

ON FOLIATIONS ADMITTING A TRANSVERSE SIMILARITY STRUCTURE

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ABSTRACT. We give a “conceptual” approach to Kourganoff’s results about foliations with a transverse similarity structure. In particular, we give a proof, understandable by the targeted community, of the very important result classifying the holonomy of the closed, non-exact Weyl structures on compact manifolds, from which arose the notion of locally conformally product structures. We also extract from the proof several results on foliations admitting locally metric transverse connections.

CONTENTS

1. Introduction	1
2. Preliminaries	3
3. Construction of a Heafliker structure	8
4. Holonomy-invariant transversal connection	10
5. Holonomy-invariant transversal similarity structure	11
6. Proofs of Theorems 2.10, 2.11, 2.13, 2.16	14
7. Closed non-exact Weyl structures	19
References	21

1. INTRODUCTION

Foliations on manifolds are studied using the structures carried by their transversals, where a transversal is a manifold which is at each point transverse to the foliation and which intersects each one of the leaves. However, foliations admitting a global connected transversal are quite rare, and a way to overcome this difficulty is to find a natural identification between local transversals meeting a same leaf. This is done using the so-called holonomy pseudo-group, which consists of germs of diffeomorphisms of the transversal obtained by sliding along leaves following a pre-defined path. Once we have this identification, the G -structures (i.e. the reductions of the frame bundle) carried by the transversal which are holonomy-invariant are of great help to understand the underlying geometry of the foliation.

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Among these structures, the Riemannian ones, defining what is called a Riemannian foliation, are probably the best understood. A very detailed presentation of this particular case can be found in the book of Molino [21]. A slightly more general case is the one of transverse similarity structures, where the Riemannian metric of the transversal is defined, only locally, up to an homothety and is preserved by the holonomy pseudo-group only up to a positive multiplicative constant, i.e. this group acts by similarities (which are often called homotheties). These structures have been studied by Kourganoff [16], who proved that the foliation is then Riemannian unless it is transversally flat. However, the analysis carried out by Kourganoff involved highly technical tools which have not been fully understood by the targeted public. The aim of this article is to provide a much more conceptual proof, avoiding technicalities while going a little bit further concerning the results that can be inferred.

Although we are essentially following the same broad lines, the whole idea of this new proof is to linearize the holonomy. Indeed, the holonomy pseudo-group can be understood infinitesimally as the parallel transport along leaves according to a special class of connections, called Bott connections. Yet, this is not sufficient in general to get the full picture. Being able to linearize the holonomy exactly means that the diffeomorphisms of the holonomy pseudo-group are completely determined by their one-jet at a point. In particular, we can analyze the geometry of the foliation by looking at its normal bundle endowed with the Bott connection induced by the transverse similarity structure. This linearization is done using a Heafliker structure [13, 17], which is more general than a foliation and corresponds to a foliation on a neighbourhood of the zero-section of the normal bundle of the initial foliation. The holonomy pseudo-group of this new foliation is then equivalent to the initial holonomy pseudo-group, but we are now in presence of a (locally) foliated bundle, which is easier to understand. The last ingredient of this analysis is then the existence of a holonomy-invariant transverse connection, which allows to linearize the holonomy of the Heafliker structure.

The first motivation to study transverse similarity structures in [16] was the so-called Belgun-Moroianu conjecture, formulated in [7]. This conjecture concerns conformal geometry, and more specifically Weyl connections on compact conformal manifolds. A Weyl connection is a generalization of the notion of Levi-Civita connection to the conformal case: it is a torsion-free connection which preserves the conformal class. The Weyl structure is said to be closed when it is locally the Levi-Civita connection of a metric in the conformal class, and exact when this property holds globally. The Belgun-Moroianu conjecture stated that a closed, non-exact Weyl connection on a compact conformal manifold is flat or has irreducible reduced holonomy. However, this was disproved by a counter-example constructed by Matveev and Nikolayevsky [19], who showed additionally that, in the analytic case, when the holonomy is reducible and non-flat, the universal cover of the compact manifold has a natural structure of a Riemannian product when endowed with a Riemannian metric canonically induced by the Weyl connection [20].

The work of Kourganoff in [16] allowed to extend this theorem to the smooth case and has been the starting point of the study of Locally Conformally Product (LCP) structures (see for example [4, 6, 10, 22]). For this reason, the comprehension of this result is really significant for the authors working on LCP structures, but, as we explained above, the technicalities of foliation theory and certain proofs left to the reader in Kourganoff's text have been an obstacle to the proper spread of this knowledge. For this reason, we try here to have a more invariant approach to the problem, using for example the tools introduced in the book of Molino [21] in order to see transverse geometry as the examination of the normal bundle of the the foliation. Yet, when speaking of the holonomy pseudo-group, some issues arise, since, as we already discussed, the holonomy is not linear. The intervention of the Heafliker structures is then a way to avoid a choice of a particular complete transversal, which led to technicalities in the original proof.

The organization of the paper goes as follow. In Section 2 we introduce the notions needed for the analysis of foliations and we state the main results of the paper about foliations and de Rham decomposition of manifolds admitting a locally metric connection. Sections 3 and 4 are devoted to the linearization of the holonomy in our particular setting by use of a Heafliker structure and well-chosen coordinates for the transversals of the foliation. In Section 5 we look at the equicontinuity domain of a foliation endowed with a similarity structure, and we provide an example showing that the results we obtain here cannot be generalized when we only have an holonomy-invariant transverse connection. The equicontinuity is the key to prove that a transverse similarity structure can be changed into a transverse Riemannian structure. Section 6 contains the proofs of the main theorems about foliations. Finally, we discuss in Section 7 an application of the previous results, reproving the striking result of [16, Theorem 1.5] about the holonomy decomposition of compact manifolds with a closed, non-exact Weyl structure.

2. PRELIMINARIES

In all this text, M is a connected manifold endowed with a foliation \mathcal{F} of codimension q . Our goal is to study the case where M is compact and the foliation \mathcal{F} admits a holonomy-invariant transversal Riemannian similarity structure $[g]$, that is $[g]$ is a class up to homothety of Riemannian metrics, defined locally on each transversal, and invariant under any holonomy transformation of \mathcal{F} . However, we do not assume that M is compact or that \mathcal{F} carries a particular structure for the moment, and we will add new assumptions throughout the text.

We recall that the holonomy pseudo-group of the foliation \mathcal{F} is the pseudo-group of diffeomorphisms of local quotient manifolds of the foliation defined by sliding along leaves. More precisely, if $x, y \in M$ are in the same leaf of \mathcal{F} and $c : [0, 1] \rightarrow M$ is a path from x to y staying in the same leaf, c defines an element of the holonomy pseudo-group by choosing two local transversals T_x and T_y at x and y respectively (i.e. T_x and T_y are manifolds which are everywhere transverse to the foliations and $x \in T_x$

and $y \in T_y$). If T_x and T_y are small enough, one can divide c into sub-paths which are each contained in the domain of a foliated chart of (M, \mathcal{F}) . In each of these domains, the sub-path of c is canonically lifted to all the leaves of the domain and sliding along the leaves just means following these lifts. Sliding the points of T_x until we reach T_y defines a local diffeomorphism from T_x to T_y . This does not depend on the chosen sequence of foliated charts we used. A more detailed discussion about the holonomy pseudo-group can be found in [21].

2.1. Foliations with transverse holonomy-invariant connection. A transverse connection ∇ on M is a linear connection on the normal bundle $N\mathcal{F} := TM/T\mathcal{F}$ of the foliation \mathcal{F} , such that for any vector fields $(X, \bar{Y}) \in T\mathcal{F} \times N\mathcal{F}$, $\nabla_X \bar{Y}$ coincides with the projection of $[X, Y]$ on $N\mathcal{F}$, where $Y \in TM$ is any representative of \bar{Y} . Such connections are also called *Bott connections*. This induces a connection on any transversal of \mathcal{F} in a natural way.

This connection is *holonomy-invariant* if for any $x, y \in M$, for any transversals S_x and S_y at x and y respectively and for any holonomy map γ sending S_x to S_y (up to a restriction of the definition sets), one has $\nabla|_{S_x} = \gamma^*(\nabla|_{S_y})$. This amounts to saying that ∇ is projectable, i.e. it projects to a connection on the local quotient manifolds of the foliation (see [21, Lemma 2.3]).

Remark 2.1. *Note that the torsion of a transverse connection is well-defined because one can construct a transverse fundamental form θ_T from $\text{Fr}(N\mathcal{F})$ to \mathbb{R}^q , also called a solder form, defined at the point $p \in \text{Fr}(N\mathcal{F})$ by:*

$$(1) \quad (\theta_T(X))_p := p^{-1}(\pi_*(X)_p) \quad \forall X \in T\text{Fr}(N\mathcal{F}),$$

where p is seen as a map from \mathbb{R}^q to $N\mathcal{F}$ and π is the canonical projection $\text{Fr}(N\mathcal{F}) \rightarrow M$. The torsion is the covariant exterior derivative of θ_T .

Definition 2.2. *We consider any Bott connection. This induces a connection of the principal bundle $\text{Fr}(N\mathcal{F})$. We consider the distribution on $\text{Fr}(N\mathcal{F})$ containing the horizontal tangent vectors which project into $T\mathcal{F}$. This distribution is involutive and it induces a foliation called the lifted foliation. This definition does not depend on the chosen connection.*

The lifted foliation is right-invariant and has the same dimension as the original foliation. It can also be defined as the distribution containing all the vectors $X \in T\text{Fr}(N\mathcal{F})$ such that $\iota_X \theta_T = 0 = \iota_X d\theta_T$, where θ_T is the fundamental transverse form. A detailed discussion can be found in [21].

Definition 2.3. *A transverse G -structure on (M, \mathcal{F}) is a G -reduction of the frame bundle of the normal bundle $N\mathcal{F}$ which is invariant by the lifted foliation (i.e. the vectors tangent to the lifted foliation are tangent to the G -reduction).*

We call a *transverse metric* on (M, \mathcal{F}) any Riemannian bundle metric on the normal bundle $N\mathcal{F} \rightarrow M$ of \mathcal{F} . A transverse Riemannian metric g_T on (M, \mathcal{F}) is said to be holonomy-invariant if, when one denotes by g the degenerate metric on TM given by the composition of the projection $TM \times TM \rightarrow N\mathcal{F} \times N\mathcal{F}$ together with g_T , then $\mathcal{L}_X g = 0$ for any $X \in \Gamma(T\mathcal{F})$. Endowed with such a structure, it exists a unique torsion-free transverse connection which preserves g_T , called the *transverse Levi-Civita connection of g_T* . This connection is projectable, i.e. holonomy-invariant. Equivalently, a transverse holonomy-invariant metric is a transverse $O(q)$ -structure.

Since Riemannian holonomy-invariant metrics are significant structures on foliations, they deserve a name:

Definition 2.4. *A Foliation admitting a transverse holonomy-invariant Riemannian structure is called a Riemannian foliation.*

Remark 2.5. *The informed reader who knows the classical book of Molino [21] should be aware of a small difference we make in the vocabulary used here. Indeed, what we call a transverse holonomy-invariant metric is what Molino simply names a transverse metric. The reason we make this difference is that we may subsequently talk about non-holonomy-invariant objects.*

We introduce a particular class of connections on manifolds:

Definition 2.6. *A linear connection ∇ on a manifold is said to be locally metric if its torsion vanishes and its reduced holonomy group $\text{Hol}_0(\nabla)$ is compact.*

We have an equivalent notion for transverse connections:

Definition 2.7. *A transverse connection is said to be locally metric if its torsion vanishes and its reduced holonomy group $\text{Hol}_0(\nabla)$ is compact.*

Remark 2.8. *A transverse locally metric connection induces a Riemannian bundle metric on the pull-back of the normal bundle $N\mathcal{F} \rightarrow M$ to the universal cover \tilde{M} of M (which is actually the normal bundle $T\tilde{M}/T\tilde{\mathcal{F}} \rightarrow \tilde{M}$ of the pulled-back foliation $\tilde{\mathcal{F}}$). Indeed, Since $\text{Hol}_0(\nabla)$ is compact, it is conjugated to a compact subgroup of the orthonormal group $O(q)$ where q is the codimension of \mathcal{F} . Consequently, denoting by $\tilde{\nabla}$ the lift of the connection, the $\tilde{\nabla}$ -holonomy bundle of an arbitrary point in the frame bundle $\text{Fr}(T\tilde{M}/T\tilde{\mathcal{F}})$ is a reduction of structure group included in $O(q)$ and it is stable by the $\tilde{\nabla}$ -holonomy. This implies $\tilde{\nabla}$ preserves an $O(q)$ -subbundle of $\text{Fr}(T\tilde{M}/T\tilde{\mathcal{F}})$, and it is also a transverse connection. In addition, this subbundle is invariant by the lifted connection. Altogether, this means that ∇ is the Levi-Civita connection of a transverse Riemannian metric g_T . If we assume moreover that this connection is irreducible, since any element γ of $\pi_1(M)$ act on $(\tilde{M}, \tilde{\mathcal{F}}, \tilde{\nabla})$ by transverse affine transformations, then $\gamma^* g_T = \lambda g_T$ for some $\lambda > 0$, and $\pi_1(M)$ acts on $(\tilde{M}, \tilde{\mathcal{F}}, g_T)$ by transverse similarities.*

The transverse metric then defined on $T\tilde{M}/T\tilde{\mathcal{F}}$ is always holonomy-invariant by definition of a transverse connection.

Of course, the same applies in the easier situation where we consider a (non-transverse) locally metric connection ∇ , i.e. there exists a metric h on the universal cover \tilde{M} of M such that the lifted connection $\tilde{\nabla}$ is the Levi-Civita connection of h .

We recall the definition of a de Rham decomposition:

Definition 2.9. A de Rham decomposition of a Riemannian manifold (M, g) is a product of manifolds $(M_0, g_0) \times \dots \times (M_p, g_p)$ isometric to M such that (M_0, g_0) is flat and the other factors are irreducible.

A direct consequence of the study of transverse similarity structures we will carry out is the following general result:

Theorem 2.10. Let \mathcal{F} be a foliation with a transverse holonomy-invariant locally metric connection on a compact manifold. If the transversal foliation has no flat factor in its local de Rham decomposition, then \mathcal{F} is a Riemannian foliation.

In the general case, consider a transversal τ , and its de Rham decomposition $\tau = \tau^0 \times \tau^*$, where τ^0 corresponds to the flat factor, and τ^* corresponds to all the other factors (we will precise what we mean exactly by this decomposition in Section 6.3). By taking the inverse images by the local submersions defining \mathcal{F} of τ^0 and τ^* , one gets foliations \mathcal{F}^0 and \mathcal{F}^* , with transversal holonomy-invariant metric connections modeled on τ^* and τ^0 respectively. For instance, \mathcal{F}^0 is obtained by saturating \mathcal{F} with τ^0 , and similarly for \mathcal{F}^* , thus \mathcal{F} appears as the intersection of \mathcal{F}^0 and \mathcal{F}^* .

Observe that the previous theorem applies to \mathcal{F}^0 , implying:

Theorem 2.11. Let \mathcal{F} be a foliation on a compact manifold together with a transverse-holonomy-invariant metric connection, and let \mathcal{F}^0 be the saturation of \mathcal{F} by the flat factor of the de Rham decomposition of the transversal connection. Then \mathcal{F}^0 is a Riemannian foliation.

2.2. Foliations with a transverse holonomy-invariant similarity structure. In the course of our analysis, we will have to deal with closures of leaves of the foliation \mathcal{F} . However, such a closure is not necessarily a manifold, since there could be a subset where the leaves accumulate. In order to overcome this technical difficulty, we consider the more general concept of *lamination*. A lamination on a topological space X is a collection of charts (U_i, φ_i) , where U_i is a covering of X , such that the maps φ_i are homeomorphisms from U_i to $V_i \times X_i$, with V_i a subset of an Euclidean space and X_i a subspace of X , and the transition maps $\varphi_i \circ \varphi_j^{-1}$ preserve the Euclidean factor.

If the topological space X admits a distance d , we define the bi-equicontinuity domain of the lamination as the set of points $x \in X$ such that there exists $\delta(x) > 0$, for any holonomy map γ from a transversal T_x of x to a transversal T_y at a point $y \in X$, for any $z_1, z_2 \in T_x$,

$$\frac{1}{\delta(x)}d(z_1, z_2) \leq d(\gamma(z_1), \gamma(z_2)) \leq \delta(x)d(z_1, z_2).$$

This set is more often called the equicontinuity domain, but we use this terminology in order to avoid confusion in the sequel, adopting the convention that *bi-equicontinuity* means that we have both the lower and the upper bound, while *equicontinuity* means we only have the upper bound. We now define precisely the structure we will be interested in, namely a transverse holonomy-invariant (Riemannian) similarity structure.

Definition 2.12. *A transverse similarity structure on the foliated manifold (M, \mathcal{F}) is a maximal open covering $\{U_i\}_{i \in I}$ of M together with, for any $i \in I$, a set of homothetic metrics $G_i := \{\lambda g_i, \lambda > 0\}$ where g_i is a transverse Riemannian metric on $(U_i, \mathcal{F}|_{U_i})$ with the property that for any $i, j \in I$, $G_i|_{U_i \cap U_j} = G_j|_{U_i \cap U_j}$. This transverse similarity structure is holonomy-invariant if the sets G_i are preserved by the holonomy pseudo-group, i.e. for any holonomy map γ defined from a connected transversal $T^i \subset U_i$ to a connected transversal $T^j \subset U_j$, $\gamma^*(g_j|_{T^j}) = \lambda g_i|_{T^i}$ for some $\lambda \in \mathbb{R}$.*

A holonomy-invariant transverse similarity structure induces a natural holonomy-invariant transverse connection. Indeed, on a sufficiently small open foliated domain on which one can choose a globally defined holonomy-invariant transverse metric of the similarity class, and the transverse Levi-Civita connection of this metric is actually independent of this choice. Consequently, if one defines a connection in such a way around each point, we just remark that these connections coincide on the intersections of the open sets because of the compatibility condition of Definition 2.12.

The main goal of our study of foliations is the following structure theorem:

Theorem 2.13. *Let \mathcal{F} be a foliation on a compact manifold, endowed with a holonomy-invariant similarity structure. If \mathcal{F} contains a closed invariant subset where it is an equicontinuous lamination, then \mathcal{F} is a Riemannian foliation.*

If \mathcal{F} is not Riemannian, then it is transversely flat, i.e. transversely modelled on $(\text{Sim}(\mathbb{R}^q), \mathbb{R}^q)$.

Remark 2.14. *Observe that we are making use above of a “metric” equicontinuity notion rather than a “topological” one, to mean that we have here Lipschitz estimates instead of just a rough uniform modulus of continuity. This was to simplify exposition, and also because of equivalence of these equicontinuity concepts in our framework of foliations endowed with a transverse holonomy-invariant connection, and again as we will see it later in Section 6.1, this is equivalent to be Riemannian (in the classical sense). In the general case, variants of “topological Riemannian” foliations were relatively recently introduced and studied from the point of view of their leaf closures and their relationships with the classical “smooth Riemannian” foliations. The history started with questions asked by E. Ghys [21, Appendix E], and some answers by Kellum [15]. As more recent references, we can quote: [1, 2, 3].*

Remark 2.15. *Theorem 2.10 says that among foliations with a transverse holonomy-invariant locally metric connection, only the flat case is relevant in the sense that it may*

have a strong non-Riemannian dynamics. Here flat means the foliation is transversally modelled on $(\text{Aff}(\mathbb{R}^q), \mathbb{R}^q)$. This is a completely open research domain, even in the dimension 0 case, that is that of compact locally flat manifolds, for which there is the huge classification conjectures by Marcus and Auslander. A more tamed situation is that of foliations with of transverse flat similarity structure, i.e. those modelled on $(\text{Sim}(\mathbb{R}^q), \mathbb{R}^q)$. The dimension 0 case is classified by Fried [11]. In the non-trivial codimension cases, most investigations concern the cases $q = 1$, that is $(\text{Aff}(\mathbb{R}), \mathbb{R})$ and $q = 2$, which actually reduces (up to orientation) to the 1-dimensional complex situation $(\text{Aff}(\mathbb{C}), \mathbb{C})$. Let us quote the following works around this problem: [5, 8, 12, 14, 18, 23, 25, 26]

2.3. De Rham decomposition. The final step of our analysis is to prove the existence of a de Rham decomposition on the universal cover \tilde{M} of a compact manifold M admitting a locally metric connection ∇ .

The main issue is that the classical de Rham theorem does not apply since this metric is not complete in general. With these notations, our main result is:

Theorem 2.16 (De Rham decomposition). *Let M be a compact conformal manifold together with a locally metric connection ∇ . Then, the Riemannian manifold (\tilde{M}, h) admits a de Rham decomposition, where h is any metric for which $\tilde{\nabla}$ is the Levi-Civita connection of $\tilde{\nabla}$ on \tilde{M} . Furthermore, the flat factor is complete.*

3. CONSTRUCTION OF A HEAFLIGER STRUCTURE

From now on, we assume that the manifold M is compact.

The usual definition of the holonomy starts with taking a complete transversal submanifold \mathcal{T} to \mathcal{F} and considering the holonomy transformations as local diffeomorphisms of \mathcal{T} . One can take for instance \mathcal{T} to be a union of small local transversals in a family of flow boxes covering M . Changing \mathcal{T} to \mathcal{T}' induces an equivalence between the two pseudo-groups of local diffeomorphisms of \mathcal{T} and \mathcal{T}' . All this, is quite delicate to formalize and manipulate. There is in particular the problem of artificial blow up of the holonomy maps due to the choice of \mathcal{T} (in particular when approaching the \mathcal{T} -boundary). There is however the nice situation where \mathcal{F} admits a supplementary foliation, and holonomy maps are viewed as local diffeomorphisms of this foliation. A strong simple and “talking” sub-case is that of foliated (also said flat) bundles where the holonomy is globally defined. Here, M fibers over a typical leaf F with fiber \mathcal{T} and the foliation \mathcal{F} is transversal to fibers. If c is a path in F with endpoints x, y , then the associated holonomy is a map $H_c : \mathcal{T}_x \rightarrow \mathcal{T}_y$ given by lifting c tangentially to $T\mathcal{F}$. In other words, $T\mathcal{F}$ as a supplementary of the vertical space of the fibration is seen as the horizontal space of a connection, and since it is integrable, this (non-linear) connection is flat, and H_c is the holonomy of this connection. The holonomy is given

by a representation $\rho : \pi_1(F) \rightarrow \text{Diff}(\mathcal{T})$ and the total space is the suspension of ρ , namely the quotient of $\tilde{F} \times \mathcal{T}$ by the diagonal action of $\pi_1(F)$.

In fact, the idea of Heafliker structures (even though motivated by other considerations), exactly allows one to maneuver to bring himself back to this situation of foliated bundles, but locally. We give the general idea in the following lines. For a more detailed expose about these structures, see for example [17, Section 1.3] and the references therein.

We endow M with an auxiliary Riemannian metric k , which will be often implicit and whose choice does not matter.

We consider $\mathcal{N} := T\mathcal{F}^\perp \simeq TM/T\mathcal{F}$, the normal bundle of \mathcal{F} . By compactness, there exists an open neighbourhood O of the zero-section of \mathcal{N} , such that the exponential Riemannian map of k , denoted by $\phi : O \rightarrow M$, once restricted to the fiber over a point $x \in M$, i.e. $\phi_x : O_x = O \cap \mathcal{N}_x \rightarrow M$, is a diffeomorphism onto a transversal \mathcal{T}_x to \mathcal{F} passing through x .

We can then consider $\hat{\mathcal{F}} := f^*\mathcal{F}$, the pull back by ϕ of \mathcal{F} . This $\hat{\mathcal{F}}$ is transversal to the fibers of the fibration $\mathcal{N} \rightarrow M$ (but only locally, i.e. along O), and thus this looks like to a foliated (flat) bundle.

The $\hat{\mathcal{F}}$ -leaves are tubular neighbourhoods of the \mathcal{F} -leaves. More precisely, \mathcal{F} is gotten as the intersection of $\hat{\mathcal{F}}$ with the 0-section (of the fibration $\mathcal{N} \rightarrow M$), and each $\hat{\mathcal{F}}$ -leaf retracts naturally to a \mathcal{F} -leaf which is its intersection with the 0-section. For this, when speaking of holonomy of $\hat{\mathcal{F}}$, we can restrict ourselves to \mathcal{F} -paths, i.e. $\hat{\mathcal{F}}$ -paths contained in M . We will in fact often identify $x \in M$ with its image 0_x by the 0-section.

The philosophy is that \mathcal{F} and $\hat{\mathcal{F}}$ have the same holonomy maps, which can be seen as local diffeomorphisms between the family of $\hat{\mathcal{F}}$ -transversals $\{O_x\}_{x \in M}$, or alternatively between the family of \mathcal{F} -transversals $\{\mathcal{T}_x\}_{x \in M}$. More precisely, if $x, y \in M$ are in the same leaf F of \mathcal{F} and c is a path in F joining x and y , the holonomy map induced by c , denoted by $\bar{H}_c : \mathcal{T}_x \rightarrow \mathcal{T}_y$ coincides with $\phi_y \circ H_c \circ \phi_x^{-1}$ defined on a neighbourhood of 0_x in the fiber O_x to O_y , where H_c is the holonomy map induced by c on $(O, \hat{\mathcal{F}})$. Observe that the domain of definition of H_c is not the full O_x , since H_c is obtained by considering horizontal lifts of c , but this does not necessarily exist for all time, e.g. O is not compact.

The *infinitesimal holonomy* map $h_c : \mathcal{N}_x \rightarrow \mathcal{N}_y$ is the derivative $d_x H_c$. It also equals the usual holonomy of the Bott connection ∇^B on the normal bundle \mathcal{N} , defined by

$$(2) \quad \nabla_X^B Y := [X, Y]^{\mathcal{N}}, \quad \forall X \in \Gamma(TM), \forall Y \in \Gamma(\mathcal{N}),$$

where the \mathcal{N} -exponent stands for the projection onto the normal bundle. Indeed, this connection is flat and its holonomy bundle through x is exactly the leaf containing x .

4. HOLONOMY-INVARIANT TRANSVERSAL CONNECTION

We now assume that (M, \mathcal{F}) carries a holonomy-invariant transverse connection ∇ , and we keep all the notations introduced in the previous section. This connection induces a connection ∇^x on each transversal \mathcal{T}_x in a natural way.

Observe that ϕ and accordingly $\hat{\mathcal{F}}$ is by no means unique, but any choice of $\hat{\mathcal{F}}$ shares with \mathcal{F} the same holonomy pseudo-group (up to equivalence in this category). We will in fact keep \mathcal{T}_x , but modify $\phi_x : O_x \rightarrow \mathcal{T}_x$ to become the exponential map for ∇^x . Indeed, the tangent space $T_x \mathcal{T}_x$ coincides with \mathcal{N}_x , and thus the exponential map for ∇^x sends a neighbourhood of 0_x (identified to x) to a neighbourhood of x in \mathcal{T}_x . So, up to a restriction of O , we can assume that the new ϕ is this exponential map.

Now, the holonomy map \bar{H}_c sends a neighbourhood of x in \mathcal{T}_x to a neighbourhood of y in \mathcal{T}_y and sends ∇^x to ∇^y because the transverse connection is holonomy-invariant. But a connection-preserving map becomes equal to its derivative in exponential coordinates. This means precisely that H_c coincides on appropriate neighbourhoods with its derivative, the infinitesimal holonomy h_c .

It is also true, conversely, that \mathcal{F} has a holonomy-invariant transversal connection if there is an associated $\hat{\mathcal{F}}$ having such a ‘‘linear’’ holonomy.

4.1. Equicontinuity domain. For any $x \in M$, we denote by $\Gamma_x(\mathcal{F})$ the set of all paths emanating from x and contained in the leaf passing through x . One can define the (infinitesimal) equicontinuity domain as the set

$$\{x \in M \mid \exists f(x) > 0, \forall c \in \Gamma_x(\mathcal{F}), \|h_c\| \leq f(x)\},$$

where the operator norm is defined by means of an auxiliary metric k . However, we will need a ‘‘bi-equicontinuity’’ condition, where we also want h_c to have a bounded contraction. To this purpose, we define \mathcal{E} to be the (infinitesimal) bi-equicontinuity domain:

$$\mathcal{E} = \{x \in M \mid \exists \delta(x) > 1, \forall u \in O_x, \forall c \in \Gamma_x(\mathcal{F}), \frac{\|u\|}{\delta(x)} \leq \|h_c(u)\| \leq \delta(x)\|u\|\}.$$

In the similarity case, on which we will focus below, this is equivalent to:

$$\frac{1}{\delta(x)} \leq \|h_c\| \leq \delta(x)$$

(or equivalently $\frac{1}{\eta(x)} \leq \det(h_c) \leq \eta(x)$ where $\eta(x)$ is a power of the previous $\delta(x)$ depending on the codimension).

It is obvious that the bi-equicontinuity domain \mathcal{E} is \mathcal{F} -invariant by definition. Our aim is to show that in the situation at hand, i.e. when the transversal has a holonomy-invariant Riemannian similarity structure, it is the whole manifold M . Since M is connected, an easy strategy is to prove that \mathcal{E} is both open and closed. The openness is just a consequence of the existence of the holonomy-invariant transverse connection.

Proposition 4.1. *[Propagation of equicontinuity] The bi-equicontinuity domain \mathcal{E} is open. Furthermore, the leaves admit compact invariant neighbourhoods in \mathcal{E} . In particular, the saturation of a compact subset in \mathcal{E} is relatively compact in \mathcal{E} .*

Proof. Let $x \in \mathcal{E}$, $\epsilon > 0$. Consider the ball $B(0_x, \epsilon)$ (with respect to the metric k) and its saturation by the holonomy pseudo-group of $\hat{\mathcal{F}}$:

$$S(x, \epsilon) := \bigcup_{c \in \Gamma_x(\mathcal{F})} H_c(B(0_x, \epsilon)).$$

Since the holonomy is linear (that is $H_c = h_c$), and by definition of $\delta(x)$,

$$S(x, \epsilon) \subset \bigcup_{y \in \mathcal{F}_x} B(0_y, \delta(x)\epsilon),$$

hence, for ϵ small one has $\overline{S(x, \epsilon)} \subset O$. The holonomy maps are thus defined for all times if one starts with a sufficiently small ball $B(0_x, \epsilon)$, and in addition the image of the saturation $S(x, \epsilon)$ by ϕ is a relatively compact \mathcal{F} -invariant neighbourhood of \mathcal{F}_x . For $u \in B(0_x, \epsilon)$ near 0_x , u in the bi-equicontinuity domain of $(O, \hat{\mathcal{F}})$ by linearity of the holonomy. Now, if as previously announced we want to restrict ourselves to the holonomy of \mathcal{F} -leaves (instead of $\hat{\mathcal{F}}$), we use $\phi : O \rightarrow M$, and see that $z = \phi(u)$ belongs to the \mathcal{F} -bi-equicontinuity domain, which is therefore open. \square

5. HOLONOMY-INVARIANT TRANSVERSAL SIMILARITY STRUCTURE

We now endow the foliation \mathcal{F} with a transverse (Riemannian) similarity structure in the sense of Definition 2.12. This in turn gives us a holonomy-invariant transverse connection ∇ as explained in the line following the definition. In particular, all the constructions and results of the previous sections still hold, and we keep the same notations. Our goal is to prove the closeness of \mathcal{E} under this assumption.

As above, we can assume that for any $x \in M$, $\phi_x : O_x \rightarrow \mathcal{T}_x$ is the exponential map of ∇^x , up to a restriction of O .

We can define a transverse Riemannian metric g on (M, \mathcal{F}) by taking around each point $x \in M$ a local transverse metric g_{V_x} on $T\mathcal{F}/TM$ belonging to the transverse similarity class and restricting it to $T_x M / T_x \mathcal{F}$. In order to insure smoothness of this family we fix a volume element on the normal bundle $\mathcal{N} \simeq N\mathcal{F} \rightarrow M$.

Note that g is not holonomy-invariant in general, since it depends strongly on the chosen volume element.

5.1. Singular metric. Let R be the curvature tensor of the bundle $\mathcal{N} \rightarrow M$ endowed with the connection ∇ (seen as a map from $\Gamma(TM) \times \Gamma(\mathcal{N})$ to $\Gamma(\mathcal{N})$). We restrict it to $\Gamma(\mathcal{N}) \times \Gamma(\mathcal{N})$ and we consider the function $w : M \rightarrow \mathbb{R}_{>0}$ which associates to $x \in M$ the norm of R with respect to g at x .

The function $w : O \rightarrow \mathbb{R}_{\geq 0}$ is continuous because it is the norm of a smooth function. In particular, since M is compact, w has a maximum $\max_M w$. The (non-smooth) transverse metric m , defined by $m := wg$, is a holonomy-invariant singular transverse metric.

Proposition 5.1. *Let $\mathcal{U} = \{x \in M, w(x) \neq 0\}$. Then, \mathcal{U} is contained in the bi-equicontinuity domain \mathcal{E} .*

Proof. Let $x \in \mathcal{U}$ and let $c : [0, 1] \ni t \mapsto c(t) \in \mathcal{F}_x$ be a path joining $x = c(0)$ to $y := c(1)$. Let $a := w(x), b := w(y)$. Assume $a > b$, say, more precisely, that $b = \inf\{w(c(t)), t \in [0, 1]\}$. The infinitesimal holonomy from $\mathcal{N}_{c(t)}$ to $\mathcal{N}_{c(s)}$ is a homothety of ratio $\sqrt{\frac{w(c(t))}{w(c(s))}}$ with respect to the metric g . In particular, since b realizes the minimum of w on c , there exists $\epsilon > 0$ (which can be taken uniform because M is compact) such that for any $t \in [0, 1]$, the $\tilde{\mathcal{F}}$ -holonomy $B(0_y, \epsilon) \rightarrow B(0_{c(t)}, \epsilon)$ is well defined and contracting.

Thus, if w vanishes somewhere on $\phi(B(0_y, \epsilon))$, then the same holds on $\phi(B(0_x, \epsilon))$. Assuming ϵ is small enough, w does not vanish on $\phi(B(0_x, \epsilon))$ neither on $\phi(B(0_y, \epsilon))$. In particular, the $\epsilon/2$ -tubular neighbourhood of the zero-section in \mathcal{N} is sent by the exponential map ϕ onto a subset of M whose closure does not meet the null locus of w . Consequently, w has a non-zero infimum on \mathcal{F}_x , and the homothety ratios $\sqrt{\frac{w(x)}{w(z)}}$ are bounded by two positive bounds for all $z \in \mathcal{F}_x$. Therefore, $x \in \mathcal{E}$. \square

5.2. Dichotomy. It remains to prove that \mathcal{E} is closed, which would lead to the following result:

Proposition 5.2. *A Foliation with a transverse holonomy-invariant similarity structure is everywhere bi-equicontinuous whence it is bi-equicontinuous at some point, that is, if \mathcal{E} is non-empty, then $\mathcal{E} = M$.*

Proof. Let $A := \bar{\mathcal{E}} - \mathcal{E}$ be the boundary of \mathcal{E} (this is an \mathcal{F} -invariant subset of M), B an ϵ -neighbourhood of A (for the metric g , and taking ϵ small enough) and $C := \mathcal{E} - B$. Thus, by Proposition 4.1 the saturation U of C is an invariant subset whose closure \bar{U} is a saturated compact subset of \mathcal{E} .

Assume by contradiction that A is non-empty. Let $x \in A$. There is a maximal transverse open ball $\phi(B(0_x, \epsilon_x))$ contained in $M - \bar{U}$, with $\epsilon_x \leq \epsilon$. Moreover, since \bar{U} is compact and does not meet the boundary of \mathcal{E} , there exists $\epsilon_0 > 0$ such that $\epsilon_x \geq \epsilon_0$. If a holonomy map is defined on $\phi(B(0_x, \epsilon_x))$, then its image is a ball $\phi(B(0_y, \epsilon'))$ with $y \in A$ and $\epsilon' \leq \epsilon_y \leq \epsilon$ (since otherwise we meet points of the invariant set U). This implies that the holonomy map has derivative of bounded distortion $\epsilon/\epsilon_x \leq \epsilon/\epsilon_0$, which does not depend on the point x .

Now, let c be a path emanating from x , i.e. $c(0) = x$, with $c(1) = y$. For small t , the holonomy from $c(0)$ to $c(t)$ is defined on the whole $\phi(B(0_x, \epsilon_x))$. Its image is contained in $\phi(B(c(t), \epsilon_{c(t)})) \subset \phi(B(c(t), \epsilon))$. Thus, this is defined for all t .

It follows that $x \in \mathcal{E}$: contradiction. This means that \mathcal{E} is closed in M , and since it is also open and M is connected, either $\mathcal{E} = M$ or $\mathcal{E} = \emptyset$. \square

5.3. Remark: non-similarity case. This dichotomy is no longer true for general foliations with a transversal connection. An example of a transversally Lorentzian foliation

of dimension 1 in a manifold of dimension 3, with a proper domain of equicontinuity, is given in [9]. We outline the construction here.

In order to give a first feeling of the example, we describe a simple construction which will not lead to a counter-example but provides the general idea. Let Σ be a closed surface and let $f : \Sigma \rightarrow \mathbb{R}^{1,1}$ be a smooth map, where $\mathbb{R}^{1,1}$ is the Minkowski plane endowed with the Lorentz metric $dx dy$. Consider the map $d : \Sigma \times \mathbb{R} \ni (z, t) \mapsto h^t(f(z)) \in \mathbb{R}^{1,1}$, with $h^t(x, y) = (e^t x, e^{-t} y)$.

Assume that f is chosen so that d is a submersion, hence we can consider the foliation defined by the level-sets of d . One has $d(z, t+n) = \phi^n(d(z, t))$, where $\phi := h^1$, so the foliation descends to a foliation of dimension 1 on $\Sigma \times \mathbb{S}^1 \simeq \Sigma \times (\mathbb{R}/\mathbb{Z})$.

In this way, $\Sigma \times \mathbb{S}^1$ has a transverse structure modelled on $\mathbb{R}^{1,1}$ and its cover $\Sigma \times \mathbb{R}$ has d as a developing map, and holonomy ϕ . More precisely the holonomy homomorphism $\pi_1(\Sigma \times \mathbb{S}^1) = \pi_1(\Sigma) \times \mathbb{Z} \rightarrow \text{Isom}(\mathbb{R}^{1,1})$ is trivial on $\pi_1(\Sigma)$ and sends the generator of \mathbb{Z} to ϕ .

Recall that d is defined as $d : \Sigma \times \mathbb{R} \ni (z, t) \mapsto h^t(f(z)) \in \mathbb{R}^{1,1}$. If we want d to be a submersion, we exactly need the image of $D_z f$ to be transversal to $V(f(z))$, where V is the vector field generating the one parameter group h^t (So $V(x, y) = (x, -y)$). In particular the boundary of $f(\Sigma)$ must be transverse to V , but this is not possible. Indeed, if we take the furthest (to 0) non-straight curve of the h -flow passing through a point $f(z)$ of $f(\Sigma)$, if $D_z f$ were transverse to V , then there would exist a point close to $f(z)$ on a more distant flow curve.

We need to modify the constriction as follows. Instead of $\mathbb{R}^{1,1}$, we will consider the 2-torus endowed with the product $\text{SL}(2, \mathbb{R}) \times \text{SL}(2, \mathbb{R})$ -action, that is the Einstein universe $\text{Eins}^{1,1}$ endowed with the Moebius group action. A model for this manifold is the compactification of the Minkowski space $\mathbb{R}^{1,1}$ together with the conformal structure given by the Lorentzian metric. More precisely, $\mathbb{R}^{1,1}$ embeds into the torus T^2 using for example the map $\varphi : \mathbb{R}^{1,1} \ni (x, y) \mapsto (\arctan x, \arctan y) \in T^2 \simeq (\mathbb{R}/\pi\mathbb{Z})^2$. The infinitesimal generator V of the one-parameter group h^t then induces a one-parameter group on the image of φ , still denoted by h^t . This one-parameter group extends uniquely to T^2 since $\varphi(\mathbb{R}^{1,1})$ is T^2 minus two circles, and is then dense. The 1-parameter group of transformations h^t has two attracting points, say A_1 and A_2 .

We claim that h^t preserves a lorentzian metric on $\text{Eins}^{1,1} - \{A_1, A_2\}$. To prove this, it is enough to find a metric on $\mathbb{R}^{1,1}$, which will be conformal to the standard one and defines a metric on $\text{Eins}^{1,1} - \{A_1, A_2\}$ by pull-back (after extension to the set where the metric is not defined). One can take $g := \frac{dx dy}{1+x^2 y^2}$, which satisfies this condition.

Now, we can find small discs around A_1 and A_2 which are transverse to the vector field generating h^t . Removing them, we get a 2-punctured torus \mathring{T}^2 with boundary $S^1 \cup S^1$. We now consider the surface Σ obtained by gluing smoothly two copies of this punctured torus along their boundaries. In order to define a suitable function f , we write $\Sigma = \mathring{T}^2 \cup \mathring{T}^2 \cup GZ$ where GZ is the gluing zone. We define f to be the canonical

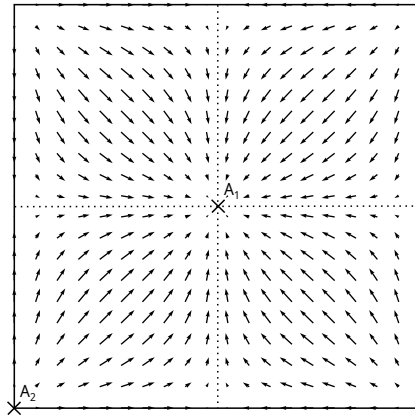


FIGURE 1. The unfolded Einstein universe, together with the vector field induced by h^t .

projection onto \mathbb{T}^2 on each copy of $\ddot{\mathbb{T}}^2$ and it is easy to see that we can choose f so that its set of critical points is just two circles, projecting onto two disjoint small circles around A_1 and A_2 (see Figure 2 below). But such circles are transverse to the vector field generating the flow h^t , hence the map $d : \Sigma \times \mathbb{R} \ni (z, t) \mapsto h^t \cdot f(z) \in \text{Eins}^{1,1}$ is a submersion.

As before, we obtain a foliation on $\Sigma \times \mathbb{S}^1$ given by the level-sets of d and with holonomy $\phi := h^1$. The holonomy morphism sends the generator of \mathbb{Z} to ϕ , as previously. The holonomy on this foliation preserves the Lorentzian metric g , hence it preserves the induced Levi-Civita connection on $\text{Eins}^{1,1}$. The equicontinuity domain is non-empty but also proper since it consists of all the points outside of the two points where the flow h^t is hyperbolic and the straight lines of the flow emanating from them (in other words, all the straight lines drawn on Figure 1).

6. PROOFS OF THEOREMS 2.10, 2.11, 2.13, 2.16

6.1. Equicontinuity implies Riemannian. So far, we have proved that a compact foliated manifold admitting a transverse holonomy-invariant similarity structure is either flat, or everywhere equicontinuous using Proposition 5.1 and Proposition 5.2. In order to complete the proof of Theorem 2.10, it remains to show that the foliation is then Riemannian.

Theorem 6.1. *A foliation with a transverse holonomy-invariant connection on a compact manifold is Riemannian once it is equicontinuous.*

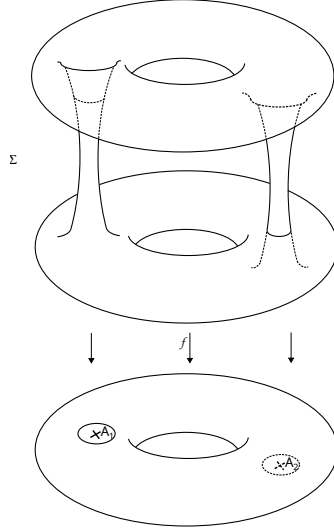


FIGURE 2. The map f from Σ to $\text{Eins}^{1,1}$. Note that the drawing is not faithful since there should be no self-intersection, but this is an immersion of Σ into \mathbb{R}^3 .

Our proof will follow closely the one of [27]. Before proceeding with the proof, we recall a few notions and a fundamental structure theorem.

Definition 6.2. *On a foliated manifold (N, \mathcal{G}) , a vector field Y is said to be foliate if for any $X \in T\mathcal{G}$ one has $[X, Y] \in T\mathcal{G}$.*

Definition 6.3. *A foliation is said to be parallelizable if it admits a transverse $\{e\}$ -structure, where $\{e\}$ is the trivial group with one element. The parallelism is said to be transversely complete if for any any vector $\bar{X} \in TM/T\mathcal{F}$ of the parallel basis, there exists a complete vector field $X \in TM$ which projects to \bar{X} .*

Theorem 6.4. [21, Theorem 4.2'] *Let (N, \mathcal{G}) be a foliation of codimension q admitting a transversely complete parallelism. Then, the closure of the leaves of \mathcal{G} define a foliation $\bar{\mathcal{G}}$ of codimension q_b which is induced by a submersion $\pi : N \rightarrow W$, where W is a manifold. Moreover, for any $z \in W$, the foliation induced by \mathcal{G} on $\bar{\mathcal{G}}_z := \pi^{-1}(z)$ has dense leaves and the space of foliate transverse vector fields of $(N_y, \mathcal{G}|_{\bar{\mathcal{G}}_z})$ is a Lie algebra of dimension $q - q_b$. In particular, if a transverse foliate vector field of the foliation \mathcal{G} tangent to $\bar{\mathcal{G}}_z$ vanishes at a point, then it is zero on all $\bar{\mathcal{G}}_z$.*

Proof of Theorem 6.1. We consider a compact manifold N^n together with a foliation of codimension q , endowed with a transverse holonomy-invariant connection. Let ω_T be the corresponding connection form on the frame bundle $\text{Fr}(TN/T\mathcal{G})$ and let θ_T be the transverse fundamental form of $\text{Fr}(TN/T\mathcal{G})$.

We consider the lifted foliation \mathcal{F}_1 on $\text{Fr}(TN/T\mathcal{G})$. It admits a transverse parallelism $\{\lambda_1, \dots, \lambda_{n^2}, u_1, \dots, u_n\}$ defined by

$$\omega_T(\lambda_i) = E_i, \theta_T(\lambda_i) = 0 \quad \text{and} \quad \omega_T(u_j) = 0, \theta_T(u_j) = e_j,$$

where (E_1, \dots, E_{n^2}) and (e_1, \dots, e_j) are the canonical bases of $\mathfrak{gl}_n(\mathbb{R})$ and \mathbb{R}^q respectively. This is a transverse parallelism because the connection is projectable (i.e. holonomy-invariant). In addition, this parallelism is complete because the λ_i 's are obviously complete and the u_j 's are represented by complete vector fields. Indeed, choose an arbitrary Riemannian metric on N and identify $TN/T\mathcal{G}$ with the orthogonal of $T\mathcal{G}$. The connection ω_T defines a connection on $T\mathcal{G}^\perp$ which can be extended to a connection ∇ on TN . Take the representative \tilde{u}_j of u_j given by the identification $T\mathcal{G}^\perp \simeq TN/T\mathcal{G}$, then its integral curves project on M to geodesics of ∇ , which are defined for all times. Consequently, the foliation \mathcal{G}_1 admits a complete transverse parallelism, and we can apply Theorem 6.4: the closures of the leaves give a foliation $\bar{\mathcal{G}}_1$ induced by a submersion $\pi : \text{Fr}(TN/T\mathcal{G}) \rightarrow W$.

In addition, the foliation \mathcal{G} has an equicontinuous holonomy pseudo-group. This implies that for any $x \in N$, the intersection of a leaf of \mathcal{G}_1 with the fiber over x is a relatively compact subset. Since N is compact, the closure of a leaf of \mathcal{G}_1 is therefore compact. The action of $G := \text{GL}_q(\mathbb{R})$ on $\text{Fr}(TN/T\mathcal{G})$ sends the closure of a leaf to the closure of a leaf, so it descends to an action on W , and this action is proper because the closures of the leaves are compact.

We now construct a fiber bundle E over W whose fiber over $z \in W$ is the set of transverse foliate vector fields tangent to $\pi^{-1}(z)$ (which is a finite-dimensional vector space of dimension independent of z according to Theorem 6.4). Since G preserves the set of foliate vector fields tangent to the closure the leaves, E admits a proper action of G .

Altogether, the vector bundles $TW \rightarrow W$ and $E \rightarrow W$ are both endowed with a proper G -action, so they admit G -invariant bundle Riemannian metrics. Summing these two metrics, we obtain a bundle Riemannian metric on $TW \oplus E \rightarrow W$ that we can pull-back to a bundle Riemannian metric on $T\text{Fr}(TN/T\mathcal{G})/T\mathcal{G}_1 \rightarrow \text{Fr}(TN/T\mathcal{G})$. The transverse metric defined this way is holonomy-invariant by definition. We reduce the fiber of $T\text{Fr}(TN/T\mathcal{G})/T\mathcal{G}_1 \rightarrow \text{Fr}(TN/T\mathcal{G})$ to the image of the horizontal vectors with respect to the connection ω_T , and the metric induced on this vector bundle is G -invariant, so it descends to a transverse holonomy-invariant metric on (N, \mathcal{G}) . \square

6.2. Proof of Theorem 2.13. We are now in a position to complete the proof of Theorem 2.13. We assume that the foliation \mathcal{F} on M admits a transverse holonomy-invariant similarity structure. If there is a non-empty closed \mathcal{F} -invariant subset of M where \mathcal{F} is an equicontinuous lamination, then the equicontinuity domain of M is non-empty, and it is the whole manifold M by Proposition 5.2. Applying Theorem 6.1, the foliation \mathcal{F} is Riemannian.

Conversely, if \mathcal{F} is not Riemannian, its equicontinuity domain must be empty by the contrapositions of Proposition 5.2 and Theorem 6.1. Using the contraposition of Proposition 5.1, the curvature of the metric $[g_T]$ is everywhere zero, i.e. it is a flat transverse metric.

6.3. Proof of Theorems 2.10 and 2.11. Here, we assume that \mathcal{F} admits a transverse holonomy-invariant locally metric connection ∇ . We first need to define precisely what we mean by the de Rham decomposition of the transversal. By definition, ∇ is a connection on the normal bundle $TM/\mathcal{F} \rightarrow M$, so there is a decomposition $N\mathcal{F} =: T'_1 \oplus \dots \oplus T'_m$ into ∇ -invariant subspaces, i.e. invariant by the holonomy of the connection ∇ . Assume first that there is no flat factor in this decomposition. The pre-images of these subspaces in TM are denoted by T_i and one has:

Lemma 6.5. *The distribution T_i is involutive.*

Proof. Let $\nabla^{\mathcal{F}}$ be any connection on $T\mathcal{F}$. One has $TM \simeq N\mathcal{F} \oplus T\mathcal{F}$ and we consider the linear connection $\nabla^0 := \nabla \oplus \nabla^{\mathcal{F}}$. Let $X, Y \in T_i$. Since ∇ is torsion-free, the torsion of ∇^0 has values in $T\mathcal{F}$ and we deduce that there is $Z \in T\mathcal{F}$ such that

$$[X, Y] = \nabla_X Y - \nabla_Y X + Z \in T_i. \quad \square$$

Let \mathcal{F}' be the foliation induced by the distribution $\bigoplus_{i \geq 2} T_i \oplus T\mathcal{F}$. The connection ∇ descends to a transverse connection ∇' on the normal bundle $TM/T\mathcal{F}'$, which is still locally metric and is preserved by the holonomy pseudo-group of the foliation $T\mathcal{F}'$. Moreover, ∇' has irreducible holonomy, and using Remark 2.8, there is a transverse holonomy-invariant Riemannian metric g'_T on the universal cover \tilde{M} of M such that the lifted connection $\tilde{\nabla}'$ is the Levi-Civita transverse connection of g'_T for the lifted foliation $\tilde{\mathcal{F}}'$. Since $\pi_1(M)$ acts by transversal similarities on $(\tilde{M}, \tilde{\mathcal{F}}', g'_T)$, g'_T induces a transverse similarity structure on M . For any holonomy map γ of the foliation \mathcal{F}' , γ lifts to a holonomy map of $\tilde{\mathcal{F}}'$ which acts as a transverse isometry, so γ preserves the similarity structure which is then holonomy-invariant. It is also non-flat since we assume there is no flat factor in the de Rham decomposition. It remains to apply Theorem 2.10 to get that \mathcal{F}' is Riemannian, and we obtain a Riemannian metric on the distribution T'_1 .

Iterating this process for all i 's one after the other and summing the metric obtained this way, we get a transverse holonomy-invariant Riemannian metric for the foliation \mathcal{F} , finishing the proof of Theorem 2.10.

To show Theorem 2.11, we remark that the pre-image of the flat distribution by the projection onto $TM/T\mathcal{F}$ is again involutive, and we replace \mathcal{F} by the foliation \mathcal{F}^0 induced by this new distribution. The transverse holonomy-invariant locally metric connection descends to the new transverse structure and has no flat factor in its de Rham decomposition. We can apply Theorem 2.10 to conclude.

6.4. Proof of Theorem 2.16. In this section, M is a compact manifold admitting a locally metric connection ∇ . By Remark 2.8, there exists metric on the universal cover \tilde{M} of M whose Levi-Civita connection is the lift $\tilde{\nabla}$ of ∇ . Let h be such a metric. Let $T\tilde{M} =: \tilde{T}_0 \oplus \tilde{T}_m$ be the decomposition of $T\tilde{M}$ into $\tilde{\nabla}$ -holonomy-invariant subspaces such that \tilde{T}_0 is flat and the \tilde{T}_i for $i \geq 1$ are irreducible. If there is only one factor, there is nothing to prove, so we assume this decomposition has at least two factors. We define two transverse foliations $\tilde{\mathcal{F}}$ and $\tilde{\mathcal{T}}$ by $T\tilde{\mathcal{F}} := \tilde{T}_m$ and $T\tilde{\mathcal{T}} := \bigoplus_{i=0}^{m-1} \tilde{T}_i$.

Since the elements of $\pi_1(M)$ act by affine transformations on $(\tilde{M}, \tilde{\nabla})$, they preserves this decomposition up to a permutation of the factors. There is a finite cover of M on which the permutations are trivial, and we can assume without loss of generality that M is this covering. Thus the distributions \tilde{T}_i descend to distributions T_i on M and the foliations $\tilde{\mathcal{F}}$ and $\tilde{\mathcal{T}}$ descend respectively to foliations \mathcal{F} and \mathcal{T} on M .

The connection ∇ descends naturally to a transverse locally-metric connection ∇^T for the foliation \mathcal{F} . This connection ∇^T is holonomy-invariant (with respect to the holonomy pseudo-group of \mathcal{F}). Indeed, for any point $x \in M$, one can take a small neighbourhood U of x such that the metric h descends to a metric g on U , and by the local de Rham theorem there exists, up to a restriction of U , an isometry $\varphi : (U, g) \rightarrow (F, g|_{TF}) \times (T, g|_{TT})$ where the product respects locally the two foliations \mathcal{F} and \mathcal{T} (i.e. $TF = T\mathcal{F}$ and $TT = T\mathcal{T}$). Moreover, ∇^T is the (transverse) Levi-Civita connection of $g|_{TT}$, so it is invariant by sliding along the leaves of \mathcal{F} in U , thus globally invariant by the holonomy pseudo-group.

Altogether, we have a compact manifold M with a foliation \mathcal{F} and a transverse locally metric holonomy-invariant connection ∇^T . In addition, ∇^T is non-flat and irreducible, so we can apply Theorem 2.10 to obtain a transverse holonomy-invariant Riemannian structure on (M, \mathcal{F}) , i.e. a holonomy invariant metric g_m on T_m . Iterating this construction by taking an arbitrary \tilde{T}_i , $i \geq 1$, for $\tilde{\mathcal{F}}$ instead of T_m , one has Riemannian metrics g_i on all T_i 's, which are invariant by sliding along the leaves of the other T_j for $0 \leq j \leq m$. The singular metrics defined this way lift to metrics \tilde{g}_i on \tilde{M} .

Assume first that the flat factor is non-trivial. We will apply the following generalization of the de Rham decomposition theorem proved in [24, Theorem 1]:

Theorem 6.6 (Ponge-Reckziegel). *Let (N, g_N) be a pseudo-Riemannian manifold with two orthogonal foliations L and K . Assume that the leaves of L are totally geodesic and geodesically complete and that the leaves of K are totally geodesic, then (N, g_N) is isomorphic to a product $(N_1, g_{N_1}) \times (N_2, g_{N_2})$ such that L and K correspond to the foliations induced respectively by N_1 and N_2 on N .*

We consider the two foliations L and K on \tilde{M} defined by $TL := \bigoplus_{i \geq 1} \tilde{T}_i$ and $TK := \tilde{T}_0$. We define a new metric h' on \tilde{M} by setting $h' = h|_{T_0} \oplus \tilde{g}_1 \oplus \dots \oplus \tilde{g}_m$. The foliations L and K are orthogonal and totally geodesic because (\tilde{M}, h') is locally a Riemannian products of integral manifolds of the two foliations. In addition, L is geodesically complete because its geodesics descend to geodesics of the induced

foliation on M together with the Riemannian structure we constructed above, and this is geodesically complete by compactness of M .

Applying Theorem 6.6, the manifold (\tilde{M}, h') is isometric to a product $(M_1, g_1) \times (M_2, g_2)$ where the foliations induced by M_1 and M_2 are respectively L and K . Moreover, (M_1, g_1) is simply-connected and complete, and the metric g_1 is, by construction, a local Riemannian product. Applying the usual de Rham decomposition theorem, we obtain that M_1 is a Riemannian product. Finally, taking the product decomposition of \tilde{M} obtained this way together with the metric induced by h on this decomposition, we obtain the de Rham decomposition we were seeking.

In the case where the flat factor is trivial, we simply apply this last step of the proof. This concludes this section.

7. CLOSED NON-EXACT WEYL STRUCTURES

In this section, we apply the result of Theorem 2.16, i.e. the existence of a de Rham decomposition, in order to prove a remarkable structure theorem for compact conformal manifolds admitting a particular connection called a *Weyl connection*. From now on, M is a compact manifold, and we endow it with a conformal structure c in the following sense:

Definition 7.1. *A conformal structure on M is set c of Riemannian metrics such that for any $g, g' \in c$, there exists a smooth function $f : M \rightarrow \mathbb{R}$ such that $g = e^{2f} g'$.*

Conformal manifolds admit a particular class of connection, called *Weyl connections* which generalize the notion of Levi-Civita connection in the conformal setting:

Definition 7.2. *A Weyl connection on the conformal manifold (M, c) is a torsion-free connection D such that D preserve the conformal structure, i.e. for any $g \in c$ there exists a 1-form θ_g on M such that $Dg = -2\theta_g \otimes g$. The 1-form θ_g is called the Lee form of D with respect to g . The triplet (M, c, D) is called a Weyl connection.*

A Weyl structure is said to be closed if its Lee form with respect to one metric, and then to all metrics in c , is closed. In this case, D is locally the Levi-Civita connection of a metric in c .

A Weyl structure is said to be exact if its Lee form with respect to one metric, and then to all metrics in c , is exact. In this case, D is the Levi-Civita connection of a metric in c .

We assume that there is a closed, non-exact Weyl connection D on the conformal manifold (M, c) . Once lifted to the universal cover \tilde{M} of M , the conformal structure c and the Weyl connection D induce a conformal structure \tilde{c} and a Weyl connection \tilde{D} on (\tilde{M}, \tilde{c}) . If g is a metric in c , the Lee form of \tilde{D} with respect to the lifted metric \tilde{g} to \tilde{M} is the pull-back $\tilde{\theta}_g$ of the Lee form θ_g to \tilde{M} , which is then exact. This means that \tilde{D} is an exact Weyl connection, and there exists a metric h , unique up to a positive multiplicative constant, such that $\tilde{D} = \nabla$ where ∇ is the Levi-Civita connection of h .

In particular, there exists $f : \tilde{M} \rightarrow \mathbb{R}$ such that $h = e^{2f}\tilde{g}$ and $\tilde{\theta}_g = df$. If we pick any $\gamma \in \pi_1(M)$, one has $\gamma^*\tilde{\theta}_g = \tilde{\theta}_g$, implying that $\gamma^*df = df$ and $\gamma^*f = f + \lambda_\gamma$ where $\lambda_\gamma \in \mathbb{R}$. Hence, $\pi_1(M)$ acts by similarities on (\tilde{M}, h) , and these similarities are not all isometries because otherwise h would descend to M , but this is impossible because D is non-exact.

The Weyl connection D is a locally metric connection of M and h is a metric on \tilde{M} whose Levi-Civita connection is \tilde{D} , so we can apply Theorem 2.16 and we infer that (\tilde{M}, h) has a de Rham decomposition $(M_0, h_0) \times \dots \times (M_p, g_p)$. We assume that (\tilde{M}, h) is neither irreducible nor flat and we denote by (N, h_N) the Riemannian product $(M_1, g_1) \times \dots \times (M_N, g_N)$.

Lemma 7.3. *The Riemannian manifold (N, h_N) is irreducible.*

Proof. By contradiction we assume that (N, h_N) is reducible, so it can be written as a product $(N_1, h_1) \times (N_2, h_2)$ where N_1 and N_2 have positive dimension. Moreover, the group $\pi_1(M)$ acts by similarities on (\tilde{M}, h) , and in particular it contains only affine maps, which then preserve the de Rham decomposition of (\tilde{M}, h) up to a permutation of the factors. Thus $\pi_1(M)$ preserve the decomposition $(M_0, g_0) \times (N, h_N)$ so it projects to a group Γ of similarities of (N, h_N) .

Let γ be a non-isometric similarity in Γ , which exists because $\pi_1(M)$ does not only contain isometries. We can assume, up to taking a power of γ , that γ preserves the decomposition $(N_1, h_1) \times (N_2, h_2)$ and it can be written as (γ_1, γ_2) where γ_1 and γ_2 act on N_1 and N_2 respectively. If (N_1, h_1) is complete, then γ_1 has a fixed point, but a manifold admitting a similarity with a fixed point is isometric to \mathbb{R}^n , which is not possible since (N, h_N) does not contain a flat factor. We can thus assume that (N_1, h_1) and (N_2, h_2) are both incomplete.

We recall that the Cauchy border of a Riemannian manifold \mathcal{N} is $\partial\mathcal{N} := \bar{\mathcal{N}} \setminus \mathcal{N}$ where $\bar{\mathcal{N}}$ is the metric completion of \mathcal{N} . The Cauchy border is preserved by any similarity of $\bar{\mathcal{N}}$, and any similarity of \mathcal{N} extends to a similarity of $\bar{\mathcal{N}}$ by density.

Since Γ acts cocompactly on N , there exist compact subsets K_1 and K_2 of N_1 and N_2 respectively such that $\Gamma \cdot (K_1 \times K_2) = N_1 \times N_2$. By compactness, there exist $0 < \alpha \leq \beta$ such that, if we denote by d_1 and d_2 the induced distances on the metric completions of N_1 and N_2 respectively, one has $\alpha \leq d_i(K_i, \partial N_i) \leq \beta$ for $i = 1, 2$. Then if we define, for $i = 1, 2$,

$$D_i := \{x \in \bar{N}_i \mid \alpha \leq d_i(\partial N_i, x) \leq \beta\}$$

one has $\Gamma \cdot (D_1 \times D_2) = N_1 \times N_2$. We choose any $(x_1, x_2) \in N_1 \times N_2$ such that $d_1(\partial N_1, x_1) = \alpha/2$ and $d_2(\partial N_2, x_2) = 2\beta$, which exists by connectedness of N and because α is a similarity of ratio different from 1. By the cocompactness of the action of Γ , there exists $\gamma \in \Gamma$ such that $(x_1, x_2) \in \gamma(K_1 \times K_2)$ and γ has ratio $\lambda > 0$. Since the Cauchy border of N is preserved by γ , one has $\lambda\alpha \leq d_1(\partial N_1, \gamma(K_1)) \leq \lambda\beta$ and $\lambda\alpha \leq d_2(\partial N_2, \gamma(K_2)) \leq \lambda\beta$, hence $\lambda \leq 1/2$ and $2 \leq \lambda$: contradiction. \square

Since we assumed that (\tilde{M}, h) is neither irreducible nor flat, N has positive dimension and (M_0, h_0) is isometric to an Euclidean space \mathbb{R}^q with $q \geq 1$. Summarizing, we proved:

Theorem 7.4. *Let (M, c, D) be a compact conformal manifold endowed with a closed non-exact Weyl structure. Let \tilde{M} be the universal cover of M and let h be the metric (unique up to a positive multiplicative constant) whose Levi-Civita connection is the lift of the Weyl connection. Then one of the following cases occurs:*

- (\tilde{M}, h) is flat;
- (\tilde{M}, h) is irreducible;
- (\tilde{M}, h) is isometric to $\mathbb{R}^q \times (N, g_N)$ where $q \geq 1$ and (N, g_N) is an irreducible Riemannian manifold.

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