1-\delta)$

$$P_{Test}(M(X) \leq 0) \quad \leq \quad P_{Train}(M(X) \leq \theta) + O\left(\frac{1}{\sqrt{N}}\left(\frac{\log N \log |\mathcal{H}|}{\theta^2 + \log 1/\delta}\right)^{\frac{1}{2}}\right)$$



How does Boosting avoid overfitting?

- As iterations proceed, impact of change is localized.
- Parameters are not jointly optimized — stagewise estimation slows down the learning process.
- Classifiers are hurt less by overfitting (Cover and Hart, 1967).
- Margin theory of Schapire and Freund, Vapnik? Disputed by Breiman (1997).
- Jury is still out!