Problem sheet # 2

Exercise 2.1 Tricks with concentration

Let X be random variable and $a \in \mathbf{R}$ such that for every $t \geqslant 0$,

$$\mathbf{P}(|X-a| \geqslant t) \leqslant C\exp(-\alpha t^2).$$

☕☕ Show the following inequalities, where C_i and $\alpha_i > 0$ depend only on C and α

1. $\mathbf{P}(|X - \mathbf{E}X| \geqslant t) \leqslant C_1 \exp(-\alpha_1 t^2)$,
2. $\mathbf{P}(|X - M_X| \geqslant t) \leqslant C_2 \exp(-\alpha_2 t^2)$, where M_X is a median of X,
3. (assuming $X \geqslant 0$) $\mathbf{P}(|X - \sqrt{\mathbf{E}X^2}| \geqslant t) \leqslant C_3 \exp(-\alpha_3 t^2)$.

Exercise 2.2 Stochastic domination

Let X, Y be random variables. ☕☕ Show that the following inequalities are equivalent

- For every $t \in \mathbf{R}$, $\mathbf{P}(X \geqslant t) \leqslant \mathbf{P}(Y \geqslant t)$,
- For every increasing function $f : \mathbf{R} \to \mathbf{R}$ such that $f(X)$ and $f(Y)$ are integrable, $\mathbf{E}f(X) \leqslant \mathbf{E}f(Y)$,
- There is a probability space Ω and random variables X', Y' defined on Ω such that X' has the same law as X, Y' as the same law as Y and $\mathbf{P}(X' \leqslant Y') = 1$.

Exercise 2.3 An alternative argument for Gaussian concentration

The goal of this exercise is to show that the following: if $G = (G_1, \dots, G_n)$ are i.i.d. $N(0,1)$ random variables and $f : \mathbf{R}^n \to \mathbf{R}$ is 1-Lipschitz, then for every $t \geqslant 0$,

$$\mathbf{P}\left(|f(G) - \mathbf{E}f(G)| \geqslant t\right) \leqslant 2e^{-\frac{2t^2}{\pi^2}}.$$

1. ☕☕ Show that we can assume that f is C^1 and $\mathbf{E}f(G) = 0$.
2. ☕☕ Let H be an independant copy of G, and for $0 \leqslant \theta \leqslant \pi/2$, define $G_\theta = G\sin(\theta) + H\cos(\theta)$. Show that for every θ, $(G_\theta, \frac{\mathrm{d}}{\mathrm{d}\theta}G_\theta)$ has the same distribution as (G, H).
3. ☕☕☕ Show that for every convex function $\psi : \mathbf{R} \to \mathbf{R}$ we have
$$\mathbf{E}\left[\psi(f(G))\right] \leqslant \mathbf{E}\left[\psi(f(G) - f(H))\right] = \mathbf{E}\left[\psi\left(\int_0^{\pi/2} \langle \nabla f(G_\theta), \frac{\mathrm{d}}{\mathrm{d}\theta}G_\theta\rangle \,\mathrm{d}\theta\right)\right] \leqslant \mathbf{E}\left[\psi\left(\frac{\pi}{2}\langle \nabla f(G), H\rangle\right)\right].$$
4. ☕☕☕ Apply the previous inequality to $\psi : x \mapsto \exp(\lambda x)$ for $\lambda \geqslant 0$, and deduce that
$$\mathbf{E}\left[\exp(\lambda f(G))\right] \leqslant \exp(\pi^2\lambda^2/8).$$
5. ☕☕ Conclude.

Exercise 2.4 Symmetric Gaussian matrices

Denote by $\lambda_1(A)$ the largest eigenvalue of a symmetric matrix A. Let G be a $n \times n$ random matrix with independent $N(0,1)$ entries and $H = \frac{G+G^t}{2}$ the symmetric part of G.

1. ☕☕☕ Using Slepian's lemma, show that $\mathbf{E}\lambda_1(H) \leqslant \sqrt{2n}$.
2. ☕ Check on a software that this result is sharp for large n. Plot the whole spectrum of H. What can you see?

Exercise 2.5 Ehrhard inequality

Denote $\Phi(t) = \mathbf{P}(X \leqslant t)$ for $X \sim N(0,1)$. The following inequality is true and called the Ehrhard inequality: for any Borel sets A, $B \subset \mathbf{R}^n$ and $t \in [0,1]$,

$$\Phi^{-1}(\gamma_n(((1-t)A + tB) \geqslant (1-t)\Phi^{-1}(\gamma_n(A)) + t\Phi^{-1}(\gamma_n(B)). \tag{1}$$

1. ☕☕ Check that there is equality when A and B are half-spaces with $A \subset B$ or $B \subset A$.

2. ☕☕☕ Deduce the Gaussian isoperimetric inequality from (1) by choosing $B = \frac{r}{t}B_2^n$ and taking $t \to 0$.

3. ☕☕☕ We are going to show that for any convex function $F : \mathbf{R}^n \to \mathbf{R}$, if G is a standard Gaussian vector in $\mathbf{R}^n$, then $M_{F(G)} \leqslant \mathbf{E}F(G)$, where $M_{F(G)}$ denotes the median.

   (a) Using (1), show that the function $g : t \mapsto \Phi^{-1}(\mathbf{P}(F(G) \leqslant t))$ is concave on $\mathbf{R}$.

   (b) Deduce that there exists $\alpha > 0$ such that $g(t) \leqslant \alpha(t - M_{F(G)})$ for every $t \in \mathbf{R}$.

   (c) Conclude that $M_{F(G)} \leqslant \mathbf{E}F(G)$.