Hypercontractivity

We work on the hypercube $\{-1,1\}^n$.

Noise operator T_ρ $\quad \rho \in [-1,1]$

For $f : \{-1,1\}^n \to \mathbb{R}$ $\quad (T_\rho f)(x) = \mathbb{E}\, f(y)$ where $y \sim N_\rho(x)$

- $\|T_\rho f\|_{L^p} \leq \|f\|_{L^p}$

$x \mapsto x^p$ convex

$$\|T_\rho f\|_{L^p}^p = \mathbb{E}_x |T_\rho f|^p = \mathbb{E}_x \Big| \mathbb{E}_{\substack{y \\ \sim \\ N_\rho(x)}} f(y) \Big|^p$$

T_ρ is contractive on L^p.

$$\leq \underbrace{\mathbb{E}_x \mathbb{E}_{y \sim N_\rho(x)}}_{y \text{ uniform on } \{-1,1\}^n} |f(y)|^p = \|f\|_{L^p}^p .$$

- Hypercontractivity

Theorem: If $1 \leq p \leq q \leq \infty$, $\quad 0 \leq \rho \leq \sqrt{\frac{p-1}{q-1}}$ then $\forall f : \{-1,1\}^n \to \mathbb{R}$ (HC)

$$\|T_\rho f\|_{L^q} \leq \|f\|_{L^p}$$

enough to check this case.

(1) Why this implies the KKL theorem?

KKL theorem: If $f : \{-1,1\}^n \to \{-1,1\}$ then

$$\max_i \mathrm{Inf}_i(f) \geq \frac{c \log n}{n} \mathrm{Var}(f)$$

$$\mathrm{Inf}_i(f) = \mathbb{P}\big(f(x^{\oplus i}) \neq f(x)\big) = \|D_i f\|_{L^2}^2 = \|D_i f\|_{L^p}^p \quad \forall p$$

$x^{\oplus i}$: x with ith bit flipped.

$$D_i f(x) = \frac{f(x^{[i \to 1]}) - f(x^{[i \to -1]})}{2}$$

Write $f = \sum_{S \subset [n]} \hat{f}_S w_S$ $\qquad \mathrm{Inf}_i f = \sum_{S \ni i} \hat{f}_S^2 .$ $\qquad D_i w_S = \begin{cases} 0 & \text{if } i \notin S \\ w_{S \setminus \{i\}} & \text{if } i \in S \end{cases}$

$$T_\rho f = \sum_{S \subset [n]} \rho^{|S|} \hat{f}_S w_S$$

We use (HC) for $\rho = \frac{1}{2}$ $\quad q = 2$ $\quad \sqrt{p-1} = \frac{1}{2} \Rightarrow p = \frac{5}{4}$

$\forall g \quad \|T_{\frac{1}{2}} g\|_{L^2} \leq \|g\|_{L^{5/4}}$. Apply this to $g = D_i f$

$$\|T_{\frac{1}{2}} D_i f\|_{L^2} \leq \|D_i f\|_{L^{5/4}} = \mathrm{Inf}_i(f)^{4/5}$$

$$\mathrm{Var}(f) = \sum_{\substack{S \neq \emptyset \\ S \subset [n]}} \hat{f}_S^2$$

For $M \in \mathbb{N}$ to be defined $\quad (M \sim \log n)$

$$\sum_{1 \leq |S| \leq M} \hat{f}_S^2 \leq \sum_{1 \leq |S| \leq M} |S| \hat{f}_S^2$$

$$\leq 4^M \sum_{\substack{\cancel{1 \leq |S| \leq M} \\ 1 \leq |S|}} 4^{-|S|} |S| \hat{f}_S^2$$

$$= 4^M \sum^n \|T_{\frac{1}{2}} D_i f\|_{L^2}^2$$

$T_{\frac{1}{2}} D_i = D_i T_{\frac{1}{2}}$

$$\|T_{\frac{1}{2}} D_i f\|_{L^2}^2 = \sum_{S \ni i} \widehat{(T_{\frac{1}{2}} f)}_S^2$$

$$= \left(\frac{1}{2}\right)^{|S|} \hat{f}_S^2$$

$$\|T_{\frac12} D_i f\|_{L^2} = \sum_{S\ni i} \left(\tfrac12\right)^{|S|} \hat f_S$$

$$= \sum_{S\ni i} \left(\frac14\right)^{|S|} \hat f_S^{\,2}$$

$1 \le |S|$

$$= 4^m \sum_{i=1}^{n} \|T_{\frac12} D_i f\|_{L^2}^2$$

$$\le 4^n \sum_i (\mathrm{Inf}_i f)^{8/5}$$

If $\exists i \quad \mathrm{Inf}_i(f) \geq \dfrac{\mathrm{Var}(f)}{n^{3/4}}$ then ok $\left(\dfrac{1}{n^{3/4}} \geq \dfrac{c \log n}{n}\right)$

Otherwise $$\sum_{1 \le |S| \le n} \hat f_S^{\,2} \le 4^n \, n \, \frac{\mathrm{Var}(f)^{8/5}}{n^{6/5}} \qquad \left[\mathrm{Var}(f) \le 1\right]$$

$$\le 4^n n^{-1/5} \mathrm{Var}(f)$$

Choose n such that $4^n n^{-1/5} = n^{-1/10}$ $[n \sim c \log n]$

$$\sum_{1 \le |S| \le M} \hat f_S^{\,2} \le n^{-1/10} \mathrm{Var}(f)$$

and since $\mathrm{Var}(f) = \sum_{1 \le |S|} \hat f_S^{\,2}$ we get that

$$\sum_{|S| > n} \hat f_S^{\,2} \geq \left(1 - \frac{1}{n^{1/10}}\right) \mathrm{Var}(f).$$

$$\max_i \mathrm{Inf}_i(f) \geq \frac1n \sum_i \mathrm{Inf}_i(f) = \frac1n \sum_S |S| \hat f_S^{\,2}$$

$$\geq \frac1n \sum_{|S|>n} |S| \hat f_S^{\,2}$$

$$\geq \frac{M}{n} \sum_{|S|>n} \hat f_S^{\,2} \geq \frac{M}{n} \cdot \frac12 \mathrm{Var}(f)$$

$$\geq \frac{c \log n}{n} \mathrm{Var}(f).$$

<u>Proof of (HC)</u> **Proof** by induction on n.

induction step : quite easy

$n=1$: very tricky

<u>inductive step</u> $n \to n+1$

more generally $[n] = I \cup J$ (disjoint union)

$$\left.\begin{array}{l}(HC)\ \{-1,1\}^I \\ (HC)\ \{-1,1\}^J\end{array}\right\} \Rightarrow (HC)\ \{-1,1\}^n$$

"tensorization property"

$q \geq p$

$f : \{-1,1\}^n \to \mathbb{R}$ $\quad f(x_I, x_J)$, $x_I \in \{-1,1\}^I$, $x_J \in \{-1,1\}^J$

$$T_\varepsilon f = T_\varepsilon^I T_\varepsilon^J f = T_\varepsilon^J T_\varepsilon^I f$$

$$\|T_\varepsilon f\|_{L^q}^q = \mathbb{E}\,|T_\varepsilon f|^q = \mathbb{E}_I \mathbb{E}_J |T_\varepsilon^J T_\varepsilon^I f|^q$$

~~then~~ for fixed x_I

$$\underset{\text{for } HC_J}{\le} \mathbb{E}_I \left(\mathbb{E}_J |T_\varepsilon^I f|^p\right)^{\frac qp}$$

$r = \dfrac qp \geq 1$

$L^r(\{-1,1\}^I)$

$$= \left\| \mathbb{E}_J |T_\varepsilon^I f|^p \right\|_{L^r}^{r = \frac qp}$$

$$\|T_\rho f\|_{L^q}^p \le \left\| \mathbb{E}_J |T_\rho^I f|^p \right\|_{L^r(I)}$$
$$\le \mathbb{E}_J \left\| |T_\rho^I f|^p \right\|_{L^r} = \mathbb{E}_J \left(\mathbb{E}_I |T_\rho^I f|^q \right)^{\frac{1}{r}}$$
$$\text{HC in } I \longrightarrow \le \mathbb{E}_J \left(\mathbb{E}_I |f|^p \right)^{\frac{q}{p}\cdot\frac{1}{r}} = \mathbb{E}_J \mathbb{E}_I |f|^p$$
$$= \|f\|_{L^p}^p .$$

$\underline{n=1}$ tricky - we only prove for $q=2$ $p=1+\rho^2$ (implies $p \ge 1+\rho^2$).

- take $f : \{-1,1\} \to \mathbb{R}$

$a = \mathbb{E} f = \dfrac{f(1)+f(-1)}{2}$ $\quad b = \dfrac{f(1)-f(-1)}{2}$ $\quad f = a w_\emptyset + b w_{\{1\}}$

$T_\rho f = a w_\emptyset + \rho b w_{\{1\}}$

$$\|T_\rho f\|_{L^2}^2 = a^2 + \rho^2 b^2 \overset{?}{\le} \|f\|_{L^p}^2$$

$$\|f\|_{L^p} = \left(\frac{|a+b|^p + |a-b|^p}{2} \right)^{\frac{1}{p}}$$

Need to prove
$$\left(a^2 + \rho^2 b^2 \right)^{\frac{1+\rho^2}{2}} \le \frac{|a+b|^{1+\rho^2} + |a-b|^{1+\rho^2}}{2}$$

We can assume $a \ge 0$, $b \ge 0$, $a \ge b$ (otherwise flip) $a=1$

$b \leftarrow x$
$$\left(1+\rho^2 x^2\right)^{\frac{1+\rho^2}{2}} \overset{?}{\le} \frac{(1+x)^{1+\rho^2} + (1-x)^{1+\rho^2}}{2} \qquad 0 \le x \le 1$$

$0 \le \alpha \le 1$ $\quad (1+t)^\alpha \le 1+\alpha t$ $\quad \forall t \ge 0$

$$\le 1 + \rho^2 x^2 \frac{1+\rho^2}{2} \le 1 + \frac{x^2}{2}\rho^2(1+\rho^2) + O(x^3)$$

$\|$ when expanded in Taylor series, all coefficients are ≥ 0.

cf
$\cosh(t) \le \exp(t^2/2)$
used in Chernoff

GAUSSIAN hypercontractivity

GAUSS space $(\mathbb{R}^n, \gamma_n)$ ← standard Gaussian measure $N(0, Id_n)$

A pair (X,Y) of random variables in $\mathbb{R}^n$ $\quad \rho \in [-1,1]$
is a pair of ρ-correlated standard Gaussians if

- (X,Y) is jointly Gaussian
- $X \sim N(0, Id_n)$, $Y \sim N(0, Id_n)$ $\quad \mathbb{E}[X_i Y_j] = \begin{cases} 0 & \text{if } i \ne j \\ \rho & \text{otherwise if } i=j. \end{cases}$

Prototype : Z_1, Z_2 iid $N(0, Id_n)$
$(Z_1, \rho Z_1 + \sqrt{1-\rho^2}\, Z_2)$ is a pair of ρ-correlated standard Gaussians.

We define the GAUSSIAN noise operators on $L^2(\gamma_n)$.

$$(U_\rho f)(x) = \mathbb{E} f(\rho x + \sqrt{1-\rho^2}\, Z)$$

We define

$f \in L^1(\gamma_n)$, $z \in \mathbb{R}^n$: $\quad (U_\rho f)(z) = \mathbb{E}_{Z \sim N(0, Id_n)} f(\rho z + \sqrt{1-\rho^2}\, Z)$

$$\mathbb{E}_{\gamma_n} U_\rho f = \mathbb{E}_{\substack{Z \sim N(0, Id_n) \\ Z' \sim N(0, Id_n) \\ Z \perp Z'}} f(\underbrace{\rho Z + \sqrt{1-\rho^2}\, Z'}_{N(0, Id_n)}) = \mathbb{E}_{\gamma_n} f .$$

$U_\rho \approx T_\rho$

$$(U_\rho U_{\rho'} f)(z) = \mathbb{E}_{\substack{Z_1, Z_2 \text{ iid} \\ N(0, Id_n)}} \left[f(\rho(\rho' z + \sqrt{1-\rho'^2}\, Z_1) + \sqrt{1-\rho^2}\, Z_2) \right]$$

$$= \mathbb{E}_{Z \sim N(0, Id_n)} \left[f(\rho\rho' z + \sqrt{1-(\rho\rho')^2}\, Z \right]$$

$U_\rho U_{\rho'} = U_{\rho\rho'}$

More usually $\rho \leftarrow e^{-t}$ $\quad P_t = U_{e^{-t}}$

$\rho \in [0,1]$ (0: complete noise, 1: no noise)

$P_t P_{t'} = P_{t+t'}$

$(P_t)_{t \geq 0}$ is called the ORNSTEIN-UHLENBECK semigroup.

As before $\|U_\rho f\|_{L^p} \leq \|f\|_{L^p}$ $\quad \forall f \in L^p$ $\; U_\rho f$ is smooth (C^∞) $\forall \rho \in]-1,1[$.

GAUSSIAN hypercontractivity theorem

$$\boxed{\begin{array}{l} \forall f \in L^p(\gamma_n) \\ q \geq p \quad 0 \leq \rho \leq \sqrt{\frac{p-1}{q-1}} \end{array} \qquad \|U_\rho f\|_{L^q} \leq \|f\|_{L^p}}$$

Proof Take f continuous, compactly supported (enough by density)

$f : \mathbb{R}^n \to \mathbb{R}$

$\forall N \in \mathbb{N}$ $\quad g_N : \{-1,1\}^{Nn} \longrightarrow \mathbb{R}$

$$g_N(x_1^1, \dots, x_N^1 ; x_1^2, \dots, x_N^2, \dots \dots, x_1^n, \dots x_N^n)$$

$$= f\left(\frac{x_1^1 + \dots + x_N^1}{\sqrt{N}}, \frac{x_1^2 + \dots + x_N^2}{\sqrt{N}}, \dots, \frac{x_1^n + \dots x_N^n}{\sqrt{N}} \right)$$

By the CLT $\lim_{N \to \infty} \|g_N\|_{L^p} = \|f\|_{L^p(\gamma_n)}$

(HC) to g_N $\quad \|T_\rho g_N\|_{L^q} \leq \|g_N\|_{L^p}$

$$\|T_\rho g_N\|_{L^q}^q = \mathbb{E}_{x_j^i} \left| \mathbb{E}_{Y_j^i \sim N_\rho(x_j^i)} f\left(\frac{Y_1^1 + \dots + Y_N^1}{\sqrt{N}}, \dots, \frac{Y_1^n + \dots + Y_N^n}{\sqrt{N}} \right) \right.$$

ρ-correlated random bits

ρ-correlated standard gaussians

[illegible] CLT

ρ-correlated random vars

$$\left(\frac{X_1^i+\dots+X_N^i}{\sqrt{N}},\ \frac{Y_1^i+\dots+Y_N^i}{\sqrt{N}}\right) \xrightarrow{CLT} \rho\text{-correlated standard gaussians}$$

$$\lim_{N\to\infty} \|T_\rho g_N\|_{L^q}^q = \mathbb{E}_{\substack{Z\sim\gamma_n\\ Z'\sim\gamma_n\\ Z \perp\!\!\!\perp Z'}} |f(\rho Z+\sqrt{1-\rho^2}\,Z')|^q = \|U_\rho f\|_{L^q}^q$$

therefore $\|U_\rho f\|_{L^q} \leqslant \|f\|_{L^p}$.

<u>Application: concentration inequality for GAUSSIAN polynomials.</u>

Z_1, Z_2, Z_3 $N(0,1)$ iid $\quad Z_1^2+4Z_1+5Z_2Z_3^4$ concentrate around its mean?

$L^2(\gamma_1)$ $\quad (1, x, x^2, x^3, \dots) \xrightarrow[\text{(GRAM-SCHMIDT)}]{\text{orthonormalization process}} (h_n)_{n\geq 0}$ HERMITE polynomials

$\uparrow$ $N(0,1)$ on $\mathbb{R}$

$$\mathbb{E}_{\gamma_1} h_i h_j = \delta_{i,j} \quad (*)$$

$$\deg h_n = n$$

<u>Consequence</u> if Z, Z' are ρ-correlated Gaussians in $\mathbb{R}$

$$\mathbb{E}[h_j(Z)h_k(Z')] = \begin{cases} 0 & j\neq k\\ \rho^j & j=k\end{cases}$$

<u>Proof</u> If $j\geq k$ realize Z' as $\rho Z+\sqrt{1-\rho^2}\,Y$ with Y $N(0,1)$, $Y \perp\!\!\!\perp Z$

$$h_k(Z') = h_k(\rho Z+\sqrt{1-\rho^2}\,Y)$$
$$= \rho^k h_k(Z) + \text{lower order terms in } Z \ (<k)$$

so $\mathbb{E}[h_j(Z)h_k(Z')] = \rho^j$ if $k=j$ by $(*)$

$= 0$ if $k<j$

In $L^2(\gamma_1)$ $\quad U_\rho h_k = \rho^k U_k$

$$\mathbb{E}[h_j U_\rho h_k] = \begin{cases} 0 & j\neq k\\ \rho^j & j=k\end{cases}$$

$n>1$ $\quad$ For $\alpha\in\mathbb{N}^n$ $\quad \alpha=(\alpha_1,\dots,\alpha_n)$

define $h_\alpha(x_1,\dots,x_n) = h_{\alpha_1}(x_1)\dots h_{\alpha_n}(x_n)$ $\quad$ HERMITE polynomials in n variables.

$(h_\alpha)_{\alpha\in\mathbb{N}^n}$ is an orthonormal basis in $L^2(\gamma_n)$

$(**)$ $U_\rho h_\alpha = \rho^{|\alpha|} h_\alpha$ $\quad |\alpha| = \alpha_1+\dots+\alpha_n$.

$\Rightarrow$ Q is a polynomial in n variables of degree k

(*) $U_\rho h_\alpha = e \ \ \ \alpha$

<u>Theorem</u> If Q is a polynomial in n variables of total degree k

<u>Then</u> $\forall q \geq 2 \quad \|Q\|_{L^q(\gamma_n)} \leq (q-1)^{k/2} \|Q\|_{L^2(\gamma_n)}$.

<u>Proof</u> $U_\rho h_\alpha = \rho^{|\alpha|} h_\alpha$ allows to define U_ρ on polynomials $\forall \rho \in \mathbb{R}$.

$U_\rho U_{\rho'} = U_{\rho\rho'} \quad \forall \rho, \rho' \in \mathbb{R}$

$$Q = U_\rho\left[U_{\frac{1}{\rho}} Q\right] \qquad p=2 \quad \rho = \frac{1}{\sqrt{q-1}}$$

$\boxed{HC} \Rightarrow \quad \|Q\|_{L^q} \leq \|U_{\frac{1}{\rho}} Q\|_{L^2}$

$$\Big\|\sum c_\alpha h_\alpha\Big\|_{L^2} = \left(\sum c_\alpha^2\right)^{1/2}$$

$$Q = \sum_\alpha c_\alpha h_\alpha \qquad U_{\frac{1}{\rho}} Q = \sum_\alpha c_\alpha \left(\frac{1}{\rho}\right)^{|\alpha|} h_\alpha$$

$$\|Q\|_{L^2} = \left(\sum_\alpha c_\alpha^2\right)^{1/2} \qquad \|U_{\frac{1}{\rho}} Q\|_{L^2} = \left(\sum_\alpha c_\alpha^2 \left(\frac{1}{\rho}\right)^{2|\alpha|}\right)^{\frac{1}{2}} = \left(\sum_\alpha c_\alpha^2 (q-1)^{|\alpha|}\right)^{\frac{1}{2}}$$

Since $\deg(Q) \leq k$, the non-vanishing c_α correspond to $|\alpha| \leq k$

so $\|U_{\frac{1}{\rho}} Q\|_{L^2} \leq (q-1)^{k/2} \left(\sum_\alpha c_\alpha^2\right)^{1/2}$

$$\|Q\|_{L^q} \leq \ldots = (q-1)^{k/2} \|Q\|_{L^2}$$

<u>Corollary</u> $Z_1 \ldots Z_n$ iid $N(0,1)$

$X = Q(Z_1, \ldots Z_n) \quad \deg Q \leq k$

Then $\quad \mathbb{P}\left(|X - \mathbb{E}X| \geq t\sqrt{\operatorname{Var} X}\right) \leq \exp\left(-\frac{k}{2e} t^{2/k}\right) \quad \forall t \geq (2e)^{k/2}$

<u>Proof</u> Identify X with $Q: (\mathbb{R}^n, \gamma_n) \longrightarrow \mathbb{R}$

Assume $\mathbb{E}X = 0 \quad \operatorname{Var} X = 1 \qquad \|X\|_{L^2} = 1$

$$\mathbb{P}(|X| \geq t) = \mathbb{P}(|X|^q \geq t^q) \overset{\text{MARKOV}}{\leq} t^{-q} \|X\|_{L^q}^q$$

$$\leq t^{-q} (q-1)^{kq/2}$$

$$\leq t^{-q} q^{kq/2}$$

$$= \left(\frac{q^{k/2}}{t}\right)^q \qquad q = \frac{t^{2/k}}{e}$$

$$q \geq 2 \iff t \geq (2e)^{k/2}$$

$$= e^{-\frac{k}{2e} t^{2/k}} \qquad \frac{q^{k/2}}{t} = e^{-\frac{k}{2}}.$$