Problem sheet # 1

The number of symbols is proportional to the amount of coffee needed to solve each question.
Do not hesitate to contact aubrun@math.univ-lyon1.fr for hints.

Exercise 1.1 A proof of the Brunn–Minkowski inequality

This exercise gives a proof of the Brunn–Minkowski inequality: if $A, B$ are nonempty subsets of $\mathbb{R}^n$ such that $A, B$ and $A + B$ are measurable, then

$$\text{vol}(A + B)^{1/n} \geq \text{vol}(A)^{1/n} + \text{vol}(B)^{1/n}. \quad (1)$$

1. Show (1) when $A$ and $B$ are boxes (=rectangular parallelepipeds whose sides are parallel to the coordinate hyperplanes).

2. Show (1) when $A$ is the union of $M$ disjoint boxes and $B$ is the union of $N$ disjoint boxes, by induction on $M + N$. For the induction step, use the following trick. Consider an axis parallel hyperplane $H$ cutting $\mathbb{R}^n$ into half-spaces $H^-$ and $H^+$, and reduce to the case where (1) $A \cap H^-$ and $A \cap H^+$ are both the union of $< M$ disjoint boxes and (2) $\frac{\text{vol}(A \cap H^-)}{\text{vol}(A)} = \frac{\text{vol}(B \cap H^+)}{\text{vol}(B)}$; then use the inequality $\text{vol}(A + B) \geq \text{vol}((A \cap H^+) + (B \cap H^+)) + \text{vol}((A \cap H^-) + (B \cap H^-))$.

3. If you know enough about measure theory, prove (1) in the general case.

Exercise 1.2 Brunn–Minkowski, variant

Show the equivalence of the following statements

1. For every non-empty compact sets $K, K'$.

2. Let $K, K'$ be a convex body (=convex compact set with non-empty interior) in $\mathbb{R}^n$ and $t \in [0, 1]$, we have $\text{vol}(tA + (1-t)B) \geq t\text{vol}(A)^t \text{vol}(B)^{1-t}$.

1 $\implies$ 2 is not hard ; for 2 $\implies$ 1 set $a = \text{vol}(A)^{1/n}$ and $b = \text{vol}(B)^{1/n}$ and write the set $A + B$ as $(a + b)[\frac{a}{a+b} + \frac{b}{a+b}]$.

Exercise 1.3 Speed of convergence for the centroid algorithm

You can use (or prove) the following result: if $X$ is a log-concave random variable, then $\mathbb{P}(X \geq \mathbb{E}X) \geq 1/e$.

1. Let $K$ be a convex body (=convex compact set with non-empty interior) in $\mathbb{R}^n$. The centroid (or barycenter) of $K$ is defined as $c(K) = \frac{1}{\text{vol}(K)} \int_K x \, dx$. Show that if $H$ is a half-space with $c(K) \in \partial H$, then $\frac{1}{2} \text{vol}(K) \leq \text{vol}(K \cap H) \leq (1 - \frac{1}{2}) \text{vol}(K)$.

2. Let $f : K \rightarrow [-B, B]$ be a convex function and $m$ be its minimum on $K$. Pick $x_0$ such that $f(x_0) = m$. Our goal is to devise an algorithm which approximates $m$. We may call an oracle which, given $x \in K$, outputs a subgradient, i.e. a linear form $\ell_x$ such that $f(y) \geq f(x) + \ell_x(y - x)$ for every $y \in K$.

(a) Why is the problem solved if $\ell_x = 0$?

(b) Define by induction a sequence $(K_j)$ of convex bodies by $K_1 = K, c_j = c(K_j)$ and $K_{j+1} = K_j \cap H$ where $H$ is the half-space $\{ y : \ell_{c_j}(y - c_j) \leq 0 \}$. Show that $\text{vol}(K_j) \leq (1 - 1/e)^j \text{vol}(K)$ and that $x_0 \in K_j$.

(c) For $0 < \varepsilon < 1$, consider the set $K(\varepsilon) = (1 - \varepsilon)x_0 + \varepsilon K$. Suppose that there is a point $y \in K(\varepsilon) \setminus K_j$. Show that $f(c_j) < f(y) \leq m + 2\varepsilon B$.

(d) Compare the volume of the sets $K(\varepsilon)$ and $K_j$ and conclude that $O(n \log(2B/\delta))$ oracle queries suffice to approximate $m$ up to additive error $\delta$.

(e) What concern could you have about the computational complexity of this algorithm?
Exercise 1.4  Kissing numbers
For \( n \geq 1 \), let \( K_n \) be the maximal number of unit balls in \( \mathbb{R}^n \) with pairwise disjoint interiors which are tangent to \( B_n \). ★ Compute \( K_1 \) and \( K_2 \), ★★ check that \( K_3 \geq 12 \) (there is equality) and ★★★ prove the bounds

\[
\left( \frac{2}{\sqrt{3}} + o(1) \right)^n \leq K_n \leq (2 + o(1))^n
\]

by considering an equivalent packing problem on \( S^{n-1} \).

Remark. The exact value of \( K_n \) is known only for \( n \in \{1, 2, 3, 4, 8, 24\} \).

Exercise 1.5  Nets and convex hull
Let \( N \subset S^{n-1} \) and \( \theta \in (0, \pi/2) \). ★★ Show that \( N \) is a \( \theta \)-net in \( (S^{n-1}, g) \) if and only if \( \cos \theta B_n \subset \text{conv } N \).

Exercise 1.6  Isoperimetry: sphere vs Euclidean space
Why can the isoperimetric inequality on \( \mathbb{R}^{n-1} \) be deduced from the isoperimetric inequality on \( S^{n-1} \)?

Exercise 1.7  Packing and covering in the discrete cube
You can use or ★★★ prove the following inequality: for integers \( 0 \leq k \leq n \) we have

\[
\frac{1}{n+1} 2^{nH(k/n)} \leq \sum_{j=0}^{k} \binom{n}{j} \leq 2^{nH(k/n)}
\]

where \( H(t) = -t \log_2 t - (1-t) \log_2 (1-t) \) is the binary entropy function.

1. ★★ Let \( Q_n = \{0,1\}^n \). For \( x, y \in Q_n \), define \( d(x,y) = \frac{1}{n} \text{card}\{i \mid x_i \neq y_i\} \). For \( \varepsilon \in (0,1) \), denote by \( N(Q_n, \varepsilon) \) and \( P(Q_n, \varepsilon) \) the covering and packing numbers for the metric space \( (Q_n, d) \). For \( 0 < \varepsilon < 1/2 \), show that

\[
1 - H(\varepsilon) \leq \limsup_{n \to \infty} n^{-1} \log_2 P(Q_n, \varepsilon) \leq 1 - H(\varepsilon/2)
\]

2. ★★ For \( 0 < \varepsilon < 1/2 \), show by a random covering argument that

\[
\lim_{n \to \infty} n^{-1} \log_2 N(Q_n, \varepsilon) = 1 - H(\varepsilon).
\]