Lemma 1 (Generation lemma). Let $\Delta \vdash t : T$,

- if $t$ is a variable $x$ then $x : T$ in $\Delta$
- if $t \equiv a \ b$ then there exists $S$ such that $\Delta \vdash a : S \to T$ and $\Delta \vdash b : S$
- if $t \equiv \lambda x. a$ with $x \not\in \text{dom}(\Delta)$ then $T = S \to U$ such that $\Delta, x \vdash a : U$

Proof. See Lectures on the Curry Howard Isomorphism for technical details. \qedhere

Exercise 1 (Subject reduction). Prove that $\lambda \to$ has the subject reduction property:

If $\Delta \vdash m : T$ then for every $m \to^*_{\beta} m'$, the typing judgement $\Delta \vdash m' : T$ holds.

1. (term substitution) Prove that if $\Delta, y : S \vdash t : T$ and $\Delta \vdash s : S$, then $\Delta \vdash t\langle s/y \rangle : T$.

Let us show the above property by induction on $t$, using the generation lemma:

- (base case) if $t$ is a variable $s$ then $x : T \in \Delta, y : S$ and so either $x \not= y$ hence $t\langle s/y \rangle = x$ and $\Delta \vdash x : T$ is a valid type derivation or $x = y$, $S = T$ hence $t\langle s/y \rangle = s$ and by hypothesis $\Delta \vdash s : T$ is a valid typing judgement.
- if $t \equiv a \ b$ then $t\langle s/y \rangle = a\langle s/y \rangle \ b\langle s/y \rangle$ and there exists $R$ such that $\Delta, y : S \vdash a : R \to T$ and $\Delta, y : S \vdash b : R$. Applying the induction hypothesis we get $\Delta \vdash a\langle s/y \rangle : R \to T$ and $\Delta \vdash b\langle s/y \rangle : R$ and so (applying the $\rightarrow_E$ rule) $\Delta \vdash t\langle s/y \rangle : S$.
- if $t \equiv \lambda x. a$, then by $\alpha$-conversion we can assume that $x \not\in \text{dom}(\Delta) \cup y$ and so there exist $R, U, T$ such that $T = R \to U$ and $\Delta, y : S, x : R \vdash a : U$. By induction hypothesis $\Delta, x : R \vdash a\langle s/y \rangle : U$ and so, applying the $\rightarrow_I$ rule, $\Delta \vdash t\langle s/y \rangle : S$ since $t\langle s/y \rangle = (\lambda x. a)\langle s/y \rangle = \lambda x. a\langle s/y \rangle$.

2. Now, show the subject reduction property.

By induction over $\beta$, let us consider the basic reduction step $m \to^*_{\beta} m_1$:

- if $m = (\lambda x. a) b$ (wlog we assume that $x \not\in \text{dom}(\Delta)$) and $m_1 = a\langle b/x \rangle$ then by generation lemma (applied twice) there exists $S$ such that $\Delta, x : S \vdash a : T$ and $\Delta \vdash b : S$. Hence, by the above substitution lemma $\Delta \vdash a\langle b/x \rangle : T$, as desired.
- if $m = a \ b$ and $m_1 = a' \ b$ then by generation lemma there exists $S$ such that $\Delta \vdash a : S \to T$ and $\Delta \vdash b : S$, and by induction hypothesis $\Delta \vdash a' : S \to T$, so, applying the $\rightarrow_E$ rule, $\Delta \vdash a' \ b : T$ is a valid type judgement.
- cases with $m = a \ b$, $m_1 = a' \ b$ and $m = \lambda x. a$, $m' = \lambda x. a'$ are similar.

3. Show that the contraposite does not hold. (Hint: $\Omega$ can help you to build a counter example) $(\lambda x. y)\Omega$ reduces to $\lambda x. x$ which is typable, but $\Omega$ is not.