SIMILARITY MANIFOLDS AND INOUE-BOMBIERI CONSTRUCTION

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Abstract. We study compact quotients of Riemannian products $\mathbb{R}^q \times N$, with N a complete, simply connected Riemannian manifold, by discrete subgroups Γ of $\mathsf{Sim}(\mathbb{R}^q) \times \mathsf{Isom}(N)$ acting freely and properly. We assume that the projection of Γ onto $\mathsf{Sim}(\mathbb{R}^q)$ contains a non-isometric similarity. This construction, when N is a symmetric space of non-compact type, is a generalization of the well-known Inoue surfaces. We prove that this situation is in fact equivalent to that of so-called LCP manifolds. We show a Bieberbach rigidity result in the case of symmetric spaces, providing a way to construct admissible groups Γ. We also consider the projection of Γ onto $\mathsf{Isom}(N)$ in the case where N has negative curvature, leading to some classification results.

Contents

1. Introduction

Inoue surfaces were constructed in 1975 by Masahisa Inoue [8] in his work on the classification of complex surfaces of Kodaira class VII and independently by Enrico Bombieri. There are defined as quotients of the complex manifold $\mathbb{C} \times \mathbb{H}$ by discrete groups of automorphisms. One can distinguish 3 types of such surfaces: S^-, S^0 and S^+. The types $S^\pm$ are well understood seeing the space $\mathbb{C} \times \mathbb{H}$ as a product $\mathbb{R}^3 \times \mathbb{R}$ on which acts a semi-direct product of the discrete Heisenberg group of dimension 3 with $\mathbb{Z}$, but our main interest here will be the class S^0. These compact manifolds are quotient of $\mathbb{C} \times \mathbb{H}$ by a discrete subgroup of $\mathsf{Sim}(\mathbb{R}^2) \times \mathsf{Isom}(\mathbb{H})$ (when seeing $\mathbb{C}$ as $\mathbb{R}^2$), where $\mathsf{Sim}(M)$ denotes the group of similarities of the Riemannian manifold M (see equation (2) below). It turns out that the structure of this quotient is very similar to the one of a recently studied class of manifolds, namely the Locally Conformally Product (LCP) manifolds.

LCP manifolds arose from the study of torsion-free connections on compact conformal manifolds. In the situation where the connection preserves the conformal structure, it is called a *Weyl connection*, and it was believed in the early days of the theory that Weyl connections on compact conformal manifolds were either flat or irreducible [3].

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However, this conjecture was disproved by Matveev and Nikolayevsky [11, 12], and the problem was closed thanks to a result of Kourganoff [9], who showed that a third and last possibility could occur. This new family was afterwards given the name of LCP manifold [5]. It consists of compact quotients of Riemannian manifolds of the form $\tilde{M} := \mathbb{R}^q \times N$, where N is simply connected, irreducible and non-complete, by a discrete subgroup Γ of $\mathsf{Sim}(\tilde{M}) \cap (\mathsf{Sim}(\mathbb{R}^q) \times \mathsf{Sim}(N))$. A new synthetic proof of this result was recently given by the authors of the present paper in [6], where the study of transversal similarity structures, playing a significant role in the analysis, was pushed further.

The relation between Inoue surfaces of type S^0 and LCP structures comes from the existence of an equivariant function supported by the manifold N on the universal cover $\mathbb{R}^q \times N$ of an LCP manifold [5, 14]. This induces a Riemannian metric on N, conformal to the original metric, for which Γ is a subgroup of $\mathsf{Sim}(\mathbb{R}^q) \times \mathsf{Isom}(N)$. Taking $q = 2$ and $N = \mathbb{H}$, we then obtain an Inoue surface of type S^0. Conversely, an Inoue surface could be given an LCP structure by taking a suitable equivariant function on $\mathbb{H}$. Consequently, Inoue surfaces share interesting properties with LCP structures. In particular, the universal cover of an LCP manifold carries a natural foliation $\tilde{\mathcal{F}}$ induced by the submersion $\mathbb{R}^q \times N \to N$, which descends to a transversely Riemannian foliation $\mathcal{F}$ on the quotient $\Gamma\backslash(\mathbb{R}^q \times N)$. The study of this foliation gives a lot of information on the structure of the quotient, and it is remarkable that the closures of its leaves are finitely covered by flat tori [9, Theorem 1.10].

In this paper, we study a class of quotient manifolds which generalizes naturally the construction of S^0 Inoue surfaces. These are compact quotients of $\mathbb{R}^q \times N$ (where N is a Riemannian complete manifold) by discrete subgroups Γ of $\mathsf{Sim}(\mathbb{R}^q) \times \mathsf{Isom}(N)$ acting properly and freely and whose projection onto $\mathsf{Sim}(\mathbb{R}^q)$ does not contain only isometries. We denote such a manifold by $Q(q, N, \Gamma)$. Our approach was first motivated by the known results for LCP manifolds and the close relationship between them and S^0 Inoue surfaces.

We first turn our attention to foliations, since they were the main tool for the understanding of LCP manifolds. One can still consider the foliation $\mathcal{F}$, defined in the same way as before, on any manifold $Q(q, N, \Gamma)$. This is a Riemannian foliation by construction, since Γ restricts to a subgroup of $\mathsf{Isom}(N)$ on N. A first question tackled by the previous observations is: are leaf closures of $\mathcal{F}$ still finitely covered by flat tori? This interrogation can actually be answered positively if one can prove a much stronger property: the class of LCP manifolds and the class of manifolds $Q(q, N, \Gamma)$ are equivalent. In the first part of this paper, we investigate this problem, and we prove that this holds by finding an equivariant function on N. More precisely we have:

Theorem 1.1 (Equivalence between LCP manifolds and quotients $Q(q, N, \Gamma)$)**.** *Let $q > 0$ and let (N, g_N) be a simply connected complete Riemannian manifold. Assume there is a discrete group $\Gamma \subset \mathsf{Sim}(\mathbb{R}^q) \times \mathsf{Isom}(N)$ acting properly discontinuously, freely and cocompactly on $\mathbb{R}^q \times N$, and such that the projection of Γ onto $\mathsf{Sim}(\mathbb{R}^q)$ does not contains only isometries. Then, denoting by P the projection of Γ onto $\mathsf{Isom}(N)$, by $\bar{P}$ the closure of P in $\mathsf{Isom}(N)$ and by $\bar{P}^0$ the identity connected component of $\bar{P}$:*

- *Γ is isomorphic to P, so the group homomorphism $\tilde{\rho} : \Gamma \to \mathbb{R}_+^*$ giving the similarity ratio of the projection onto $\mathsf{Sim}(\mathbb{R}^q)$ induces a group homomorphism $\rho : P \to \mathbb{R}_+^*$;*
- *there exists a $\bar{P}$-equivariant (for the morphism ρ) function e^f on N. In particular, Γ acts by similarities, not all isometries, on $\mathbb{R}^q \times (N, e^{2f} g_N)$;*

- *the group $\bar{P}^0$ is abelian and $\Gamma_0 = \Gamma \cap (\mathsf{Sim}(\mathbb{R}^q) \times \bar{P}^0)$ is a lattice in $\mathbb{R}^q \times \bar{P}^0$;*
- *the foliation $\tilde{\mathcal{F}}$ induced by the submersion $\mathbb{R}^q \times N \to N$ descends to a transversely Riemannian foliation $\mathcal{F}$ on $\Gamma \backslash (\mathbb{R}^q \times N)$ and the closures of the leaves of $\mathcal{F}$ are finitely covered by flat tori.*

Having this new understanding of the manifolds $Q(q, N, \Gamma)$, we would like to know how to construct examples, and in particular how can one find admissible discrete groups Γ. In order to continue in this direction, we restrict our setting to the case where N is a homogeneous manifold, so that its isometry group acts transitively on it. Finding a discrete cocompact group acting on $\mathbb{R}^q \times N$ can be done by finding a connected subgroup of $\mathsf{Sim}(\mathbb{R}^q) \times \mathsf{Isom}(N)$ acting properly and transitively on the product and taking a lattice in this group. The question is then: are all possible groups Γ lattices of a connected Lie subgroup of $\mathsf{Sim}(\mathbb{R}^q) \times \mathsf{Isom}(N)$ acting properly? This last property, which can be formulated on any homogeneous manifold, is called the *Bieberbach rigidity*. This concept of rigidity is for example equivalent in $\mathrm{Aff}(\mathbb{R}^n)$ to the Auslander conjecture, stating that affine crystallographic groups are virtually solvable (see [7, Sections 1.3 and 1.4] and the references therein for additional details).

The second part of the paper is devoted to the proof of the Bieberbach rigidity on the universal cover of $Q(q, N, \Gamma)$ in the case where N is a symmetric space of non-compact type. Notice that adding a compact factor to this symmetric space would not alter the result, so it is justified to restrict our analysis to this setting. The proof relies on the possibility to imbed the isometry group of N into a linear group, so we can use the algebraic group theory to conclude. We obtain the following result:

Theorem 1.2 (Bieberbach rigidity). *Let (N, g_N) be a complete, simply connected Riemannian symmetric space of non-compact type and let $q > 0$ be an integer. Let Γ be a subgroup of $\mathsf{Sim}(\mathbb{R}^q) \times \mathsf{Isom}(N, g_N)$ acting properly discontinuously, freely and cocompactly on $\mathbb{R}^q \times N$. Then, up to taking a finite index subgroup of Γ, there exists a connected subgroup L of $\mathsf{Sim}(\mathbb{R}^q) \times \mathsf{Isom}(N, g_N)$ acting properly and transitively on $\mathbb{R}^q \times N$ such that Γ is a lattice of L.*

The main goal would be to classify all possible quotients $Q(q, N, \Gamma)$, i.e. knowing which manifolds N and groups Γ could occur in this construction. The Bieberbach rigidity is a first significant step for the construction of lattices Γ in the symmetric case, and the next one would be to classify the possible manifolds N. We study in the last part of the paper the case where N is a Hadamard manifold of strictly negative curvature.

A first talkative example is the hyperbolic space $\mathbb{H}^n$, for which we give a description of the projection of Γ onto $\mathsf{Isom}(\mathbb{H}^n)$. We then focus on the general case of Hadamard manifolds with strictly negative curvature, using the same approach as in the case of the hyperbolic space. In this setting we obtain the following classification result:

Theorem 1.3 (Classification in the case of negative curvature). *Let N^n be a complete Riemannian manifold of strictly negative curvature and let $q > 0$ be an integer. Assume there is a discrete group $\Gamma \subset \mathsf{Sim}(\mathbb{R}^q) \times \mathsf{Isom}(N)$ acting properly, freely and cocompactly on $\mathbb{R}^q \times N$ and such that the projection of Γ onto $\mathsf{Sim}(\mathbb{R}^q)$ does not contain only isometries. Then, one has an isometry*

$$N \simeq (\mathbb{R} \times \mathbb{R}^{n-1}, dt^2 + \langle S_t \cdot, S_t \cdot \rangle) \tag{1}$$

where $S_t \in \mathsf{GL}(\mathbb{R}^{n-1})$ for all $t \in \mathbb{R}$. Denoting by $\bar{P}$ the closure of the projection of Γ onto $\mathsf{Isom}(N)$ and by $\bar{P}^0$ its identity connected component, $\bar{P}^0$ is exactly the set of translations of the factor $\mathbb{R}^{n-1}$ in the identification (1).

If moreover N is a homogeneous space, then either it is the hyperbolic space $\mathbb{H}^n$ or $\mathsf{Isom}(N)$ fixes a unique point at infinity and there exists an endomorphism A of $\mathbb{R}^{n-1}$ such that $S_t = e^{tA}$ in (1).

The present paper is divided into three parts. In Section 2 we prove Theorem 1.1, in Section 3 we investigate Bieberbach rigidity and we prove Theorem 1.2 and we finally consider the situation where N has negative curvature in Section 4, where we prove Theorem 1.3.

2. Compact quotient of product manifold by similarities

We first recall that a similarity (or homothety) between two Riemannian manifolds (M_1, g_1) and (M_2, g_2) is a diffeomorphism $\phi : M_1 \to M_2$ satisfying

$$\phi^* g_2 = \lambda^2 g_1 \tag{2}$$

for a positive real number λ called the ratio of the similarity. The set of similarities from a Riemannian manifold (M, g) to itself is denoted by $\mathsf{Sim}(M, g)$. The set of isometries of (M, g), i.e. the similarities with ratio 1, is denoted by $\mathsf{Isom}(M, g)$. We will often drop the metric g in these notations when there is no possible confusion.

The identity connected component of a Lie group G is denoted by G^0.

We consider the Riemannian product $\tilde{M} := \mathbb{R}^q \times (N, g_N)$ where $q \geq 1$, (N, g_N) is a simply connected complete Riemannian manifold. Let Γ be a subgroup of $\mathsf{Sim}(\mathbb{R}^q) \times \mathsf{Isom}(N, g_N)$ acting freely, properly discontinuously and cocompactly on $\tilde{M}$. In particular, $M := \Gamma \backslash \tilde{M}$ is a compact manifold. We define P as the projection of Γ onto $\mathsf{Isom}(N)$ and we denote respectively by $\bar{P}$ and $\bar{P}^0$ the closure of P in $\mathsf{Isom}(N, g_N)$ for the compact-open topology and its identity connected component.

We assume that there is at least one strict similarity in the projection of Γ onto $\mathsf{Sim}(\mathbb{R}^q)$, i.e. a similarity with ratio different from 1. In this case, we know that P is isomorphic to $\pi_1(M)$, i.e. the projection $\pi_1(M) \to P$ is injective using the proof of [5, Lemma 2.10], which is copied on the one of [9, Lemma 4.17]. We denote by $\varphi : P \to \pi_1(M)$ this isomorphism. Let $\rho_0 : \mathsf{Sim}(\mathbb{R}^q) \to \mathbb{R}_+^*$ be the group homomorphism giving the ratio of a similarity and we define $\rho := \rho_0 \circ \varphi$.

Our goal is to prove the following proposition:

Proposition 2.1. *There exists a smooth positive function $f : N \to \mathbb{R}$ which is P-equivariant, i.e. for any $p \in P$ one has $p^* f = \rho(p) f$.*

We start with a technical lemma:

Lemma 2.2. *There exists $\delta > 0$ such that for any $p \in P$ with $\rho(p) \neq 1$ and for any $x \in N$, $d_N(p(x), x) \geq \delta$ where d_N stands for the Riemannian distance on N.*

Proof. The group Γ preserves the product decomposition $\tilde{M} \simeq \mathbb{R}^q \times N$, so the two transverse foliations given by the product structure induce transverse foliations $\mathcal{F}$ and $\mathcal{G}$ on M. In addition, the group Γ projects to isometries on the second factor N, thus the Riemannian exponential of N descends to a map $\Xi : T\mathcal{G} \to M$ and the Riemannian metric g_N descends to a Riemannian bundle metric on $T\mathcal{G} \to M$. By compactness, there exists $\delta > 0$ such that for any $y \in M$, Ξ is injective on the open ball of radius δ of

$T_y\mathcal{G}$ (it suffices to find a finite covering of M by open subsets which are the projection of subsets of the form $B \times V$ with B a ball of $\mathbb{R}^q$ and V a small ball of N).

Now, let $p \in P$ with $\rho(p) \neq 1$. The associated map $\varphi(p)$ can be written as $(\phi, p) \in \mathrm{Sim}(\mathbb{R}^q) \times \mathrm{Isom}(N)$ and ϕ has a unique fixed point $a \in \mathbb{R}^q$. Let $x \in N$ and let $B_N(x,\delta)$ be the image of the open ball of radius δ in T_xN by the Riemannian exponential map of N. We have $(\phi,p)(a,x) = (a,p(x))$. But the previous discussion implies that the restriction of the projection $\tilde{M} \to M$ to $\{a\} \times B_N(x,\delta)$ is injective. Consequently, $(a,p(x)) \notin \{a\} \times B_N(x,\delta)$ i.e. $p(x) \notin B_N(x,\delta)$ and $d_N(x,p(x)) \geq \delta$. □

From the previous technical lemma, we would like to infer that the group homomorphism ρ can be extended to a group homomorphism $\bar{P} \to \mathbb{R}_+^*$ by continuity. In order to do so, it is sufficient to prove the following result:

Lemma 2.3. *For any $p \in \bar{P}^0 \cap P$, one has $\rho(p) = 1$.*

Proof. By Lemma 2.2, there exists an open neighbourhood U of id in $\mathrm{Isom}(N)$ such that $\rho(U \cap P) = \{1\}$, and we can assume this neighbourhood to be symmetric, i.e. $U = U^{-1}$ by replacing it by $U \cap U^{-1}$. This means that for any $p \in P$, $\rho((p \cdot U) \cap P) = \{\rho(p)\}$.

We define E as the set of all $p_0 \in \bar{P}^0$ such that there is a neighbourhood V of p_0 in $\bar{P}^0$ satisfying $\rho(V \cap P) = \{1\}$. The set E is open by definition and non-empty because $\mathrm{id} \in E$. We claim that E is also closed. Indeed, if $p_0 \in \bar{P}^0$ is not in E, then there exists $p \in p_0 \cdot U$ such that $\rho(p) \neq 1$, and $p_0 \in p \cdot U$ because U is symmetric. But $\rho((p \cdot U) \cap \bar{P}^0) = \{\rho(p)\}$, which means that for any $p_0' \in p \cdot U$ and any neighbourhood V of p_0', $P \cap V \cap (p \cdot U) \neq \emptyset$ because P is dense in $\bar{P}^0$, and for any $p' \in P \cap V \cap (p \cdot U)$, $\rho(p') = \rho(p) \neq 1$, so p_0' is not in E.

The set E is open, non-empty and closed in the connected set $\bar{P}^0$, thus $E = \bar{P}^0$ and the lemma follows. □

Corollary 2.4. *The group homomorphism $\rho : P \to \mathbb{R}_+^*$ extends uniquely to a continuous group homomorphism $\tilde{\rho} : \bar{P} \to \mathbb{R}_+^*$.*

Proof. This is a direct consequence of Lemma 2.3. □

We can now prove Proposition 2.1:

Proof of Proposition 2.1. By assumption, the group P acts cocompactly on N. Let then K be a compact subset of N such that $\bar{P} \cdot K = N$ and let f_0 be a non-negative function with compact support such that $f_0|_K = 1$. Let μ be the Haar-measure of $\bar{P}$ and we define for any $x \in N$:

$$f(x) := \int_{\bar{P}} \tilde{\rho}(p)^{-1}(p^* f_0)(x) d\mu(p). \tag{3}$$

The function f is well-defined because $\bar{P}$ acts properly on N and it is positive because for any $x \in N$ there exists $p \in \bar{P}$ such that $p(x) \in K$, so $(p^* f_0)(x) = 1$. Moreover, the function is smooth by construction. It remains to prove that it is equivariant. One has for any $p' \in P$:

$$\begin{aligned}(p'^* f)(x) &= p'^* \int_{\bar{P}} \tilde{\rho}(p)^{-1}(p^* f_0)(x) d\mu(p) = \int_{\bar{P}} \tilde{\rho}(p)^{-1}(p'^* p^* f_0)(x) d\mu(p) \\ &= \int_{\bar{P}} \tilde{\rho}(pp')^{-1} \tilde{\rho}(p')((pp')^* f_0)(x) d\mu(p) \\ &= \tilde{\rho}(p') \int_{\bar{P}} \tilde{\rho}(p)^{-1} \tilde{\rho}(p')(p^* f_0)(x) d\mu(p) \\ &= \rho(p') f(x).\end{aligned}$$

□

Corollary 2.5. *There exists a metric g'_N on N such that $\Gamma \subset \mathsf{Sim}(\tilde{M}) \cap (\mathsf{Sim}(\mathbb{R}^q) \times \mathsf{Sim}(N, g'_N))$.*

Proof. Taking f to be the function given by Proposition 2.1, we define $g'_N := f^2 g_N$, and we easily verify that this metric has the desired property. □

This last corollary means that we can equivalently see Γ as a subgroup of similarities of $(\tilde{M}, g_{\mathbb{R}^q} + g'_N)$ preserving the decomposition $\mathbb{R}^q \times N$, acting freely, properly discontinuously and cocompactly on $\tilde{M}$ and containing a non-isometric similarity. The two settings are actually equivalent, since in the latter case, there exists an equivariant function on N as shown in [5, Proposition 3.6] and in a simpler way in [14]. Note that in the general setting stated in the papers studying the case $\Gamma \in \mathsf{Sim}(\tilde{M})$, it is assumed that N is an irreducible (non-complete) Riemannian manifold. However, the irreducibility is not relevant for the results we use here. In particular, we can apply the analysis done in [9], and more precisely we have:

Corollary 2.6. *The group $\bar{P}^0$ is abelian.*

Proof. This is a consequence of the previous discussion and of [9, Lemma 4.1]. □

Remark 2.7. *We consider the foliation induced by the submersion $\tilde{M} \simeq \mathbb{R}^q \times N \to N$. Noticing that Γ preserves the decomposition $\mathbb{R}^q \times N$, this foliation descends to a foliation $\mathcal{F}$ on $M = \Gamma \backslash \tilde{M}$. Applying* [9, Theorem 1.10]*, the closure of the leaves of $\mathcal{F}$ are finitely covered by flat tori. More precisely, by* [9, Lemma 4.18] *there is a subgroup Γ_0 of Γ such that $\Gamma_0 = \Gamma \cap (\mathsf{Sim}(\mathbb{R}^q) \times \bar{P}^0)$ which is abelian and is a lattice in $\mathbb{R}^q \times \bar{P}^0$.*

The combination of Corollary 2.5, Corollary 2.6 and Remark 2.7 implies Theorem 1.1.

3. Bieberbach rigidity

We still consider the setting introduced in Section 2, namely we have a Riemannian product $\tilde{M} := \mathbb{R}^q \times (N, g_N)$ on which a group $\Gamma \leqslant \mathrm{Sim}(\mathbb{R}^q) \times \mathsf{Isom}(N, g_N)$ acts freely, properly discontinuously and cocompactly. We assume moreover that N is a symmetric space of non-compact type, i.e. that it has non-positive curvature and that its de Rham decomposition has no Euclidean factor.

In this context, a classical question is the following: is it possible to find a connected group $L \leqslant \mathsf{Sim}(\mathbb{R}^q) \times \mathsf{Isom}(N)$ acting properly on $\mathbb{R}^q \times N$, such that $\Gamma \leqslant L$ and Γ is a lattice in L? This property is called *Bieberbach rigidity*. This section is devoted to the proof of the Bieberbach rigidity in the case at hand. Yet, we will prove it up to a finite covering of $M := \tilde{\Gamma} \backslash M$ or equivalently up to taking a finite index subgroup of Γ, since this is false in the general case.

The Riemannian manifold (N, g_N) is symmetric, thus homogeneous, so its isometry group $\mathsf{Isom}(N)$ has finitely many connected components. Up to taking a finite index subgroup of Γ, we can assume that $P \leqslant \mathsf{Isom}(N)^0$, where P is the group introduced in Section 2, hence $\bar{P} \leqslant \mathsf{Isom}(N)^0$. We consider H, the normalizer of $\bar{P}^0$ in $\mathsf{Isom}(N)^0$, and in particular $\bar{P} \leqslant H$ since $\bar{P}^0$ is the identity connected component of $\bar{P}$.

The group $\mathsf{Isom}(N)^0$ is the identity connected component of the isometry group of a symmetric space of non-compact type, thus it has a trivial center and it is semi-simple. Consequently, $\mathsf{Isom}(N)^0$ is isomorphic to its image in $\mathsf{GL}(\mathfrak{g})$ by the Ad map. We denote by $\mathfrak{p}^0$ the Lie algebra of $\bar{P}^0$. One has

$$H = \{g \in \mathsf{Isom}(N)^0,\ \mathrm{Ad}_g(\mathfrak{p}^0) = \mathfrak{p}^0\}. \tag{4}$$

The image of $\mathsf{Isom}(N)^0$ by Ad is an algebraic subgroup of $\mathsf{GL}(\mathfrak{g})$, and (4) shows that Ad_H, the image of H by Ad, is an algebraic subgroup of $\mathsf{GL}(\mathfrak{g})$ because it is defined by polynomial equations. Hence, Ad_H is an algebraic variety over $\mathbb{R}$, thus it has finitely many connected components and up to taking a finite index subgroup of Γ, $\bar{P}$ is contained in H^0. In addition, we have the following property:

Lemma 3.1. *The group P is cocompact in H^0.*

Proof. The Riemannian manifold (N, g_N) is a homogeneous space, so it can be written as a a quotient G/K where K is the isotropy group of an arbitrary point of N. In particular, K is compact and we know that P acts cocompactly on N, which means that $P\backslash G/K$ is compact, so $P\backslash G$ is compact. Moreover, we have the inclusions $P \subset H^0 \subset G$, and we infer that $P\backslash H^0$ is compact. □

In order to prove that the group we construct will act transitively, we also need:

Lemma 3.2. *The group H^0 acts transitively on N.*

Proof. Since N is a symmetric space of non-compact type, $\mathsf{Isom}(N)$ is semi-simple and $N \simeq \mathsf{Isom}(N)/K$ where K is the maximal compact subgroup of $\mathsf{Isom}(N)$. The group $\mathsf{Isom}(N)^0$ acts transitively on the compact manifold $\mathsf{Isom}(N)^0/H$ and H has a finite number of connected components, so by [13, Corollary 2] there is a compact subgroup of $\mathsf{Isom}(N)^0$ acting transitively on $\mathsf{Isom}(N)^0/H$, thus K acts transitively on $\mathsf{Isom}(N)^0/H$. Consequently, H acts transitively on N and so does H^0. □

Up to taking a finite index subgroup of Γ, we can assume that the projection of Γ onto $\mathsf{Sim}(\mathbb{R}^q)$ preserves the orientation. By construction, H^0 normalizes the abelian group $\bar{P}^0$. In particular, H^0 acts on $\bar{P}^0$ by conjugation, and this action can be viewed as a matrix group action on the Lie algebra $\mathfrak{p}^0$ of $\bar{P}^0$. We fix a basis $\mathcal{B}$ of $\mathfrak{p}^0$ and we introduce the map φ which to any $h \in H^0$ associates the absolute value of the determinant of $\mathfrak{p}^0 \ni x \mapsto \mathrm{Ad}_h x$ in the basis $\mathcal{B}$. Let $\mathsf{Sim}^+(\mathbb{R}^q)$ be the orientation-preserving similarities of $\mathbb{R}^q$ and let $\rho : \mathsf{Sim}^+(\mathbb{R}^q) \to \mathbb{R}_+^*$ be the map giving the ratio of a similarity of $\mathbb{R}^q$. We consider the subgroup of $\mathsf{Sim}(\mathbb{R}^q) \times \mathsf{Isom}(N)$ defined as

$$L := \{(s, h) \in \mathsf{Sim}^+(\mathbb{R}^q) \times H^0,\ \rho(s) = \varphi(h)^{-1/q}\}. \tag{5}$$

The group L is obviously connected, since if we take an element (s, h) of L, there is a continuous path from (s, h) to $(\rho(s)\mathrm{Id}, h)$ by connectedness of the direct Euclidean group $\mathbb{R}^q \rtimes \mathsf{SO}(q)$, and it remains to remark that H^0 is connected and $\varphi(\cdot)^{-1/q}$ is continuous, so $H^0 \ni h \mapsto (\varphi(h)^{-1/q}, h)$ has a connected image. We claim that:

Lemma 3.3. *The group L contains Γ.*

Proof. By Remark 2.7 (or by [9, Lemma 4.18]), there is a normal subgroup Γ_0 of Γ which is a lattice in $\mathbb{R}^q \times \bar{P}^0$. The group Γ acts by conjugation on $\mathbb{R}^q \times \bar{P}^0$ and preserves

Γ_0, so this action can be viewed as a matrix group action on the Lie algebra $\mathbb{R}^q \times \mathfrak{p}^0$ of $\mathbb{R}^q \times \bar{P}^0$. The preimage γ_0 of Γ_0 is a lattice in $\mathbb{R}^q \times \mathfrak{p}^0$, so this matrix group is a subgroup of $\mathsf{GL}(\mathbb{Z}^{q+m})$ in a basis $\mathcal{B}_0$ of γ_0 (which is also a basis of $\mathbb{R}^q \times \mathfrak{p}^0$). Given $(s, p) \in \Gamma \subset \mathsf{Sim}^+(\mathbb{R}^q) \times G$, the matrix of $\mathrm{Ad}_{(s,m)}$ in $\mathcal{B}_0$ has determinant ± 1, and so does its matrix in $(\mathcal{B}_q, \mathcal{B})$ where $\mathcal{B}_q$ is the canonical basis of $\mathbb{R}^q$ and $\mathcal{B}$ is the basis of $\mathfrak{p}^0$ introduced above. We deduce that the ratio of s is $\varphi(p)^{-1/q}$, so (s, p) is in L. □

Finally, Γ and L have the desired properties for the Bieberbach rigidity:

Lemma 3.4. *The group Γ is a lattice in L and L acts properly on $\tilde{M}$.*

Proof. The group L normalizes $\mathbb{R}^q \times \bar{P}^0$, so $L/(\mathbb{R}^q \times \bar{P}^0)$ is a group isomorphic to $\mathsf{SO}(q) \times (H^0/\bar{P}^0)$. Since $\bar{P}$ is cocompact in H^0 by Lemma 3.1, $\bar{P}/\bar{P}^0$ acts cocompactly on $L/(\mathbb{R}^q \times \bar{P}^0)$. It remains to remark that Γ_0 acts cocompactly on $\mathbb{R}^q \times \bar{P}^0$ to prove that Γ acts cocompactly on L.

To prove that L acts properly on $\tilde{M}$, we observe that, since $\mathbb{R}^q \subset L$ acts properly on $\mathbb{R}^q$, is it sufficient to prove that $L/\mathbb{R}^q$ acts properly on N. But $L/\mathbb{R}^q$ is isomorphic to $\mathsf{SO}(q) \times H^0$, which acts properly on N because $\mathsf{SO}(q)$ is compact, and H^0 acts properly on N as a closed subgroup of the isometry group of the complete Riemannian manifold (N, g_N). □

Lemma 3.2, Lemma 3.3 and Lemma 3.4 together imply Theorem 1.2.

4. Classification results for N of negative curvature

In this section we keep the same setting as in Section 2, and we assume that N has negative curvature. We would like to know what are the restrictions on N. We first begin with a talkative example, namely the one of the real hyperbolic space $\mathbb{H}^n$. The study of this particular case will provide us with the intuition we need to analyze manifolds of negative curvature later on.

We recall that a Hadamard manifold M admits a border at infinity that we will denote by ∂M, defined as the set of geodesic rays quotiented by the relation $\alpha \sim \beta \Leftrightarrow [0, +\infty) \ni t \mapsto d(\alpha(t), \beta(t))$ is bounded. The isometries of M are classified into three groups: the elliptic isometries, which have a fixed point in M, the hyperbolic isometries, which fix a complete geodesic in M and acts by a non-trivial translation on it, and the parabolic isometries, which are all the remaining isometries. When the curvature of M is bounded from above by a negative constant, hyperbolic isometries are the ones with exactly two fixed points on ∂M and no fixed point in M, and parabolic isometries are the ones with exactly one fixed point in ∂M and no fixed point in M. For a more complete presentation, see [2] for example.

4.1. **The case of the real hyperbolic space.** As we already emphasized, the quotients we study in this paper can be viewed as a generalization of the Inoue surfaces. We recall briefly the construction of the Inoue surfaces of type S^0. Let A be a real 3×3 matrix in $\mathsf{GL}_3(\mathbb{Z})$ with two complex conjugate eigenvalues α and $\bar{\alpha}$ and one real eigenvalue $\lambda > 1$ (take for example the companion matrix of the polynomial $X^3 - X^2 + 3X - 1$). We consider the manifold $\tilde{M} := \mathbb{R}^3 \times \mathbb{R}_+^*$ on which acts the group

$$\Gamma := \mathbb{Z}^3 \rtimes \langle \mathbb{R}^3 \times \mathbb{R}_+^* \ni (X, x) \mapsto (AX, \lambda x) \rangle \quad (6)$$

where $\mathbb{Z}^3$ is the canonical lattice of $\mathbb{R}^3$ and we used the notation $\langle F \rangle$ to denote the group generated by the family F. The manifold $M := \Gamma \backslash \tilde{M}$ is an Inoue surface of type S^0. The vector space $\mathbb{R}^3$ decomposes as the direct sum $\ker(A - \alpha)(A - \bar{\alpha}) \oplus \ker(A - \lambda) \simeq$

$\mathbb{R}^2 \oplus \mathbb{R}$ and there exists a scalar product b on $\mathbb{R}^2$ such that the restriction of A to $\mathbb{R}^2$ is a b-similarity of ratio $|\alpha| = \lambda^{-1/2}$. We denote by (x, y) the coordinates of $\ker(A-\lambda)\times\mathbb{R}_+^* \simeq \mathbb{R}\times\mathbb{R}_+^*$ and we endow $\tilde{M}$ with the metric $h = b + y^{-2}(dx^2+dy^2)$. One has $(\tilde{M}, h) \simeq (\mathbb{R}^2, b) \times (\mathbb{R}\times\mathbb{R}_+^*, y^{-2}(dx^2+dy^2)) \simeq \mathbb{C}\times\mathbb{H}$, and Γ is a subgroup of $\mathsf{Sim}(\mathbb{C}) \times \mathsf{Isom}(\mathbb{H})$.

Using the same idea, we can construct examples where the universal cover is $\mathbb{R}^q \times \mathbb{H}^n$, $\mathbb{H}^n$ denoting the hyperbolic space of dimension n. We give here the construction:

Example 4.1. *We consider $\tilde{M} := \mathbb{R}^{q+n-1}\times\mathbb{R}_+^*$ and we pick a matrix $A \in \mathsf{GL}_{q+n-1}(\mathbb{Z})$ such that there is a decomposition $\mathbb{R}^{n+1} =: E \oplus F$ with $\dim(E) = q$, two scalar products b_E and b_F on E and F respectively and a real number $\lambda > 0$ such that $A|_E \in \lambda O(b_E)$ and $A|_F \in \lambda^{-q/\dim(F)}O(b_F)$. We let the group*

$$\Gamma := \mathbb{Z}^{q+n-1} \rtimes \langle \mathbb{R}^{q+n-1}\times\mathbb{R}_+^* \ni (x,t) \mapsto (Ax, \lambda^{-1}t)\rangle \tag{7}$$

acting on $\tilde{M}$. We endow $\tilde{M}$ with the metric

$$h := b_E + \frac{1}{t^2}(b_F + dt^2). \tag{8}$$

One has $(\tilde{M}, h) \simeq \mathbb{R}^q \times \mathbb{H}^n$. The group Γ is a subgroup of $\mathsf{Sim}(\mathbb{R}^q)\times\mathsf{Isom}(\mathbb{H}^n)$ and it acts properly, freely and cocompactly on $\tilde{M}$.

In the case $q = 2$ and $n = 3$, we can take the matrix

$$A = \begin{pmatrix} 2 & 1 & & \\ 1 & 1 & & \\ & & 2 & 1 \\ & & 1 & 1 \end{pmatrix}, \tag{9}$$

which is diagonalizable with two eigenvalues λ and λ^{-1} of multiplicity 2.

Remark 4.2. *From Example 4.1 we can formulate an interesting question on matrices with integer coefficients: what are the matrices of $\mathsf{GL}_p(\mathrm{Z})$ (for p an integer) such that there exists a decomposition $\mathbb{R}^p =: E \oplus F$ satisfying the conditions of the example? A first result in this direction was given in* [10, Proposition 3]*, showing that when $\dim(E) = 1$ one has $p \in \{2, 3\}$. We remark that these matrices induce an Anosov diffeomorphism on the torus with a single Lyapunov exponent.*

Notice also that we can always construct suitable matrices in the cases $q = n-1$ and $q = 2(n-1)$ taking block matrices as in (9).

The key in Example 4.1 is the existence of the matrix A. Actually, we can prove:

Claim **4.3.** *The existence of such a matrix is a necessary and sufficient condition for the existence of a group $\Gamma \subset \mathsf{Sim}(\mathbb{R}^q)\times\mathsf{Isom}(\mathbb{H}^n)$ acting freely, properly discontinuously and cocompactly on $\mathbb{R}^q \times \mathbb{H}^n$.*

The sufficient part comes from the construction in Example 4.1, so it remains to prove that it is necessary. We recall that in this setting, the group $\bar{P}$ has been defined in Section 2 as the closure of the projection of Γ onto $\mathsf{Isom}(\mathbb{H}^n)$ and $\bar{P}^0$ is its identity connected component.

We first prove a lemma which holds in full generality:

Lemma 4.4. *The group $\bar{P}^0$ is not compact.*

Proof. By contradiction, we assume that $\bar{P}^0$ is compact. We know that the group Γ_0 introduced in Remark 2.7 is a lattice in $\mathbb{R}^q \times \bar{P}^0$. By compactness of $\bar{P}^0$, the projection Γ_0' of Γ_0 onto $\mathbb{R}^q$ is discrete because Γ_0 is discrete. Consequently, Γ_0' is a lattice of $\mathbb{R}^q$ which must be preserved by the action of Γ by conjugation on $\mathbb{R}^q$. But this is possible only if all the similarities in $\Gamma|_{\mathbb{R}^q}$ are isometries, because otherwise we could find non-zero vectors of arbitrarily small size inside Γ_0'. This is a contradiction because we assumed that $\Gamma|_{\mathbb{R}^q}$ contains at least one similarity of ratio different from 1. □

Lemma 4.5. *All the elements of $\bar{P}$ fix a common point of $\partial\mathbb{H}^n$ and $\bar{P}^0$ does not contain hyperbolic isometries.*

Proof. We study the fixed points of the elements of $\bar{P}^0$ and we use the classification of isometries in hyperbolic spaces.

First, assume that there is a parabolic element p_0 in $\bar{P}^0$, i.e. p_0 fixes exactly one point $x \in \partial\mathbb{H}^n$ on the boundary. Since $\bar{P}^0$ is abelian, any element of $\bar{P}^0$ must fix x. Now, pick any element $p \in \bar{P}$. We know that $\bar{P}^0$ is a normal subgroup of $\bar{P}$, so $p^{-1}p_0p$ is in $\bar{P}^0$ and $p^{-1}p_0px = x$, thus $p_0px = px$ which implies $px = x$.

We now assume that there is a hyperbolic element p_0 in $\bar{P}^0$, i.e. P_0 fixes exactly two points x_1 and x_2, lying on the boundary. Again, since $\bar{P}^0$ is abelian and connected, any element of $\bar{P}^0$ fixes these two points. Let $p \in \bar{P}$, so we have as before $p_0px_i = px_i$ for $i = 1, 2$. Up to taking a subgroup of Γ of index 2 in the construction, we can assume that $px_i = x_i$. Up to conjugation, the group of isometries of $\mathbb{H}^n$ fixing two points of the boundary is $\mathsf{O}(n-1) \rtimes \mathbb{R}_+^*$ when using the model $\mathbb{H}^n \simeq (\mathbb{R}^{n-1} \times \mathbb{R}_+^*, \frac{1}{x_n^2}(dx_1^2 + \ldots + dx_n^2))$. But this group does not act cocompactly on $\mathbb{H}^n$, so this case is impossible.

The last case remaining is the one where $\bar{P}^0$ contains only elliptic elements, i.e. all elements have a fixed point in $\mathbb{H}^n$. This means that any element of $\bar{P}^0$ is contained in the isotropy group of a point, which is compact. Since $\bar{P}^0$ is abelian and connected, and therefore the product of a torus with $\mathbb{R}^m$ for some m, this implies that $m = 0$ and $\bar{P}^0$ is compact, because otherwise some elements would not be contained in a compact subgroup. This case is impossible because of Lemma 4.4. □

Thanks to Lemma 4.5, we can assume that $\bar{P}$ fixes the point at infinity in $\overline{\mathbb{H}^n}$ when we use the model $\mathbb{H}^n \simeq (\mathbb{R}^{n-1} \times \mathbb{R}_+^*, \frac{1}{x_n^2}(dx_1^2 + \ldots + dx_n^2))$. The isometries in $\mathsf{Isom}(\mathbb{H}^n)$ preserving the point at infinity actually lie in the group $\mathsf{Isom}(\mathbb{R}^{n-1}) \rtimes \mathbb{R}_+^*$ where $\mathsf{Isom}(\mathbb{R}^{n-1})$ acts on the first factor and $\mathbb{R}_+^*$ is the group of positive scalar $n \times n$ matrices.

The group $\bar{P}$ acts cocompactly on $\mathbb{H}^n$ and writing $\mathcal{H} := \bar{P} \cap \mathsf{Isom}(\mathbb{R}^{n-1})$ we have:

$$\mathbb{H}^n/\bar{P} \simeq (\mathbb{H}^n/\mathcal{H})/(\bar{P}/\mathcal{H}).$$

Moreover, $(\mathbb{H}^n/\mathcal{H}) \simeq (\mathbb{R}^{n-1}/\mathcal{H}) \times \mathbb{R}_+^*$ and $\bar{P}/\mathcal{H}$ acts freely on the factor $\mathbb{R}_+^*$ because if an element $\bar{\gamma} \in \bar{P}/\mathcal{H} \subset \mathbb{R}_+^*$ has a fixed point in $\mathbb{R}_+^*$, then it is the identity. Consequently, $\mathbb{H}^n/\bar{P}$ is compact only if $\mathbb{R}^{n-1}/\mathcal{H}$ is compact. In addition, $\bar{P}$ acts cocompactly on $\mathbb{H}^n$, so it must contains a non-trivial similarity p of $\mathbb{R}^n$ of ratio $0 < \lambda < 1$. The map p restricts to a similarity $\bar{p}$ of $\mathbb{R}^{n-1}$ and this restriction acts by conjugation on $\mathcal{H}$ because it is a normal subgroup of $\bar{P}$. Thus, if K is a compact subset of $\mathbb{R}^{n-1}$ such that $\mathcal{H} \cdot K = \mathbb{R}^{n-1}$, one has

$$\mathcal{H} \cdot \bar{p}(K) = \bar{p}\mathcal{H}\bar{p}^{-1} \cdot \bar{p}(K) = \bar{p}\mathcal{H} \cdot K = \bar{p}(\mathbb{R}^{n-1}) = \mathbb{R}^{n-1}.$$

The diameter of $\bar{p}(K)$ is λ times the diameter of K (for any fixed Euclidean norm on $\mathbb{R}^{n-1}$). Iterating this process, we can take the compact K arbitrarily small, so, for any

$y \in \mathbb{R}^{n-1}$, $\mathcal{H} \cdot y$ is dense in $\mathbb{R}^{n-1}$. Since the action of $\mathcal{H}$ is proper, this implies that $\mathcal{H} \cdot y = \mathbb{R}^{n-1}$.

We infer that $\mathcal{H}$ acts transitively on $\mathbb{R}^{n-1}$. It follows that the connected component of the identity in $\mathcal{H}$ already acts transitively on $\mathbb{R}^{n-1}$. The group $\bar{P}^0$ is contained in $\mathsf{Isom}(\mathbb{R}^{n-1})$ because $\bar{P}^0$ contains only parabolic elements, hence $\bar{P}^0$ is the identity connected component of $\mathcal{H} = \bar{P} \cap \mathsf{Isom}(\mathbb{R}^{n-1})$, and it acts transitively on $\mathbb{R}^{n-1}$. In order to conclude that $\bar{P}^0 = \mathbb{R}^{n-1}$, we need to prove:

Lemma 4.6. *Let $m > 0$ be an integer. The only connected abelian subgroup of* $\mathsf{Isom}(\mathbb{R}^m)$ *acting transitively on $\mathbb{R}^m$ is the group of translations $\mathbb{R}^m$.*

Proof. Let G be such a subgroup of $\mathsf{Isom}(\mathbb{R}^m)$. Any element of G is of the form $x \mapsto Rx + t$ where $(R, t) \in \mathsf{SO}(n) \times \mathbb{R}^m$. Since G is abelian, the linear parts of the elements of G commute, so they are all contained in a maximal torus of $\mathsf{SO}(n)$. This implies that, up to conjugation, we can assume that the linear part of the elements of G are contained in the product group $\prod_{i=1}^{m/2} \mathsf{O}(2)$ if m is even or $\prod_{i=1}^{m/2} \mathsf{O}(2) \times \{1\}$ if m is odd. Consequently, G is a subgroup of $\prod_{i=1}^{m/2} E^+(2)$ if m is even or $\prod_{i=1}^{m/2} E^+(2) \times \mathbb{R}$ if m is odd, where $E^+(2)$ is the group of direct isometries of $\mathbb{R}^2$ and $\mathbb{R}$ acts by translations on $\mathbb{R}$. The projection of G onto each one of the $E^+(2)$ factors is abelian and acts transitively on $\mathbb{R}^2$ because G acts transitively on $\mathbb{R}^m$. It is then sufficient to prove the lemma for the case $m = 2$.

We now assume that $m = 2$. Let $I : x \mapsto Rx + t$ be an element of G. If R is the identity, the isometry is a translation. If R is a non-trivial rotation, then up to applying a translation we can assume that $0_{\mathbb{R}^2}$ is a fixed point of I, so $t = 0$. Now, since G acts transitively on $\mathbb{R}^2$, for any $t' \in \mathbb{R}^2$ there exists $R' \in \mathsf{SO}(2)$ such that $x \mapsto R'x + t'$ is in G. But the isometry $x \mapsto Rx$ should commute with this map, giving $Rt' = t'$, and this is true for any $t' \in \mathbb{R}^2$, so R is the identity.

It follows that G contains only translations, and it contains all of them because it acts transitively on $\mathbb{R}^m$. ☐

We proved the following:

Proposition 4.7. *If $(N, g_N) = \mathbb{H}^n \simeq (\mathbb{R}^{n-1} \times \mathbb{R}^*_+, \frac{1}{x_n^2}(dx_1^2 + \ldots + dx_n^2))$, then up to conjugation the group $\bar{P}$ preserves the point at infinity and $\bar{P}^0 = \mathbb{R}^{n-1}$.*

Now, assume we are in the case $\mathbb{R}^q \times N = \mathbb{R}^q \times \mathbb{H}^n$. We know that the subgroup Γ_0 of Γ defined in Theorem 1.1 is a lattice of $\mathbb{R}^q \times \bar{P}^0 \simeq \mathbb{R}^q \times \mathbb{R}^{n-1}$. Let $\gamma \in \Gamma$ which projects to a non-isometric similarity on $\mathsf{Sim}(\mathbb{R}^q)$. The matrix of the action of γ by conjugation of $\mathbb{R}^q \times \mathbb{R}^{n-1}$ is an element A of $\mathsf{GL}_{q+n-1}(\mathbb{Z})$ when seen in a basis of the lattice Γ_0. Moreover, A restricts to $E := \mathbb{R}^q$ and to $F := \mathbb{R}^{n-1}$ and is a similarity of ratio λ on E and a similarity of ratio $\lambda^{-q/(n-1)}$ on F. This proves Claim 4.3.

4.2. **The general case of negative curvature.** In this section we examine a more general framework, using an analysis similar to the one we followed in the previous section. We assume that the manifold N^n has negative sectional curvature, and since $\bar{P}$ acts cocompactly on N by isometries, this implies that the curvature is bounded from above by a strictly negative constant. The isometries of N fall in exactly one of the following classes: elliptic, hyperbolic and parabolic. The sectional curvature being bounded from above by a negative constant, an elliptic isometry is one which admits a fixed point, a hyperbolic isometry is one with no fixed point in N and exactly two fixed points on the boundary, and a parabolic isometry is an isometry with no fixed point in N and exactly one fixed point on the boundary of N.

We can prove the same result as in Lemma 4.5 for the hyperbolic space:

Lemma 4.8. *All the elements of $\bar{P}$ fix a common point of ∂N and $\bar{P}^0$ does not contain hyperbolic isometries.*

Proof. The proof is the same as for Lemma 4.5, except to prove that $\bar{P}^0$ does not contain hyperbolic isometries. Assume that there exists a hyperbolic element $p_0 \in \bar{P}^0$ fixing x_1 and x_2 on the boundary of N. We prove in the same way as in the proof of Lemma 4.5 that, up to taking a group of finite index, all the elements in $\bar{P}$ fix x_1 and x_2, so they preserve the geodesic between these two points. But the group of isometries preserving a geodesic in N does not act cocompactly on N, because the image of a compact by this group stays at bounded distance from the geodesic. □

By Lemma 4.8, the group $\bar{P}$ fixes a point $x \in \partial N$. Let H be a horosphere centered at x and let f be a Busemann function at x. For any $y \in N$, there is a unique geodesic γ_y joining y and x, and $H \cap \gamma_y$ is reduced to a point denoted by $\eta(y)$. The map $\eta \times f : N \to \mathbb{R} \times H$, $y \mapsto (\eta(y), f(y))$ is a homeomorphism [15, Proposition 11]. Let $\mathcal{H}$ be the stabilizer of H in $\bar{P}$, i.e. the subset of non-hyperbolic isometries. This is a normal subgroup of $\bar{P}$, so we have

$$N/\bar{P} \simeq (N/\mathcal{H})/(\bar{P}/\mathcal{H}).$$

In addition, using the homeomorphism $N \simeq \mathbb{R} \times H$ one has $N/\mathcal{H} \simeq \mathbb{R} \times H/\mathcal{H}$. The group $\bar{P}/\mathcal{H}$ acts freely on the factor $\mathbb{R}$ because if $p \in \bar{P}$ stabilizes a horosphere centered at x, then it stabilizes all the horospheres centered at x, so p is in $\mathcal{H}$. Since $N/\bar{P}$ is compact, we deduce that $H/\mathcal{H}$ is compact.

Following the analysis we have done for the hyperbolic space, we would like to prove that $H/\mathcal{H}$ is a point. Since $\bar{P}$ acts cocompactly on N, the group $\bar{P}/G$ is non-trivial, and there exists a hyperbolic isometry p in $\bar{P}$. This isometry preserves the horospheres centered at x, and using again the homeomorphism $N \simeq \mathbb{R} \times H$ it can be written as $p =: (p_1, p_2)$ where $p_1 : \mathbb{R} \to \mathbb{R}$ and $p_2 : H \to H$. We know that H is a C^2-hypersurface in N and $\mathcal{H}$ is a subgroup of the isometries of H. In order to prove that $H/\mathcal{H}$ is a point, it is sufficient to prove that p_2 contracts the distances and to proceed as in the case of the hyperbolic space. Up to taking p^{-1} instead of p, we can assume that x is the repulsive point of the hyperbolic isometry p.

Lemma 4.9. *For any $a \in H$ and $X \in T_a H$, one has $\|d_a p_2(X)\| < \|X\|$.*

Proof. Let $a \in H$ and let $X \in T_a H \setminus \{0\}$. We observe that p_2 can be viewed as the composition of p together with the sliding along geodesics joining points of $p(H)$ to the fixed point $x \in \partial N$ until we reach H. Consequently, if we take an injective curve $c : (-\epsilon, \epsilon) \to H$ such that $c(0) = a$ and $c'(0) = X$, then $p \circ c$ is a curve on $p(H)$ and sliding along the geodesics induces a map

$$r : (-\epsilon, \epsilon) \times [0, +\infty) \to N,\ (t, u) \mapsto \gamma_{c(t)}(u) \tag{10}$$

where for any $y \in N$, γ_y is the unit speed geodesic joining y and x. If α is the distance between $p(H)$ and H, then $r(\cdot, \alpha)$ is a curve on H.

One has $d_a p_2(X) = d_{(0,\alpha)} r(\partial/\partial t)$ and we remark that $J(u) := d_{(0,u)} r(\partial/\partial t)$ is a Jacobi field along the geodesic $\gamma := r(0, \cdot)$. Denoting by R the Riemann curvature tensor of N and since there is a constant $k > 0$ such that the sectional curvature of

N is bounded from above by $-k$, one has, following the same computations as in [1, Lemma IV.2.2]:

$$\begin{aligned}\|J\|'' &= \frac{1}{\|J\|^3}(-\langle R(J,\dot\gamma),\dot\gamma,J\rangle\|J\|^2 + \|J'\|^2\|J\|^2 - \langle J',J\rangle^2)\\ &\geq -\frac{1}{\|J\|}\langle R(J,\dot\gamma),\dot\gamma,J\rangle \geq \frac{k}{\|J\|}(\|J\|^2 - \langle J,\dot\gamma\rangle^2).\end{aligned}$$

We know that $J(0) = d_a p(X) \neq 0$ and J never vanishes because the family of geodesics $\gamma_{c(\cdot)}$ meet at the point $x \in \partial N$. In addition, J is never tangent to γ because for any u, $r(\cdot,u)$ is an injective curve on a horosphere transverse to γ. We conclude that for any $u \in [0,\alpha]$ one has $\|J\|(u) > 0$ and $\|J\|$ is a strictly convex function.

The geodesics $\gamma_{c(\cdot)}$ all meet at the repulsive point of p, namely $x \in \partial N$, so $\lim_{u\to+\infty} J(u) < +\infty$, thus $\|J\|$ is a strictly decreasing function and we have

$$\|X\| = \|d_a p(X)\| = \|J(0)\| > \|J(\alpha)\| = \|d_a p_2(X)\|$$

which concludes the proof. □

The quotient $H/\mathcal{H}$ is compact and by Lemma 4.9, H admits a contraction mapping p_2 which acts on $\mathcal{H}$ by conjugation. We proceed as in the case of the hyperbolic space and we pick a compact K of H such that $\mathcal{H}\cdot K = H$. Then one has:

$$\mathcal{H}\cdot p_2(K) = H, \tag{11}$$

so the compact K can be taken to have arbitrary small diameter. This implies that, since $\mathcal{H}$ acts properly, $\mathcal{H}$ acts transitively on H and the connected component of the identity of $\mathcal{H}$, namely $\bar P^0 \cap \mathcal{H} = \bar P^0$, acts transitively on H. We know that $\bar P^0$ is an abelian group of isometries and H is simply connected, so H is a flat manifold isometric to $\mathbb{R}^p$ for some $p > 0$ and it is a smooth submanifold of N. The identification $N \simeq \mathbb{R}\times H$ is therefore a diffeomorphism. We can actually tell more: if γ is the axis of the hyperbolic isometry p, then $\bar P^0\gamma = N$ and the metric along γ determines completely the metric on N. The geodesic γ is orthogonal to H, so one has that $N \to N/\bar P^0$ is a Riemannian submersion. Moreover, the same proof as in Section 4.1 shows that $\bar P^0$ is exactly the group of translations of the factor $\mathbb{R}^{n-1}$. Thus, the Riemannian manifold N is isometric to $(\mathbb{R}\times\mathbb{R}^{n-1}, dt^2 + \langle S_t\cdot, S_t\cdot\rangle)$ where $S_t \in \mathsf{GL}(\mathbb{R}^{n-1})$ and $\langle\cdot,\cdot\rangle$ is the standard scalar product on $\mathbb{R}^{n-1}$. Up to a linear transformation of the factor $\mathbb{R}^{n-1}$, we can assume that $S_0 = \mathrm{Id}$.

We now assume that N is a homogeneous manifold. Using a result of Chen [4, Theorem 4.1], either N is a symmetric space of rank 1 or Isom(N) fixes a point of ∂N. Assume first that N is not a symmetric space, so the whole isometry group fixes $x \in \partial N$ because it is the only fixed point of the elements of $\bar P^0$. The subgroup of Isom(N) preserving the axis γ is closed because γ is closed in N. By transitivity of the action of Isom(N), it contains uncountably many elements acting freely on γ, thus it is non-discrete and its identity component contains a one-parameter subgroup φ_s acting by translations along γ. Using the diffeomorphism $N \simeq \mathbb{R}\times\mathbb{R}^{n-1}$, where γ is identified to the set $\mathbb{R}\times\{0\}$, this one-parameter subgroup is of the form $\varphi_s : (t,a) \mapsto (t+s, A_s a)$ where A_s is a linear endomorphism of $\mathbb{R}^{n-1}$, because φ_s preserves the horospheres centered at x and permutes the geodesic rays going to x. The map $s \mapsto A_s$ is a one-parameter group of invertible matrices, so $A_s = e^{-sA}$ for some matrix A. Taking $X, Y \in T\mathbb{R}^{n-1}$ and since φ_s is a one-parameter group of isometries one has for any

$t \in \mathbb{R}$:

$$\langle S_t X, S_t Y\rangle = \langle S_0 A_{-t} X, S_0 A_{-t} Y\rangle = \langle e^{tA} X, e^{tA} Y\rangle. \tag{12}$$

It remains to consider the case where N is a symmetric space of rank 1. Since N is simply connected, it is either the real hyperbolic space $\mathbb{H}^n$, the complex hyperbolic space $\mathbb{CH}^{n/2}$, the quaternionic hyperbolic space $\mathbb{HH}^{n/4}$ or the octonionic hyperbolic plane $\mathbb{C}a\mathbb{H}^2$ up to rescaling. But we know that the horospheres centered at $x \in \partial N$ are isometric to flat manifolds. The horospheres of $\mathbb{CH}^{n/2}$ and $\mathbb{HH}^{n/4}$ are isometric to the Heisenberg group of dimension $n-1$ with a left-invariant metric and the horospheres of $\mathbb{C}a\mathbb{H}^2$ are isometric to a 2-steps nilpotent Lie group with a left-invariant metric. The only possibility is then $N = \mathbb{H}^n$.

Summarizing, we proved Theorem 1.3.

Remark 4.10. *In the non-homogeneous case, since there is a hyberbolic isometry p in $\bar{P}$, the matrices S_t have a periodicity. Indeed, $p =: (p_1, p_2)$ (using the notations introduced above) acts as a linear map A on the factor $\mathbb{R}^{n-1}$ and it is an isometry, so $\langle S_{t+c}A\cdot, S_{t+c}A\cdot\rangle = \langle S_t\cdot, S_t\cdot\rangle$ where c is the translation part of p along its axis.*

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