Greedy algorithms for computing the Birkhoff decomposition

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Summary

A square matrix is doubly stochastic if it has nonnegative entries, and the sum of entries in any given row and any given column is equal to one. A permutation matrix is a square matrix with a single nonzero of value one in each row and each column. Birkhoff's Theorem, sometimes called the Birkhoff–von Neumann (BvN) theorem, states that any doubly stochastic matrix can be written as a convex combination of permutation matrices [1]. The BvN decomposition of a given doubly stochastic matrix is not unique, and some applications require finding the sparsest representation in which we seek the minimum number of permutation matrices. The problem of finding a BvN decomposition with the smallest number of permutation matrices is shown to be NP-hard [2] and effective heuristics are of significant interest. This internship will investigate a new family of algorithms for computing BvN decompositions based on greedy algorithms found in the sparse approximation literature [3].

[1] G. Birkhoff. Tres observaciones sobre el algebra lineal. Univ. Nac. Tucumán Rev. Ser. A, (5):147–150, 1946.

[2] F. Dufossé and B. Uçar. Notes on Birkhoff–von Neumann decomposition of doubly stochastic matrices. Linear Algebra and its Applications, 497:108–115, 2016.

[3] S. Foucart and H. Rauhut, A Mathematical Introduction to Compressive Sensing, Birkhäuser New York, NY 2013.

The internship will last 5--6 months and start in the first half of 2024 (typically February--July). It is part of the GABI project (http://perso.ens-lyon.fr/bora.ucar/gabi/). The internship will be carried out at LIP and CREATIS laboratories in Lyon, while the official location is LIP, ENS de Lyon.

The initial plans are as follows:
- Propose, implement, and test a greedy algorithm for the Birkhoff decomposition with the squared Euclidean norm as a loss function.
- Compare this algorithm with greedy algorithms built from the Manhattan distance (ℓ_1 norm).

From there, depending on the preference of the intern, we may focus on:
- proving bounds on the approximation error and recovery of the true underlying number of permutations.
- proposing algorithmic strategies to improve the performance of the greedy approaches, for instance by exploring the combinatorial aspect of the problem

Requirements
The ideal candidate is interested in the design, analysis, and implementation of algorithms and experimenting with them reproducibly. Experience in at least one of the following (Julia, Matlab, Python, C/C++) is required; **working** knowledge of linear algebra and continuous optimization algorithms is required.

Knowledge, skills and competences to be developed
- Sparse approximation theory
- Greedy algorithms for sparse approximation
- Numerical optimization (Linear programming, Quadratic Programming, Convex Programming)
- Development of numerical algorithms
- Scientific evaluation of algorithm performance

Monitoring plan
Weekly meetings with the supervisors.

Evaluation plan
The algorithms and codes will be developed with the standard practices of using version-controlling systems — this way the progress will be overseen. In the end, the demonstrated effects will set the success.

Payment
The standard internship gratification is available via the GABI project funded by Fédération Informatique de Lyon (FIL).