Problem sheet # 1

The number of ☕ symbols is proportional to the amount of coffee needed to solve each question.
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Exercise 1.1 A proof of the Brunn–Minkowski inequality

This exercise gives a proof of the Brunn–Minkowski inequality: if A, B are nonempty subsets of $\mathbf{R}^n$ such that A, B and $A+B$ are measurable, then

$$\mathrm{vol}(A+B)^{1/n} \geqslant \mathrm{vol}(A)^{1/n} + \mathrm{vol}(B)^{1/n}. \tag{1}$$

1. ☕ Show (1) when A and B are boxes (=rectangular parallelepipeds whose sides are parallel to the coordinate hyperplanes).
2. ☕☕☕ Show (1) when A is the union of M disjoint boxes and B is the union of N disjoint boxes, by induction on $M+N$. For the induction step, use the following trick. Consider an axis parallel hyperplane H cutting $\mathbf{R}^n$ into half-spaces H^- and H^+, and reduce to the case where (1) $A \cap H^-$ and $A \cap H^+$ are both the union of $< M$ disjoint boxes and (2) $\frac{\mathrm{vol}(A\cap H^-)}{\mathrm{vol}(A)} = \frac{\mathrm{vol}(B\cap H^-)}{\mathrm{vol}(B)}$; then use the inequality $\mathrm{vol}(A+B) \geqslant \mathrm{vol}((A\cap H^+)+(B\cap H^+)) + \mathrm{vol}((A\cap H^-)+(B\cap H^-))$.
3. ☕☕ If you know enough about measure thoery, prove (1) in the general case.

Exercise 1.2 Brunn–Minkoswki, variant

☕ Show the equivalence of the following statements

1. For every non-empty compact sets A, B in $\mathbf{R}^n$, we have $\mathrm{vol}(A+B)^{1/n} \geqslant \mathrm{vol}(A)^{1/n} + \mathrm{vol}(B)^{1/n}$.
2. For every compact sets A, B in $\mathbf{R}^n$ and $t \in]0,1[$, we have $\mathrm{vol}(tA+(1-t)B) \geqslant \mathrm{vol}(A)^t\,\mathrm{vol}(B)^{1-t}$.

$1 \implies 2$ is not hard ; for $2 \implies 1$ set $a = \mathrm{vol}(A)^{1/n}$ and $b = \mathrm{vol}(B)^{1/n}$ and write the set $A+B$ as $(a+b)[\frac{a}{a+b}\frac{A}{a} + \frac{b}{a+b}\frac{B}{b}]$

Exercise 1.3 Speed of convergence for the centroid algorithm

You can use (or ☕☕☕☕ prove) the following result: if X is a log-concave random variable, then $\mathbf{P}(X \geqslant \mathbf{E}X) \geqslant 1/e$.

1. Let K be a convex body (=convex compact set with non-empty interior) in $\mathbf{R}^n$. The centroid (or barycenter) of K is defined as $\mathrm{c}(K) = \frac{1}{\mathrm{vol}\,K}\int_K x\,\mathrm{d}x$. ☕☕ Show that if H is a half-space with $\mathrm{c}(K) \in \partial H$, then $\frac{1}{e}\mathrm{vol}(K) \leqslant \mathrm{vol}(K\cap H) \leqslant \left(1-\frac{1}{e}\right)\mathrm{vol}(K)$.
2. Let $f : K \to [-B, B]$ a convex function and m be its minimum on K. Pick x_0 such that $f(x_0) = m$.

   Our goal is to devise an algorithm which approximates m. We may call an oracle which, given $x \in K$, outputs a *subgradient*, i.e. a linear form ℓ_x such that $f(y) \geqslant f(x) + \ell_x(y-x)$ for every $y \in K$.

   (a) ☕ Why is the problem solved if $\ell_x = 0$?

   (b) Define by induction a sequence (K_j) of convex bodies by $K_1 = K$, $c_j = \mathrm{c}(K_j)$ and $K_{j+1} = K_j \cap H$ where H is the half-space $\{y : \ell_{c_j}(y-c_j) \leqslant 0\}$. ☕☕ Show that $\mathrm{vol}(K_j) \leqslant (1-1/e)^j\,\mathrm{vol}(K)$ and that $x_0 \in K_j$.

   (c) For $0 < \varepsilon < 1$, consider the set $K_{(\varepsilon)} = (1-\varepsilon)x_0 + \varepsilon K$. Suppose that there is a point $y \in K_{(\varepsilon)} \setminus K_j$. ☕☕ Show that $f(c_j) < f(y) \leqslant m + 2\varepsilon B$.

   (d) ☕☕ Compare the volume of the sets $K_{(\varepsilon)}$ and K_j and conclude that that $O(n\log(2B/\delta))$ oracle queries suffice to approximate m up to additive error δ.

   (e) ☕☕☕ What concern could you have about the computational complexity of this algorithm?

Exercise 1.4 Kissing numbers

For $n \geqslant 1$, let K_n be the maximal number of unit balls in $\mathbf{R}^n$ with pairwise disjoint interiors which are tangent to B_n. ☕ Compute K_1 and K_2, ☕☕ check that $K_3 \geqslant 12$ (there is equality) and ☕☕ prove the bounds

$$\left(\frac{2}{\sqrt{3}} + o(1)\right)^n \leqslant K_n \leqslant (2 + o(1))^n$$

by considering an equivalent packing problem on S^{n-1}.

Remark. The exact value of K_n is known only for $n \in \{1, 2, 3, 4, 8, 24\}$.

Exercise 1.5 Nets and convex hull

Let $\mathcal{N} \subset S^{n-1}$ and $\theta \in (0, \pi/2)$. ☕☕ Show that $\mathcal{N}$ is a θ-net in (S^{n-1}, g) if and only if $(\cos\theta)B_n \subset \operatorname{conv}\mathcal{N}$.

Exercise 1.6 Isoperimetry: sphere vs Euclidean space

☕☕☕ Why can the isoperimetric inequality on $\mathbf{R}^{n-1}$ be deduced from the isoperimetric inequality on S^{n-1}?

Exercise 1.7 Packing and covering in the discrete cube

You can use or ☕☕☕ prove the following inequality: for integers $0 \leqslant k \leqslant n$ we have

$$\frac{1}{n+1} 2^{nH(k/n)} \leqslant \sum_{j=0}^{k} \binom{n}{j} \leqslant 2^{nH(k/n)}$$

where $H(t) = -t \log_2 t - (1-t)\log_2(1-t)$ is the binary entropy function.

1. ☕☕ Let $Q_n = \{0, 1\}^n$. For x, $y \in Q_n$, define $d(x, y) = \frac{1}{n} \operatorname{card}\{i \;\; x_i \neq y_i\}$. For $\varepsilon \in (0, 1)$, denote by $N(Q_n, \varepsilon)$ and $P(Q_n, \varepsilon)$ the covering and packing numbers for the metric space (Q_n, d). For $0 < \varepsilon < 1/2$, show that

$$1 - H(\varepsilon) \leqslant \limsup_{n\to\infty} n^{-1} \log_2 P(Q_n, \varepsilon) \leqslant 1 - H(\varepsilon/2)$$

2. ☕☕ For $0 < \varepsilon < 1/2$, show by a random covering argument that

$$\lim_{n\to\infty} n^{-1} \log_2 N(Q_n, \varepsilon) = 1 - H(\varepsilon).$$