Examples for the Package GeneralizedFourierSeries

April 2026

```
> restart;
```

Load the package:

```
> read "GeneralizedFourierSeries.mpl";
> with(GeneralizedFourierSeries);
```

[*DiffeqtoRec*, *DiffoptoPair*, *DiffoptoRec*, *FunctiontoRec*]

Version of the package:

```
> GeneralizedFourierSeries:-version;
```

1.0

Display timings (increase to also display order):

```
> infolevel['FractionsOfRecurrenceOperators']:=1:
```

The package computes recurrences for the coefficients of expansions on bases of classical orthogonal polynomials. In some cases, it is also possible to solve these recurrences almost directly. We first list a few values used when computing the initial conditions in these recurrences.

Hermite polynomials

```
> hnH:=Pi^(1/2)*2^n*n!:lcoeffHn:=2^n:
```

Laguerre polynomials

```
> hnL:=GAMMA(n+alpha+1)/n!:lcoeffLn:=(-1)^n/n!:
```

Jacobi polynomials

```
> hnJ:=2^(alpha+beta+1)*GAMMA(n+alpha+1)*GAMMA(n+beta+1)/(2*n+alpha+
  beta+1)/GAMMA(n+alpha+beta+1)/n!:
```

Bessel polynomials

```
> lcoeffyn:=pochhammer(a+n-1,n)/2^n:
```

Chebyshev

```
> hnT:=proc(n) if n=0 then Pi else Pi/2 fi end:
```

Start clock:

```
> T0:=time():
```

Simple expansions

exp(x) (Examples 6.1 & 6.4 in the article)

```
> for Basis in [ChebyshevT(n,x),HermiteH(n,x),GegenbauerC(n,lambda,x)]
  do
```

```
     FunctiontoRec(exp(x),Basis,u,algo="mixed") od;
FunctiontoRec: time: 0.012 sec.
```
$$u(n) + (-2n-2)\,u(n+1) - u(n+2)$$
```
FunctiontoRec: time: 0.011 sec.
```
$$u(n) + (-2n-2)\,u(n+1)$$
```
FunctiontoRec: time: 0.06 sec.
```
$$(-\lambda - n - 2)\,u(n) + 2\,(\lambda+n)\,(\lambda+n+2)\,u(n+1) + (\lambda+n)\,u(n+2)$$

(1-x^2)^(-1/4) (Example 7.6)

```
> FunctiontoRec((1-x^2)^(-1/4),ChebyshevT(n,x),c);
FunctiontoRec: time: 0.017 sec.
```
$$(1+2n)\,c(n) + (-3-2n)\,c(n+2)$$

arccos(x) (Exs 7.1 & 7.7)

```
> DiffoptoPair((1-x^2)*Dx^2-x*Dx,Dx,x,n,ChebyshevT(n,x));
```
$$[OrePoly(-2n^2), OrePoly(2)]$$
```
> DiffeqtoRec((1-x^2)*((1-x^2)*diff(y(x),x,x)-x*diff(y(x),x)),y(x),c
  (n),ChebyshevT(n,x));
```
$$-n^2\,c(n) + (2n^2+8n+8)\,c(n+2) - (n+4)^2\,c(n+4)$$

Relation between polynomials (Sec. 9.1)

Connection coefficients (Sec. 9.1.1)

Trivial connections (incl. Ex. 6.2)

```
> L:=[LegendreP(n,x),ChebyshevT(n,x),HermiteH(n,x),LaguerreL(n,alpha,
  x),GegenbauerC(n,lambda,x),JacobiP(n,alpha,beta,x)];
```
$$L := [\mathrm{LegendreP}(n,x), \mathrm{ChebyshevT}(n,x), \mathrm{HermiteH}(n,x), \mathrm{LaguerreL}(n,\alpha,x), \mathrm{GegenbauerC}(n,\lambda,x), \mathrm{JacobiP}(n,\alpha,\beta,x)]$$
```
> for pol in L do FunctiontoRec(subs(n=k,pol),pol,u) assuming
  k::nonnegint,alpha>-1,beta>-1,lambda>-1/2,lambda<>0 od;
FunctiontoRec: time: 0.049 sec.
```
$$(k-n)\,u(n)$$
```
FunctiontoRec: time: 0.037 sec.
```
$$(k-n)\,(k+n)\,u(n)$$
```
FunctiontoRec: time: 0.021 sec.
```
$$(k-n)\,u(n)$$
```
FunctiontoRec: time: 0.034 sec.
```
$$(k-n)\,u(n)$$

```
FunctiontoRec: time: 0.147 sec.
```

$$(k-n)\,u(n)$$

```
FunctiontoRec: time: 0.76 sec.
```

$$(k-n)\,(k+1+\alpha+\beta+n)\,u(n)$$

Laguerre Eq. (12) in Howell (1937)

```
> FunctiontoRec(LaguerreL(m,alpha,x),LaguerreL(nu,beta,x),u);
FunctiontoRec: time: 0.028 sec.
```

$$(m-\nu)\,u(\nu)+(-m+\beta+1+\nu-\alpha)\,u(1+\nu)$$

```
> rsolve(%,u(nu)) assuming nu>=0,nu<=m,m::nonnegint;
```

$$\frac{\Gamma(1+m)\,\Gamma(-m+\beta+1-\alpha)\,(-1)^{\nu}\,u(0)}{\Gamma(m-\nu+1)\,\Gamma(-m+\beta+1+\nu-\alpha)}$$

```
> subs(u(0)=solve(eval(%,nu=m)=1,u(0)),%);
```

$$\frac{(-1)^{\nu}\,\Gamma(\beta+1-\alpha)}{\Gamma(m-\nu+1)\,\Gamma(-m+\beta+1+\nu-\alpha)\,(-1)^{m}}$$

It may be worth observing that Godoy et al. 1997 obtain a recurrence of order 2 instead of 1, using an expansion on the basis of the derivatives of the polynomials.

Jacobi to Jacobi (Godoy, Ronveaux, Zarzo, Area 1997, 3.3.2; see also Ismail 2005 Thm. 9.1.1)

```
> rec:=FunctiontoRec(JacobiP(n,a,b,x),JacobiP(m,c,d,x),u) assuming
  a>-1,b>-1,c>-1,d>-1,n::nonnegint,m::nonnegint;
FunctiontoRec: time: 0.657 sec.
```

$$\begin{aligned} rec :=\ & -(c+d+2m+5)\,(m+2+c+d)\,(c+d+2m+4)\,(m+c+d+1)\,(m-n)\,(m+n \\ & +a+b+1)\,u(m)-(c+d+2m+5)\,(m+2+c+d)\,(acd+2acm-acn+ad^2 \\ & +2adm+adn+2am^2-bc^2-bcd-2bcm-bcn-2bdm+bdn-2bm^2-c^2m \\ & -cm^2-cn^2+d^2m+dm^2+dn^2+2ac+4ad+6am-4bc-2bd-6bm-c^2 \\ & -3cm-cn+d^2+3dm+dn+4a-4b-2c+2d)\,(c+d+2m+1)\,u(1+m)-(c \\ & +d+2m+1)\,(d+m+2)\,(c+m+2)\,(2m+2+c+d)\,(c+d+m+n+3)\,(-c+a \\ & +b-d-m-2+n)\,u(m+2) \end{aligned}$$

It does not match the recurrence in their article (also of order 2), again because their Jacobi polynomials are monic. Here is a check of our recurrence:

```
> nn:=6:
> P:=expand(JacobiP(nn,a,b,x)):
> for i from nn to 0 by -1 do pol:=expand(JacobiP(i,c,d,x)); c[i]:=
  normal(coeff(P,x,i)/coeff(pol,x,i)); P:=expand(P-c[i]*pol) od:
> for j from 0 to nn do
     j,normal(eval(eval(rec,[n=nn,m=0]),[seq(u(i)=c[i],i=0..nn),seq(u
  (i)=0,i=nn+1..nn+4)]))
  od;
```

$$0,\ 0$$

$$1, 0$$
$$2, 0$$
$$3, 0$$
$$4, 0$$
$$5, 0$$
$$6, 0$$

A variant is in (Ismail-Simeonov 2012 eq. (6.4)).

```
> i:='i':j:='j':c:='c':
```

Jacobi to Jacobi special case (Cor. 9.1.2 Ismail 2005)

The order of the recurrence becomes 1 in that case

```
> rec:=FunctiontoRec(JacobiP(n,a,b,x),JacobiP(k,a,c,x),u) assuming
  a>-1,b>-1,c>-1,n::nonnegint;
```

```
FunctiontoRec: time: 0.486 sec.
```

$$rec := (a+c+2k+3)(k+a+c+1)(k-n)(k+n+a+b+1)u(k) - (a+k+1)(a+c+2k+1)(b-c-k+n-1)(a+c+k+n+2)u(k+1)$$

```
> sol:=rsolve(rec,u(k)) assuming n::posint,k::nonnegint,k<=n;
```

$$sol := \left((a+c+2k+1)u(0)\Gamma(k+n+a+b+1)\Gamma(k+a+c+1)\Gamma(-b+c-n+1)\Gamma(a+c+n+2)\Gamma(a+1)\Gamma(n+1)\right) / \left(\Gamma(2+a+c)\Gamma(k-b+c-n+1)\Gamma(a+c+k+n+2)\Gamma(n+a+b+1)\Gamma(a+k+1)\Gamma(1-k+n)\right)$$

```
> eval(%,k=n);
```

$$\left((a+c+2n+1)u(0)\Gamma(2n+a+b+1)\Gamma(n+a+c+1)\Gamma(-b+c-n+1)\Gamma(a+c+n+2)\Gamma(a+1)\Gamma(n+1)\right) / \left(\Gamma(2+a+c)\Gamma(c+1-b)\Gamma(a+c+2n+2)\Gamma(n+a+b+1)\Gamma(a+n+1)\right)$$

```
> simplify(%) assuming n::posint;
```

$$\left(u(0)(n+a+c+1)\Gamma(2n+a+b+1)\Gamma(n+a+c+1)^2\Gamma(-b+c-n+1)\Gamma(a+1)\Gamma(n+1)\right) / \left(\Gamma(a+c+2n+1)\Gamma(2+a+c)\Gamma(c+1-b)\Gamma(n+a+b+1)\Gamma(a+n+1)\right)$$

```
> solve(%=1,u(0));
```

$$\frac{\Gamma(a+c+2n+1)\Gamma(2+a+c)\Gamma(c+1-b)\Gamma(n+a+b+1)\Gamma(a+n+1)}{\Gamma(2n+a+b+1)\Gamma(n+a+c+1)^2\Gamma(-b+c-n+1)\Gamma(a+1)\Gamma(n+1)(n+a+c+1)}$$

```
> subs(u(0)=%,sol);
```

$$\left((a+c+2k+1)\Gamma(a+c+2n+1)\Gamma(c+1-b)\Gamma(a+n+1)\Gamma(k+n+a+b+1)\Gamma(k+a+c+1)\Gamma(a+c+n+2)\right) / \left(\Gamma(2n+a+b+1)\Gamma(n+a+c+1)^2(n+a+c+1)\Gamma(k-b+c-n+1)\Gamma(a+c+k+n+2)\Gamma(a+k+1)\Gamma(1-k+n)\right)$$

```
> sol:=simplify(%) assuming n::posint,k::nonnegint,k<=n;
```

$$sol := \left((a+c+2k+1)\Gamma(a+c+2n+1)\Gamma(c+1-b)\Gamma(a+n+1)\Gamma(k+n+a+b+1)\Gamma(k+a+c+1)\right) / \left(\Gamma(2n+a+b+1)\Gamma(n+a+c+1)\Gamma(k-b+c-n+1)\Gamma(a+c+k+n+2)\Gamma(a+k+1)\Gamma(1-k+n)\right)$$

Gegenbauer to Gegenbauer (Ismail 2005 Thm. 9.2.1)

```
> lcoeffGn:=1/GAMMA(lambda)*2^n*GAMMA(n+lambda)/GAMMA(n+1);
```

$$lcoeffGn := \frac{2^n\,\Gamma(\lambda+n)}{\Gamma(\lambda)\,\Gamma(n+1)}$$

```
> rec:=FunctiontoRec(GegenbauerC(n,lambda,x),GegenbauerC(k,nu,x),u);
FunctiontoRec: time: 0.06 sec.
```

$$rec := -(\nu+k+2)\,(k-n)\,(k+2\lambda+n)\,u(k)+(\nu+k)\,(k+n+2\nu+2)\,(k-2\lambda+2\nu+2-n)\,u(k+2)$$

Initial conditions are easily obtained for the leading coefficient and the one below, so we revert the recurrence:

```
> eval(rec,u=proc(t) v(n-t) end);
```

$$-(\nu+k+2)\,(k-n)\,(k+2\lambda+n)\,v(n-k)+(\nu+k)\,(k+n+2\nu+2)\,(k-2\lambda+2\nu+2-n)\,v(n-k-2)$$

```
> subs(k=n-k,%);
```

$$(\nu+n-k+2)\,k\,(2n-k+2\lambda)\,v(k)+(\nu+n-k)\,(2n-k+2\nu+2)\,(-k-2\lambda+2\nu+2)\,v(k-2)$$

```
> rec2:=%:
> rsolve({rec2,v(0)=lcoeffGn/eval(lcoeffGn,lambda=nu),v(1)=0},v(k))
  assuming n::posint,k::posint,k<=n/2;
```

$$\begin{cases} \dfrac{(k-\nu-n)\,\Gamma\left(-n-\nu+\frac{k}{2}\right)\Gamma(-\lambda-n+1)\,\Gamma\left(\frac{k}{2}+\lambda-\nu\right)\Gamma(\lambda+n)\,\Gamma(\nu)}{\Gamma(1-\nu-n)\,\Gamma\left(\frac{k}{2}-\lambda-n+1\right)\Gamma(\lambda-\nu)\,\Gamma\left(\frac{k}{2}+1\right)\Gamma(\lambda)\,\Gamma(\nu+n)} & k::even \\ 0 & k::odd \end{cases}$$

Gegenbauer to Chebyshev (Area, Dimitrov, Godoy, Ronveaux 2004).

```
> FunctiontoRec(GegenbauerC(n,lambda,s*x),ChebyshevT(m,x),u);
FunctiontoRec: time: 0.031 sec.
```

$$s^2\,(3+m)\,(m-n)\,(m+2\lambda+n)\,u(m)+2\,(m+2)\,(2ns^2\lambda+m^2s^2+n^2s^2+2\lambda s^2+4s^2m-2m^2+2s^2-8m-6)\,u(m+2)-(1+m)\,s^2\,(m+n+4)\,(-m+2\lambda-4+n)\,u(m+4)$$

```
> rec:=%:
```

Check:

```
> nn:=10:
> P:=expand(GegenbauerC(nn,lambda,x)):P:=expand(subs(x=s*x,%/lcoeff
  (%,x))):
> for i from nn to 0 by -1 do pol:=expand(ChebyshevT(i,x)); c[i]:=
  normal(coeff(P,x,i)/coeff(pol,x,i)); P:=expand(P-c[i]*pol) od:
```

Note the change for u(0)

```
> for j from 0 to nn do
    j,normal(eval(eval(rec,[n=nn,m=j]),[u(0)=c[0]*2,seq(u(i)=c[i],i=
  1..nn),seq(u(i)=0,i=nn+1..nn+4)]))
  od;
```

$$0, 0$$
$$1, 0$$
$$2, 0$$
$$3, 0$$
$$4, 0$$
$$5, 0$$
$$6, 0$$
$$7, 0$$
$$8, 0$$
$$9, 0$$
$$10, 0$$

```
> i:='i':j:='j':
```

Comparison with their formula (3.5):

```
> Am:=(-1)^((n-m)/2)/m!/((n-m)/2)!*s^m/2^(3*n-m)*pochhammer(2*lambda,
  n)*pochhammer(n+2*lambda,n)*n!/pochhammer(lambda+1/2,n)/pochhammer
  (lambda+(n+m)/2,(n-m)/2)/pochhammer(lambda,n)*hypergeom([(m-n)/2,(2*
  lambda+m+n)/2],[m+1],s^2);
```

$$Am := \left((-1)^{\frac{n}{2}-\frac{m}{2}}\, s^m\, \mathrm{pochhammer}(2\,\lambda, n)\, \mathrm{pochhammer}(2\,\lambda + n, n)\, n!\, \mathrm{hypergeom}\left(\left[\frac{m}{2} - \frac{n}{2}, \lambda + \frac{m}{2} + \frac{n}{2} \right], [1+m], s^2 \right) \right) \Big/ \left(m!\, \left(\frac{n}{2} - \frac{m}{2} \right)!\, 2^{3n-m}\, \mathrm{pochhammer}\left(\lambda + \frac{1}{2}, n \right) \mathrm{pochhammer}\left(\lambda + \frac{m}{2} + \frac{n}{2}, \frac{n}{2} - \frac{m}{2} \right) \mathrm{pochhammer}(\lambda, n) \right)$$

```
> um:=Am/2^(m-1); # they use monic ChebyshevT
```

$$um := \left((-1)^{\frac{n}{2}-\frac{m}{2}}\, s^m\, \mathrm{pochhammer}(2\,\lambda, n)\, \mathrm{pochhammer}(2\,\lambda + n, n)\, n!\, \mathrm{hypergeom}\left(\left[\frac{m}{2} - \frac{n}{2}, \lambda + \frac{m}{2} + \frac{n}{2} \right], [1+m], s^2 \right) \right) \Big/ \left(m!\, \left(\frac{n}{2} - \frac{m}{2} \right)!\, 2^{3n-m}\, \mathrm{pochhammer}\left(\lambda + \frac{1}{2}, n \right) \mathrm{pochhammer}\left(\lambda + \frac{m}{2} + \frac{n}{2}, \frac{n}{2} - \frac{m}{2} \right) \mathrm{pochhammer}(\lambda, n)\, 2^{m-1} \right)$$

```
> normal(expand(convert(series(eval(um,[n=10,m=4]),s,infinity),
  polynom))/c[4]);
```

$$1$$

```
> normal(expand(convert(series(eval(um,[n=10,m=2]),s,infinity),
  polynom))/c[2]);
```

$$1$$

```
> normal(expand(convert(series(eval(um,[n=10,m=0]),s,infinity),
  polynom))/c[0]);
```

$$2$$

Associated Hermite to Hermite (Ronveaux, Zarzo, Godoy 1995)

The associated Hermite polynomials satisfy a linear differential equation of order 4 (Ronveaux 1988):

```
> deq := diff(y(x), x, x, x, x) + (-4*x^2 + 8*c + 4*n)*diff(y(x), x,
  x) - 12*x*diff(y(x), x) + 4*n*(n + 2)*y(x);
```

$$deq := \frac{d^4}{dx^4} y(x) + \left(-4x^2 + 8c + 4n\right)\left(\frac{d^2}{dx^2} y(x)\right) - 12x\left(\frac{d}{dx} y(x)\right) + 4n(n+2)y(x)$$

```
> DiffeqtoRec(subs(n=n-1,deq),y(x),u(m),HermiteH(m,x));
```

$$\left(-m^2 + n^2 - 2m - 1\right)u(m) + 4(1+m)\left(2mc - m^2 + mn + 4c - 5m + 2n - 6\right)u(m+2)$$

RoZaGo95 give the case c=1:

```
> DiffeqtoRec(subs(n=n-1,c=1,deq),y(x),u(m),HermiteH(m,x)) assuming
  n::nonnegint;
```

$$(1+m-n)\left((-m-n-1)u(m) - 4(m+2)(1+m)u(m+2)\right)$$

Note the factor m-n+1 showing that at m=n-1, the recurrence cannot be used to compute the next coefficient. This is expected since the next coefficient is 0.

The recurrence in RoZaGo95 has a different constant factor in the term u(m): they consider monic Hermite polynomials, whereas we use the classical ones with leading coefficient 2^n.

Associated Laguerre to Laguerre (Ronveaux, Zarzo, Godoy 1995)

```
> deq:=n*(n+2)*y(x)+(3*n+3*alpha+6*c-3*x)*diff(y(x),x)+(-alpha^2+2*
  alpha*x+4*c*x+2*n*x
  -x^2+4)*diff(diff(y(x),x),x)+5*x*diff(diff(diff(y(x),x),x),x)+x^2*
  diff(diff(
  diff(diff(y(x),x),x),x),x);
```

$$deq := n(n+2)y(x) + (3n + 3\alpha + 6c - 3x)\left(\frac{d}{dx} y(x)\right) + \left(-\alpha^2 + 2\alpha x + 4cx + 2nx - x^2 + 4\right)\left(\frac{d^2}{dx^2} y(x)\right) + 5x\left(\frac{d^3}{dx^3} y(x)\right) + x^2\left(\frac{d^4}{dx^4} y(x)\right)$$

```
> DiffeqtoRec(deq,y(x),u(m),LaguerreL(m,alpha,x));
```

$$\left(-(m-2)^2 + n^2 - 6m + 4 + 2n\right)u(m) + \left(-4(m-2)c + 3(m-2)^2 - 2(m-2)n - n^2 - 14c + 21m - 6 - 9n\right)u(1+m) + \left(4c\alpha - 2\alpha(m-2) + 2n\alpha + 4(m-2)c - 2(m-2)^2 + 2(m-2)n - 8\alpha + 16c - 16m + 8n\right)u(m+2)$$

Note that they give an order 4 recurrence for the special case c=1, where we still get an order 2 recurrence:

```
> DiffeqtoRec(subs(n=n-1,c=1,deq),y(x),u(m),LaguerreL(m,alpha,x));
```

$$(1+m-n)\left((-m-n-1)u(m) + (3m+n+4)u(1+m) + (-2\alpha - 4 - 2m)u(m+2)\right)$$

Jacobi to Hermite (Godoy et al. 1997)

```
> FunctiontoRec(JacobiP(n,a,b,x),HermiteH(m,x),u);
FunctiontoRec: time: 0.035 sec.
```

$$(m-n)\,(m+n+a+b+1)\,u(m)+2\,(1+m)\,(a-b)\,u(1+m)+2\,(1+m)\,(m+2)\,(2\,m+a+b+1)\,u(m+2)+4\,(1+m)\,(m+2)\,(3+m)\,(m+4)\,u(m+4)$$

Hermite to Jacobi (Godoy et al. 1997)

```
> FunctiontoRec(HermiteH(m,x),JacobiP(n,a,b,x),u);
```

```
FunctiontoRec: time: 0.463 sec.
```

$$8\,(a+b+2\,n+8)\,(a+b+2\,n+9)\,(n+4+a+b)\,(a+b+2\,n+6)\,(a+b+2\,n+7)\,(n+3+a+b)\,(n+2+a+b)\,(n+a+b+1)\,(m-n)\,u(n)+4\,(2\,n+a+b+1)\,(a+b+2\,n+8)\,(a+b+2\,n+9)\,(n+4+a+b)\,(a+b+2\,n+7)\,(n+3+a+b)\,(n+2+a+b)\,(a-b)\,(2+4\,m+a+b-2\,n)\,u(n+1)+(2\,n+a+b+1)\,(2\,n+2+a+b)\,(n+3+a+b)\,(a+b+2\,n+8)\,(a+b+2\,n+9)\,(n+4+a+b)\,(a^4+4\,a^3\,b+8\,a^3\,n+6\,a^2\,b^2+24\,a^2\,b\,n+24\,a^2\,n^2+4\,a\,b^3+24\,a\,b^2\,n+48\,a\,b\,n^2+32\,a\,n^3+b^4+8\,b^3\,n+24\,b^2\,n^2+32\,b\,n^3+16\,n^4+24\,a^3+48\,a^2\,b+8\,a^2\,m+112\,a^2\,n+48\,a\,b^2-32\,a\,b\,m+224\,a\,b\,n-16\,a\,m\,n+232\,n^2\,a+24\,b^3+8\,b^2\,m+112\,b^2\,n-16\,b\,m\,n+232\,n^2\,b-16\,m\,n^2+160\,n^3+137\,a^2+202\,a\,b-40\,a\,m+516\,n\,a+137\,b^2-40\,b\,m+516\,n\,b-80\,m\,n+556\,n^2+342\,a+342\,b-96\,m+780\,n+360)\,u(n+2)-4\,(b+n+3)\,(a+n+3)\,(a+b+2\,n+3)\,(2\,n+a+b+1)\,(2\,n+2+a+b)\,(a+b+2\,n+9)\,(n+4+a+b)\,(a-b)\,(12+4\,m+3\,a+3\,b+2\,n)\,u(n+3)+8\,(b+n+4)\,(a+n+4)\,(b+n+3)\,(a+n+3)\,(a+b+2\,n+3)\,(a+b+2\,n+4)\,(2\,n+a+b+1)\,(2\,n+2+a+b)\,(5+m+n+a+b)\,u(n+4)$$

Jacobi to Laguerre (ibid.)

```
> FunctiontoRec(JacobiP(n,a,b,x),LaguerreL(m,alpha,x),u);
```

```
FunctiontoRec: time: 0.027 sec.
```

$$-(m-n)\,(m+n+a+b+1)\,u(m)+(a\,\alpha+3\,a\,m-2\,n\,a+b\,\alpha+2\,\alpha\,m+3\,b\,m-2\,n\,b+4\,m^2-2\,n^2+4\,a+2\,\alpha+2\,b+10\,m-2\,n+6)\,u(1+m)+(-2\,a\,\alpha-3\,a\,m+n\,a-\alpha^2-2\,b\,\alpha-6\,\alpha\,m-3\,b\,m+n\,b-6\,m^2+n^2-7\,a-11\,\alpha-5\,b-24\,m+n-23)\,u(m+2)+(\alpha+3+m)\,(2\,\alpha+a+b+4\,m+10)\,u(3+m)-(\alpha+3+m)\,(\alpha+4+m)\,u(m+4)$$

Laguerre to Jacobi (ibid.)

```
> FunctiontoRec(LaguerreL(m,alpha,x),JacobiP(n,a,b,x),u);
```

```
FunctiontoRec: time: 0.125 sec.
```

$$4\,(a+b+2\,n+8)\,(a+b+2\,n+9)\,(n+4+a+b)\,(a+b+2\,n+6)\,(a+b+2\,n+7)\,(n+3+a+b)\,(n+2+a+b)\,(n+a+b+1)\,(m-n)\,u(n)+2\,(2\,n+a+b+1)\,(n+2+a+b)\,(n+3+a+b)\,(a+b+2\,n+7)\,(a+b+2\,n+8)\,(a+b+2\,n+9)\,(n+4+a+b)\,(a^2\,\alpha+a^2\,n+2\,a\,\alpha\,b+4\,a\,\alpha\,n+2\,a\,b\,n+4\,n^2\,a+\alpha\,b^2+4\,\alpha\,b\,n+4\,\alpha\,n^2+b^2\,n+4\,n^2\,b+4\,n^3+2\,a^2+8\,a\,\alpha+2\,a\,b+4\,a\,m+10\,n\,a+8\,b\,\alpha+16\,n\,\alpha-4\,b\,m+14\,n\,b+20\,n^2+10\,a+12\,\alpha+6\,b+28\,n+12)\,u(n+1)-(2\,n+a+b+1)\,(2\,n+2+a$$

$$+ b)\,(n+3+a+b)\,(a+b+2n+8)\,(a+b+2n+9)\,(n+4+a+b)\,\left(a^4 - 2\,\alpha\,a^3 + 2\,a^3\,b + 4\,a^3\,n - 2\,a^2\,\alpha\,b - 8\,a^2\,\alpha\,n + 4\,a^2\,b\,n + 4\,a^2\,n^2 + 2\,a\,\alpha\,b^2 - 8\,a\,\alpha\,n^2 - 2\,a\,b^3 - 4\,a\,b^2\,n + 2\,\alpha\,b^3 + 8\,\alpha\,b^2\,n + 8\,\alpha\,b\,n^2 - b^4 - 4\,b^3\,n - 4\,b^2\,n^2 + 10\,a^3 - 20\,a^2\,\alpha + 18\,a^2\,b - 4\,a^2\,m + 32\,a^2\,n - 40\,a\,\alpha\,n - 6\,a\,b^2 + 16\,a\,b\,m + 8\,a\,b\,n + 8\,a\,m\,n + 12\,n^2\,a + 20\,\alpha\,b^2 + 40\,\alpha\,b\,n - 14\,b^3 - 4\,b^2\,m - 24\,b^2\,n + 8\,b\,m\,n - 4\,n^2\,b + 8\,m\,n^2 + 45\,a^2 - 42\,a\,\alpha + 44\,a\,b + 20\,a\,m + 72\,n\,a + 42\,b\,\alpha - 37\,b^2 + 20\,b\,m - 8\,n\,b + 40\,m\,n + 12\,n^2 + 96\,a + 12\,b + 48\,m + 60\,n + 72\right)\,u(n+2) + 2\,(b+n+3)\,(a+n+3)\,(a+b+2n+3)\,(2n+a+b+1)\,(2n+2+a+b)\,(n+4+a+b)\,(a+b+2n+9)\,\left(a^3 - a^2\,\alpha + 3\,a^2\,b + 5\,a^2\,n - 2\,a\,\alpha\,b - 4\,a\,\alpha\,n + 3\,a\,b^2 + 10\,a\,b\,n + 8\,n^2\,a - \alpha\,b^2 - 4\,\alpha\,b\,n - 4\,\alpha\,n^2 + b^3 + 5\,b^2\,n + 8\,n^2\,b + 4\,n^3 + 13\,a^2 - 12\,a\,\alpha + 32\,a\,b - 4\,a\,m + 50\,n\,a - 12\,b\,\alpha - 24\,n\,\alpha + 19\,b^2 + 4\,b\,m + 54\,n\,b + 40\,n^2 + 68\,a - 32\,\alpha + 92\,b + 128\,n + 128\right)\,u(n+3) + 4\,(b+n+4)\,(a+n+4)\,(b+n+3)\,(a+n+3)\,(a+b+2n+3)\,(a+b+2n+4)\,(2n+a+b+1)\,(2n+2+a+b)\,(5+m+n+a+b)\,u(n+4)$$

Linearization coefficients (Sec. 9.1.2)

Hermite (Feldheim 1938 (1.4 & 1.5) also sec. 3.2 in Lewanowicz 1996)

```
> rec:=FunctiontoRec(HermiteH(m,x)*HermiteH(n,x),HermiteH(nu,x),u);
FunctiontoRec: time: 0.031 sec.
```

$$rec := (m+n-\nu)\,u(\nu) + (\nu+2-n+m)\,(-\nu-2-n+m)\,u(2+\nu)$$

Initial conditions are easily obtained for the leading coefficient and the one below, so we revert the recurrence:

```
> eval(rec,u=proc(nu) v(m+n-nu) end);
```

$$(m+n-\nu)\,v(m+n-\nu) + (\nu+2-n+m)\,(-\nu-2-n+m)\,v(m+n-2-\nu)$$

```
> subs(nu=m+n-k,%);
```

$$k\,v(k) + (2\,m-k+2)\,(-2\,n+k-2)\,v(k-2)$$

```
> rec2:=%:
> rsolve({rec2,v(0)=lcoeffHn*subs(n=m,lcoeffHn)/subs(n=m+n,lcoeffHn),
  v(1)=0},v(k)) assuming n::posint,m::posint,k::even;
```

$$\frac{\Gamma(n+1)\,\Gamma(1+m)\,2^{\frac{k}{2}}}{\Gamma\left(1-\frac{k}{2}+n\right)\,\Gamma\left(1-\frac{k}{2}+m\right)\,\Gamma\left(\frac{k}{2}+1\right)}$$

Special case: Hn^2

```
> co2:=eval(%,m=n);
```

$$co2 := \frac{\Gamma(n+1)^2\, 2^{\frac{k}{2}}}{\Gamma\left(1-\frac{k}{2}+n\right)^2\, \Gamma\left(\frac{k}{2}+1\right)}$$

Laguerre (sec. 3.3 in Lewanowicz 96)

```
> FunctiontoRec(LaguerreL(i,alpha,x)*LaguerreL(j,alpha,x),LaguerreL
  (k,alpha,x),u);
```

```
FunctiontoRec: time: 0.077 sec.
```

$$2\,(k+2)\,(k+1)\,(i+j-k)\,u(k) - (k+2)\left(i^2-2\,ij+4\,ik+j^2+4\,jk-5\,k^2+9\,i+9\,j-15\,k-10\right)u(k+1) + \left(\alpha\,i^2-2\,\alpha\,ij+\alpha\,j^2-\alpha\,k^2+2\,i^2\,k-4\,ijk+2\,ik^2+2\,j^2\,k+2\,jk^2-4\,k^3-4\,\alpha\,k+5\,i^2-10\,ij+11\,ik+5\,j^2+11\,jk-28\,k^2-4\,\alpha+15\,i+15\,j-65\,k-50\right)u(k+2) - (k+3-j+i)\,(-k-3-j+i)\,(\alpha+3+k)\,u(3+k)$$

Lewanowicz 96 has a recurrence of order 2 by differentiation:

```
> st:=time():
> deq:=gfun:-holexprtodiffeq(LaguerreL(i,alpha,x)*LaguerreL(j,alpha,
  x),y(x),false):
> collect(DiffeqtoRec(diff(deq,x),y(x),u(k),LaguerreL(k,alpha,x)),u,
  factor);
```

$$-2\,(k+2)\,(k+1)\,(i+j-k)\,u(k) + (k+2)\left(i^2-2\,ij+2\,ik+j^2+2\,jk-3\,k^2+3\,i+3\,j-7\,k-4\right)u(k+1) - (k+2-j+i)\,(-k-2-j+i)\,(\alpha+2+k)\,u(k+2)$$

```
> time()-st;
```

0.086

Generalization of the previous one: Eq. (1.07) in Feldheim (1940)

```
> rec:=FunctiontoRec(LaguerreL(m,alpha,x)*LaguerreL(n,beta,x),
  LaguerreL(s,alpha+beta,x),u);
```

```
FunctiontoRec: time: 0.478 sec.
```

$$rec := 4\,(s+3)\,(s+2)\,(s+1)\,(m+n-s)\,(\alpha-\beta+2\,m-2\,n)\,u(s) - 2\,(s+3)\,(s+2)\left(4\,\alpha^2\,m+2\,\alpha^2\,n-4\,\alpha^2\,s-2\,\alpha\,\beta\,m+2\,\alpha\,\beta\,n+11\,\alpha\,m^2-6\,\alpha\,mn-2\,\alpha\,ms-5\,\alpha\,n^2+18\,\alpha\,ns-9\,\alpha\,s^2-2\,\beta^2\,m-4\,\beta^2\,n+4\,\beta^2\,s+5\,\beta\,m^2+6\,\beta\,mn-18\,\beta\,ms-11\,\beta\,n^2+2\,\beta\,ns+9\,\beta\,s^2+2\,m^3-6\,m^2\,n+16\,m^2\,s+6\,mn^2-18\,s^2\,m-2\,n^3-16\,n^2\,s+18\,ns^2-4\,\alpha^2+13\,\alpha\,m+33\,n\,\alpha-33\,\alpha\,s+4\,\beta^2-33\,\beta\,m-13\,\beta\,n+33\,\beta\,s+46\,m^2-66\,ms-46\,n^2+66\,ns-24\,\alpha+24\,\beta-48\,m+48\,n\right)u(s+1) + (s+3)\left(5\,\alpha^3\,m+\alpha^3\,n-5\,\alpha^3\,s+\alpha^2\,\beta\,m+5\,\alpha^2\,\beta\,n-9\,\alpha^2\,\beta\,s+19\,\alpha^2\,m^2-18\,\alpha^2\,mn+12\,\alpha^2\,ms-\alpha^2\,n^2+28\,\alpha^2\,ns-31\,\alpha^2\,s^2-5\,\alpha\,\beta^2\,m-\alpha\,\beta^2\,n+9\,\alpha\,\beta^2\,s+20\,\alpha\,\beta\,m^2-56\,\alpha\,\beta\,ms-20\,\alpha\,\beta\,n^2+56\,\alpha\,\beta\,ns+8\,\alpha\,m^3-24\,\alpha\,m^2\,n+72\,\alpha\,m^2\,s+24\,\alpha\,mn^2-48\,\alpha\,mns-48\,\alpha\,ms^2-8\,\alpha\,n^3-24\,\alpha\,n^2\,s+96\,\alpha\,ns^2-32\,\alpha\,s^3-\beta^3\,m-5\,\beta^3\,n+5\,\beta^3\,s+\beta^2\,m^2+18\,\beta^2\,mn-28\,\beta^2\,ms-19\,\beta^2\,n^2$$

$$
\begin{aligned}
&- 12\,\beta^2\,n\,s + 31\,\beta^2\,s^2 + 8\,\beta\,m^3 - 24\,\beta\,m^2\,n + 24\,\beta\,m^2\,s + 24\,\beta\,m\,n^2 + 48\,\beta\,m\,n\,s - 96\,\beta\,m\,s^2 \\
&- 8\,\beta\,n^3 - 72\,\beta\,n^2\,s + 48\,\beta\,n\,s^2 + 32\,\beta\,s^3 + 16\,m^3\,s - 48\,m^2\,n\,s + 48\,m^2\,s^2 + 48\,m\,n^2\,s \\
&- 64\,m\,s^3 - 16\,n^3\,s - 48\,n^2\,s^2 + 64\,n\,s^3 - 10\,\alpha^3 - 18\,\alpha^2\,\beta + 62\,\alpha^2\,m + 78\,\alpha^2\,n - 162\,\alpha^2\,s \\
&+ 18\,\alpha\,\beta^2 - 128\,\alpha\,\beta\,m + 128\,\alpha\,\beta\,n + 234\,\alpha\,m^2 - 144\,\alpha\,m\,n - 208\,\alpha\,m\,s - 90\,\alpha\,n^2 \\
&+ 532\,\alpha\,n\,s - 266\,\alpha\,s^2 + 10\,\beta^3 - 78\,\beta^2\,m - 62\,\beta^2\,n + 162\,\beta^2\,s + 90\,\beta\,m^2 + 144\,\beta\,m\,n \\
&- 532\,\beta\,m\,s - 234\,\beta\,n^2 + 208\,\beta\,n\,s + 266\,\beta\,s^2 + 48\,m^3 - 144\,m^2\,n + 324\,m^2\,s + 144\,m\,n^2 \\
&- 532\,s^2\,m - 48\,n^3 - 324\,n^2\,s + 532\,n\,s^2 - 200\,\alpha^2 - 176\,\alpha\,m + 728\,n\,\alpha - 728\,\alpha\,s + 200\,\beta^2 \\
&- 728\,\beta\,m + 176\,\beta\,n + 728\,\beta\,s + 552\,m^2 - 1456\,m\,s - 552\,n^2 + 1456\,n\,s - 648\,\alpha + 648\,\beta \\
&- 1296\,m + 1296\,n\big)\,u(s+2) + \big(-\alpha^4\,m + \alpha^4\,s - \alpha^3\,\beta\,m - \alpha^3\,\beta\,n + 6\,\alpha^3\,\beta\,s - 5\,\alpha^3\,m^2 \\
&+ 6\,\alpha^3\,m\,n - 12\,\alpha^3\,m\,s - \alpha^3\,n^2 - 4\,\alpha^3\,n\,s + 17\,\alpha^3\,s^2 + \alpha^2\,\beta^2\,m - \alpha^2\,\beta^2\,n - 9\,\alpha^2\,\beta\,m^2 \\
&+ 6\,\alpha^2\,\beta\,m\,n + 24\,\alpha^2\,\beta\,m\,s + 3\,\alpha^2\,\beta\,n^2 - 40\,\alpha^2\,\beta\,n\,s + 27\,\alpha^2\,\beta\,s^2 - 4\,\alpha^2\,m^3 + 12\,\alpha^2\,m^2\,n \\
&- 50\,\alpha^2\,m^2\,s - 12\,\alpha^2\,m\,n^2 + 60\,\alpha^2\,m\,n\,s + 10\,\alpha^2\,m\,s^2 + 4\,\alpha^2\,n^3 - 10\,\alpha^2\,n^2\,s - 54\,\alpha^2\,n\,s^2 \\
&+ 44\,\alpha^2\,s^3 + \alpha\,\beta^3\,m + \alpha\,\beta^3\,n - 6\,\alpha\,\beta^3\,s - 3\,\alpha\,\beta^2\,m^2 - 6\,\alpha\,\beta^2\,m\,n + 40\,\alpha\,\beta^2\,m\,s + 9\,\alpha\,\beta^2\,n^2 \\
&- 24\,\alpha\,\beta^2\,n\,s - 27\,\alpha\,\beta^2\,s^2 - 8\,\alpha\,\beta\,m^3 + 24\,\alpha\,\beta\,m^2\,n - 40\,\alpha\,\beta\,m^2\,s - 24\,\alpha\,\beta\,m\,n^2 \\
&+ 132\,\alpha\,\beta\,m\,s^2 + 8\,\alpha\,\beta\,n^3 + 40\,\alpha\,\beta\,n^2\,s - 132\,\alpha\,\beta\,n\,s^2 - 24\,\alpha\,m^3\,s + 72\,\alpha\,m^2\,n\,s - 84\,\alpha\,m^2\,s^2 \\
&- 72\,\alpha\,m\,n^2\,s + 72\,\alpha\,m\,n\,s^2 + 80\,\alpha\,m\,s^3 + 24\,\alpha\,n^3\,s + 12\,\alpha\,n^2\,s^2 - 112\,\alpha\,n\,s^3 + 28\,\alpha\,s^4 + \beta^4\,n \\
&- \beta^4\,s + \beta^3\,m^2 - 6\,\beta^3\,m\,n + 4\,\beta^3\,m\,s + 5\,\beta^3\,n^2 + 12\,\beta^3\,n\,s - 17\,\beta^3\,s^2 - 4\,\beta^2\,m^3 + 12\,\beta^2\,m^2\,n \\
&+ 10\,\beta^2\,m^2\,s - 12\,\beta^2\,m\,n^2 - 60\,\beta^2\,m\,n\,s + 54\,\beta^2\,m\,s^2 + 4\,\beta^2\,n^3 + 50\,\beta^2\,n^2\,s - 10\,\beta^2\,n\,s^2 \\
&- 44\,\beta^2\,s^3 - 24\,\beta\,m^3\,s + 72\,\beta\,m^2\,n\,s - 12\,\beta\,m^2\,s^2 - 72\,\beta\,m\,n^2\,s - 72\,\beta\,m\,n\,s^2 + 112\,\beta\,m\,s^3 \\
&+ 24\,\beta\,n^3\,s + 84\,\beta\,n^2\,s^2 - 80\,\beta\,n\,s^3 - 28\,\beta\,s^4 - 24\,m^3\,s^2 + 72\,m^2\,n\,s^2 - 32\,m^2\,s^3 - 72\,m\,n^2\,s^2 \\
&+ 56\,m\,s^4 + 24\,n^3\,s^2 + 32\,n^2\,s^3 - 56\,n\,s^4 + 3\,\alpha^4 + 18\,\alpha^3\,\beta - 48\,\alpha^3\,m - 15\,\alpha^3\,n + 114\,\alpha^3\,s \\
&+ 69\,\alpha^2\,\beta\,m - 132\,\alpha^2\,\beta\,n + 180\,\alpha^2\,\beta\,s - 181\,\alpha^2\,m^2 + 210\,\alpha^2\,m\,n + 32\,\alpha^2\,m\,s - 29\,\alpha^2\,n^2 \\
&- 378\,\alpha^2\,n\,s + 455\,\alpha^2\,s^2 - 18\,\alpha\,\beta^3 + 132\,\alpha\,\beta^2\,m - 69\,\alpha\,\beta^2\,n - 180\,\alpha\,\beta^2\,s - 152\,\alpha\,\beta\,m^2 \\
&+ 878\,\alpha\,\beta\,m\,s + 152\,\alpha\,\beta\,n^2 - 878\,\alpha\,\beta\,n\,s - 84\,\alpha\,m^3 + 252\,\alpha\,m^2\,n - 624\,\alpha\,m^2\,s - 252\,\alpha\,m\,n^2 \\
&+ 504\,\alpha\,m\,n\,s + 798\,\alpha\,m\,s^2 + 84\,\alpha\,n^3 + 120\,\alpha\,n^2\,s - 1170\,\alpha\,n\,s^2 + 390\,\alpha\,s^3 - 3\,\beta^4 + 15\,\beta^3\,m \\
&+ 48\,\beta^3\,n - 114\,\beta^3\,s + 29\,\beta^2\,m^2 - 210\,\beta^2\,m\,n + 378\,\beta^2\,m\,s + 181\,\beta^2\,n^2 - 32\,\beta^2\,n\,s - 455\,\beta^2\,s^2 \\
&- 84\,\beta\,m^3 + 252\,\beta\,m^2\,n - 120\,\beta\,m^2\,s - 252\,\beta\,m\,n^2 - 504\,\beta\,m\,n\,s + 1170\,\beta\,m\,s^2 + 84\,\beta\,n^3 \\
&+ 624\,\beta\,n^2\,s - 798\,\beta\,n\,s^2 - 390\,\beta\,s^3 - 168\,m^3\,s + 504\,m^2\,n\,s - 372\,m^2\,s^2 - 504\,m\,n^2\,s \\
&+ 780\,m\,s^3 + 168\,n^3\,s + 372\,n^2\,s^2 - 780\,n\,s^3 + 189\,\alpha^3 + 297\,\alpha^2\,\beta - 17\,\alpha^2\,m - 663\,\alpha^2\,n \\
&+ 1565\,\alpha^2\,s - 297\,\alpha\,\beta^2 + 1454\,\alpha\,\beta\,m - 1454\,\alpha\,\beta\,n - 1166\,\alpha\,m^2 + 888\,\alpha\,m\,n + 2636\,\alpha\,m\,s \\
&+ 278\,\alpha\,n^2 - 4080\,\alpha\,n\,s + 2040\,\alpha\,s^2 - 189\,\beta^3 + 663\,\beta^2\,m + 17\,\beta^2\,n - 1565\,\beta^2\,s - 278\,\beta\,m^2
\end{aligned}
$$

$$
\begin{aligned}
&- 888\,\beta\,m\,n + 4080\,\beta\,m\,s + 1166\,\beta\,n^2 - 2636\,\beta\,n\,s - 2040\,\beta\,s^2 - 296\,m^3 + 888\,m^2\,n \\
&- 1444\,m^2\,s - 888\,m\,n^2 + 4080\,s^2\,m + 296\,n^3 + 1444\,n^2\,s - 4080\,n\,s^2 + 1788\,\alpha^2 + 2874\,\alpha\,m \\
&- 4746\,n\,\alpha + 4746\,\alpha\,s - 1788\,\beta^2 + 4746\,\beta\,m - 2874\,\beta\,n - 4746\,\beta\,s - 1872\,m^2 + 9492\,m\,s \\
&+ 1872\,n^2 - 9492\,n\,s + 4140\,\alpha - 4140\,\beta + 8280\,m - 8280\,n\big)\,u(s+3) - (\alpha+\beta+4 \\
&+ s)\,\big(\alpha^3\,\beta - 3\,\alpha^3\,m + \alpha^3\,n + 3\,\alpha^3\,s + 5\,\alpha^2\,\beta\,m - 7\,\alpha^2\,\beta\,n + 11\,\alpha^2\,\beta\,s - 11\,\alpha^2\,m^2 \\
&+ 18\,\alpha^2\,m\,n - 4\,\alpha^2\,m\,s - 7\,\alpha^2\,n^2 - 4\,\alpha^2\,n\,s + 15\,\alpha^2\,s^2 - \alpha\,\beta^3 + 7\,\alpha\,\beta^2\,m - 5\,\alpha\,\beta^2\,n \\
&- 11\,\alpha\,\beta^2\,s - 4\,\alpha\,\beta\,m^2 + 48\,\alpha\,\beta\,m\,s + 4\,\alpha\,\beta\,n^2 - 48\,\alpha\,\beta\,n\,s - 8\,\alpha\,m^3 + 24\,\alpha\,m^2\,n \\
&- 32\,\alpha\,m^2\,s - 24\,\alpha\,m\,n^2 + 48\,\alpha\,m\,n\,s + 28\,\alpha\,m\,s^2 + 8\,\alpha\,n^3 - 16\,\alpha\,n^2\,s - 36\,\alpha\,n\,s^2 + 12\,\alpha\,s^3 \\
&- \beta^3\,m + 3\,\beta^3\,n - 3\,\beta^3\,s + 7\,\beta^2\,m^2 - 18\,\beta^2\,m\,n + 4\,\beta^2\,m\,s + 11\,\beta^2\,n^2 + 4\,\beta^2\,n\,s - 15\,\beta^2\,s^2 \\
&- 8\,\beta\,m^3 + 24\,\beta\,m^2\,n + 16\,\beta\,m^2\,s - 24\,\beta\,m\,n^2 - 48\,\beta\,m\,n\,s + 36\,\beta\,m\,s^2 + 8\,\beta\,n^3 + 32\,\beta\,n^2\,s \\
&- 28\,\beta\,n\,s^2 - 12\,\beta\,s^3 - 16\,m^3\,s + 48\,m^2\,n\,s - 8\,m^2\,s^2 - 48\,m\,n^2\,s + 24\,m\,s^3 + 16\,n^3\,s + 8\,n^2\,s^2 \\
&- 24\,n\,s^3 + 12\,\alpha^3 + 44\,\alpha^2\,\beta - 20\,\alpha^2\,m - 18\,\alpha^2\,n + 124\,\alpha^2\,s - 44\,\alpha\,\beta^2 + 190\,\alpha\,\beta\,m \\
&- 190\,\alpha\,\beta\,n - 134\,\alpha\,m^2 + 192\,\alpha\,m\,n + 224\,\alpha\,m\,s - 58\,\alpha\,n^2 - 300\,\alpha\,n\,s + 150\,\alpha\,s^2 - 12\,\beta^3 \\
&+ 18\,\beta^2\,m + 20\,\beta^2\,n - 124\,\beta^2\,s + 58\,\beta\,m^2 - 192\,\beta\,m\,n + 300\,\beta\,m\,s + 134\,\beta\,n^2 - 224\,\beta\,n\,s \\
&- 150\,\beta\,s^2 - 64\,m^3 + 192\,m^2\,n - 76\,m^2\,s - 192\,m\,n^2 + 300\,s^2\,m + 64\,n^3 + 76\,n^2\,s - 300\,n\,s^2 \\
&+ 256\,\alpha^2 + 446\,\alpha\,m - 626\,n\,\alpha + 626\,\alpha\,s - 256\,\beta^2 + 626\,\beta\,m - 446\,\beta\,n - 626\,\beta\,s - 180\,m^2 \\
&+ 1252\,m\,s + 180\,n^2 - 1252\,n\,s + 872\,\alpha - 872\,\beta + 1744\,m - 1744\,n\big)\,u(s+4) + 2\,(\beta - m \\
&+ n + s + 5)\,(\alpha + m - n + s + 5)\,(\alpha + \beta + 4 + s)\,(\alpha + \beta + 5 + s)\,(\alpha - \beta + 2\,m \\
&- 2\,n)\,u(s+5)
\end{aligned}
$$

Check:

```
> m:=5:n:=3:P:=expand(LaguerreL(m,alpha,x)*LaguerreL(n,beta,x)):
> for deg from m+n by -1 to 0 do ll:=expand(LaguerreL(deg,alpha+beta,
  x));c[deg]:=coeff(P,x,deg)/coeff(ll,x,deg); P:=expand(P-c[deg]*ll)
  od:
> L:=[seq(u(i)=c[i],i=0..m+n)]:
> seq(expand(subs(s=i,L,rec)),i=0..3);
```

$$0, 0, 0, 0$$

```
> m:='m':n:='n':
```

Laguerre * exp (Ismail 2005 Eq. (9.3.4))

```
> f:=exp(-2*x)*LaguerreL(m,0,x)*LaguerreL(n,0,x);
```

$$f := \mathrm{e}^{-2\,x}\,\mathrm{LaguerreL}(m,x)\,\mathrm{LaguerreL}(n,x)$$

```
> FunctiontoRec(f,LaguerreL(k,0,x),u);
FunctiontoRec: time: 0.058 sec.
```

$$
\begin{aligned}
&-2\,(k+1)\,(k+m+n+2)\,u(k) + \big(13\,k^2 + 10\,m\,k + 10\,n\,k + m^2 - 2\,m\,n + n^2 + 53\,k + 17\,m \\
&+ 17\,n + 54\big)\,u(k+1) + \big(-31\,k^2 - 16\,m\,k - 16\,n\,k - m^2 + 2\,m\,n - n^2 - 158\,k - 38\,m
\end{aligned}
$$

$$-38\,n-204\big)\,u(k+2)+2\,(3+k)\,(16\,k+4\,m+4\,n+49)\,u(3+k)-12\,(3+k)\,(k+4)\,u(k+4)$$

```
> f:=exp(-2*a*x)*LaguerreL(n,alpha,x)*LaguerreL(m,alpha,x);
```

$$f := \mathrm{e}^{-2\,a\,x}\,\mathrm{LaguerreL}(n,\alpha,x)\,\mathrm{LaguerreL}(m,\alpha,x)$$

```
> FunctiontoRec(f,LaguerreL(k,alpha,x),u);
FunctiontoRec: time: 0.108 sec.
```

$$-4\,a\,(a-1)\,(2\,a-1)^2\,(3+k)\,(k+2)\,(k+1)\,u(k)+2\,(2\,a-1)\,(3+k)\,(k+2)\,\big(12\,a^3\,\alpha+24\,a^3\,k+60\,a^3-10\,a^2\,\alpha-28\,a^2\,k-72\,a^2+4\,a\,k+2\,a\,m+2\,n\,a+14\,a+k-m-n+1\big)\,u(k+1)-(3+k)\,\big(48\,a^4\,\alpha^2+240\,a^4\,\alpha\,k+240\,a^4\,k^2+720\,a^4\,\alpha+1440\,a^4\,k-32\,a^3\,\alpha^2-256\,a^3\,\alpha\,k-320\,a^3\,k^2+2208\,a^4-784\,a^3\,\alpha-1952\,a^3\,k+44\,a^2\,\alpha\,k+16\,a^2\,\alpha\,m+16\,a^2\,\alpha\,n+84\,a^2\,k^2+32\,a^2\,k\,m+32\,a^2\,k\,n-3040\,a^3+160\,a^2\,\alpha+572\,a^2\,k+96\,a^2\,m+96\,a^2\,n+8\,a\,\alpha\,k-8\,a\,\alpha\,m-8\,a\,\alpha\,n+16\,a\,k^2-24\,a\,k\,m-24\,a\,k\,n+976\,a^2+14\,a\,\alpha+60\,a\,k-74\,a\,m-74\,n\,a-5\,k^2+4\,m\,k+4\,n\,k+m^2-2\,m\,n+n^2+38\,a-25\,k+13\,m+13\,n-30\big)\,u(k+2)+\big(16\,a^4\,\alpha^3+192\,a^4\,\alpha^2\,k+480\,a^4\,\alpha\,k^2+320\,a^4\,k^3+672\,a^4\,\alpha^2+3360\,a^4\,\alpha\,k+3360\,a^4\,k^2-96\,a^3\,\alpha^2\,k-384\,a^3\,\alpha\,k^2-320\,a^3\,k^3+5936\,a^4\,\alpha+11872\,a^4\,k-344\,a^3\,\alpha^2-2736\,a^3\,\alpha\,k-3408\,a^3\,k^2-8\,a^2\,\alpha^2\,k+8\,a^2\,\alpha^2\,m+8\,a^2\,\alpha^2\,n+48\,a^2\,\alpha\,k\,m+48\,a^2\,\alpha\,k\,n+16\,a^2\,k^3+48\,a^2\,k^2\,m+48\,a^2\,k^2\,n+14112\,a^4-4920\,a^3\,\alpha-12208\,a^3\,k-20\,a^2\,\alpha^2+72\,a^2\,\alpha\,k+168\,a^2\,\alpha\,m+168\,a^2\,\alpha\,n+252\,a^2\,k^2+336\,a^2\,k\,m+336\,a^2\,k\,n+24\,a\,\alpha\,k^2-16\,a\,\alpha\,k\,m-16\,a\,\alpha\,k\,n+32\,a\,k^3-24\,a\,k^2\,m-24\,a\,k^2\,n-14704\,a^3+252\,a^2\,\alpha+1180\,a^2\,k+592\,a^2\,m+592\,a^2\,n+148\,a\,\alpha\,k-58\,a\,\alpha\,m-58\,a\,\alpha\,n+306\,a\,k^2-172\,a\,k\,m-172\,a\,k\,n-\alpha\,k^2+\alpha\,m^2-2\,\alpha\,m\,n+\alpha\,n^2-4\,k^3+2\,k^2\,m+2\,k^2\,n+2\,k\,m^2-4\,k\,m\,n+2\,k\,n^2+1736\,a^2+226\,a\,\alpha+974\,a\,k-310\,a\,m-310\,n\,a-6\,\alpha\,k-40\,k^2+15\,m\,k+15\,n\,k+7\,m^2-14\,m\,n+7\,n^2+1030\,a-9\,\alpha-133\,k+28\,m+28\,n-147\big)\,u(3+k)-(\alpha+4+k)\,\big(48\,a^4\,\alpha^2+240\,a^4\,\alpha\,k+240\,a^4\,k^2+960\,a^4\,\alpha+1920\,a^4\,k-96\,a^3\,\alpha\,k-160\,a^3\,k^2+3888\,a^4-400\,a^3\,\alpha-1312\,a^3\,k-28\,a^2\,\alpha\,k+16\,a^2\,\alpha\,m+16\,a^2\,\alpha\,n-36\,a^2\,k^2+32\,a^2\,k\,m+32\,a^2\,k\,n-2720\,a^3-96\,a^2\,\alpha-244\,a^2\,k+128\,a^2\,m+128\,a^2\,n+4\,a\,\alpha\,k+20\,a\,k^2-8\,a\,k\,m-8\,a\,k\,n-408\,a^2+14\,a\,\alpha+152\,a\,k-34\,a\,m-34\,n\,a-k^2+m^2-2\,m\,n+n^2+290\,a-8\,k-16\big)\,u(k+4)+2\,a\,(\alpha+4+k)\,(\alpha+5+k)\,\big(24\,a^3\,\alpha+48\,a^3\,k+216\,a^3-16\,a^2\,k-76\,a^2-2\,a\,\alpha-12\,a\,k+4\,a\,m+4\,n\,a-50\,a+2\,k+9\big)\,u(k+5)-4\,a^2\,(\alpha+4+k)\,(\alpha+5+k)\,(\alpha+6+k)\,(2\,a-1)\,(2\,a+1)\,u(6+k)$$

```
> f:='f':
```

Gegenbauer (sec. 3.4 in Lewanowicz96)

```
> FunctiontoRec(GegenbauerC(i,nu,x)*GegenbauerC(j,nu,x),GegenbauerC
  (k,nu,x),u) assuming nu>-1/2,nu<>0;
```

```
FunctiontoRec: time: 0.287 sec.
```

$$(k+2)\,(k+1)\,(i+j-k)\,(i-j-k-2\,\nu)\,(i+j+k+4\,\nu)\,(i-j+k+2\,\nu)\,(\nu+k+3)\,(\nu+k+4)\,u(k)-2\,(i^4\,k^2+2\,i^4\,k\,\nu+2\,i^4\,\nu^2+4\,i^3\,k^2\,\nu+8\,i^3\,k\,\nu^2+8\,i^3\,\nu^3-2\,i^2\,j^2\,k^2-4\,i^2\,j^2\,k\,\nu-4\,i^2\,j^2\,\nu^2-4\,i^2\,j\,k^2\,\nu-8\,i^2\,j\,k\,\nu^2-8\,i^2\,j\,\nu^3-2\,i^2\,k^4-8\,i^2\,k^3\,\nu-4\,i^2\,k^2\,\nu^2+8\,i^2\,k\,\nu^3+8\,i^2\,\nu^4-4\,i\,j^2\,k^2\,\nu-8\,i\,j^2\,k\,\nu^2-8\,i\,j^2\,\nu^3-8\,i\,j\,k^2\,\nu^2-16\,i\,j\,k\,\nu^3-16\,i\,j\,\nu^4-4\,i\,k^4\,\nu-16\,i\,k^3\,\nu^2-16\,i\,k^2\,\nu^3+j^4\,k^2+2\,j^4\,k\,\nu+2\,j^4\,\nu^2+4\,j^3\,k^2\,\nu+8\,j^3\,k\,\nu^2+8\,j^3\,\nu^3-2\,j^2\,k^4-8\,j^2\,k^3\,\nu-4\,j^2\,k^2\,\nu^2+8\,j^2\,k\,\nu^3+8\,j^2\,\nu^4-4\,j\,k^4\,\nu-16\,j\,k^3\,\nu^2-16\,j\,k^2\,\nu^3+k^6+6\,k^5\,\nu+10\,k^4\,\nu^2-8\,k^2\,\nu^4+4\,i^4\,k+3\,i^4\,\nu+16\,i^3\,k\,\nu+12\,i^3\,\nu^2-8\,i^2\,j^2\,k-6\,i^2\,j^2\,\nu-16\,i^2\,j\,k\,\nu-12\,i^2\,j\,\nu^2-16\,i^2\,k^3-54\,i^2\,k^2\,\nu-28\,i^2\,k\,\nu^2+12\,i^2\,\nu^3-16\,i\,j^2\,k\,\nu-12\,i\,j^2\,\nu^2-32\,i\,j\,k\,\nu^2-24\,i\,j\,\nu^3-32\,i\,k^3\,\nu-108\,i\,k^2\,\nu^2-88\,i\,k\,\nu^3+4\,j^4\,k+3\,j^4\,\nu+16\,j^3\,k\,\nu+12\,j^3\,\nu^2-16\,j^2\,k^3-54\,j^2\,k^2\,\nu-28\,j^2\,k\,\nu^2+12\,j^2\,\nu^3-32\,j\,k^3\,\nu-108\,j\,k^2\,\nu^2-88\,j\,k\,\nu^3+12\,k^5+67\,k^4\,\nu+108\,k^3\,\nu^2+28\,k^2\,\nu^3-32\,k\,\nu^4+3\,i^4+12\,i^3\,\nu-6\,i^2\,j^2-12\,i^2\,j\,\nu-46\,i^2\,k^2-116\,i^2\,k\,\nu-52\,i^2\,\nu^2-12\,i\,j^2\,\nu-24\,i\,j\,\nu^2-92\,i\,k^2\,\nu-232\,i\,k\,\nu^2-128\,i\,\nu^3+3\,j^4+12\,j^3\,\nu-46\,j^2\,k^2-116\,j^2\,k\,\nu-52\,j^2\,\nu^2-92\,j\,k^2\,\nu-232\,j\,k\,\nu^2-128\,j\,\nu^3+59\,k^4+292\,k^3\,\nu+412\,k^2\,\nu^2+128\,k\,\nu^3-32\,\nu^4-56\,i^2\,k-72\,i^2\,\nu-112\,i\,k\,\nu-144\,i\,\nu^2-56\,j^2\,k-72\,j^2\,\nu-112\,j\,k\,\nu-144\,j\,\nu^2+152\,k^3+616\,k^2\,\nu+656\,k\,\nu^2+144\,\nu^3-24\,i^2-48\,i\,\nu-24\,j^2-48\,j\,\nu+216\,k^2+624\,k\,\nu+368\,\nu^2+160\,k+240\,\nu+48)\,(\nu+k+4)\,(\nu+k)\,u(k+2)+(k+2+2\,\nu)\,(k+3+2\,\nu)\,(k+4-j+i)\,(-k-4-j+i)\,(i+j+k+2\,\nu+4)\,(2\,\nu-4-k+j+i)\,(\nu+k+1)\,(\nu+k)\,u(k+4)$$

Lewanowicz 1996 obtains a recurrence of order 2 by differentiation:

```
> st:=time():
> deq:=gfun:-holexprtodiffeq(GegenbauerC(i,nu,x)*GegenbauerC(j,nu,x),
  y(x),false):
> collect(DiffeqtoRec(diff(deq,x),y(x),u(k),GegenbauerC(k,nu,x)),u,
  factor) assuming nu>-1/2,nu<>0;
```

$$(\nu+k+2)\,(k+2)\,(k+1)\,(i+j-k)\,(i-j-k-2\,\nu)\,(i+j+k+4\,\nu)\,(i-j+k+2\,\nu)\,u(k)-(\nu+k)\,(k+1+2\,\nu)\,(k+2\,\nu)\,(k+2-j+i)\,(-k-2-j+i)\,(i+j+k+2\,\nu+2)\,(2\,\nu-2-k+j+i)\,u(k+2)$$

```
> time()-st;
```

0.302

Special case for nu=1:

```
> DiffeqtoRec(subs(nu=1,diff(deq,x)),y(x),u(k),GegenbauerC(k,1,x))
  assuming nu>-1/2,nu<>0 assuming i::nonnegint,j::nonnegint,
```

```
k::nonnegint,i+j<=k;
```

$$(i+j-k)\,(u(k)-u(k+2))$$

Jacobi

```
> FunctiontoRec(JacobiP(i,alpha,beta,x)*JacobiP(j,alpha,beta,x),
  JacobiP(k,alpha,beta,x),u) assuming alpha>-1,beta>-1,i::nonnegint,
  j::nonnegint;
```

```
FunctiontoRec: time: 8.612 sec.
```

[Length of output exceeds limit of 50000]

Use the derivative instead:

```
> st:=time():
> deq:=gfun:-holexprtodiffeq(JacobiP(i,alpha,beta,x)*JacobiP(j,alpha,
  beta,x),y(x),false):
> collect(DiffeqtoRec(diff(deq,x),y(x),u(k),JacobiP(k,alpha,beta,x)),
  u,factor) assuming alpha>-1,beta>-1;
```

$$-(k+\alpha+\beta+1)\,(k+2)\,(k+1)\,(i+j-k)\,(k+1+j-i+\beta+\alpha)\,(2\alpha+2\beta+i+j+k+2)\,(k+1-j+i+\beta+\alpha)\,(\alpha+\beta+2k+4)\,(\alpha+\beta+2k+5)\,u(k)-(k+\alpha+\beta+1)\,(k+2)\,(\alpha-\beta)\,(\alpha^2 i^2-2\alpha^2 ij+2\alpha^2 ik+\alpha^2 j^2+2\alpha^2 jk-3\alpha^2 k^2+2\alpha\beta i^2-4\alpha\beta ij+4\alpha\beta ik+2\alpha\beta j^2+4\alpha\beta jk-6\alpha\beta k^2+2\alpha i^3-2\alpha i^2 j+2\alpha i^2 k-2\alpha ij^2+2\alpha ik^2+2\alpha j^3+2\alpha j^2 k+2\alpha jk^2-6\alpha k^3+\beta^2 i^2-2\beta^2 ij+2\beta^2 ik+\beta^2 j^2+2\beta^2 jk-3\beta^2 k^2+2\beta i^3-2\beta i^2 j+2\beta i^2 k-2\beta ij^2+2\beta ik^2+2\beta j^3+2\beta j^2 k+2\beta jk^2-6\beta k^3+i^4-2i^2 j^2+2i^2 k^2+j^4+2j^2 k^2-3k^4+3\alpha^2 i+3\alpha^2 j-7\alpha^2 k+6\alpha\beta i+6\alpha\beta j-14\alpha\beta k+5\alpha i^2-4\alpha ij+8\alpha ik+5\alpha j^2+8\alpha jk-25\alpha k^2+3\beta^2 i+3\beta^2 j-7\beta^2 k+5\beta i^2-4\beta ij+8\beta ik+5\beta j^2+8\beta jk-25\beta k^2+2i^3-2i^2 j+6i^2 k-2ij^2+2ik^2+2j^3+6j^2 k+2jk^2-18k^3-4\alpha^2-8\alpha\beta+7\alpha i+7\alpha j-33\alpha k-4\beta^2+7\beta i+7\beta j-33\beta k+5i^2-2ij+6ik+5j^2+6jk-39k^2-14\alpha-14\beta+4i+4j-36k-12)\,(\alpha+\beta+2k+5)\,(\alpha+\beta+2k+1)\,u(k+1)-(k+\alpha+\beta+1)\,(\alpha+\beta+2k+1)\,(2k+2+\alpha+\beta)\,(k+2-j+i)\,(-k-2-j+i)\,(\beta+k+2)\,(\alpha+2+k)\,(\alpha+\beta+i+j+k+3)\,(-k-1+j+i+\beta+\alpha)\,u(k+2)$$

```
> time()-st;
```

$$9.763$$

Generalized Linearization Problems (Ronveaux, Hounkonnou, Belmehdi 1995)

```
> oldinfo:=infolevel['FractionsOfRecurrenceOperators']:
> infolevel['FractionsOfRecurrenceOperators']:=2:
> linquad:=proc(p1,p2,p3,u) #i,j,k are global
```

```
  local res,st,order,ii,deq,rec;
      print(sprintf("Expansion of the product %a * %a in the basis
  %a",p1,subs(i=j,p2),subs(i=k,p3)));
      FunctiontoRec(p1*subs(i=j,p2),subs(i=k,p3),u);
  end:
> L:=[HermiteH(i,x),LaguerreL(i,alpha,x),JacobiP(i,a,b,x)]:
> linquad(L[2],L[2],L[1],u):
  linquad(L[1],L[1],L[2],u):
  linquad(L[1],L[1],L[3],u):
  linquad(L[2],L[2],L[3],u):
  linquad(L[3],L[3],L[1],u):
  linquad(L[3],L[3],L[2],u):
  linquad(L[1],L[3],L[1],u):
  linquad(L[2],L[3],L[2],u):
```

"Expansion of the product LaguerreL(i,alpha,x) * LaguerreL(j,alpha,x) in the basis HermiteH(k,x)"

```
DiffoptoPair: pair with degrees 9/2
DiffoptoRec: order recurrence: 9
FunctiontoRec: time: 0.33 sec.
```

"Expansion of the product HermiteH(i,x) * HermiteH(j,x) in the basis LaguerreL(k,alpha,x)"

```
DiffoptoPair: pair with degrees 9/6
DiffoptoRec: order recurrence: 9
FunctiontoRec: time: 0.041 sec.
```

"Expansion of the product HermiteH(i,x) * HermiteH(j,x) in the basis JacobiP(k,a,b,x)"

```
DiffoptoPair: pair with degrees 12/10
DiffoptoRec: order recurrence: 12
FunctiontoRec: time: 1.433 sec.
```

"Expansion of the product LaguerreL(i,alpha,x) * LaguerreL(j,alpha,x) in the basis JacobiP(k,a,b,x)"

```
DiffoptoPair: pair with degrees 12/10
DiffoptoRec: order recurrence: 12
FunctiontoRec: time: 2.082 sec.
```

"Expansion of the product JacobiP(i,a,b,x) * JacobiP(j,a,b,x) in the basis HermiteH(k,x)"

```
DiffoptoPair: pair with degrees 12/2
DiffoptoRec: order recurrence: 12
FunctiontoRec: time: 0.106 sec.
```

"Expansion of the product JacobiP(i,a,b,x) * JacobiP(j,a,b,x) in the basis LaguerreL(k,alpha,x)"

```
DiffoptoPair: pair with degrees 12/6
DiffoptoRec: order recurrence: 12
FunctiontoRec: time: 0.181 sec.
```

"Expansion of the product HermiteH(i,x) * JacobiP(j,a,b,x) in the basis HermiteH(k,x)"

```
DiffoptoPair: pair with degrees 20/10
DiffoptoRec: order recurrence: 20
FunctiontoRec: time: 0.802 sec.
```

"Expansion of the product LaguerreL(i,alpha,x) * JacobiP(j,a,b,x) in the basis LaguerreL(k,alpha,x)"

```
DiffoptoPair: pair with degrees 22/12
DiffoptoPair: order denom left horner: 4
DiffoptoPair: reduced pair with degrees 20/10
```

```
DiffoptoRec: order recurrence: 20
FunctiontoRec: time: 4.893 sec.
> infolevel['FractionsOfRecurrenceOperators']:=oldinfo:
> L:='L':
```

Other relations between polynomials (Sec. 9.1.3)

Chebyshev (Godoy et al. 1997, 4.1.1.3)

```
> FunctiontoRec(ChebyshevT(n,a*t+b),ChebyshevT(m,t),u);
FunctiontoRec: time: 0.025 sec.
```

$$a^2\,(3+m)\,(m-n)\,(n+m)\,u(m)+2\,(1+m)\,(3+m)\,a\,b\,(2\,m+1)\,u(1+m)+2\,(m+2)\,\left(a^2\,m^2+n^2\,a^2+2\,b^2\,m^2+4\,a^2\,m+8\,b^2\,m+2\,a^2+6\,b^2-2\,m^2-8\,m-6\right)\,u(m+2)+2\,(1+m)\,(3+m)\,a\,b\,(2\,m+7)\,u(3+m)+(1+m)\,a^2\,(m+n+4)\,(m+4-n)\,u(m+4)$$

Hermite (Godoy et al. 1997, 4.1.1.1) (also in Feldheim)

Addition formula (2.3 in Feldheim 1938)

```
> rec:=FunctiontoRec(HermiteH(n,t+y),HermiteH(m,t),u); # Feldheim
  (2.3)
FunctiontoRec: time: 0.014 sec.
```

$$rec := (-m+n)\,u(m)-2\,(1+m)\,y\,u(1+m)$$

Initial condition from the leading coefficient.

```
> rsolve(rec,u(m)) assuming n::posint,m::posint,n<=m;
```

$$\frac{\Gamma(n+1)\,2^{-m}\,y^{-m}\,u(0)}{\Gamma(1-m+n)\,\Gamma(1+m)}$$

```
> eval(%,m=n)=1;
```

$$2^{-n}\,y^{-n}\,u(0)=1$$

```
> co:=subs(isolate(%,u(0)),%%);
```

$$co := \frac{\Gamma(n+1)\,2^{-m}\,y^{-m}}{\Gamma(1-m+n)\,\Gamma(1+m)\,2^{-n}\,y^{-n}}$$

```
> convert(%,binomial);
```

$$\frac{2^{-m}\,y^{-m}\,\binom{n}{m}}{2^{-n}\,y^{-n}}$$

```
> combine(%,power);
```

$$2^{-m+n}\,y^{-m+n}\,\binom{n}{m}$$

Product formula (eq. (4) in Feldheim 1940)

```
> rec:=FunctiontoRec(HermiteH(n,y/a),HermiteH(nu,y),u);
FunctiontoRec: time: 0.017 sec.
```

$$rec := (-\nu+n)\,u(\nu)+2\,(1+\nu)\,(a-1)\,(a+1)\,(2+\nu)\,u(2+\nu)$$

Initial condition from the leading term, so we revert the recurrence:

```
> eval(rec,u=proc(nu) v(n-nu) end);
```

$$(-\nu+n)\,v(-\nu+n)+2\,(1+\nu)\,(a-1)\,(a+1)\,(2+\nu)\,v(n-2-\nu)$$

```
> subs(nu=n-k,%);
```

$$k\,v(k)+2\,(1-k+n)\,(a-1)\,(a+1)\,(2+n-k)\,v(k-2)$$

```
> rsolve({%,v(0)=1/a^n,v(1)=0},v(k)) assuming k<n,n::posint;
```

$$\begin{cases} \dfrac{(-1)^{\frac{k}{2}}\,(a-1)^{\frac{k}{2}}\,(a+1)^{\frac{k}{2}}\,\Gamma(n+1)}{\Gamma(1-k+n)\,\Gamma\left(\frac{k}{2}+1\right)\,a^n} & k::even \\ 0 & k::odd \end{cases}$$

```
> eval(%,k=2*r) assuming r::nonnegint;
```

$$\frac{(-1)^r\,(a-1)^r\,(a+1)^r\,\Gamma(n+1)}{\Gamma(1-2\,r+n)\,\Gamma(r+1)\,a^n}$$

Inversion formula (Feldheim 1940, eq. (4b))

```
> rec:=FunctiontoRec(2^n*x^n,HermiteH(k,x),u);
FunctiontoRec: time: 0.015 sec.
```

$$rec := (k-n)\,u(k)+2\,(k+2)\,(k+1)\,u(k+2)$$

Again, invert to reason on the leading coefficient

```
> eval(rec,u=proc(nu) v(n-nu) end);
```

$$(k-n)\,v(n-k)+2\,(k+2)\,(k+1)\,v(n-k-2)$$

```
> subs(k=n-k,%);
```

$$-k\,v(k)+2\,(2+n-k)\,(1-k+n)\,v(k-2)$$

```
> rsolve({%,v(0)=1,v(1)=0},v(k)) assuming k<n,n::posint;
```

$$\begin{cases} \dfrac{\Gamma(n+1)}{\Gamma(1-k+n)\,\Gamma\left(\frac{k}{2}+1\right)} & k::even \\ 0 & k::odd \end{cases}$$

```
> eval(%,k=2*r) assuming r::nonnegint;
```

$$\frac{\Gamma(n+1)}{\Gamma(1-2\,r+n)\,\Gamma(r+1)}$$

```
> FunctiontoRec(HermiteH(n,t^2),HermiteH(m,t),u); # Feldheim 1940 eq.
  (45)
FunctiontoRec: time: 0.026 sec.
```

$$(-m+2\,n)\,u(m)+\left(-8\,m^2+12\,m\,n-36\,m+36\,n-40\right)u(m+2)-4\,(m+4)\,\left(6\,m^2-6\,m\,n+41\,m-24\,n+73\right)u(m+4)-8\,(m+4)\,(5+m)\,(m+6)\,(4\,m-2\,n+17)\,u(m+6)-16\,(m+4)\,(5+m)\,(m+6)\,(7+m)\,(m+8)\,u(m+8)$$

```
> FunctiontoRec(x^k*HermiteH(n,x),HermiteH(m,x),u); # Godoy et al.
  1997, eq. (27)
```

```
FunctiontoRec: time: 0.021 sec.
```

$$(k+n-m)\,u(m)+\left(2k^2-4m^2+4mn+4k-18m+10n-20\right)u(m+2)-4(3+m)(m+4)(k+m-n+4)\,u(m+4)$$

Another product (eq. (68))

```
> rec:=FunctiontoRec(HermiteH(n,(y+z)/sqrt(2))*HermiteH(n,(y-z)/sqrt
  (2)),HermiteH(k,z),u);
FunctiontoRec: time: 0.099 sec.
```

$$rec := (2n-k)\,u(k)+\left(4ky^2+2k^2-4nk+8y^2+4n-8\right)u(k+2)+4(k+4)\left(2ky^2+k^2-2nk+10y^2+12k-16n+29\right)u(k+4)-8(k+4)(k+5)(6+k)(k-2n+5)\,u(6+k)$$

Monomial to Laguerre (Eq. (13b) in Feldheim (1940a))

```
> rec:=FunctiontoRec(x^n,LaguerreL(m,alpha,x),u);
FunctiontoRec: time: 0.377 sec.
```

$$rec := (m-n)\,u(m)+\left(-\alpha-1-m\right)u(1+m)$$

Initial conditions are easily obtained for the leading coefficient, so we revert the recurrence:

```
> eval(rec,u=proc(nu) v(n-nu) end);
```

$$(m-n)\,v(-m+n)+\left(-\alpha-1-m\right)v(n-1-m)$$

```
> subs(m=n-k,%);
```

$$-k\,v(k)+\left(-\alpha-1-n+k\right)v(k-1)$$

```
> rec2:=%:
> rsolve({rec2,v(0)=1/lcoeffLn},v(k)) assuming n::posint,alpha>-1;
```

$$\frac{\Gamma(n+\alpha+1)\,(-1)^k\,n!}{\Gamma(1+\alpha+n-k)\,\Gamma(k+1)\,(-1)^n}$$

```
> convert(%,binomial)
```

$$\frac{n!\,(-1)^k\binom{n+\alpha}{k}}{(-1)^n}$$

Relation Laguerre-Hermite (https://dlmf.nist.gov/18.7.E19)

```
> rec:=FunctiontoRec(HermiteH(2*n,sqrt(x)),LaguerreL(m,-1/2,x),u);
FunctiontoRec: time: 0.027 sec.
```

$$rec := (m-n)\,u(m)$$

This means that the coefficient is 0 except for n=m, where it is

```
> subs(n=2*n,lcoeffHn)/lcoeffLn;
```

$$\frac{2^{2n}\,n!}{(-1)^n}$$

Same in the other direction

```
> rec:=FunctiontoRec(LaguerreL(n,-1/2,y^2),HermiteH(k,y),u);
```

```
FunctiontoRec: time: 0.016 sec.
```

$$rec := (-2n+k)\,u(k)$$

Variant – p. 238 in Feldheim (1940a)

```
> rec:=FunctiontoRec(y*LaguerreL(n,-1/2,y^2),HermiteH(k,y),u);
```

```
FunctiontoRec: time: 0.019 sec.
```

$$rec := (2n+1-k)\,u(k) + \left(-4k^2+8nk-18k+20n-14\right)u(k+2) - 4\,(3+k)\,(k+4)\,(k-2n+5)\,u(k+4)$$

```
> subs(k=2*n+1-k,rec);
```

$$k\,u(2n+1-k) + \left(-4\,(2n+1-k)^2+8n\,(2n+1-k)-16n-32+18k\right)u(2n+3-k) - 4\,(4+2n-k)\,(2n+5-k)\,(6-k)\,u(2n+5-k)$$

```
> eval(%,u=proc(i) v(2*n+1-i) end);
```

$$k\,v(k) + \left(-4\,(2n+1-k)^2+8n\,(2n+1-k)-16n-32+18k\right)v(k-2) - 4\,(4+2n-k)\,(2n+5-k)\,(6-k)\,v(k-4)$$

```
> rec2:=%:
> ini:={v(0)=lcoeffLn/subs(n=2*n+1,lcoeffHn)};
```

$$ini := \left\{v(0) = \frac{(-1)^n}{n!\,2^{1+2n}}\right\}$$

```
> rsolve({rec2} union ini,v(k)) assuming k::even,2*n+1>=k:
> subs(sin(Pi*n)=0,%);
```

$$\begin{cases} \dfrac{(-1)^n}{2\,n!\,\left(2^n\right)^2} & \dfrac{k}{2}=0 \\ v(2) & \dfrac{k}{2}=1 \\ \dfrac{\left(4n^2-1\right)(2n-3)\left(\dfrac{v(2)}{2}-\dfrac{n\,(-1)^n\,4^{-n}}{\Gamma(n+1)}\right)4^{\frac{k}{2}}\,\Gamma\left(-n+\dfrac{k}{2}-\dfrac{1}{2}\right)\left(2n-\dfrac{k}{2}+1\right)}{4\,\Gamma\left(-n+\dfrac{5}{2}\right)k\left(\dfrac{k}{2}-1\right)} & otherwise \end{cases}$$

Scaled Hermite to Laguerre – Eq. (41) in Feldheim (1940a)

```
> rec:=FunctiontoRec(HermiteH(n,y/mu),LaguerreL(m,alpha,y),u);
```

```
FunctiontoRec: time: 0.021 sec.
```

$$rec := (2m - 2n)\,u(m) + (-2\alpha - 6 - 6m + 4n)\,u(1+m) + (-\mu^2 + 4\alpha + 6m - 2n + 12)\,u(m+2) + (-2\alpha - 6 - 2m)\,u(3+m)$$

Check:

```
> n:=10:P:=expand(HermiteH(n,y/mu)):
> for deg from n by -1 to 0 do ll:=expand(LaguerreL(deg,alpha,y));c
  [deg]:=coeff(P,y,deg)/coeff(ll,y,deg); P:=expand(P-c[deg]*ll) od:
> L:=[seq(u(i)=c[i],i=0..n)]:
> seq(expand(subs(m=i,L,rec)),i=0..7);
```

$$0, 0, 0, 0, 0, 0, 0, 0$$

```
> n:='n':
```

Same in the other direction – Eq. (42)

```
> rec:=FunctiontoRec(LaguerreL(n,alpha,lambda*y),HermiteH(m,y),u);
```

```
FunctiontoRec: time: 0.016 sec.
```

$$rec := -\lambda\,(m-n)\,u(m) + 2\,(\alpha + 1 + m)\,(1+m)\,u(1+m) - 2\,(1+m)\,\lambda\,(m+2)\,u(m+2) + 4\,(1+m)\,(m+2)\,(3+m)\,u(3+m)$$

Check:

```
> n:=10:P:=expand(LaguerreL(n,alpha,lambda*y)):
> for deg from n by -1 to 0 do ll:=expand(HermiteH(deg,y));c[deg]:=
  coeff(P,y,deg)/coeff(ll,y,deg); P:=expand(P-c[deg]*ll) od:
> L:=[seq(u(i)=c[i],i=0..n)]:
> seq(expand(subs(m=i,L,rec)),i=0..7);
```

$$0, 0, 0, 0, 0, 0, 0, 0$$

```
> n:='n':
```

Monomials to Jacobi & Gegenbauer (Luke 1969, chap. 8)

Eq (1) p. 274

```
> lambda:=alpha+beta+1;
```

$$\lambda := \alpha + \beta + 1$$

```
> hnJ:=2^(alpha+beta+1)*GAMMA(n+alpha+1)*GAMMA(n+beta+1)/(2*n+alpha+
  beta+1)/GAMMA(n+alpha+beta+1)/n!;
```

$$hnJ := \frac{2^{\alpha+\beta+1}\,\Gamma(n+\alpha+1)\,\Gamma(n+\beta+1)}{(2n+\alpha+\beta+1)\,\Gamma(n+\alpha+\beta+1)\,n!}$$

(28) p. 277

```
> rec:=FunctiontoRec(x^m,JacobiP(n,alpha,beta,x),u);
```

```
FunctiontoRec: time: 0.343 sec.
```

$$rec := 2\,(\alpha+\beta+2n+4)\,(\alpha+\beta+2n+5)\,(n+2+\alpha+\beta)\,(n+\alpha+\beta+1)\,(m-n)\,u(n) + (2n+\alpha+\beta+1)\,(\alpha+\beta+2n+5)\,(n+2+\alpha+\beta)\,(\alpha-\beta)\,(2+2m+\alpha+\beta)\,u(n$$

$$+1)-2\,(n+2+\beta)\,(n+2+\alpha)\,(2n+\alpha+\beta+1)\,(2n+2+\alpha+\beta)\,(m+3+n+\alpha+\beta)\,u(n+2)$$

Check

```
> sol:=2^n*m!*GAMMA(n+lambda)/(m-n)!/GAMMA(2*n+lambda)*hypergeom([n-
  m,alpha+n+1],[2*n+lambda+1],2);
```

$$sol := \frac{2^n\, m!\,\Gamma(n+\alpha+\beta+1)\,\mathrm{hypergeom}([-m+n,\,n+\alpha+1],[2n+2+\alpha+\beta],2)}{(m-n)!\,\Gamma(2n+\alpha+\beta+1)}$$

```
> simplify(eval(rec,u=unapply(sol,n)));
```

$$0$$

(30) p. 277

```
> rec:=FunctiontoRec((1-x)^m,JacobiP(n,alpha,beta,x),u);
```

```
FunctiontoRec: time: 0.356 sec.
```

$$rec := -(\alpha+\beta+2n+3)\,(n+\alpha+\beta+1)\,(m-n)\,u(n)-(n+\alpha+1)\,(2n+\alpha+\beta+1)\,(n+\alpha+\beta+2+m)\,u(n+1)$$

Initial condition:

```
> Int((1-x)^m*(1-x)^alpha*(1+x)^beta,x=-1..1);
```

$$\int_{-1}^{1}(1-x)^m\,(1-x)^\alpha\,(1+x)^\beta\,dx$$

```
> IntegrationTools[Change](%,x=2*u-1);
```

$$2^{1+\beta}\int_0^1 u^\beta\,(2-2u)^{\alpha+m}\,du$$

this is a beta integral

```
> simplify(value(%)/eval(hnJ,n=0));
```

$$\frac{2^m\,\Gamma(\alpha+\beta+2)\,\Gamma(\alpha+1+m)}{\Gamma(\alpha+\beta+m+2)\,\Gamma(1+\alpha)}$$

Explicit form for the coefficient:

```
> C:=rsolve({rec,u(0)=%},u(n)) assuming m::posint;
```

$$C := \frac{2^m\,(2n+\alpha+\beta+1)\,(-1)^n\,\Gamma(n+\alpha+\beta+1)\,\Gamma(\alpha+1+m)\,\Gamma(1+m)}{\Gamma(n+\alpha+\beta+2+m)\,\Gamma(n+\alpha+1)\,\Gamma(1+m-n)}$$

```
> lambda:='lambda':
```

(29) p. 277

```
> hnG:=2^(1-2*lambda)*Pi*GAMMA(n+2*lambda)/(n+lambda)/GAMMA(lambda)
  ^2/n!;
```

$$hnG := \frac{2^{1-2\lambda}\,\pi\,\Gamma(2\lambda+n)}{(\lambda+n)\,\Gamma(\lambda)^2\,n!}$$

```
> rec:=FunctiontoRec(x^m,GegenbauerC(n,lambda,x),u);
```

```
FunctiontoRec: time: 0.033 sec.
```

$$rec := (\lambda+n+2)\,(m-n)\,u(n)-(\lambda+n)\,(m+2+2\lambda+n)\,u(n+2)$$

Initial condition for even m: write x^(2m)=(1-x^2-1)^m, expand and sum termwise:

```
> Int((1-x^2)^(lambda+i-1/2),x=-1..1);
```

$$\int_{-1}^{1} \left(-x^2+1\right)^{\lambda+i-\frac{1}{2}}\,dx$$

```
> IntegrationTools[Change](%,x=2*u-1);
```

$$2\int_{0}^{1} \left(-4\,u^2+4\,u\right)^{\lambda+i-\frac{1}{2}}\,du$$

```
> value(%)/eval(hnG,n=0);
```

$$\frac{\Gamma\left(\lambda+i+\frac{1}{2}\right)\lambda\,\Gamma(\lambda)^2}{\sqrt{\pi}\,\Gamma(\lambda+i+1)\,2^{1-2\lambda}\Gamma(2\,\lambda)}$$

```
> sum(binomial(m,i)*(-1)^(m-i)*%,i=0..m);
```

$$\frac{(-1)^m\binom{2\,m}{m}}{4^m\binom{\lambda+m}{m}}$$

```
> rsolve({subs(m=2*m,rec),u(0)=%,u(1)=0},u(n)) assuming m::posint;
```

$$\begin{cases} \dfrac{(-1)^m\,(\lambda+n)\,4^{-m}\,\Gamma(\lambda)\,\Gamma(2\,m+1)}{\Gamma\left(\frac{n}{2}+\lambda+m+1\right)\Gamma\left(1-\frac{n}{2}+m\right)} & n::even \\ 0 & n::odd \end{cases}$$

```
> eval(%,n=2*n) assuming n::posint;
```

$$\frac{(-1)^m\,(\lambda+2\,n)\,4^{-m}\,\Gamma(\lambda)\,\Gamma(2\,m+1)}{\Gamma(n+\lambda+m+1)\,\Gamma(1+m-n)}$$

```
> lambda:='lambda':
```

Generating Functions (Sec. 9.2)

Hermite (Feldheim 1938, 1.1)

```
> rec:=FunctiontoRec(exp(2*a*x-a^2),HermiteH(nu,x),u);
FunctiontoRec: time: 0.013 sec.
```

$$rec := -2\,a\,u(\nu)+2\,(1+\nu)\,u(1+\nu)$$

Initial condition:

```
> int(exp(-x^2)*exp(2*a*x-a^2),x=-infinity..infinity)/eval(hnH,n=0);
```

$$1$$

Coefficient:

```
> rsolve({rec,u(0)=%},u(nu));
```

$$\frac{a^{\nu}}{\Gamma(1+\nu)}$$

Hermite 2 (PrBrMa86 5.12.1.1)

```
> F:=(1+4*t^2)^(-3/2)*(1+2*x*t+4*t^2)*exp(4*x^2*t^2/(1+4*t^2));
```

$$F := \frac{\left(4t^2+2xt+1\right) e^{\frac{4x^2t^2}{4t^2+1}}}{\left(4t^2+1\right)^{3/2}}$$

```
> rec:=FunctiontoRec(F,HermiteH(k,x),u);
FunctiontoRec: time: 0.031 sec.
```

$$rec := -4t^3u(k) - 4t^2\left(4t^2+1\right)u(k+1) - 2t\left(4kt^2+16t^2-k-1\right)u(k+2) + 2\left(4t^2+1\right)(3+k)u(3+k) + 4t(3+k)(k+4)u(k+4)$$

```
> int(exp(-x^2)*F,x=-infinity..infinity)/eval(hnH,n=0) assuming t>0,
  t<1/100;
```

$$1$$

```
> int(exp(-x^2)*HermiteH(1,x)*F,x=-infinity..infinity)/eval(hnH,n=1)
  assuming t>0,t<1/100;
```

$$t$$

```
> int(exp(-x^2)*HermiteH(2,x)*F,x=-infinity..infinity)/eval(hnH,n=2)
  assuming t>0,t<1/100;
```

$$t^2$$

```
> int(exp(-x^2)*HermiteH(3,x)*F,x=-infinity..infinity)/eval(hnH,n=3)
  assuming t>0,t<1/100;
```

$$t^3$$

```
> pp:=gfun:-rectoproc({rec,seq(u(i)=t^i,i=0..3)},u(k),list,evalfun=
  expand):
> LL:=pp(20);
```

$$LL := \left[1, t, t^2, t^3, \frac{t^4}{2}, \frac{t^5}{2}, \frac{t^6}{6}, \frac{t^7}{6}, \frac{t^8}{24}, \frac{t^9}{24}, \frac{t^{10}}{120}, \frac{t^{11}}{120}, \frac{t^{12}}{720}, \frac{t^{13}}{720}, \frac{t^{14}}{5040}, \frac{t^{15}}{5040}, \frac{t^{16}}{40320}, \frac{t^{17}}{40320}, \frac{t^{18}}{362880}, \frac{t^{19}}{362880}, \frac{t^{20}}{3628800}\right]$$

```
> LREtools[MinimalRecurrence]({rec,seq(u(i)=t^i,i=0..3)},u(k));
```

$$(k+2)(k-1)u(k+2) + 2tu(k+1) - 2kt^2u(k) = 0, \text{"Relation holds for"}, 0 \le k, \text{"Initial terms"}, \{u(0)=1, u(1)=t, u(3)=t^3\}$$

Gegenbauer (PrBrMa86 5.13.1.1)

```
> F:=(1-2*t*x+t^2)^(-nu);
```

$$F := \left(t^2-2xt+1\right)^{-\nu}$$

```
> gfun:-holexprtodiffeq(F,y(x));
```

$$\left\{\left(t^2-2\,x\,t+1\right)\left(\frac{\mathrm{d}}{\mathrm{d}x}\,y(x)\right)-2\,\nu\,t\,y(x),y(0)=\left(t^2+1\right)^{-\nu}\right\}$$

```
> rec:=FunctiontoRec(F,GegenbauerC(k,nu,x),u) assuming nu>-1/2,nu<>0;
FunctiontoRec: time: 0.042 sec.
```

$$rec:=2\,t\,u(k)+2\left(-t^2-1\right)u(k+1)+2\,t\,u(k+2)$$

```
> LREtools[hypergeomsols](rec,u(k),{},output=basis);
```

$$\left[t^k,\left(\frac{1}{t}\right)^k\right]$$

Only one solution corresponds to generating functions.

Gegenbauer 2 (PrBrMa86 5.13.1.3)

```
> F:=GAMMA(nu+1/2)*exp(x*t)*(t/2*sqrt(1-x^2))^(1/2-nu)*BesselJ
  (nu-1/2,t*sqrt(1-x^2));
```

$$F:=\Gamma\left(\nu+\frac{1}{2}\right)\mathrm{e}^{x\,t}\left(\frac{t\sqrt{-x^2+1}}{2}\right)^{\frac{1}{2}-\nu}\mathrm{BesselJ}\left(\nu-\frac{1}{2},t\sqrt{-x^2+1}\right)$$

```
> rec:=FunctiontoRec(F,GegenbauerC(k,nu,x),u) assuming nu>-1/2,nu<>0;
FunctiontoRec: time: 0.142 sec.
```

$$\begin{aligned}rec:=&-\left(\nu+k+3\right)\left(k+2\right)\left(k+1\right)t\,u(k)+\left(\nu+k+3\right)\left(k+2\right)\left(k^2+2\,k\,\nu-t^2+k\right.\\&\left.+2\,\nu\right)u(k+1)-2\,t\left(k^2+2\,k\,\nu+2\,\nu^2+4\,k+3\,\nu+3\right)u(k+2)+\left(\nu+k+1\right)\left(k+2\right.\\&\left.+2\,\nu\right)\left(k^2+2\,k\,\nu-t^2+7\,k+8\,\nu+12\right)u(3+k)+\left(\nu+k+1\right)\left(k+2+2\,\nu\right)\left(k+3\right.\\&\left.+2\,\nu\right)t\,u(k+4)\end{aligned}$$

```
> LREtools[hypergeomsols](rec,u(k),{},output=basis);
```

$$\left[\frac{t^k}{\Gamma\left(k+2\,\nu\right)},\left(-\frac{1}{t}\right)^k\Gamma\left(k+1\right)\right]$$

Gegenbauer 3 (PrBrMa86 5.13.1.5)

```
> F:=(1-t*x)^(-mu)*hypergeom([mu/2,(mu+1)/2],[nu+1/2],t^2*(x^2-1)/(1-
  t*x)^2);
```

$$F:=\left(-x\,t+1\right)^{-\mu}\mathrm{hypergeom}\left(\left[\frac{\mu}{2},\frac{\mu}{2}+\frac{1}{2}\right],\left[\nu+\frac{1}{2}\right],\frac{t^2\left(x^2-1\right)}{\left(-x\,t+1\right)^2}\right)$$

```
> rec:=FunctiontoRec(F,GegenbauerC(k,nu,x),u) assuming nu>-1/2,nu<>0;
FunctiontoRec: time: 0.572 sec.
```

$$\begin{aligned}rec:=&\left(\nu+k+3\right)\left(k+2\right)\left(k+1\right)\left(k+\mu\right)t\,u(k)-\left(\nu+k+3\right)\left(k+2\right)\left(3\,k^2\,t^2+2\,k\,\mu\,t^2\right.\\&\left.+4\,k\,\nu\,t^2-\mu^2\,t^2+4\,\mu\,\nu\,t^2+7\,k\,t^2+3\,\mu\,t^2+4\,\nu\,t^2+k^2+2\,k\,\nu+4\,t^2+k+2\,\nu\right)u(k+1)\\&+2\,t\left(k^4\,t^2+4\,k^3\,\nu\,t^2-k^2\,\mu^2\,t^2+2\,k^2\,\mu\,\nu\,t^2+5\,k^2\,\nu^2\,t^2-2\,k\,\mu^2\,\nu\,t^2+4\,k\,\mu\,\nu^2\,t^2+2\,k\,\nu^3\,t^2\right.\\&\left.-\mu^2\,\nu^2\,t^2+2\,\mu\,\nu^3\,t^2+8\,k^3\,t^2+24\,k^2\,\nu\,t^2-4\,k\,\mu^2\,t^2+8\,k\,\mu\,\nu\,t^2+20\,k\,\nu^2\,t^2-4\,\mu^2\,\nu\,t^2\right.\end{aligned}$$

$$+ 8\,\mu\,\nu^2\,t^2 + 4\,\nu^3\,t^2 + 2\,k^4 + 8\,k^3\,\nu + 10\,k^2\,\nu^2 + 23\,k^2\,t^2 + 4\,k\,\nu^3 + 46\,k\,\nu\,t^2 - 3\,\mu^2\,t^2 + 6\,\mu\,\nu\,t^2 + 20\,\nu^2\,t^2 + 16\,k^3 + k^2\,\mu + 49\,k^2\,\nu + 2\,k\,\mu\,\nu + 42\,k\,\nu^2 + 28\,k\,t^2 + 2\,\mu\,\nu^2 + 8\,\nu^3 + 28\,\nu\,t^2 + 46\,k^2 + 4\,k\,\mu + 96\,k\,\nu + 3\,\mu\,\nu + 44\,\nu^2 + 12\,t^2 + 56\,k + 3\,\mu + 60\,\nu + 24\big)\,u(k+2) - (\nu + k + 1)\,(k + 2 + 2\,\nu)\,\big(3\,k^2\,t^2 - 2\,k\,\mu\,t^2 + 8\,k\,\nu\,t^2 - \mu^2\,t^2 + 4\,\nu^2\,t^2 + 17\,k\,t^2 - 5\,\mu\,t^2 + 22\,\nu\,t^2 + k^2 + 2\,k\,\nu + 24\,t^2 + 7\,k + 8\,\nu + 12\big)\,u(3+k) + (\nu + k + 1)\,(k + 3 + 2\,\nu)\,(k + 2 + 2\,\nu)\,(k - \mu + 2\,\nu + 4)\,t\,u(k+4)$$

```
> LREtools[hypergeomsols](rec,u(k),{},output=basis);
```

$$\left[\frac{t^k\,\Gamma(k+\mu)}{\Gamma(k+2\,\nu)},\ \frac{\left(\frac{1}{t}\right)^k\,\Gamma(k+1)}{\Gamma(k-\mu+2\,\nu+1)}\right]$$

Laguerre

```
> FunctiontoRec((1-t)^(-alpha-1)*exp(-t*x/(1-t)),LaguerreL(m,alpha,
  x),u);
FunctiontoRec: time: 0.024 sec.
```

$$-t\,u(m) + u(1+m)$$

```
> FunctiontoRec(BesselJ(a,2*sqrt(t*y))*exp(t)*(t*y)^(-a/2),LaguerreL
  (n,a,y),u);
FunctiontoRec: time: 0.033 sec.
```

$$t\,u(n) + (-a - 1 - n)\,u(n+1)$$

Jacobi (PrBrMa86 5.14.1.1)

```
> rho:=sqrt(1-2*t*x+t^2);
```

$$\rho := \sqrt{t^2 - 2\,x\,t + 1}$$

```
> F:=2^(alpha+beta)/rho*(1-t+rho)^(-alpha)*(1+t+rho)^(-beta);
```

$$F := \frac{2^{\alpha+\beta}\left(1 - t + \sqrt{t^2 - 2\,x\,t + 1}\right)^{-\alpha}\left(1 + t + \sqrt{t^2 - 2\,x\,t + 1}\right)^{-\beta}}{\sqrt{t^2 - 2\,x\,t + 1}}$$

```
> rec:=FunctiontoRec(F,JacobiP(k,alpha,beta,x),u) assuming alpha>-1,
  beta>-1:
FunctiontoRec: time: 44.921 sec.
> collect(rec,u,degree);
```

$$18\,u(k) + 20\,u(3+k) + 20\,u(6+k) + 19\,u(7+k) + 18\,u(8+k) + 19\,u(k+1) + 20\,u(k+2) + 20\,u(k+4) + 20\,u(k+5)$$

```
> collect(rec,u,proc(v) degree(v,k) end);
```

$$12\,u(k) + 12\,u(3+k) + 12\,u(6+k) + 12\,u(7+k) + 12\,u(8+k) + 12\,u(k+1) + 12\,u(k+2) + 12\,u(k+4) + 12\,u(k+5)$$

```
> collect(rec,u,proc(v) degree(v,beta) end);
```

$$10\,u(k) + 10\,u(3+k) + 10\,u(6+k) + 10\,u(7+k) + 10\,u(8+k) + 10\,u(k+1) + 10\,u(k+2)$$

$$+ 10\,u(k+4) + 10\,u(k+5)$$

```
> collect(rec,u,proc(v) degree(v,alpha) end);
```

$$10\,u(k) + 10\,u(3+k) + 10\,u(6+k) + 10\,u(7+k) + 10\,u(8+k) + 10\,u(k+1) + 10\,u(k+2) + 10\,u(k+4) + 10\,u(k+5)$$

```
> collect(rec,u,proc(v) degree(v,t) end);
```

$$4\,u(k) + 6\,u(3+k) + 6\,u(6+k) + 5\,u(7+k) + 4\,u(8+k) + 5\,u(k+1) + 6\,u(k+2) + 6\,u(k+4) + 6\,u(k+5)$$

```
> expand(eval(rec,u=unapply(t^k,k)));
```

$$0$$

Special case alpha+beta+1=0

```
> FunctiontoRec(op(subs(beta=-1-alpha,[F,JacobiP(k,alpha,beta,x)])),
  u) assuming alpha>-1;
FunctiontoRec: time: 3.163 sec.
```

[Length of output exceeds limit of 50000]

Chebyshev expansions (Sec. 9.3)

exp(x)

```
> rec:=FunctiontoRec(exp(x),ChebyshevT(m,x),u);
FunctiontoRec: time: 0.012 sec.
```

$$rec := u(m) + (-2-2\,m)\,u(1+m) - u(m+2)$$

Check:

```
> seq(hnT(i),i=0..5);
```

$$\pi, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}$$

```
> for i from 0 to 5 do c[i]:=evalf(1/hnT(i)*Int(exp(x)/sqrt(1-x^2)*
  expand(orthopoly[T](i,x)),x=-1..1)) od;
```

$$c_0 := 1.266065878$$

$$c_1 := 1.130318208$$

$$c_2 := 0.2714953394$$

$$c_3 := 0.04433684984$$

$$c_4 := 0.005474240442$$

$$c_5 := 0.0005429263122$$

```
> subs([u(0)=c[0]*2,seq(u(i)=c[i],i=1..5)],[seq(eval(rec,m=i),i=0..3)
  ]);
```

$$[6. \times 10^{-10}, 1.6 \times 10^{-10}, -4.2 \times 10^{-11}, -1.22 \times 10^{-11}]$$

x*exp(x) (Lewanowicz 1991 p. 305)

```
> f:=x*exp(x);
```

$$f := x\,e^{x}$$

```
> rec1:=FunctiontoRec(f,ChebyshevT(n,x),u);;
FunctiontoRec: time: 0.016 sec.
```

$$rec1 := -u(n) + 2\,n\,u(n+1) + (2\,n+8)\,u(n+3) + u(n+4)$$

```
> gfun:-holexprtodiffeq(f,y(x),false);
```

$$x\left(\frac{d}{dx}\,y(x)\right) - y(x)\,x - y(x)$$

```
> dop1:=DEtools[de2diffop](%,y(x),[Dx,x]);
```

$$dop1 := x\,Dx - x - 1$$

```
> Q:=1/x^2*(x*(x^2-1)*Dx^2+(2*x^2+1)*Dx+(x-1));
```

$$Q := \frac{x\,(x^2-1)\,Dx^2 + (2\,x^2+1)\,Dx + x - 1}{x^2}$$

```
> Ps:=DEtools[mult](Q,dop1,[Dx,x]);
```

$$Ps := (x^2-1)\,Dx^3 + (-x^2+3\,x+1)\,Dx^2 + (-4\,x+1)\,Dx - 3$$

```
> rec2:=DiffoptoRec(Ps,Dx,x,n,ChebyshevT(n,x));
```

$$rec2 := OrePoly(-(n-2)^2 - 7\,n + 1, 2\,(n-2)^3 + 18\,(n-2)^2 + 54\,n - 54, (n-2)^2 + 5\,n - 3)$$

```
> rec2:=collect(add(op(i,%)*u(n+i-1),i=1..nops(rec2)),u,factor);
```

$$rec2 := (-n^2 - 3\,n - 3)\,u(n) + 2\,(n+1)^3\,u(n+1) + (n^2+n+1)\,u(n+2)$$

Elliptic E(x) (Lewanowicz 1976, Ex. 2.1)

```
> FunctiontoRec(EllipticE(x),ChebyshevT(m,x),u);
FunctiontoRec: time: 0.021 sec.
```

$$(m-1)\,(1+m)\,u(m) + (-4\,m-8)\,u(m+2) - (3+m)\,(5+m)\,u(m+4)$$

Lommel s_{\mu,\nu}(x) (Lewanowicz 1976, Ex. 3.1)

```
> f:=(a*x)^(1-mu)*LommelS1(mu,nu,a*x);
```

$$f := (a\,x)^{1-\mu}\,\mathrm{LommelS1}(\mu, \nu, a\,x)$$

```
> st:=time():
```

The differential equation is obtained by interpolation from 4 values:

```
> for i to 4 do deq[i]:=gfun:-holexprtodiffeq(eval(f,mu=i),y(x)) od:
> deq:=collect(interp([$1..4],[seq(deq[i],i=1..4)],mu),[diff(y(x),x,
  x),diff(y(x),x),y(x)],factor);;
```

$$deq := x^2\left(\frac{d^2}{dx^2}\,y(x)\right) + x\,(2\,\mu-1)\left(\frac{d}{dx}\,y(x)\right) + (a^2\,x^2 + \mu^2 - \nu^2 - 2\,\mu + 1)\,y(x) - a^2\,x^2$$

Consider the homogeneous part of the equation:

```
> hom:=deq-eval(deq,y(x)=0):
> DiffeqtoRec(hom,y(x),u(m),ChebyshevT(m,x));
```

$$(5+m)\,a^2\,u(m)+\left(4\,m^3+8\,m^2\,\mu+4\,m\,\mu^2-4\,m\,\nu^2+2\,a^2+28\,m^2+48\,m\,\mu+20\,\mu^2-20\,\nu^2+44\,m+40\,\mu+20\right)u(m+2)+\left(-2\,a^2\,m+8\,m^3-8\,m\,\mu^2+8\,m\,\nu^2-8\,a^2+96\,m^2+32\,m\,\mu-32\,\mu^2+32\,\nu^2+344\,m+128\,\mu+352\right)u(m+4)+\left(4\,m^3-8\,m^2\,\mu+4\,m\,\mu^2-4\,m\,\nu^2-2\,a^2+68\,m^2-80\,m\,\mu+12\,\mu^2-12\,\nu^2+364\,m-168\,\mu+588\right)u(m+6)+a^2\,(3+m)\,u(m+8)$$

```
> time()-st;
```

$$0.159$$

This is (up to a change of variables) Lewanowicz' formula.

```
> f:='f':
```

Other Chebyshev expansions

```
> rec:=FunctiontoRec((1-x^2)^(-1/4),ChebyshevT(m,x),u);
FunctiontoRec: time: 0.014 sec.
```

$$rec := (2\,m+1)\,u(m)+(-3-2\,m)\,u(m+2)$$

```
> FunctiontoRec(arccos(x),ChebyshevT(n,x),u);
FunctiontoRec: time: 0.02 sec.
```

$$-2\,u(n)\,n$$

```
> DiffoptoRec((1-x^2)^2*Dx^2-(1-x^2)*x*Dx,Dx,x,n,ChebyshevT(n,x));
```

$$OrePoly\left(-n^2,0,2\,n^2+8\,n+8,0,-(n+4)^2\right)$$

```
> FunctiontoRec(arctan(x),ChebyshevT(n,x),u);
FunctiontoRec: time: 0.011 sec.
```

$$u(n)\,n+(6\,n+12)\,u(n+2)+(n+4)\,u(n+4)$$

```
> FunctiontoRec(erf(x),ChebyshevT(n,x),u);
FunctiontoRec: time: 0.013 sec.
```

$$n\,(n+3)\,u(n)+2\,(n+2)^3\,u(n+2)-(n+1)\,(n+4)\,u(n+4)$$

```
> FunctiontoRec(arctanh(x),ChebyshevT(n,x),u);
FunctiontoRec: time: 0.016 sec.
```

$$2\,u(n)\,n+2\,(-n-2)\,u(n+2)$$

Generating function:

```
> rec:=FunctiontoRec(exp(z*u)*cos(u*sqrt(1-z^2)),ChebyshevT(n,z),a);
FunctiontoRec: time: 0.039 sec.
```

$$rec := -u\,n\,a(n)+\left(n^2-u^2+n\right)a(n+1)-2\,u\,a(n+2)+\left(n^2-u^2+7\,n+12\right)a(n+3)+u\,(n+4)\,a(n+4)$$

```
> normal(expand(eval(%,a=unapply(u^n/n!,n))));
```

$$0$$

Fourier expansions (Sec. 9.4) from PBM vol. 1. sec. 5.4.2

ln(2*sin(x/2)) (eq. 9.)

```
> F:=ln(2*sin(x/2));
```

$$F := \ln\left(2 \sin\left(\frac{x}{2}\right)\right)$$

```
> Gt:=subs(x=arccos(t),F);
```

$$Gt := \ln\left(2 \sin\left(\frac{\arccos(t)}{2}\right)\right)$$

```
> G:=convert(Gt,exp);
```

$$G := \ln\left(-\mathrm{I}\left(\sqrt{t+\mathrm{I}\sqrt{-t^2+1}} - \frac{1}{\sqrt{t+\mathrm{I}\sqrt{-t^2+1}}}\right)\right)$$

```
> simplify(%);
```

$$\ln\left(\frac{-\mathrm{I}\,t+\sqrt{-t^2+1}+\mathrm{I}}{\sqrt{t+\mathrm{I}\sqrt{-t^2+1}}}\right)$$

```
> FunctiontoRec(%,ChebyshevT(n,t),u);
FunctiontoRec: time: 0.053 sec.
```

$$(n+1)\,u(n+1) + (-n-2)\,u(n+2)$$

```
> rsolve(%,u(n));
```

$$charfcn_0(n)\,u(0)$$

```
> deq:=gfun:-holexprtodiffeq(G,y(t),false);
```

$$deq := -1 + (2t-2)\left(\frac{\mathrm{d}}{\mathrm{d}t}\,y(t)\right)$$

```
> gfun:-diffeqtohomdiffeq(%,y(t));
```

$$-2\,\frac{\mathrm{d}}{\mathrm{d}t}\,y(t) + (2-2t)\left(\frac{\mathrm{d}^2}{\mathrm{d}t^2}\,y(t)\right)$$

```
> rec:=DiffeqtoRec((1-t^2)*%,y(t),u(k),ChebyshevT(k,t));
```

$$rec := -k^2\,u(k) + (k^2-1)\,u(k+1) + (k^2+6k+8)\,u(k+2) - (3+k)^2\,u(3+k)$$

```
> ini:=seq(u(i)=int(F*cos(i*x),x=0..2*Pi),i=0..3);
```

$$ini := u(0)=0,\ u(1)=-\pi,\ u(2)=-\frac{\pi}{2},\ u(3)=-\frac{\pi}{3}$$

```
> rsolve({rec,ini},u(k));
```

$$\begin{cases} 0 & k=0 \\ -\dfrac{\pi}{k} & otherwise \end{cases}$$

```
> deqhom:=deq-eval(deq,y(t)=0);
```

$$deqhom := (2t-2)\left(\frac{\mathrm{d}}{\mathrm{d}t}\,y(t)\right)$$

```
> DiffeqtoRec((1-t^2)*deqhom,y(t),u(k),ChebyshevT(k,t));
```

$$2k\,u(k) + 2\,(-2k-2)\,u(k+1) + 2\,(2k+6)\,u(3+k) + 2\,(-k-4)\,u(k+4)$$

```
> normal(eval(%,u=unapply(-1/k,k)));
```

$$0$$

Another situation where analytic properties play a role

```
> F:=x^2;
```

$$F := x^2$$

```
> G:=subs(x=arccos(t),F);
```

$$G := \arccos(t)^2$$

```
> FunctiontoRec(G,ChebyshevT(n,t),u);
FunctiontoRec: time: 0.026 sec.
```

$$u(n+1)$$

```
> deq:=gfun:-holexprtodiffeq(G,y(t),false);
```

$$deq := (t^2-1)\left(\frac{d^3}{dt^3}\,y(t)\right) + 3\,t\left(\frac{d^2}{dt^2}\,y(t)\right) + \frac{d}{dt}\,y(t)$$

```
> rec:=DiffeqtoRec((1-t^2)*deq,y(t),u(k),ChebyshevT(k,t));
```

$$rec := k^3\,u(k) - (k^2+4\,k+4)\,(k+2)\,u(k+2)$$

Check:

```
> ini:=seq(u(i)=int(F*cos(i*x),x=-Pi..Pi),i=0..10);
```

$$ini := u(0) = \frac{2\,\pi^3}{3}, u(1) = -4\,\pi, u(2) = \pi, u(3) = -\frac{4\,\pi}{9}, u(4) = \frac{\pi}{4}, u(5) = -\frac{4\,\pi}{25}, u(6) = \frac{\pi}{9},$$

$$u(7) = -\frac{4\,\pi}{49}, u(8) = \frac{\pi}{16}, u(9) = -\frac{4\,\pi}{81}, u(10) = \frac{\pi}{25}$$

```
> subs([ini],[seq(eval(rec,k=i),i=0..8)]);
```

$$[-8\,\pi, 8\,\pi, -8\,\pi, 8\,\pi, -8\,\pi, 8\,\pi, -8\,\pi, 8\,\pi, -8\,\pi]$$

```
> rec:=DiffeqtoRec((1-t^2)^2*deq,y(t),u(k),ChebyshevT(k,t));
```

$$rec := k^3\,u(k) + (-3\,k^3 - 18\,k^2 - 36\,k - 24)\,u(k+2) + (3\,k^3 + 36\,k^2 + 144\,k + 192)\,u(k+4)$$
$$- (k^2 + 12\,k + 36)\,(6+k)\,u(6+k)$$

```
> subs([ini],[seq(eval(rec,k=i),i=0..4)]);
```

$$[0, 0, 0, 0, 0]$$

```
> LREtools[MinimalRecurrence](rec,u(k),{ini});
```

$$(k+1)^2\,u(k+1) + k^2\,u(k) = 0, \text{"Relation holds for"}, 1 \le k, \text{"Initial terms"}, \{u(1) = -4\,\pi\}$$

```
> recmin:=%[1];
```

$$recmin := (k+1)^2\,u(k+1) + k^2\,u(k) = 0$$

```
> rsolve({rec,seq(u(i)=subs([ini],u(i)),i=0..5)},u(k));
```

$$\begin{cases} \frac{2\,\pi^3}{3} & k = 0 \\ \frac{4\,(-1)^k\,\pi}{k^2} & otherwise \end{cases}$$

Same with x^4:

```
> F:=x^4;
```

$$F := x^4$$

```
> G:=subs(x=arccos(t),F);
```

$$G := \arccos(t)^4$$

```
> FunctiontoRec(G,ChebyshevT(n,t),u);
FunctiontoRec: time: 0.054 sec.
```

$$u(n+1)$$

```
> deq:=gfun:-holexprtodiffeq(G,y(t),false);
```

$$deq := (t^4 - 2\,t^2 + 1)\left(\frac{d^5}{dt^5}\,y(t)\right) + (10\,t^3 - 10\,t)\left(\frac{d^4}{dt^4}\,y(t)\right) + (25\,t^2 - 10)\left(\frac{d^3}{dt^3}\,y(t)\right) + 15\,t\left(\frac{d^2}{dt^2}\,y(t)\right) + \frac{d}{dt}\,y(t)$$

```
> series((1-t^2)^2*diff(G,t$5),t=-1,3);
```

$$-\frac{105\,\pi^3\,\sqrt{2}}{2\,(t+1)^{5/2}} + O\left(\frac{1}{(t+1)^{3/2}}\right)$$

```
> series((1-t^2)^5*diff(G,t$5),t=-1,6);
```

$$-420\,\pi^3\,\sqrt{2}\,\sqrt{t+1} + O\left((t+1)^{3/2}\right)$$

```
> rec:=DiffeqtoRec((1-t^2)^3*deq,y(t),u(k),ChebyshevT(k,t));
```

$$rec := k^5\,u(k) + (-5\,k^5 - 50\,k^4 - 200\,k^3 - 400\,k^2 - 400\,k - 160)\,u(k+2) + (10\,k^5 + 200\,k^4 + 1600\,k^3 + 6400\,k^2 + 12800\,k + 10240)\,u(k+4) + (-10\,k^5 - 300\,k^4 - 3600\,k^3 - 21600\,k^2 - 64800\,k - 77760)\,u(6+k) + (5\,k^5 + 200\,k^4 + 3200\,k^3 + 25600\,k^2 + 102400\,k + 163840)\,u(8+k) - (k^4 + 40\,k^3 + 600\,k^2 + 4000\,k + 10000)\,(k+10)\,u(k+10)$$

Check:

```
> ini:=seq(u(i)=int(F*cos(i*x),x=-Pi..Pi),i=0..15);
```

$$ini := u(0) = \frac{2\,\pi^5}{5}, u(1) = -8\,\pi^3 + 48\,\pi, u(2) = 2\,\pi^3 - 3\,\pi, u(3) = -\frac{8}{9}\,\pi^3 + \frac{16}{27}\,\pi, u(4) = \frac{1}{2}\,\pi^3 - \frac{3}{16}\,\pi, u(5) = -\frac{8}{25}\,\pi^3 + \frac{48}{625}\,\pi, u(6) = \frac{2}{9}\,\pi^3 - \frac{1}{27}\,\pi, u(7) = -\frac{8}{49}\,\pi^3 + \frac{48}{2401}\,\pi, u(8) = \frac{1}{8}\,\pi^3 - \frac{3}{256}\,\pi, u(9) = -\frac{8}{81}\,\pi^3 + \frac{16}{2187}\,\pi, u(10) = \frac{2}{25}\,\pi^3 - \frac{3}{625}\,\pi, u(11) = -\frac{8}{121}\,\pi^3 + \frac{48}{14641}\,\pi, u(12) = \frac{1}{18}\,\pi^3 - \frac{1}{432}\,\pi, u(13) = -\frac{8}{169}\,\pi^3 + \frac{48}{28561}\,\pi,$$

$$u(14) = \frac{2}{49}\,\pi^3 - \frac{3}{2401}\,\pi,\ u(15) = -\frac{8}{225}\,\pi^3 + \frac{16}{16875}\,\pi$$

```
> subs([ini],[seq(eval(rec,k=i),i=0..5)]);
```

$$[0, 0, 0, 0, 0, 0]$$

```
> L:=gfun:-rectoproc({rec,ini},u(k),list)(100):
> gfun:-listtorec(L,u(k))[1];
```

$$\left\{k^5\left(\pi^2 k^2 + 2\pi^2 k + \pi^2 - 6\right)u(k) + k\left(\pi^2 k^6 + 4\pi^2 k^5 + 6\pi^2 k^4 + 4\pi^2 k^3 + \pi^2 k^2 - 6k^4 - 24k^3 - 36k^2 - 24k - 6\right)u(k+1),\ u(0) = \frac{2\pi^5}{5},\ u(1) = -8\pi^3 + 48\pi\right\}$$

```
> collect(op(1,%),u,factor);
```

$$k^5\left(\pi^2 k^2 + 2\pi^2 k + \pi^2 - 6\right)u(k) + k\,(k+1)^4\left(\pi^2 k^2 - 6\right)u(k+1)$$

```
> rsolve({rec,ini},u(k));
```

$$\begin{cases} \dfrac{2\pi^5}{5} & k = 0 \\ \dfrac{8\,(-1)^k\,\pi\left(\pi^2 k^2 - 6\right)}{k^4} & \text{otherwise} \end{cases}$$

A simple trigonometric series

```
> sum((-1)^k/k^2*cos(k*x),k=1..infinity) assuming x>Pi,x<2*Pi;
```

$$\frac{11}{12}\,\pi^2 + \frac{1}{4}\,x^2 - \pi x$$

```
> f:=subs(x=arccos(t),%);
```

$$f := \frac{11\,\pi^2}{12} + \frac{\arccos(t)^2}{4} - \pi\arccos(t)$$

```
> simplify(%) assuming t::real;
```

$$\frac{23\,\pi^2}{48} + \frac{3\,\pi\arcsin(t)}{4} + \frac{\arcsin(t)^2}{4}$$

```
> rec:=FunctiontoRec(f,ChebyshevT(k,t),u);
FunctiontoRec: time: 0.031 sec.
```

$$rec := u(k+1)$$

```
> deq:=gfun:-holexprtodiffeq(f,y(t),false);
```

$$deq := \left(t^2 - 1\right)\left(\frac{d^3}{dt^3}\,y(t)\right) + 3t\left(\frac{d^2}{dt^2}\,y(t)\right) + \frac{d}{dt}\,y(t)$$

```
> rec:=DiffeqtoRec((1-t^2)*deq,y(t),u(k),ChebyshevT(k,t));
```

$$rec := k^3\,u(k) - \left(k^2 + 4k + 4\right)(k+2)\,u(k+2)$$

```
> eval(rec,u=unapply((-1)^k/k^2,k));
```

$$(-1)^k k - \frac{(k^2+4k+4)(-1)^{k+2}}{k+2}$$

```
> expand(%);
```

$$(-1)^k k - \frac{(-1)^k k^2}{k+2} - \frac{4(-1)^k k}{k+2} - \frac{4(-1)^k}{k+2}$$

```
> rec:=DiffeqtoRec((1-t^2)^2*deq,y(t),u(k),ChebyshevT(k,t));
```

$$rec := k^3 u(k) + (-3k^3 - 18k^2 - 36k - 24)\,u(k+2) + (3k^3 + 36k^2 + 144k + 192)\,u(k+4) - (k^2 + 12k + 36)(6+k)\,u(6+k)$$

```
> eval(rec,u=unapply((-1)^k/k^2,k));
```

$$(-1)^k k + \frac{(-3k^3 - 18k^2 - 36k - 24)(-1)^{k+2}}{(k+2)^2} + \frac{(3k^3 + 36k^2 + 144k + 192)(-1)^{k+4}}{(k+4)^2} - \frac{(k^2+12k+36)(-1)^{6+k}}{6+k}$$

```
> expand(%);
```

$$(-1)^k k - \frac{3(-1)^k k^3}{(k+2)^2} - \frac{18(-1)^k k^2}{(k+2)^2} - \frac{36(-1)^k k}{(k+2)^2} - \frac{24(-1)^k}{(k+2)^2} + \frac{3(-1)^k k^3}{(k+4)^2} + \frac{36(-1)^k k^2}{(k+4)^2} + \frac{144(-1)^k k}{(k+4)^2} + \frac{192(-1)^k}{(k+4)^2} - \frac{(-1)^k k^2}{6+k} - \frac{12(-1)^k k}{6+k} - \frac{36(-1)^k}{6+k}$$

```
> normal(%);
```

$$0$$

Another one

```
> sum(cos(k*x)/(1-2*k^2),k=0..infinity) assuming x>0,x<2*Pi;
```

$$\frac{\pi \cos\left(\frac{\sqrt{2}(\pi - x)}{2}\right)\sqrt{2} + 2\sin\left(\frac{\pi\sqrt{2}}{2}\right)}{4\sin\left(\frac{\pi\sqrt{2}}{2}\right)}$$

```
> simplify(%,trig);
```

$$\frac{\pi \cos\left(\frac{\sqrt{2}(\pi - x)}{2}\right)\sqrt{2}\csc\left(\frac{\pi\sqrt{2}}{2}\right)}{4} + \frac{1}{2}$$

```
> f:=subs(x=arccos(t),%);
```

$$f := \frac{\pi \cos\left(\frac{\sqrt{2}(\pi - \arccos(t))}{2}\right)\sqrt{2}\csc\left(\frac{\pi\sqrt{2}}{2}\right)}{4} + \frac{1}{2}$$

```
> simplify(f);
```

$$\frac{\pi \cos\left(\frac{\sqrt{2}\,(\pi + 2\arcsin(t))}{4}\right)\sqrt{2}\,\csc\left(\frac{\pi\sqrt{2}}{2}\right)}{4} + \frac{1}{2}$$

```
> convert(%,exp);
```

$$\frac{I\pi\left(\frac{e^{\frac{I}{4}\sqrt{2}\,(\pi+2\arcsin(t))}}{2} + \frac{e^{-\frac{I}{4}\sqrt{2}\,(\pi+2\arcsin(t))}}{2}\right)\sqrt{2}}{2\left(e^{\frac{I}{2}\pi\sqrt{2}} - e^{-\frac{I}{2}\pi\sqrt{2}}\right)} + \frac{1}{2}$$

```
> deq:=gfun:-holexprtodiffeq(%,y(t),false);
```

$$deq := (4t^2-4)\left(\frac{d^2}{dt^2}y(t)\right) + 4\left(\frac{d}{dt}y(t)\right)t - 2y(t) + 1$$

```
> deqhom:=deq-eval(deq,y(t)=0);
```

$$deqhom := (4t^2-4)\left(\frac{d^2}{dt^2}y(t)\right) + 4\left(\frac{d}{dt}y(t)\right)t - 2y(t)$$

```
> deq:=gfun:-diffeqtohomdiffeq(deq,y(t));
```

$$deq := 2\frac{d}{dt}y(t) + 12t\left(\frac{d^2}{dt^2}y(t)\right) + (4t^2-4)\left(\frac{d^3}{dt^3}y(t)\right)$$

```
> rec:=DiffeqtoRec(deq,y(t),u(k),ChebyshevT(k,t));
```

$$rec := u(k+1)$$

```
> rec:=DiffeqtoRec((1-t^2)*deq,y(t),u(k),ChebyshevT(k,t));
```

$$rec := k(2k^2-1)u(k) - (2k^2+8k+7)(k+2)u(k+2)$$

```
> normal(eval(rec,u=unapply(1/(1-2*k^2),k)));
```

$$2$$

```
> rec:=DiffeqtoRec((1-t^2)^2*deq,y(t),u(k),ChebyshevT(k,t));
```

$$rec := k(2k^2-1)u(k) + (-6k^3-36k^2-69k-42)u(k+2) + (6k^3+72k^2+285k + 372)u(k+4) - (2k^2+24k+71)(6+k)u(6+k)$$

```
> normal(eval(rec,u=unapply(1/(1-2*k^2),k)));
```

$$0$$

```
> DiffeqtoRec((1-t^2)*deqhom,y(t),u(k),ChebyshevT(k,t));
```

$$(-2k^2+1)u(k) + (4k^2+16k+14)u(k+2) + (-2k^2-16k-31)u(k+4)$$

```
> normal(eval(%,u=unapply(1/(1-2*k^2),k)));
```

$$0$$

Koepf - Chiadjeu Example 4.9

```
> st:=time():
> f:=cos(5*t)*ln(2+cos(5*t));
```

$$f := \cos(5t)\ln(2+\cos(5t))$$

```
> g:=eval(f,t=arccos(x)/5);
```

$$g := x \ln(2 + x)$$

```
> deq:=gfun:-holexprtodiffeq(g,y(x),false);
```

$$deq := (x^2 + 2x)\left(\frac{d}{dx} y(x)\right) - y(x)\, x - x^2 - 2\, y(x)$$

```
> deqhom:=deq-eval(deq,y(x)=0);
```

$$deqhom := (x^2 + 2x)\left(\frac{d}{dx} y(x)\right) - y(x)\, x - 2\, y(x)$$

```
> rec:=DiffeqtoRec(deqhom,y(x),u(n),ChebyshevT(n,x));
```

$$rec := (n-1)\, u(n) + 4n\, u(n+1) + (2n+4)\, u(n+2) + (4n+16)\, u(n+3) + (n+5)\, u(n+4)$$

```
> time()-st;
```

$$0.047$$

Koepf and Chiadjeu obtain a 2nd order LRE.

```
> normal(expand(eval(rec,u=unapply((-2+sqrt(3))^n*(sqrt(3)+2*n)/(n+1)
  /(n-1),n))));
```

$$0$$

```
> LREtools[hypergeomsols](rec,u(n),{},output=basis);
```

$$\left[\frac{(-I)^n}{(n+1)\,(n-1)}, \frac{I^n}{(n+1)\,(n-1)}, \frac{\left(-2-\sqrt{3}\right)^n \left(2n-\sqrt{3}\right)}{(n+1)\,(n-1)}, \frac{\left(-2+\sqrt{3}\right)^n \left(\sqrt{3}+2n\right)}{(n+1)\,(n-1)}\right]$$

Koepf - Chiadjeu ex. 2.5

```
> f:=arctan(2+cos(t)*exp(I*t));
```

$$f := \arctan\left(2 + \cos(t)\, e^{I t}\right)$$

```
> G:=subs(t=arccos(t),%);
```

$$G := \arctan\left(2 + \cos(\arccos(t))\, e^{I \arccos(t)}\right)$$

```
> simplify(G);
```

$$\arctan\left(I\sqrt{-t^2+1}\; t + t^2 + 2\right)$$

```
> G:=%:
> deq:=gfun:-holexprtodiffeq(G,y(t),false);
```

$$deq := (261\, t^7 - 181\, t^5 - 97\, t^3 + 217\, t)\left(\frac{d}{dt} y(t)\right) + (87\, t^8 + 22\, t^6 - 80\, t^4 - 54\, t^2 + 25)\left(\frac{d^2}{dt^2} y(t)\right) - 14\, t^2 - 14$$

```
> deqhom:=deq-eval(deq,y(t)=0);
```

$$deqhom := (261\,t^7 - 181\,t^5 - 97\,t^3 + 217\,t)\left(\frac{d}{dt}\,y(t)\right) + (87\,t^8 + 22\,t^6 - 80\,t^4 - 54\,t^2 + 25)\left(\frac{d^2}{dt^2}\,y(t)\right)$$

```
> deq:=gfun:-diffeqtohomdiffeq(deq,y(t));
```

$$deq := (-18270\,t^8 - 17976\,t^6 + 14028\,t^4 + 7112\,t^2 - 3038)\left(\frac{d}{dt}\,y(t)\right) + (-10962\,t^9 - 12096\,t^7 + 4284\,t^5 + 2800\,t^3 - 826\,t)\left(\frac{d^2}{dt^2}\,y(t)\right) + (-1218\,t^{10} - 1526\,t^8 + 812\,t^6 + 1876\,t^4 + 406\,t^2 - 350)\left(\frac{d^3}{dt^3}\,y(t)\right)$$

```
> rec:=DiffeqtoRec(deq,y(t),u(k),ChebyshevT(k,t));
```

$$rec := -87\,k\,(k^2 + 6\,k + 8)\,u(k) + (-1480\,k^3 - 15204\,k^2 - 49784\,k - 50592)\,u(k+2) + (-9348\,k^3 - 133500\,k^2 - 626256\,k - 967296)\,u(k+4) + (-25720\,k^3 - 480148\,k^2 - 2938248\,k - 5899680)\,u(6+k) + (-29130\,k^3 - 699120\,k^2 - 5561520\,k - 14663040)\,u(8+k) + (-25720\,k^3 - 754412\,k^2 - 7326472\,k - 23543520)\,u(k+10) - 12\,(12+k)\,(779\,k^2 + 16919\,k + 91432)\,u(12+k) - 4\,(12+k)\,(k+14)\,(370\,k + 4339)\,u(k+14) - 87\,(12+k)\,(k+14)\,(k+16)\,u(k+16)$$

Laguerre expansions (Sec. 9.5)

Eq. (18) in Howell (1937)

```
> c:='c':
> FunctiontoRec(exp(c*x),LaguerreL(n,alpha,x),u);
FunctiontoRec: time: 0.019 sec.
```

$$-c\,u(n) + (c-1)\,u(n+1)$$

```
> for i from 0 to 5 do i,simplify(1/eval(hnL,n=i)*int(exp(-x+c*x)*
  x^alpha*expand(LaguerreL(i,alpha,x)),x=0..infinity)) assuming c<1,
  alpha>-1 od;
```

$$0,\ (-c+1)^{-\alpha-1}$$

$$1,\ (-c+1)^{-\alpha-1} - (-c+1)^{-2-\alpha}$$

$$2,\ c^2\,(-c+1)^{-3-\alpha}$$

$$3,\ -c^3\,(-c+1)^{-4-\alpha}$$

$$4,\ c^4\,(-c+1)^{-5-\alpha}$$

$$5,\ -c^5\,(-c+1)^{-6-\alpha}$$

```
> f:=exp(x-x/lambda)/lambda^(n+alpha+1)*LaguerreL(n,alpha,x/lambda);
```

$$f := \frac{e^{x-\frac{x}{\lambda}}\ \mathrm{LaguerreL}\left(n, \alpha, \frac{x}{\lambda}\right)}{\lambda^{n+\alpha+1}}$$

```
> rec:=FunctiontoRec(f,LaguerreL(r,alpha,x),u) assuming alpha>-1;
FunctiontoRec: time: 0.03 sec.
```

$$rec := -(\lambda - 1)\ (r+1)\ u(r) + (n-r-1)\ u(r+1)$$

By orthogonality, the coefficient is 0 for r<n. For r=n, we get 1. Check:

```
> for i from 0 to 5 do simplify(1/eval(hnL,n=i)*int(exp(-x/lambda)*
  x^alpha/lambda^(i+alpha+1)*LaguerreL(i,alpha,x/lambda)*LaguerreL(i,
  alpha,x),x=0..infinity)) assuming lambda>0, alpha>-1 od;
```

$$1$$
$$1$$
$$1$$
$$1$$
$$1$$
$$1$$

```
> eval(subs(r=n+k,u=(s->v(s-n)),rec));
```

$$-(\lambda - 1)\ (k+n+1)\ v(k) + (-k-1)\ v(k+1)$$

```
> rsolve({%,v(0)=1},v(k)) assuming k::nonnegint;
```

$$\frac{\Gamma(k+n+1)\ (-\lambda+1)^k}{\Gamma(n+1)\ \Gamma(k+1)}$$

```
> subs(k=r-n,%);
```

$$\frac{\Gamma(r+1)\ (-\lambda+1)^{r-n}}{\Gamma(n+1)\ \Gamma(r-n+1)}$$

Howell gives the expansion for 0<lambda<2. Beyond that, we have

```
> i:=4:1/eval(hnL,n=i+2)*int(x^alpha*exp(-x/lambda)*expand(LaguerreL
  (i,alpha,x/lambda)*LaguerreL(i+2,alpha,x)),x=0..infinity) assuming
  alpha>-1,lambda>0;
```

$$\frac{1}{\Gamma(7+\alpha)}\left(720\left(-\frac{1141\,\lambda^{1+\alpha}\Gamma(\alpha+4)\,\alpha^3}{1080} - \frac{10\,\lambda^{1+\alpha}\Gamma(\alpha+4)\,\lambda^3}{3}\right.\right.$$
$$- \frac{413\,\lambda^{1+\alpha}\Gamma(\alpha+4)\,\alpha^2}{216} - 30\,\lambda^{1+\alpha}\Gamma(\alpha+4)\,\lambda^2 - \frac{9\,\lambda^{1+\alpha}\Gamma(\alpha+4)\,\alpha}{5} - 18\,\lambda^{1+\alpha}\Gamma(\alpha$$
$$+4)\,\lambda + \frac{\lambda^{1+\alpha}\Gamma(\alpha+3)\,\alpha^8}{2880} + \frac{7\,\lambda^{1+\alpha}\Gamma(\alpha+3)\,\alpha^7}{720} + \frac{167\,\lambda^{1+\alpha}\Gamma(\alpha+3)\,\alpha^6}{1440}$$
$$+ \frac{553\,\lambda^{1+\alpha}\Gamma(\alpha+3)\,\alpha^5}{720} + \frac{8869\,\lambda^{1+\alpha}\Gamma(\alpha+3)\,\alpha^4}{2880} + \frac{343\,\lambda^{1+\alpha}\Gamma(\alpha+3)\,\alpha^3}{45}$$
$$\left.+ \frac{2713\,\lambda^{1+\alpha}\Gamma(\alpha+3)\,\alpha^2}{240} + \frac{15\,\lambda^{1+\alpha}\Gamma(\alpha+3)\,\lambda^2}{2} + \frac{91\,\lambda^{1+\alpha}\Gamma(\alpha+3)\,\alpha}{10}\right.$$

$$+ 24\,\lambda^{1+\alpha}\,\Gamma(\alpha+3)\,\lambda - \frac{\lambda^{\alpha+2}\,\Gamma(\alpha+6)\,\alpha^5}{2880} - \frac{\lambda^{\alpha+2}\,\Gamma(\alpha+6)\,\alpha^4}{144} - \frac{\lambda^{\alpha+2}\,\Gamma(\alpha+6)\,\lambda^4}{20}$$
$$- \frac{31\,\lambda^{\alpha+2}\,\Gamma(\alpha+6)\,\alpha^3}{576} - \frac{5\,\lambda^{\alpha+2}\,\Gamma(\alpha+6)\,\lambda^3}{2} - \frac{29\,\lambda^{\alpha+2}\,\Gamma(\alpha+6)\,\alpha^2}{144} - 10\,\lambda^{\alpha+2}\,\Gamma(\alpha$$
$$+ 6)\,\lambda^2 - \frac{29\,\lambda^{\alpha+2}\,\Gamma(\alpha+6)\,\alpha}{80} - 5\,\lambda^{\alpha+2}\,\Gamma(\alpha+6)\,\lambda - \frac{\lambda^{\alpha+4}\,\Gamma(\alpha+8)\,\alpha^3}{864}$$
$$- \frac{\lambda^{\alpha+4}\,\Gamma(\alpha+8)\,\lambda^3}{180} - \frac{5\,\lambda^{\alpha+4}\,\Gamma(\alpha+8)\,\alpha^2}{288} - \frac{3\,\lambda^{\alpha+4}\,\Gamma(\alpha+8)\,\lambda^2}{20}$$
$$- \frac{37\,\lambda^{\alpha+4}\,\Gamma(\alpha+8)\,\alpha}{432} - \frac{5\,\lambda^{\alpha+4}\,\Gamma(\alpha+8)\,\lambda}{12} - \frac{\lambda^{\alpha+6}\,\Gamma(\alpha+10)\,\alpha}{2880}$$
$$- \frac{\lambda^{\alpha+6}\,\Gamma(\alpha+10)\,\lambda}{1080} + \frac{\lambda^{1+\alpha}\,\Gamma(\alpha+5)\,\alpha^6}{17280} + \frac{7\,\lambda^{1+\alpha}\,\Gamma(\alpha+5)\,\alpha^5}{5760}$$
$$+ \frac{35\,\lambda^{1+\alpha}\,\Gamma(\alpha+5)\,\alpha^4}{3456} + \frac{5\,\lambda^{1+\alpha}\,\Gamma(\alpha+5)\,\lambda^4}{8} + \frac{49\,\lambda^{1+\alpha}\,\Gamma(\alpha+5)\,\alpha^3}{1152}$$
$$+ \frac{40\,\lambda^{1+\alpha}\,\Gamma(\alpha+5)\,\lambda^3}{3} + \frac{203\,\lambda^{1+\alpha}\,\Gamma(\alpha+5)\,\alpha^2}{2160} + \frac{45\,\lambda^{1+\alpha}\,\Gamma(\alpha+5)\,\lambda^2}{2}$$
$$+ \frac{49\,\lambda^{1+\alpha}\,\Gamma(\alpha+5)\,\alpha}{480} + 4\,\lambda^{1+\alpha}\,\Gamma(\alpha+5)\,\lambda + \frac{\lambda^{\alpha+3}\,\Gamma(7+\alpha)\,\alpha^4}{1152} + \frac{\lambda^{\alpha+3}\,\Gamma(7+\alpha)\,\lambda^4}{720}$$
$$+ \frac{\lambda^{\alpha+3}\,\Gamma(7+\alpha)\,\alpha^3}{64} + \frac{\lambda^{\alpha+3}\,\Gamma(7+\alpha)\,\lambda^3}{5} + \frac{119\,\lambda^{\alpha+3}\,\Gamma(7+\alpha)\,\alpha^2}{1152}$$
$$+ \frac{15\,\lambda^{\alpha+3}\,\Gamma(7+\alpha)\,\lambda^2}{8} + \frac{19\,\lambda^{\alpha+3}\,\Gamma(7+\alpha)\,\alpha}{64} + \frac{20\,\lambda^{\alpha+3}\,\Gamma(7+\alpha)\,\lambda}{9}$$
$$+ \frac{\lambda^{\alpha+5}\,\Gamma(\alpha+9)\,\alpha^2}{1152} + \frac{\lambda^{\alpha+5}\,\Gamma(\alpha+9)\,\lambda^2}{240} + \frac{11\,\lambda^{\alpha+5}\,\Gamma(\alpha+9)\,\alpha}{1152} + \frac{\lambda^{\alpha+5}\,\Gamma(\alpha+9)\,\lambda}{30}$$
$$- \frac{\lambda^{1+\alpha}\,\Gamma(\alpha+2)\,\alpha^9}{4320} - \frac{\lambda^{1+\alpha}\,\Gamma(\alpha+2)\,\alpha^8}{144} - \frac{13\,\lambda^{1+\alpha}\,\Gamma(\alpha+2)\,\alpha^7}{144}$$
$$- \frac{2\,\lambda^{1+\alpha}\,\Gamma(\alpha+2)\,\alpha^6}{3} - \frac{1477\,\lambda^{1+\alpha}\,\Gamma(\alpha+2)\,\alpha^5}{480} - \frac{147\,\lambda^{1+\alpha}\,\Gamma(\alpha+2)\,\alpha^4}{16}$$
$$- \frac{3823\,\lambda^{1+\alpha}\,\Gamma(\alpha+2)\,\alpha^3}{216} - \frac{761\,\lambda^{1+\alpha}\,\Gamma(\alpha+2)\,\alpha^2}{36} + \frac{6349\,\lambda^{1+\alpha}\,\alpha^2\,\Gamma(1+\alpha)}{720}$$
$$+ \frac{7\,\lambda^{1+\alpha}\,\alpha^8\,\Gamma(1+\alpha)}{288} + \frac{109\,\lambda^{1+\alpha}\,\alpha^7\,\Gamma(1+\alpha)}{576} + \frac{\lambda^{1+\alpha}\,\alpha^{10}\,\Gamma(1+\alpha)}{17280}$$
$$+ \frac{31\,\lambda^{1+\alpha}\,\alpha^9\,\Gamma(1+\alpha)}{17280} + \frac{68\,\lambda^{1+\alpha}\,\alpha\,\Gamma(1+\alpha)}{15} + \frac{11615\,\lambda^{1+\alpha}\,\alpha^4\,\Gamma(1+\alpha)}{1728}$$

$$+\frac{8389\,\lambda^{1+\alpha}\,\alpha^3\,\Gamma(1+\alpha)}{864}+\frac{599\,\lambda^{1+\alpha}\,\alpha^6\,\Gamma(1+\alpha)}{640}+\frac{5887\,\lambda^{1+\alpha}\,\alpha^5\,\Gamma(1+\alpha)}{1920}$$

$$-\frac{212\,\lambda^{1+\alpha}\,\Gamma(\alpha+2)\,\alpha}{15}-6\,\lambda^{1+\alpha}\,\Gamma(\alpha+2)\,\lambda-\lambda^{1+\alpha}\,\Gamma(\alpha+2)\,\alpha^6\,\lambda$$

$$-\frac{1477\,\lambda^{1+\alpha}\,\Gamma(\alpha+2)\,\alpha^5\,\lambda}{320}-\frac{441\,\lambda^{1+\alpha}\,\Gamma(\alpha+2)\,\alpha^4\,\lambda}{32}-\frac{3823\,\lambda^{1+\alpha}\,\Gamma(\alpha+2)\,\alpha^3\,\lambda}{144}$$

$$-\frac{761\,\lambda^{1+\alpha}\,\Gamma(\alpha+2)\,\alpha^2\,\lambda}{24}-\frac{106\,\lambda^{1+\alpha}\,\Gamma(\alpha+2)\,\alpha\,\lambda}{5}-12\,\lambda^{\alpha+2}\,\Gamma(\alpha+6)\,\alpha\,\lambda^2$$

$$-6\,\lambda^{\alpha+2}\,\Gamma(\alpha+6)\,\alpha\,\lambda-\frac{\lambda^{\alpha+4}\,\Gamma(\alpha+8)\,\alpha^3\,\lambda^3}{4320}-\frac{\lambda^{\alpha+4}\,\Gamma(\alpha+8)\,\alpha^3\,\lambda^2}{480}$$

$$-\frac{\lambda^{\alpha+4}\,\Gamma(\alpha+8)\,\alpha^2\,\lambda^3}{480}-\frac{\lambda^{\alpha+4}\,\Gamma(\alpha+8)\,\alpha^3\,\lambda}{288}-\frac{13\,\lambda^{\alpha+4}\,\Gamma(\alpha+8)\,\alpha^2\,\lambda^2}{480}$$

$$-\frac{13\,\lambda^{\alpha+4}\,\Gamma(\alpha+8)\,\alpha\,\lambda^3}{2160}-\frac{5\,\lambda^{\alpha+4}\,\Gamma(\alpha+8)\,\alpha^2\,\lambda}{96}-\frac{9\,\lambda^{\alpha+4}\,\Gamma(\alpha+8)\,\alpha\,\lambda^2}{80}$$

$$-\frac{37\,\lambda^{\alpha+4}\,\Gamma(\alpha+8)\,\alpha\,\lambda}{144}-\frac{\lambda^{\alpha+6}\,\Gamma(\alpha+10)\,\alpha\,\lambda}{4320}+\frac{\lambda^{1+\alpha}\,\Gamma(\alpha+5)\,\alpha^6\,\lambda^4}{1152}$$

$$+\frac{\lambda^{1+\alpha}\,\Gamma(\alpha+5)\,\alpha^6\,\lambda^3}{216}+\frac{7\,\lambda^{1+\alpha}\,\Gamma(\alpha+5)\,\alpha^5\,\lambda^4}{384}+\frac{\lambda^{1+\alpha}\,\Gamma(\alpha+5)\,\alpha^6\,\lambda^2}{192}$$

$$+\frac{\lambda^{1+\alpha}\,\Gamma(\alpha+5)\,\alpha^5\,\lambda^3}{9}+\frac{175\,\lambda^{1+\alpha}\,\Gamma(\alpha+5)\,\alpha^4\,\lambda^4}{1152}+\frac{\lambda^{1+\alpha}\,\Gamma(\alpha+5)\,\alpha^6\,\lambda}{720}$$

$$+\frac{25\,\lambda^{1+\alpha}\,\Gamma(\alpha+5)\,\alpha^5\,\lambda^2}{192}+\frac{235\,\lambda^{1+\alpha}\,\Gamma(\alpha+5)\,\alpha^4\,\lambda^3}{216}+\frac{245\,\lambda^{1+\alpha}\,\Gamma(\alpha+5)\,\alpha^3\,\lambda^4}{384}$$

$$+\frac{\lambda^{1+\alpha}\,\Gamma(\alpha+5)\,\alpha^5\,\lambda}{30}+\frac{257\,\lambda^{1+\alpha}\,\Gamma(\alpha+5)\,\alpha^4\,\lambda^2}{192}+\frac{50\,\lambda^{1+\alpha}\,\Gamma(\alpha+5)\,\alpha^3\,\lambda^3}{9}$$

$$+\frac{203\,\lambda^{1+\alpha}\,\Gamma(\alpha+5)\,\alpha^2\,\lambda^4}{144}+\frac{47\,\lambda^{1+\alpha}\,\Gamma(\alpha+5)\,\alpha^4\,\lambda}{144}+\frac{1391\,\lambda^{1+\alpha}\,\Gamma(\alpha+5)\,\alpha^3\,\lambda^2}{192}$$

$$+\frac{841\,\lambda^{1+\alpha}\,\Gamma(\alpha+5)\,\alpha^2\,\lambda^3}{54}+\frac{49\,\lambda^{1+\alpha}\,\Gamma(\alpha+5)\,\alpha\,\lambda^4}{32}+\frac{5\,\lambda^{1+\alpha}\,\Gamma(\alpha+5)\,\alpha^3\,\lambda}{3}$$

$$+\frac{697\,\lambda^{1+\alpha}\,\Gamma(\alpha+5)\,\alpha^2\,\lambda^2}{32}+\frac{68\,\lambda^{1+\alpha}\,\Gamma(\alpha+5)\,\alpha\,\lambda^3}{3}+\frac{841\,\lambda^{1+\alpha}\,\Gamma(\alpha+5)\,\alpha^2\,\lambda}{180}$$

$$+\frac{69\,\lambda^{1+\alpha}\,\Gamma(\alpha+5)\,\alpha\,\lambda^2}{2}+\frac{34\,\lambda^{1+\alpha}\,\Gamma(\alpha+5)\,\alpha\,\lambda}{5}+\frac{\lambda^{\alpha+3}\,\Gamma(7+\alpha)\,\alpha^4\,\lambda^4}{17280}$$

$$+\frac{\lambda^{\alpha+3}\,\Gamma(7+\alpha)\,\alpha^4\,\lambda^3}{720}+\frac{\lambda^{\alpha+3}\,\Gamma(7+\alpha)\,\alpha^3\,\lambda^4}{1728}+\frac{\lambda^{\alpha+3}\,\Gamma(7+\alpha)\,\alpha^4\,\lambda^2}{192}$$

$$+\frac{\lambda^{\alpha+3}\,\Gamma(7+\alpha)\,\alpha^3\,\lambda^3}{48}+\frac{7\,\lambda^{\alpha+3}\,\Gamma(7+\alpha)\,\alpha^2\,\lambda^4}{3456}+\frac{\lambda^{\alpha+3}\,\Gamma(7+\alpha)\,\alpha^4\,\lambda}{216}$$
$$+\frac{3\,\lambda^{\alpha+3}\,\Gamma(7+\alpha)\,\alpha^3\,\lambda^2}{32}+\frac{\lambda^{\alpha+3}\,\Gamma(7+\alpha)\,\alpha^2\,\lambda^3}{9}-\frac{25\,\lambda^{1+\alpha}\,\Gamma(\alpha+4)\,\alpha^6\,\lambda^3}{864}$$
$$-\frac{\lambda^{1+\alpha}\,\Gamma(\alpha+4)\,\alpha^7\,\lambda}{480}-\frac{3\,\lambda^{1+\alpha}\,\Gamma(\alpha+4)\,\alpha^6\,\lambda^2}{32}-\frac{259\,\lambda^{1+\alpha}\,\Gamma(\alpha+4)\,\alpha^5\,\lambda^3}{864}$$
$$-\frac{9\,\lambda^{1+\alpha}\,\Gamma(\alpha+4)\,\alpha^6\,\lambda}{160}-\frac{307\,\lambda^{1+\alpha}\,\Gamma(\alpha+4)\,\alpha^5\,\lambda^2}{288}-\frac{1435\,\lambda^{1+\alpha}\,\Gamma(\alpha+4)\,\alpha^4\,\lambda^3}{864}$$
$$-\frac{307\,\lambda^{1+\alpha}\,\Gamma(\alpha+4)\,\alpha^5\,\lambda}{480}-\frac{635\,\lambda^{1+\alpha}\,\Gamma(\alpha+4)\,\alpha^4\,\lambda^2}{96}-\frac{1141\,\lambda^{1+\alpha}\,\Gamma(\alpha+4)\,\alpha^3\,\lambda^3}{216}$$
$$-\frac{127\,\lambda^{1+\alpha}\,\Gamma(\alpha+4)\,\alpha^4\,\lambda}{32}-\frac{1741\,\lambda^{1+\alpha}\,\Gamma(\alpha+4)\,\alpha^3\,\lambda^2}{72}-\frac{2065\,\lambda^{1+\alpha}\,\Gamma(\alpha+4)\,\alpha^2\,\lambda^3}{216}$$
$$-\frac{1741\,\lambda^{1+\alpha}\,\Gamma(\alpha+4)\,\alpha^3\,\lambda}{120}-\frac{1249\,\lambda^{1+\alpha}\,\Gamma(\alpha+4)\,\alpha^2\,\lambda^2}{24}-9\,\lambda^{1+\alpha}\,\Gamma(\alpha+4)\,\alpha\,\lambda^3$$
$$-\frac{1249\,\lambda^{1+\alpha}\,\Gamma(\alpha+4)\,\alpha^2\,\lambda}{40}-61\,\lambda^{1+\alpha}\,\Gamma(\alpha+4)\,\alpha\,\lambda^2-\frac{183\,\lambda^{1+\alpha}\,\Gamma(\alpha+4)\,\alpha\,\lambda}{5}$$
$$+\frac{\lambda^{1+\alpha}\,\Gamma(\alpha+3)\,\alpha^8\,\lambda^2}{1152}+\frac{\lambda^{1+\alpha}\,\Gamma(\alpha+3)\,\alpha^8\,\lambda}{720}+\frac{7\,\lambda^{1+\alpha}\,\Gamma(\alpha+3)\,\alpha^7\,\lambda^2}{288}$$
$$+\frac{29\,\lambda^{1+\alpha}\,\Gamma(\alpha+3)\,\alpha^7\,\lambda}{720}+\frac{167\,\lambda^{1+\alpha}\,\Gamma(\alpha+3)\,\alpha^6\,\lambda^2}{576}+\frac{361\,\lambda^{1+\alpha}\,\Gamma(\alpha+3)\,\alpha^6\,\lambda}{720}$$
$$+\frac{553\,\lambda^{1+\alpha}\,\Gamma(\alpha+3)\,\alpha^5\,\lambda^2}{288}+\frac{2519\,\lambda^{1+\alpha}\,\Gamma(\alpha+3)\,\alpha^5\,\lambda}{720}+\frac{8869\,\lambda^{1+\alpha}\,\Gamma(\alpha+3)\,\alpha^4\,\lambda^2}{1152}$$
$$+\frac{5387\,\lambda^{1+\alpha}\,\Gamma(\alpha+3)\,\alpha^4\,\lambda}{360}+\frac{343\,\lambda^{1+\alpha}\,\Gamma(\alpha+3)\,\alpha^3\,\lambda^2}{18}+\frac{7229\,\lambda^{1+\alpha}\,\Gamma(\alpha+3)\,\alpha^3\,\lambda}{180}$$
$$+\frac{2713\,\lambda^{1+\alpha}\,\Gamma(\alpha+3)\,\alpha^2\,\lambda^2}{96}+\frac{1981\,\lambda^{1+\alpha}\,\Gamma(\alpha+3)\,\alpha^2\,\lambda}{30}+\frac{91\,\lambda^{1+\alpha}\,\Gamma(\alpha+3)\,\alpha\,\lambda^2}{4}$$
$$+\frac{304\,\lambda^{1+\alpha}\,\Gamma(\alpha+3)\,\alpha\,\lambda}{5}-\frac{\lambda^{\alpha+2}\,\Gamma(\alpha+6)\,\alpha^5\,\lambda^4}{2880}-\frac{\lambda^{\alpha+2}\,\Gamma(\alpha+6)\,\alpha^5\,\lambda^3}{288}$$
$$-\frac{\lambda^{\alpha+2}\,\Gamma(\alpha+6)\,\alpha^4\,\lambda^4}{180}-\frac{\lambda^{\alpha+2}\,\Gamma(\alpha+6)\,\alpha^5\,\lambda^2}{144}-\frac{5\,\lambda^{\alpha+2}\,\Gamma(\alpha+6)\,\alpha^4\,\lambda^3}{72}$$
$$-\frac{19\,\lambda^{\alpha+2}\,\Gamma(\alpha+6)\,\alpha^3\,\lambda^4}{576}-\frac{\lambda^{\alpha+2}\,\Gamma(\alpha+6)\,\alpha^5\,\lambda}{288}-\frac{11\,\lambda^{\alpha+2}\,\Gamma(\alpha+6)\,\alpha^4\,\lambda^2}{72}$$
$$-\frac{155\,\lambda^{\alpha+2}\,\Gamma(\alpha+6)\,\alpha^3\,\lambda^3}{288}-\frac{13\,\lambda^{\alpha+2}\,\Gamma(\alpha+6)\,\alpha^2\,\lambda^4}{144}-\frac{11\,\lambda^{\alpha+2}\,\Gamma(\alpha+6)\,\alpha^4\,\lambda}{144}$$

$$-\frac{191\,\lambda^{\alpha+2}\,\Gamma(\alpha+6)\,\alpha^3\,\lambda^2}{144}-\frac{145\,\lambda^{\alpha+2}\,\Gamma(\alpha+6)\,\alpha^2\,\lambda^3}{72}-\frac{9\,\lambda^{\alpha+2}\,\Gamma(\alpha+6)\,\alpha\,\lambda^4}{80}$$
$$-\frac{191\,\lambda^{\alpha+2}\,\Gamma(\alpha+6)\,\alpha^3\,\lambda}{288}-\frac{409\,\lambda^{\alpha+2}\,\Gamma(\alpha+6)\,\alpha^2\,\lambda^2}{72}-\frac{29\,\lambda^{\alpha+2}\,\Gamma(\alpha+6)\,\alpha\,\lambda^3}{8}$$
$$-\frac{409\,\lambda^{\alpha+2}\,\Gamma(\alpha+6)\,\alpha^2\,\lambda}{144}-\frac{\lambda^{1+\alpha}\,\Gamma(\alpha+4)\,\alpha^7\,\lambda^3}{864}-\frac{\lambda^{1+\alpha}\,\Gamma(\alpha+4)\,\alpha^7\,\lambda^2}{288}$$
$$+\frac{\lambda^{\alpha+5}\,\Gamma(\alpha+9)\,\alpha^2\,\lambda^2}{2880}+\frac{\lambda^{\alpha+5}\,\Gamma(\alpha+9)\,\alpha^2\,\lambda}{720}+\frac{7\,\lambda^{\alpha+5}\,\Gamma(\alpha+9)\,\alpha\,\lambda^2}{2880}$$
$$+\frac{\lambda^{\alpha+5}\,\Gamma(\alpha+9)\,\alpha\,\lambda}{72}-\frac{\lambda^{1+\alpha}\,\Gamma(\alpha+2)\,\alpha^9\,\lambda}{2880}-\frac{\lambda^{1+\alpha}\,\Gamma(\alpha+2)\,\alpha^8\,\lambda}{96}$$
$$-\frac{13\,\lambda^{1+\alpha}\,\Gamma(\alpha+2)\,\alpha^7\,\lambda}{96}+\frac{5\,\lambda^{\alpha+3}\,\Gamma(7+\alpha)\,\alpha\,\lambda^4}{1728}+\frac{19\,\lambda^{\alpha+3}\,\Gamma(7+\alpha)\,\alpha^3\,\lambda}{216}$$
$$+\frac{119\,\lambda^{\alpha+3}\,\Gamma(7+\alpha)\,\alpha^2\,\lambda^2}{192}+\frac{\lambda^{\alpha+3}\,\Gamma(7+\alpha)\,\alpha\,\lambda^3}{4}+\frac{67\,\lambda^{\alpha+3}\,\Gamma(7+\alpha)\,\alpha^2\,\lambda}{108}$$
$$+\frac{57\,\lambda^{\alpha+3}\,\Gamma(7+\alpha)\,\alpha\,\lambda^2}{32}+\frac{52\,\lambda^{\alpha+3}\,\Gamma(7+\alpha)\,\alpha\,\lambda}{27}-\frac{\lambda^{1+\alpha}\,\Gamma(\alpha+4)\,\alpha^7}{4320}$$
$$-\frac{5\,\lambda^{1+\alpha}\,\Gamma(\alpha+4)\,\alpha^6}{864}-\frac{259\,\lambda^{1+\alpha}\,\Gamma(\alpha+4)\,\alpha^5}{4320}-\frac{287\,\lambda^{1+\alpha}\,\Gamma(\alpha+4)\,\alpha^4}{864}$$
$$-\frac{2\,\lambda^{1+\alpha}\,\Gamma(\alpha+4)}{3}+3\,\lambda^{1+\alpha}\,\Gamma(\alpha+3)-\frac{\lambda^{\alpha+2}\,\Gamma(\alpha+6)}{4}-\frac{5\,\lambda^{\alpha+4}\,\Gamma(\alpha+8)}{36}$$
$$-\frac{\lambda^{\alpha+6}\,\Gamma(\alpha+10)}{480}+\frac{\lambda^{1+\alpha}\,\Gamma(\alpha+5)}{24}+\frac{5\,\lambda^{\alpha+3}\,\Gamma(7+\alpha)}{16}+\frac{5\,\lambda^{\alpha+5}\,\Gamma(\alpha+9)}{192}$$
$$-4\,\lambda^{1+\alpha}\,\Gamma(\alpha+2)+\lambda^{1+\alpha}\,\Gamma(1+\alpha)+\frac{\lambda^{7+\alpha}\,\Gamma(11+\alpha)}{17280}\Bigg)\Bigg)$$

```
> factor(normal(expand(%)));
```

$$15\,(\lambda-1)^2\,\lambda^5\,\lambda^{\alpha}$$

Hermite expansions (Sec. 9.6)

PBM86 5.12.1.3

```
> F:=exp(2*x*t-t^2)*HermiteH(n,x-t);
```

$$F := e^{-t^2+2xt}\,\mathrm{HermiteH}(n, x-t)$$

```
> rec:=FunctiontoRec(F,HermiteH(k,x),u);
FunctiontoRec: time: 0.023 sec.
```

$$rec := 2\,t\,u(k) + 2\,(n-1-k)\,u(k+1)$$

PBM86 5.12.1.6

```
> F:=exp(t^2)*sin(2*x*t);
```

$$F := e^{t^2}\sin(2\,x\,t)$$

```
> rec:=FunctiontoRec(F,HermiteH(k,x),u);
FunctiontoRec: time: 0.017 sec.
```

$$rec := 4\,t^2\,u(k) + 4\,(k+2)\,(k+1)\,u(k+2)$$

PBM86 5.12.1.11

```
> F:=sqrt(Pi/2)*erf(2*t*x/sqrt(1-4*t^2));
```

$$F := \frac{\sqrt{2}\,\sqrt{\pi}\,\operatorname{erf}\left(\frac{2\,t\,x}{\sqrt{-4\,t^2+1}}\right)}{2}$$

```
> rec:=FunctiontoRec(F,HermiteH(k,x),u);
FunctiontoRec: time: 0.04 sec.
```

$$rec := -8\,k\,t^2\,u(k) - 4\,(k+2)\,(k+1)\,u(k+2)$$

PBM86 5.12.1.13 (sgn function)

```
> st:=time():
> rec:=DiffeqtoRec(x*diff(y(x),x),y(x),u(k),HermiteH(k,x));
```

$$rec := k\,u(k) + 2\,(k+2)\,(k+1)\,u(k+2)$$

```
> time()-st;
```

$$0.010$$

```
> LREtools[mhypergeomsols](rec=0,u(k));
```

$$\left[\left[2,\ \left\{\frac{(-1)^k\,k!}{k\,(2\,k)!}\right\}\right]\right]$$

```
> 1/eval(hnH,n=1)*int(exp(-x^2)*abs(x),x=-infinity..infinity);
```

$$\frac{1}{2\sqrt{\pi}}$$

```
> rsolve({rec,u(0)=0,u(1)=%},u(k));
```

$$\begin{cases} 0 & k::even \\ \dfrac{(-1)^{\frac{k}{2}-\frac{1}{2}}\,4^{-\frac{k}{2}+\frac{1}{2}}}{2\,k\,\Gamma\left(\frac{k}{2}+\frac{1}{2}\right)\sqrt{\pi}} & k::odd \end{cases}$$

```
> simplify(%) assuming k::odd;
```

$$\frac{-\mathrm{I}\,(-1)^{\frac{k}{2}}\,2^{-k}}{k\sqrt{\pi}\;\Gamma\left(\frac{k}{2}+\frac{1}{2}\right)}$$

```
> subs(k=2*m+1,%);
```

$$\frac{-\mathrm{I}\,(-1)^{\frac{1}{2}+m}\,2^{-2m-1}}{(2m+1)\sqrt{\pi}\;\Gamma(1+m)}$$

```
> simplify(%) assuming m::nonnegint;
```

$$\frac{(-1)^m\,4^{-m}}{\sqrt{\pi}\;(4m+2)\,\Gamma(1+m)}$$

PBM86 5.12.1.22

```
> F:=GAMMA(3/4)/(1-64*t^2)^(1/2)*(4*x^2*t)^(1/4)*exp(64*x^2*t^2/(64*
  t^2-1))*BesselI(-1/4,8*x^2*t/(1-64*t^2));
```

$$F := \frac{\Gamma\left(\frac{3}{4}\right)4^{1/4}\left(tx^2\right)^{1/4}\mathrm{e}^{\frac{64x^2t^2}{64t^2-1}}\,\mathrm{BesselI}\left(-\frac{1}{4},\frac{8x^2t}{-64t^2+1}\right)}{\sqrt{-64t^2+1}}$$

```
> rec:=FunctiontoRec(F,HermiteH(k,x),u);
FunctiontoRec: time: 0.046 sec.
```

$$rec := 64t^2u(k)-4\,(3+k)\,(k+4)\,u(k+4)$$

Since only the dependence in x is considered, the same result is obtained for the simpler

```
> f:=sqrt(x)*exp(-x^2*t^2/(1-t^2))*BesselI(-1/4,t*x^2/(1-t^2));
```

$$f := \sqrt{x}\;\mathrm{e}^{-\frac{x^2t^2}{-t^2+1}}\,\mathrm{BesselI}\left(-\frac{1}{4},\frac{tx^2}{-t^2+1}\right)$$

```
> FunctiontoRec(f,HermiteH(k,x),u);
FunctiontoRec: time: 0.03 sec.
```

$$t^2u(k)-4\,(3+k)\,(k+4)\,u(k+4)$$

```
> asympt(f,x,2) assuming t>0,t<1;
```

$$\left(\frac{\sqrt{2}\,\sqrt{\frac{1}{x}}}{2\sqrt{\frac{t\pi}{-t^2+1}}}+\mathrm{O}\left(\left(\frac{1}{x}\right)^{5/2}\right)\right)\mathrm{e}^{\frac{tx^2}{t+1}}$$

```
> normal(-1+t/(t+1));
```

$$-\frac{1}{t+1}$$

PBM86 5.12.3.1

```
> ini:='ini':
> F:=sin(2*a*x)/x;
```

$$F := \frac{\sin(2\,a\,x)}{x}$$

```
> rec:=FunctiontoRec(F,HermiteH(k,x),u);
FunctiontoRec: time: 0.022 sec.
```

$$rec := 2\,a^2\,u(k) + 2\,(k+2)\,\left(2\,a^2 + k + 3\right)\,u(k+2) + 4\,(k+2)\,(3+k)\,(k+4)\,u(k+4)$$

```
> for i from 0 to 3 do ini[i]:=1/eval(hnH,n=i)*int(F*expand(HermiteH
  (i,x))*exp(-x^2),x=-infinity..infinity) od;
```

$$ini_0 := \sqrt{\pi}\ \mathrm{erf}(a)$$

$$ini_1 := 0$$

$$ini_2 := \frac{\displaystyle\int_{-\infty}^{\infty} \frac{\sin(2\,a\,x)\,\left(4\,x^2-2\right)\,\mathrm{e}^{-x^2}}{x}\,\mathrm{d}x}{8\,\sqrt{\pi}}$$

$$ini_3 := 0$$

```
> int(x^k*(4*x^2 - 2)*exp(-x^2)/x, x = -infinity .. infinity)
  assuming k::posint;
```

$$-\left((-1)^k - 1\right)\,(k-1)\,\Gamma\left(\frac{k}{2}\right)$$

```
> simplify(subs(k=2*m+1,%)) assuming m::nonnegint;
```

$$4\,m\,\Gamma\left(\frac{1}{2} + m\right)$$

```
> sum((2*a)^(2*m+1)*(-1)^m/(2*m+1)!*%,m=0..infinity);
```

$$-\frac{2\,a^3\,\pi}{\left(a^2\right)^{3/2}} + \frac{4\,a\,\sqrt{\pi}}{\mathrm{e}^{a^2}} + \frac{2\,a^3\,\pi\,\mathrm{erfc}\left(\sqrt{a^2}\right)}{\left(a^2\right)^{3/2}}$$

```
> simplify(%) assuming a>0;
```

$$4\,a\,\sqrt{\pi}\ \mathrm{e}^{-a^2} - 2\,\pi\,\mathrm{erf}(a)$$

```
> vali2:=%/eval(hnH,n=2):
```

Check:

```
> evalf(subs(a=1/3,ini[2])),evalf(subs(a=1/3,vali2));
```

$$-0.01155437442,\ -0.01155437453$$

```
> ini[2]:=vali2:
> i:='i':
> sol:=rsolve({rec,seq(u(i)=ini[i],i=0..3)},u(k)) assuming a>0,
  k::even;
```

$$sol := \frac{\left(-\frac{1}{4}\right)^{\frac{k}{2}} \left(\sqrt{\pi}\ \mathrm{erf}(a) + \frac{\sqrt{\pi}\ \left(\mathrm{erfc}(a)\ \Gamma\left(\frac{k}{2}+\frac{1}{2}\right) - \Gamma\left(\frac{k}{2}+\frac{1}{2}, a^2\right)\right)}{e^0\ \Gamma\left(\frac{k}{2}+\frac{1}{2}\right)}\right)}{\Gamma\left(\frac{k}{2}+1\right)}$$

```
> sol:=convert(sol,erf);
```

$$sol := \frac{\left(-\frac{1}{4}\right)^{\frac{k}{2}} \left(\sqrt{\pi}\ \mathrm{erf}(a) + \frac{\sqrt{\pi}\ \left((1-\mathrm{erf}(a))\ \Gamma\left(\frac{k}{2}+\frac{1}{2}\right) - \Gamma\left(\frac{k}{2}+\frac{1}{2}, a^2\right)\right)}{\Gamma\left(\frac{k}{2}+\frac{1}{2}\right)}\right)}{\Gamma\left(\frac{k}{2}+1\right)}$$

```
> sol:=simplify(%);
```

$$sol := \frac{\left(-\Gamma\left(\frac{k}{2}+\frac{1}{2}, a^2\right) + \Gamma\left(\frac{k}{2}+\frac{1}{2}\right)\right) (-1)^{\frac{k}{2}}\ 2^{-k}\ \sqrt{\pi}}{\Gamma\left(\frac{k}{2}+\frac{1}{2}\right) \Gamma\left(\frac{k}{2}+1\right)}$$

```
> sol:=subs(k=2*k,sol);
```

$$sol := \frac{\left(-\Gamma\left(k+\frac{1}{2}, a^2\right) + \Gamma\left(k+\frac{1}{2}\right)\right) (-1)^{k}\ 2^{-2k}\ \sqrt{\pi}}{\Gamma\left(k+\frac{1}{2}\right) \Gamma(k+1)}$$

```
> GAMMA(2*k+1)=expand(GAMMA(2*k+1));
```

$$\Gamma(2k+1) = \frac{k\ (2^k)^2\ \Gamma(k)\ \Gamma\left(k+\frac{1}{2}\right)}{\sqrt{\pi}}$$

```
> sol:=subs(isolate(%,sqrt(Pi)),sol);
```

$$sol := \frac{\left(-\Gamma\left(k+\frac{1}{2}, a^2\right) + \Gamma\left(k+\frac{1}{2}\right)\right) (-1)^{k}\ 2^{-2k}\ k\ (2^k)^2\ \Gamma(k)}{\Gamma(2k+1)\ \Gamma(k+1)}$$

```
> sol:=subs(GAMMA(k+1)=expand(GAMMA(k+1)),sol);
```

$$sol := \frac{\left(-\Gamma\left(k+\frac{1}{2}, a^2\right) + \Gamma\left(k+\frac{1}{2}\right)\right) (-1)^{k}\ 2^{-2k}\ (2^k)^2}{\Gamma(2k+1)}$$

```
> sol:=combine(%,power);
```

$$sol := \frac{(-1)^{k} \left(-\Gamma\left(k+\frac{1}{2}, a^2\right) + \Gamma\left(k+\frac{1}{2}\right)\right)}{\Gamma(2k+1)}$$

Gegenbauer expansions from PrBrMa 1986 5.13.1

5.13.1.7 sgn function

```
> DiffoptoRec(x*Dx,Dx,x,k,GegenbauerC(k,nu,x));
```

$$OrePoly(-k\,(\nu+k+2),\,0,\,-(\nu+k)\,(k+2+2\,\nu))$$

```
> rec:=add(op(i,%)*u(k+i-1),i=1..nops(op(1,%)));
```

$$rec := -k\,(\nu+k+2)\,u(k)-(\nu+k)\,(k+2+2\,\nu)\,u(k+2)$$

```
> collect(rec,u,factor);
```

$$-k\,(\nu+k+2)\,u(k)-(\nu+k)\,(k+2+2\,\nu)\,u(k+2)$$

```
> rsolve(rec,u(k));
```

$$\begin{cases} charfcn_0(k)\,u(0) & k::even \\ \dfrac{(-1)^{\frac{k}{2}-\frac{1}{2}}\,(\nu+k)\,u(1)\,\Gamma\left(\nu+\frac{3}{2}\right)\Gamma\left(\frac{k}{2}\right)}{\sqrt{\pi}\,(1+\nu)\,\Gamma\left(\frac{k}{2}+1+\nu\right)} & k::odd \end{cases}$$

5.13.1.9

```
> F:=(1-2*t*x+t^2)^(-n/2-nu)*GegenbauerC(n,nu,(x-t)/sqrt(1-2*t*x+t^2)
  );
```

$$F := \left(t^2-2\,x\,t+1\right)^{-\frac{n}{2}-\nu}\,\mathrm{GegenbauerC}\left(n,\nu,\frac{x-t}{\sqrt{t^2-2\,x\,t+1}}\right)$$

```
> rec:=FunctiontoRec(F,GegenbauerC(k,nu,x),u) assuming nu>-1/2,nu<>0;
FunctiontoRec: time: 1.141 sec.
```

$$\begin{aligned} rec := {} & -(\nu+k+3)\,(k+2)\,(k+1)\,(k+n+2\,\nu)\,t^2\,u(k)+(\nu+k+3)\,t\,(k+2)\,\left(k^2\,t^2\right. \\ & +2\,k\,\nu\,t^2+t^2\,k+2\,\nu\,t^2+3\,k^2+2\,n\,k+8\,k\,\nu-n^2+4\,\nu^2+7\,k+3\,n+10\,\nu+4\Big)\,u(k+1) \\ & +\left(-4\,k^4\,t^2-16\,k^3\,\nu\,t^2-20\,k^2\,\nu^2\,t^2-8\,k\,\nu^3\,t^2-32\,k^3\,t^2-2\,k^2\,n\,t^2-102\,k^2\,\nu\,t^2-4\,k\,n\,\nu\,t^2\right. \\ & -92\,k\,\nu^2\,t^2-4\,n\,\nu^2\,t^2-24\,\nu^3\,t^2-2\,k^4-8\,k^3\,\nu+2\,k^2\,n^2+4\,k^2\,n\,\nu-10\,k^2\,\nu^2-92\,k^2\,t^2 \\ & +4\,k\,n^2\,\nu+8\,k\,n\,\nu^2-8\,k\,n\,t^2-4\,k\,\nu^3-208\,k\,\nu\,t^2+2\,n^2\,\nu^2+4\,n\,\nu^3-6\,n\,\nu\,t^2-100\,\nu^2\,t^2 \\ & -16\,k^3-48\,k^2\,\nu+8\,k\,n^2+16\,k\,n\,\nu-40\,k\,\nu^2-112\,t^2\,k+8\,n^2\,\nu+16\,n\,\nu^2-6\,n\,t^2-8\,\nu^3 \\ & -132\,\nu\,t^2-46\,k^2-92\,k\,\nu+6\,n^2+12\,n\,\nu-40\,\nu^2-48\,t^2-56\,k-56\,\nu-24\Big)\,u(k+2)+(\nu \\ & +k+1)\,t\,(k+2+2\,\nu)\,\left(k^2\,t^2+2\,k\,\nu\,t^2+7\,t^2\,k+8\,\nu\,t^2+3\,k^2-2\,n\,k+4\,k\,\nu-n^2-4\,n\,\nu\right. \\ & +12\,t^2+17\,k-5\,n+12\,\nu+24\Big)\,u(3+k)-(\nu+k+1)\,(k+3+2\,\nu)\,(k+2+2\,\nu)\,(k \\ & -n+4)\,t^2\,u(k+4) \end{aligned}$$

Special case for nu=1:

```
> FunctiontoRec(op(subs(nu=1,[F,GegenbauerC(k,nu,x)])),u);
```

```
FunctiontoRec: time: 0.231 sec.
```

$$-t^2\,(k+1)\,(k+n+2)\,u(k)+t\left(k^2\,t^2+3\,t^2\,k+3\,k^2+2\,n\,k-n^2+2\,t^2+15\,k+3\,n+18\right)u(k+1)+\left(-4\,k^2\,t^2-24\,t^2\,k-2\,n\,t^2-2\,k^2+2\,n^2-38\,t^2-12\,k+4\,n-16\right)u(k+2)+t\left(k^2\,t^2+9\,t^2\,k+3\,k^2-2\,n\,k-n^2+20\,t^2+21\,k-9\,n+36\right)u(3+k)-t^2\,(k+5)\,(k-n+4)\,u(k+4)$$

Bessel polynomials identities (Sec.9.7) from Al-Salam (1957)

Relation with Laguerre polynomials (Eq. 2.5)

```
> rec:=FunctiontoRec(n!*(-x/2)^n*LaguerreL(n,-2*n-a-1,2/x),Bessely(k,
  a+2,x),u) assuming k::nonnegint,n::nonnegint,a+2+k>-1,-2*n-a-1/2>-1;
FunctiontoRec: time: 0.201 sec.
```

$$rec := (k-n)\,(k+a+n+1)\,u(k)$$

Connection (Eq. 8.2)

```
> rec:=FunctiontoRec(Bessely(m,a,x),Bessely(n,b,x),u);
FunctiontoRec: time: 0.094 sec.
```

$$rec := -(b+n-1)\,(b+1+2\,n)\,(m-n)\,(m+a+n-1)\,u(n)-(b+2\,n-1)\,(n+1)\,(b+m+n)\,(-b+a-n-1+m)\,u(n+1)$$

Generating functions

Eq. 2.11

```
> F:=exp((1-(1-2*x*t)^(1/2))/x);
```

$$F := \mathrm{e}^{\frac{1-\sqrt{-2\,x\,t+1}}{x}}$$

```
> rec:=FunctiontoRec(F,Bessely(k,2,x),u);
FunctiontoRec: time: 0.108 sec.
```

$$rec := (7+2\,k)\,t^2\,k\,u(k)-(7+2\,k)\,t\left(3\,k^2-t^2+9\,k+6\right)u(k+1)+(k+2)\,(3+k)\left(4\,k^2-8\,t^2+20\,k+21\right)u(k+2)+(2\,k+3)\,t\left(3\,k^2-t^2+21\,k+36\right)u(3+k)+(2\,k+3)\,t^2\,(k+5)\,u(k+4)$$

```
> normal(expand(eval(rec,u=unapply(t^(k+1)/(k+1)!,k))));
```

$$0$$

Eq. 2.14

```
> F:=(1-2*x*t)^(-1/2)*(1/2+1/2*(1-2*x*t)^(1/2))^(-a)*exp((1-(1-2*x*t)
  ^(1/2))/x);
```

$$F := \frac{\left(\frac{1}{2} + \frac{\sqrt{-2xt+1}}{2}\right)^{-a} e^{\frac{1-\sqrt{-2xt+1}}{x}}}{\sqrt{-2xt+1}}$$

```
> rec:=FunctiontoRec(F,Bessely(k,a+2,x),u) assuming a+2+k>-1,
  k::nonnegint;
```

```
FunctiontoRec: time: 26.603 sec.
```

[Length of output exceeds limit of 50000]

```
> normal(expand(eval(rec,u=unapply(t^k/k!,k))));
```

$$0$$

Eq. 6.12

```
> F:=(1-t)^(-1-b)*hypergeom([(1+b)/2,(2+b)/2],[],2*x*t/(1-t)^2);
```

$$F := (1-t)^{-1-b}\,\text{hypergeom}\left(\left[1+\frac{b}{2}, \frac{1}{2}+\frac{b}{2}\right], [\,], \frac{2xt}{(1-t)^2}\right)$$

```
> rec:=FunctiontoRec(F,Bessely(n,b+2,x),u) assuming b+2+n>-1,
  n::nonnegint;
```

```
FunctiontoRec: time: 0.149 sec.
```

$$rec := t\,(b+n+2)\,(1+b+n)\,u(n) - (n+1)\,(b+n+2)\,(t^2+1)\,u(n+1) + (n+2)\,(n+1)\,t\,u(n+2)$$

```
> LREtools[hypergeomsols](rec,u(n),{},output=basis);
```

$$\left[\frac{t^n\,\Gamma(1+b+n)}{\Gamma(n+1)}, \frac{\left(\frac{1}{t}\right)^n \Gamma(1+b+n)}{\Gamma(n+1)}\right]$$

Eq. (7.5)

```
> rec:=FunctiontoRec((-x/2)^s,Bessely(n,b+2,x),u);
```

```
FunctiontoRec: time: 0.045 sec.
```

$$rec := (1+b+n)\,(b+3+2n)\,(-s+n)\,u(n) - (b+1+2n)\,(n+1)\,(s+b+n+2)\,u(n+1)$$

```
> sol:=rsolve(rec,u(n)) assuming n::nonnegint,s::nonnegint,n<=s;
```

$$sol := \frac{(-1)^n\,(b+1+2n)\,\Gamma(s+b+2)\,\Gamma(1+b+n)\,\Gamma(s+1)\,u(0)}{\Gamma(2+b)\,\Gamma(s+b+n+2)\,\Gamma(1-n+s)\,\Gamma(n+1)}$$

```
> solve(subs(n=s,sol)=(-1)^s/2^s/subs(n=s,a=b+2,convert(lcoeffyn,
  GAMMA)),u(0));
```

$$\frac{\Gamma(2+b)\,\Gamma(2s+b+2)\,\Gamma(1)}{\Gamma(s+b+2)\,\Gamma(b+1+2s)\,(b+1+2s)}$$

```
> sol:=subs(u(0)=%,sol);
```

$$sol := \frac{(-1)^n\,(b+1+2\,n)\,\Gamma(1+b+n)\,\Gamma(s+1)\,\Gamma(2\,s+b+2)}{\Gamma(s+b+n+2)\,\Gamma(1-n+s)\,\Gamma(n+1)\,\Gamma(b+1+2\,s)\,(b+1+2\,s)}$$

```
> sol:=simplify(sol,GAMMA);
```

$$sol := \frac{(-1)^n\,(b+1+2\,n)\,\Gamma(1+b+n)\,\Gamma(s+1)}{\Gamma(s+b+n+2)\,\Gamma(1-n+s)\,\Gamma(n+1)}$$

Check:

```
> collect(expand(add(eval(%,[n=i,s=5])*Bessely(i,b+2,x),i=0..5)),x,
  normal);
```

$$-\frac{x^5}{32}$$

```
> convert(sol,factorial);
```

$$\frac{(-1)^n\,(b+1+2\,n)\,(b+n)!\,s!}{(s+b+1+n)!\,(s-n)!\,n!}$$

Linearization by Atia Zeng 2012

```
> FunctiontoRec(Bessely(n,2,x/2)*Bessely(m,2,x/2),Bessely(k,2,x),u);
FunctiontoRec: time: 0.243 sec.
```

$$(2\,k+13)\,(2\,k+11)\,(k+4)\,(9+2\,k)\,(-n+1+m+k)\,(-n-m+k)\,(k+m+n+2)\,(n+1-m+k)\,u(k)+(9+2\,k)\,(13\,k^4-14\,k^2\,m^2-14\,k^2\,n^2+m^4-2\,m^2\,n^2+n^4+110\,k^3-14\,k^2\,m-14\,k^2\,n-74\,k\,m^2-74\,k\,n^2+2\,m^3-2\,m^2\,n-2\,m\,n^2+2\,n^3+317\,k^2-74\,m\,k-74\,n\,k-79\,m^2-2\,m\,n-79\,n^2+376\,k-80\,m-80\,n+156)\,(2\,k+13)\,(2\,k+11)\,(2\,k+1)\,u(k+1)+(207\,k^5-126\,k^3\,m^2-126\,k^3\,n^2-k\,m^4+2\,k\,m^2\,n^2-k\,n^4+2935\,k^4-126\,k^3\,m-126\,k^3\,n-1182\,k^2\,m^2-1182\,k^2\,n^2-2\,k\,m^3+2\,k\,m^2\,n+2\,k\,m\,n^2-2\,k\,n^3-9\,m^4+18\,m^2\,n^2-9\,n^4+15993\,k^3-1182\,k^2\,m-1182\,k^2\,n-3413\,k\,m^2+2\,k\,m\,n-3413\,k\,n^2-18\,m^3+18\,m^2\,n+18\,m\,n^2-18\,n^3+41813\,k^2-3412\,m\,k-3412\,n\,k-2889\,m^2+18\,m\,n-2889\,n^2+52512\,k-2880\,m-2880\,n+25380)\,(2\,k+13)\,(2\,k+11)\,(2\,k+1)\,u(k+2)+2\,(2\,k+3)\,(2\,k+1)\,(2\,k+13)\,(2\,k+11)\,(139\,k^4-50\,k^2\,m^2-50\,k^2\,n^2-m^4+2\,m^2\,n^2-n^4+1946\,k^3-50\,k^2\,m-50\,k^2\,n-350\,k\,m^2-350\,k\,n^2-2\,m^3+2\,m^2\,n+2\,m\,n^2-2\,n^3+10109\,k^2-350\,m\,k-350\,n\,k-581\,m^2+2\,m\,n-581\,n^2+23086\,k-580\,m-580\,n+19524)\,u(3+k)+(207\,k^5-126\,k^3\,m^2-126\,k^3\,n^2-k\,m^4+2\,k\,m^2\,n^2-k\,n^4+4310\,k^4-126\,k^3\,m-126\,k^3\,n-1464\,k^2\,m^2-1464\,k^2\,n^2-2\,k\,m^3+2\,k\,m^2\,n+2\,k\,m\,n^2-2\,k\,n^3+2\,m^4-4\,m^2\,n^2+2\,n^4+35243\,k^3-1464\,k^2\,m-1464\,k^2\,n-5387\,k\,m^2+2\,k\,m\,n-5387\,k\,n^2+4\,m^3-4\,m^2\,n-4\,m\,n^2+4\,n^3+141160\,k^2-5386\,m\,k-5386\,n\,k-6302\,m^2-4\,m\,n-6302\,n^2+276316\,k-6304\,m-6304\,n+211080)\,(2\,k+13)\,(2\,k+3)\,(2\,k+1)\,u(k+4)+(13\,k^4-14\,k^2\,m^2-14\,k^2\,n^2+m^4-2\,m^2\,n^2+n^4+254\,k^3-14\,k^2\,m-14\,k^2\,n-122\,k\,m^2-122\,k\,n^2$$

$$+ 2\,m^3 - 2\,m^2\,n - 2\,m\,n^2 + 2\,n^3 + 1829\,k^2 - 122\,m\,k - 122\,n\,k - 247\,m^2 - 2\,m\,n - 247\,n^2 + 5728\,k - 248\,m - 248\,n + 6540\big)\,(2\,k+13)\,(2\,k+5)\,(2\,k+3)\,(2\,k+1)\,u(k+5) + (-n + 5 - m + k)\,(n+6-m+k)\,(-n+6+m+k)\,(n+7+m+k)\,(2\,k+3)\,(2\,k+1)\,(2\,k+5)\,(3+k)\,u(6+k)$$

Jacobi expansions from PrBrMa (Sec. 9.8) 1986 5.14.1 & 5.14.5

```
> rho:=sqrt(1-2*t*x+t^2);
```

$$\rho := \sqrt{t^2 - 2\,x\,t + 1}$$

5.14.1.3

```
> alpha:='alpha':
> F:=1/alpha*((1-t+rho)/(1+t+rho))^(-alpha)-1/alpha;
```

$$F := \frac{\left(\dfrac{1 - t + \sqrt{t^2 - 2\,x\,t + 1}}{1 + t + \sqrt{t^2 - 2\,x\,t + 1}}\right)^{-\alpha}}{\alpha} - \frac{1}{\alpha}$$

```
> rec:=FunctiontoRec(F,JacobiP(k,alpha,-alpha,x),u) assuming
  alpha>-1,-alpha>-1,k::nonnegint;
FunctiontoRec: time: 1.033 sec.
```

$$rec := -t^2\,(\alpha\,t-1)\,(3+k)\,k\,(9+2\,k)\,(k+2)\,(k+4)^2\,(k+1)^2\,(2\,k+11)\,(k+5)\,u(k) - t\,(3 + k)\,(2\,k+11)\,(k+5)\,(9+2\,k)\,(k+4)\,(k+2)\,(k+1)\,\big(\alpha^3\,k\,t^3 - \alpha\,k^3\,t^3 + 4\,\alpha^3\,t^3 - 6\,\alpha\,k^2\,t^3 - 5\,\alpha^2\,k\,t^2 - 5\,\alpha\,k^3\,t - 9\,\alpha\,k\,t^3 + 3\,k^3\,t^2 - 10\,\alpha^2\,t^2 - 38\,\alpha\,k^2\,t - 4\,\alpha\,t^3 + 23\,k^2\,t^2 - 83\,\alpha\,k\,t + 3\,k^3 + 54\,t^2\,k - 54\,\alpha\,t + 21\,k^2 + 40\,t^2 + 42\,k + 24\big)\,u(k+1) + (3+k)\,(2\,k + 11)\,(k+5)\,(k+4)\,\big(8\,\alpha^4\,k^2\,t^4 + 14\,\alpha^3\,k^4\,t^3 - 12\,\alpha^2\,k^4\,t^4 - 14\,\alpha\,k^6\,t^3 + 4\,k^6\,t^4 + 48\,\alpha^4\,k\,t^4 + 163\,\alpha^3\,k^3\,t^3 - 132\,\alpha^2\,k^3\,t^4 - 235\,\alpha\,k^5\,t^3 + 68\,k^5\,t^4 + 54\,\alpha^4\,t^4 + 654\,\alpha^3\,k^2\,t^3 - 34\,\alpha^2\,k^4\,t^2 - 521\,\alpha^2\,k^2\,t^4 - 16\,\alpha\,k^6\,t - 1578\,\alpha\,k^4\,t^3 + 22\,k^6\,t^2 + 467\,k^4\,t^4 + 1065\,\alpha^3\,k\,t^3 - 375\,\alpha^2\,k^3\,t^2 - 885\,\alpha^2\,k\,t^4 - 280\,\alpha\,k^5\,t - 5421\,\alpha\,k^3\,t^3 + 383\,k^5\,t^2 + 1657\,k^3\,t^4 + 594\,\alpha^3\,t^3 - 1445\,\alpha^2\,k^2\,t^2 - 540\,\alpha^2\,t^4 - 1964\,\alpha\,k^4\,t - 10062\,\alpha\,k^2\,t^3 + 4\,k^6 + 2700\,k^4\,t^2 + 3204\,k^2\,t^4 - 2310\,\alpha^2\,k\,t^2 - 7058\,\alpha\,k^3\,t - 9588\,\alpha\,k\,t^3 + 64\,k^5 + 9856\,k^3\,t^2 + 3204\,k\,t^4 - 1296\,\alpha^2\,t^2 - 13716\,\alpha\,k^2\,t - 3672\,\alpha\,t^3 + 407\,k^4 + 19637\,k^2\,t^2 + 1296\,t^4 - 13698\,\alpha\,k\,t + 1310\,k^3 + 20238\,t^2\,k - 5508\,\alpha\,t + 2241\,k^2 + 8424\,t^2 + 1926\,k + 648\big)\,u(k+2) + (2\,k+3)\,(2\,k+11)\,(k+5)\,(k+4)\,\big(2\,\alpha^5\,k^2\,t^4 - 4\,\alpha^3\,k^4\,t^4 + 2\,\alpha\,k^6\,t^4 + 14\,\alpha^5\,k\,t^4 - 56\,\alpha^3\,k^3\,t^4 + 42\,\alpha\,k^5\,t^4 + 18\,\alpha^5\,t^4 - 18\,\alpha^4\,k^2\,t^3 - 14\,\alpha^3\,k^4\,t^2 - 282\,\alpha^3\,k^2\,t^4 + 28\,\alpha^2\,k^4\,t^3 + 14\,\alpha\,k^6\,t^2 + 360\,\alpha\,k^4\,t^4 - 10\,k^6\,t^3$$

$$-126\,\alpha^4 k t^3 - 196\,\alpha^3 k^3 t^2 - 602\,\alpha^3 k t^4 + 392\,\alpha^2 k^3 t^3 + 294\,\alpha k^5 t^2 + 1610\,\alpha k^3 t^4 - 210\,k^5 t^3$$
$$- 210\,\alpha^4 t^3 - 988\,\alpha^3 k^2 t^2 - 450\,\alpha^3 t^4 + 14\,\alpha^2 k^4 t + 2026\,\alpha^2 k^2 t^3 + 4\,\alpha k^6 + 2533\,\alpha k^4 t^2$$
$$+ 3954\,\alpha k^2 t^4 - 10\,k^6 t - 1820\,k^4 t^3 - 2114\,\alpha^3 k t^2 + 196\,\alpha^2 k^3 t + 4578\,\alpha^2 k t^3 + 84\,\alpha k^5$$
$$+ 11452\,\alpha k^3 t^2 + 5040\,\alpha k t^4 - 210\,k^5 t - 8330\,k^3 t^3 - 1608\,\alpha^3 t^2 + 998\,\alpha^2 k^2 t + 3810\,\alpha^2 t^3$$
$$+ 721\,\alpha k^4 + 28635\,\alpha k^2 t^2 + 2592\,\alpha t^4 - 1818\,k^4 t - 21230\,k^2 t^3 + 2184\,\alpha^2 k t + 3234\,\alpha k^3$$
$$+ 37520\,\alpha k t^2 - 8302\,k^3 t - 28560\,k t^3 + 1728\,\alpha^2 t + 7987\,\alpha k^2 + 20112\,\alpha t^2 - 21084\,k^2 t$$
$$- 15840\,t^3 + 10290\,\alpha k - 28224\,k t + 5400\,\alpha - 15552\,t)\,u(3+k) - (\alpha + 4 + k)\,(\alpha - k$$
$$- 4)\,(2k+3)\,(k+5)\,(8\,\alpha^4 k^2 t^4 + 14\,\alpha^3 k^4 t^3 - 12\,\alpha^2 k^4 t^4 - 14\,\alpha k^6 t^3 + 4\,k^6 t^4 + 64\,\alpha^4 k t^4$$
$$+ 229\,\alpha^3 k^3 t^3 - 204\,\alpha^2 k^3 t^4 - 353\,\alpha k^5 t^3 + 100\,k^5 t^4 + 110\,\alpha^4 t^4 + 1347\,\alpha^3 k^2 t^3 - 34\,\alpha^2 k^4 t^2$$
$$- 1277\,\alpha^2 k^2 t^4 - 16\,\alpha k^6 t - 3643\,\alpha k^4 t^3 + 22\,k^6 t^2 + 1027\,k^4 t^4 + 3338\,\alpha^3 k t^3 - 577\,\alpha^2 k^3 t^2$$
$$- 3469\,\alpha^2 k t^4 - 392\,\alpha k^5 t - 19653\,\alpha k^3 t^3 + 541\,k^5 t^2 + 5539\,k^3 t^4 + 2890\,\alpha^3 t^3 - 3566\,\alpha^2 k^2 t^2$$
$$- 3410\,\alpha^2 t^4 - 3924\,\alpha k^4 t - 58313\,\alpha k^2 t^3 + 4\,k^6 + 5465\,k^4 t^2 + 16525\,k^2 t^4 - 9443\,\alpha^2 k t^2$$
$$- 20494\,\alpha k^3 t - 90022\,\alpha k t^3 + 104\,k^5 + 28994\,k^3 t^2 + 25825\,k t^4 - 8940\,\alpha^2 t^2 - 58754\,\alpha k^2 t$$
$$- 56410\,\alpha t^3 + 1107\,k^4 + 85101\,k^2 t^2 + 16500\,t^4 - 87480\,\alpha k t + 6166\,k^3 + 130857\,t^2 k$$
$$- 52800\,\alpha t + 18929\,k^2 + 82260\,t^2 + 30330\,k + 19800)\,u(k+4) - (2k+3)\,(\alpha + 4 + k)\,(\alpha$$
$$- k - 4)\,(\alpha - k - 5)\,(\alpha + 5 + k)\,(2k+5)\,t\,(6+k)\,(\alpha^3 k t^3 - \alpha k^3 t^3 + 3\,\alpha^3 t^3 - 15\,\alpha k^2 t^3$$
$$- 5\,\alpha^2 k t^2 - 5\,\alpha k^3 t - 72\,\alpha k t^3 + 3\,k^3 t^2 - 25\,\alpha^2 t^2 - 67\,\alpha k^2 t - 108\,\alpha t^3 + 40\,k^2 t^2 - 286\,\alpha k t$$
$$+ 3\,k^3 + 173\,t^2 k - 380\,\alpha t + 42\,k^2 + 240\,t^2 + 189\,k + 270)\,u(k+5) + (2k+3)\,(\alpha + 4$$
$$+ k)\,(\alpha - k - 4)\,(\alpha - k - 5)\,(\alpha + 5 + k)\,(2k+5)\,t^2\,(\alpha + 6 + k)\,(\alpha - k - 6)\,(\alpha\,t$$
$$- 1)\,(3+k)\,(7+k)\,u(6+k)$$

```
> LREtools[hypergeomsols](rec,u(k),{},output=basis);
```

$$\left[\frac{t^k}{k+1},\ \frac{\left(\frac{1}{t}\right)^k \Gamma(k)\,\Gamma(k+1)}{\Gamma(k+1-\alpha)\,\Gamma(\alpha+1+k)}\right]$$

Check:

```
> expand(eval(rec,u=unapply(t^(k+1)/(k+1),k)));
```

$$0$$

5.14.1.4

```
> F:=2^(2*alpha)/alpha*(1-t+rho)^(-alpha)*(1+t+rho)^(-alpha);
```

$$F := \frac{2^{2\alpha}\left(1 - t + \sqrt{t^2 - 2xt + 1}\right)^{-\alpha}\left(1 + t + \sqrt{t^2 - 2xt + 1}\right)^{-\alpha}}{\alpha}$$

```
> rec:=FunctiontoRec(F,JacobiP(k,alpha,alpha,x),u) assuming alpha>-1:
FunctiontoRec: time: 2.669 sec.
```

```
> LREtools[hypergeomsols](rec,u(k),{},output=basis);
```

$$\left[\frac{t^k}{\alpha+k}, \frac{t^k\,\Gamma(k+2\,\alpha+1)\,\Gamma(k+1)}{\Gamma(\alpha+2+k)\,\Gamma(\alpha+1+k)}, \frac{\left(\frac{1}{t}\right)^k}{\alpha+k}, \frac{\left(\frac{1}{t}\right)^k\,\Gamma(k+2\,\alpha+1)\,\Gamma(k+1)}{\Gamma(\alpha+2+k)\,\Gamma(\alpha+1+k)}\right]$$

```
> expand(normal(eval(rec,u=unapply(t^k/(k+alpha),k))));
```

$$0$$

5.14.1.5

```
> F:=2^alpha/alpha*(1-t+rho)^(-alpha);
```

$$F := \frac{2^\alpha\left(1-t+\sqrt{t^2-2\,x\,t+1}\right)^{-\alpha}}{\alpha}$$

```
> rec:=FunctiontoRec(F,JacobiP(k,alpha,-1,x),u) assuming alpha>-1;
FunctiontoRec: time: 0.595 sec.
```

$$rec := -(\alpha+k)\,t\,u(k)+\left(t^2+1\right)(\alpha+1+k)\,u(k+1)-(\alpha+2+k)\,t\,u(k+2)$$

Special case for alpha=-1/2

```
> FunctiontoRec(subs(alpha=-1/2,F),JacobiP(k,-1/2,-1,x),u);
FunctiontoRec: time: 0.168 sec.
```

$$-t\,(2\,k-1)\,u(k)+\left(t^2+1\right)(2\,k+1)\,u(k+1)-(2\,k+3)\,t\,u(k+2)$$

```
> LREtools[hypergeomsols](rec,u(k),{},output=basis);
```

$$\left[\frac{t^k}{\alpha+k}, \frac{\left(\frac{1}{t}\right)^k}{\alpha+k}\right]$$

Only the first one makes sense for a generating series.

```
> expand(normal(eval(rec,u=unapply(t^k/(k+alpha),k))));
```

$$0$$

5.14.1.6

```
> ini:='ini':
> F:=(1+t)^(-alpha-beta-1)*hypergeom([(alpha+beta+1)/2,(alpha+beta+2)
  /2],[beta+1],(2*t*x+2*t)/(t+1)^2);
```

$$F := (t+1)^{-\alpha-\beta-1}\,\mathrm{hypergeom}\left(\left[\frac{\alpha}{2}+\frac{\beta}{2}+1, \frac{\alpha}{2}+\frac{\beta}{2}+\frac{1}{2}\right], [1+\beta], \frac{2\,x\,t+2\,t}{(t+1)^2}\right)$$

```
> rec:=FunctiontoRec(F,JacobiP(k,alpha,beta,x),u) assuming alpha>-1,
  beta>-1;
FunctiontoRec: time: 0.413 sec.
```

$$rec := -(k+2+\alpha+\beta)\,(k+\alpha+\beta+1)\,t\,u(k)+(\beta+k+1)\left(t^2+1\right)(k+2+\alpha+\beta)\,u(k+1)-(\beta+k+2)\,(\beta+k+1)\,t\,u(k+2)$$

Initial conditions from Prudnikov, Brychkov, Marichev vol. 3 2.21.1.24:

```
> eq:=Int(x^(alpha-1)*(y-x)^(beta-1)*hypergeom([a,b],[c],-omega*x),x=
```

```
0..y)=Beta(alpha,beta)*y^(alpha+beta-1)*hypergeom([a,b,alpha],[c,
alpha+beta],-omega*y):
> IntegrationTools[Change](eq,x=1+u):
> subs([alpha=B+1,beta=A+1,y=2,u=x],%):
> neweq:=subs(a=(alpha+beta+2)/2,b=(alpha+beta+1)/2,c=beta+1,omega=
-2*t/(1+t)^2,A=alpha+k,B=beta+ell,%);
```

$$neweq := \int_{-1}^{1} (1+x)^{\beta+\ell}\,(1-x)^{\alpha+k}\,\mathrm{hypergeom}\left(\left[\frac{\alpha}{2}+\frac{\beta}{2}+1,\ \frac{\alpha}{2}+\frac{\beta}{2}+\frac{1}{2}\right],\ [1+\beta],\ \frac{2\,t\,(1+x)}{(t+1)^2}\right) dx = \mathrm{B}(\alpha+1+k,\ \beta+\ell+1)\ 2^{\beta+\ell+1+\alpha+k}\,\mathrm{hypergeom}\left(\left[\frac{\alpha}{2}+\frac{\beta}{2}+1,\ \frac{\alpha}{2}+\frac{\beta}{2}+\frac{1}{2},\ \beta+\ell+1\right],\ [1+\beta,\ \beta+\ell+2+\alpha+k],\ \frac{4\,t}{(t+1)^2}\right)$$

```
> expand(JacobiP(0,alpha,beta,x));
```

$$1$$

```
> ini[0]:=eval(op(2,neweq),[k=0,ell=0])*(1+t)^(-alpha-beta-1)/eval
(hnJ,[n=0]);
```

$$ini_0 := \frac{1}{\Gamma(1+\alpha)\,\Gamma(1+\beta)}\left(\mathrm{B}(1+\alpha,\ 1+\beta)\ \mathrm{hypergeom}\left(\left[\frac{\alpha}{2}+\frac{\beta}{2}+1,\ \frac{\alpha}{2}+\frac{\beta}{2}+\frac{1}{2}\right],\ [\alpha+\beta+2],\ \frac{4\,t}{(t+1)^2}\right)(t+1)^{-\alpha-\beta-1}\,(\alpha+\beta+1)\,\Gamma(\alpha+\beta+1)\right)$$

```
> ini[0]:=simplify(%) assuming alpha>-1,beta>-1,t>-1,t<1;
```

$$ini_0 := 1$$

```
> expand(JacobiP(1,alpha,beta,x)=(alpha+1)/2*(1+x)-(beta+1)/2*(1-x));
```

$$\frac{1}{2}\,\alpha+\frac{1}{2}\,\alpha\,x+\frac{1}{2}\,\beta\,x-\frac{1}{2}\,\beta+x=\frac{1}{2}\,\alpha+\frac{1}{2}\,\alpha\,x+\frac{1}{2}\,\beta\,x-\frac{1}{2}\,\beta+x$$

```
> evalb(%);
```

$$true$$

```
> ini[1]:=((alpha+1)/2*eval(op(2,neweq),[k=0,ell=1])-(beta+1)/2*eval
(op(2,neweq),[k=1,ell=0]))*(1+t)^(-alpha-beta-1)/eval(hnJ,n=1);
```

$$ini_1 := \frac{1}{2^{\alpha+\beta+1}\,\Gamma(\alpha+2)\,\Gamma(\beta+2)}\left(\left(\frac{1}{2}\left((1+\alpha)\,\mathrm{B}(1+\alpha,\beta+2)\,2^{\alpha+\beta+2}\,\mathrm{hypergeom}\left(\left[\beta+2,\ \frac{\alpha}{2}+\frac{\beta}{2}+1,\ \frac{\alpha}{2}+\frac{\beta}{2}+\frac{1}{2}\right],\ [1+\beta,\ \beta+3+\alpha],\ \frac{4\,t}{(t+1)^2}\right)\right)-\frac{1}{2}\left((1+\beta)\,\mathrm{B}(1+\beta,\alpha+2)\,2^{\alpha+\beta+2}\,\mathrm{hypergeom}\left(\left[\frac{\alpha}{2}+\frac{\beta}{2}+1,\ \frac{\alpha}{2}+\frac{\beta}{2}+\frac{1}{2}\right],\ [\beta+3+\alpha],\ \frac{4\,t}{(t+1)^2}\right)\right)\right)(t+1)^{-\alpha-\beta-1}\,(\beta+3+\alpha)\,\Gamma(\alpha+\beta+2)\right)$$

```
> ini[1]:=simplify(%) assuming alpha>-1,beta>-1,t>-1,t<1;
```

$$ini_1 := \frac{t\,(\alpha+\beta+1)}{1+\beta}$$

```
> rsolve({rec,u(0)=ini[0],u(1)=ini[1]},u(k));
```

$$\frac{t^k\,\Gamma(k+\alpha+\beta+1)\,\Gamma(1+\beta)}{\Gamma(\alpha+\beta+1)\,\Gamma(\beta+k+1)}$$

5.14.1.7

```
> F:=GAMMA(alpha+1)*GAMMA(beta+1)*(2/t)^((alpha+beta)/2)*(x-1)^(-
  alpha/2)*(x+1)^(-beta/2)*BesselI(alpha,2*t*sqrt(x-1))*BesselI(beta,
  2*t*sqrt(x+1));
```

$$F := \Gamma(1+\alpha)\,\Gamma(1+\beta)\left(\frac{2}{t}\right)^{\frac{\alpha}{2}+\frac{\beta}{2}}(x-1)^{-\frac{\alpha}{2}}(1+x)^{-\frac{\beta}{2}}\,\mathrm{BesselI}\left(\alpha,\,2\,t\sqrt{x-1}\right)\mathrm{BesselI}\left(\beta,\,2\,t\sqrt{1+x}\right)$$

```
> rec:=FunctiontoRec(F,JacobiP(k,alpha,beta,x),u) assuming alpha>-1,
  beta>-1:
FunctiontoRec: time: 8.508 sec.
```

Special cases

```
> for total in [-1,0,1] do FunctiontoRec(op(subs(beta=total-alpha,[F,
  JacobiP(k,alpha,beta,x)])),u) assuming alpha>-1 od;
FunctiontoRec: time: 1.896 sec.
```

$$2\left(-8t^2+2\alpha+1\right)t^2(3+k)(k+2)(k+1)k(7+2k)(k+4)(9+2k)(k+5)u(k)+(3+k)(k+2)\left(-48\alpha^2k^3t^2+16k^5t^2+64kt^6+4\alpha^3k^3-296\alpha^2k^2t^2-4\alpha k^5-48\alpha k^3t^2-80\alpha kt^4+88k^4t^2+224t^6+18\alpha^3k^2+6\alpha^2k^3-512\alpha^2kt^2-22\alpha k^4-296\alpha k^2t^2-120\alpha t^4-2k^5+144k^3t^2-40kt^4+14\alpha^3k+27\alpha^2k^2-264\alpha^2t^2-30\alpha k^3-512\alpha kt^2-11k^4+168k^2t^2-60t^4+21\alpha^2k-5\alpha k^2-264\alpha t^2-16k^3+240t^2k+7\alpha k-7k^2+144t^2\right)(k+5)(k+4)(9+2k)u(k+1)-2(3+k)\left(8\alpha^4k^4-8\alpha^2k^6-80\alpha^2k^2t^4-16k^4t^4+72\alpha^4k^3+16\alpha^3k^4+44\alpha^3k^2t^2-96\alpha^2k^5-360\alpha^2kt^4-8\alpha k^6-44\alpha k^4t^2-80\alpha k^2t^4-64\alpha t^6-232k^3t^4+226\alpha^4k^2+144\alpha^3k^3+178\alpha^3kt^2-454\alpha^2k^4+66\alpha^2k^2t^2-360\alpha^2t^4-96\alpha k^5-406\alpha k^3t^2-360\alpha kt^4+4k^6-22k^4t^2-984k^2t^4-32t^6+294\alpha^4k+452\alpha^3k^2+168\alpha^3t^2-1080\alpha^2k^3+267\alpha^2kt^2-462\alpha k^4-1440\alpha k^2t^2-360\alpha t^4+48k^5-203k^3t^2-1488kt^4+132\alpha^4+588\alpha^3k-1356\alpha^2k^2+252\alpha^2t^2-1152\alpha k^3-2283\alpha kt^2+232k^4-731k^2t^2-720t^4+264\alpha^3-846\alpha^2k-1582\alpha k^2-1332\alpha t^2+576k^3-1186t^2k-204\alpha^2-1140\alpha k+772k^2-708t^2-336\alpha+528k+144\right)(k+5)(k+4)(9+2k)u(k+2)-2(k+5)(k+4)\left(-96\alpha^4k^4t^2+64\alpha^2k^6t^2+128\alpha^2k^2t^6+32k^8t^2+128k^4t^6\right)$$

$$+ 24\,\alpha^5 k^4 - 1152\,\alpha^4 k^3 t^2 - 48\,\alpha^3 k^6 - 192\,\alpha^3 k^4 t^2 - 224\,\alpha^3 k^2 t^4 + 1152\,\alpha^2 k^5 t^2 + 768\,\alpha^2 k t^6$$
$$+ 24\,\alpha k^8 + 64\,\alpha k^6 t^2 + 224\,\alpha k^4 t^4 + 128\,\alpha k^2 t^6 + 768 k^7 t^2 + 1536 k^3 t^6 + 288\,\alpha^5 k^3 + 60\,\alpha^4 k^4$$
$$- 4904\,\alpha^4 k^2 t^2 - 864\,\alpha^3 k^5 - 2304\,\alpha^3 k^3 t^2 - 1344\,\alpha^3 k t^4 - 72\,\alpha^2 k^6 + 8624\,\alpha^2 k^4 t^2$$
$$- 336\,\alpha^2 k^2 t^4 + 864\,\alpha^2 t^6 + 576\,\alpha k^7 + 1152\,\alpha k^5 t^2 + 2688\,\alpha k^3 t^4 + 768\,\alpha k t^6 + 12 k^8$$
$$+ 7800 k^6 t^2 + 112 k^4 t^4 + 6624 k^2 t^6 + 1242\,\alpha^5 k^2 + 720\,\alpha^4 k^3 - 8688\,\alpha^4 k t^2 - 6324\,\alpha^3 k^4$$
$$- 9808\,\alpha^3 k^2 t^2 - 1800\,\alpha^3 t^4 - 1296\,\alpha^2 k^5 + 34368\,\alpha^2 k^3 t^2 - 2016\,\alpha^2 k t^4 + 5946\,\alpha k^6$$
$$+ 8720\,\alpha k^4 t^2 + 11832\,\alpha k^2 t^4 + 864\,\alpha t^6 + 288 k^7 + 43632 k^5 t^2 + 1344 k^3 t^4 + 12096 k t^6$$
$$+ 2268\,\alpha^5 k + 3105\,\alpha^4 k^2 - 5328\,\alpha^4 t^2 - 24048\,\alpha^3 k^3 - 17376\,\alpha^3 k t^2 - 9546\,\alpha^2 k^4$$
$$+ 76592\,\alpha^2 k^2 t^2 - 2700\,\alpha^2 t^4 + 34452\,\alpha k^5 + 35520\,\alpha k^3 t^2 + 22608\,\alpha k t^4 + 2985 k^6$$
$$+ 146272 k^4 t^2 + 5972 k^2 t^4 + 7776 t^6 + 1458\,\alpha^5 + 5670\,\alpha^4 k - 50004\,\alpha^3 k^2 - 10656\,\alpha^3 t^2$$
$$- 36792\,\alpha^2 k^3 + 89760\,\alpha^2 k t^2 + 122418\,\alpha k^4 + 81496\,\alpha k^2 t^2 + 15300\,\alpha t^4 + 17442 k^5$$
$$+ 298560 k^3 t^2 + 11640 k t^4 + 3645\,\alpha^4 - 53784\,\alpha^3 k - 78111\,\alpha^2 k^2 + 42696\,\alpha^2 t^2 + 272808\,\alpha k^3$$
$$+ 98448\,\alpha k t^2 + 62802 k^4 + 357816 k^2 t^2 + 8100 t^4 - 23328\,\alpha^3 - 86346\,\alpha^2 k + 371790\,\alpha k^2$$
$$+ 48024\,\alpha t^2 + 142560 k^3 + 225360 t^2 k - 38637\,\alpha^2 + 282852\,\alpha k + 199017 k^2 + 54864 t^2$$
$$+ 91854\,\alpha + 156006 k + 52488)\,u(3+k) + 2\,(\alpha+4+k)\,(\alpha-k-3)\,(8\,\alpha^4 k^4 - 8\,\alpha^2 k^6$$
$$- 80\,\alpha^2 k^2 t^4 - 16 k^4 t^4 + 120\,\alpha^4 k^3 + 16\,\alpha^3 k^4 + 44\,\alpha^3 k^2 t^2 - 192\,\alpha^2 k^5 - 600\,\alpha^2 k t^4 - 8\,\alpha k^6$$
$$- 44\,\alpha k^4 t^2 - 80\,\alpha k^2 t^4 - 64\,\alpha t^6 - 152 k^3 t^4 + 658\,\alpha^4 k^2 + 240\,\alpha^3 k^3 + 350\,\alpha^3 k t^2 - 1894\,\alpha^2 k^4$$
$$+ 66\,\alpha^2 k^2 t^2 - 1080\,\alpha^2 t^4 - 192\,\alpha k^5 - 650\,\alpha k^3 t^2 - 600\,\alpha k t^4 + 4 k^6 - 22 k^4 t^2 - 264 k^2 t^4$$
$$- 32 t^6 + 1554\,\alpha^4 k + 1316\,\alpha^3 k^2 + 684\,\alpha^3 t^2 - 9816\,\alpha^2 k^3 + 525\,\alpha^2 k t^2 - 1902\,\alpha k^4$$
$$- 3636\,\alpha k^2 t^2 - 1080\,\alpha t^4 + 96 k^5 - 325 k^3 t^2 + 912 k t^4 + 1320\,\alpha^4 + 3108\,\alpha^3 k - 28140\,\alpha^2 k^2$$
$$+ 1026\,\alpha^2 t^2 - 9936\,\alpha k^3 - 9165\,\alpha k t^2 + 952 k^4 - 1829 k^2 t^2 + 2160 t^4 + 2640\,\alpha^3 - 42210\,\alpha^2 k$$
$$- 28798\,\alpha k^2 - 8802\,\alpha t^2 + 4992 k^3 - 4670 t^2 k - 25800\,\alpha^2 - 43764\,\alpha k + 14596 k^2 - 4572 t^2$$
$$- 27120\,\alpha + 22560 k + 14400)\,(k+5)\,(2k+3)\,u(k+4) + (2k+3)\,(\alpha-k-3)\,(\alpha-k$$
$$- 4)\,(\alpha+5+k)\,(\alpha+4+k)\,(-48\,\alpha^2 k^3 t^2 + 16 k^5 t^2 + 64 k t^6 + 4\,\alpha^3 k^3 - 568\,\alpha^2 k^2 t^2$$
$$- 4\,\alpha k^5 - 48\,\alpha k^3 t^2 - 80\,\alpha k t^4 + 392 k^4 t^2 + 160 t^6 + 54\,\alpha^3 k^2 + 6\,\alpha^2 k^3 - 2144\,\alpha^2 k t^2$$
$$- 98\,\alpha k^4 - 568\,\alpha k^2 t^2 - 360\,\alpha t^4 - 2 k^5 + 3792 k^3 t^2 - 40 k t^4 + 230\,\alpha^3 k + 81\,\alpha^2 k^2$$
$$- 2520\,\alpha^2 t^2 - 942\,\alpha k^3 - 2144\,\alpha k t^2 - 49 k^4 + 17976 k^2 t^2 - 180 t^4 + 300\,\alpha^3 + 345\,\alpha^2 k$$
$$- 4423\,\alpha k^2 - 2520\,\alpha t^2 - 472 k^3 + 41424 t^2 k + 450\,\alpha^2 - 10085\,\alpha k - 2225 k^2 + 36720 t^2$$
$$- 8850\,\alpha - 5100 k - 4500)\,u(k+5) - 2\,(-8 t^2 + 2\,\alpha + 1)\,(\alpha-k-3)\,(\alpha-k-4)\,(\alpha$$
$$- k - 5)\,(\alpha+6+k)\,(\alpha+5+k)\,(\alpha+4+k)\,t^2\,(2k+5)\,(2k+3)\,u(6+k)$$

FunctiontoRec: time: 0.928 sec.

$$-4\,(2k+11)\,(k+5)\,(k+4)\,(9+2k)\,(3+k)\,(k+2)\,(k+1)^2\,t^4\,u(k) - 2\,(3+k)\,(k+2)\left(3\,\alpha^2\,k^2 - k^4 - 4\,t^4 + 11\,\alpha^2\,k - 4\,k^3 + 10\,\alpha^2 - 8\,k^2 - 13\,k - 10\right)t^2\,(2k+11)\,(k+5)\,(k+4)\,(9+2k)\,u(k+1) - (3+k)\left(4\,\alpha^4\,k^4 - 4\,\alpha^2\,k^6 - 40\,\alpha^2\,k^2\,t^4 - 8\,k^4\,t^4 + 44\,\alpha^4\,k^3 - 60\,\alpha^2\,k^5 - 220\,\alpha^2\,k\,t^4 - 132\,k^3\,t^4 + 171\,\alpha^4\,k^2 - 367\,\alpha^2\,k^4 - 252\,\alpha^2\,t^4 + 4\,k^6 - 648\,k^2\,t^4 + 279\,\alpha^4\,k - 1183\,\alpha^2\,k^3 + 60\,k^5 - 1112\,k\,t^4 + 162\,\alpha^4 - 2133\,\alpha^2\,k^2 + 363\,k^4 - 612\,t^4 - 2043\,\alpha^2\,k + 1139\,k^3 - 810\,\alpha^2 + 1962\,k^2 + 1764\,k + 648\right)(k+5)\,(k+4)\,(2k+11)\,u(k+2) + 4\left(3\,\alpha^4\,k^2 - 2\,\alpha^2\,k^4 - 4\,\alpha^2\,t^4 - k^6 - 4\,k^2\,t^4 + 21\,\alpha^4\,k - 28\,\alpha^2\,k^3 - 21\,k^5 - 28\,k\,t^4 + 35\,\alpha^4 - 157\,\alpha^2\,k^2 - 176\,k^4 - 44\,t^4 - 413\,\alpha^2\,k - 749\,k^3 - 415\,\alpha^2 - 1683\,k^2 - 1834\,k - 700\right)t^2\,(k+5)\,(k+4)\,(2k+11)\,(2k+3)\,u(3+k) + (\alpha+4+k)\,(\alpha-k-4)\left(4\,\alpha^4\,k^4 - 4\,\alpha^2\,k^6 - 40\,\alpha^2\,k^2\,t^4 - 8\,k^4\,t^4 + 68\,\alpha^4\,k^3 - 108\,\alpha^2\,k^5 - 340\,\alpha^2\,k\,t^4 - 92\,k^3\,t^4 + 423\,\alpha^4\,k^2 - 1207\,\alpha^2\,k^4 - 672\,\alpha^2\,t^4 + 4\,k^6 - 228\,k^2\,t^4 + 1135\,\alpha^4\,k - 7133\,\alpha^2\,k^3 + 108\,k^5 + 468\,k\,t^4 + 1100\,\alpha^4 - 23448\,\alpha^2\,k^2 + 1203\,k^4 + 1488\,t^4 - 40510\,\alpha^2\,k + 7065\,k^3 - 28600\,\alpha^2 + 23025\,k^2 + 39375\,k + 27500\right)(k+5)\,(2k+3)\,u(k+4) - 2\,t^2\,(\alpha-k-5)\,(\alpha+5+k)\,(\alpha+4+k)\,(\alpha-k-4)\left(3\,\alpha^2\,k^2 - k^4 - 4\,t^4 + 31\,\alpha^2\,k - 24\,k^3 + 80\,\alpha^2 - 218\,k^2 - 883\,k - 1340\right)(2k+5)\,(2k+3)\,u(k+5) + 4\,t^4\,(\alpha+4+k)\,(\alpha-k-4)\,(\alpha-k-5)\,(\alpha+5+k)\,(\alpha+6+k)\,(\alpha-k-6)\,(2k+5)\,(2k+3)\,u(6+k)$$

FunctiontoRec: time: 0.897 sec.

$$-2\,(k+5)\,(9+2k)\,(6+k)\,(2k+11)\,(k+4)\,(3+k)\,(k+2)\,(k+1)\,t^2\left(8\,t^2 + 2\,\alpha - 1\right)u(k) - (2k+11)\,(6+k)\,(k+5)\,(3+k)\,(k+4)\left(48\,\alpha^2\,k^3\,t^2 - 16\,k^5\,t^2 - 64\,k\,t^6 + 4\,\alpha^3\,k^3 + 440\,\alpha^2\,k^2\,t^2 - 4\,\alpha\,k^5 - 48\,\alpha\,k^3\,t^2 - 80\,\alpha\,k\,t^4 - 168\,k^4\,t^2 - 288\,t^6 + 30\,\alpha^3\,k^2 - 6\,\alpha^2\,k^3 + 1248\,\alpha^2\,k\,t^2 - 42\,\alpha\,k^4 - 440\,\alpha\,k^2\,t^2 - 200\,\alpha\,t^4 + 2\,k^5 - 656\,k^3\,t^2 + 40\,k\,t^4 + 62\,\alpha^3\,k - 45\,\alpha^2\,k^2 + 1120\,\alpha^2\,t^2 - 158\,\alpha\,k^3 - 1248\,\alpha\,k\,t^2 + 21\,k^4 - 1288\,k^2\,t^2 + 100\,t^4 + 36\,\alpha^3 - 93\,\alpha^2\,k - 267\,\alpha\,k^2 - 1120\,\alpha\,t^2 + 80\,k^3 - 1440\,t^2\,k - 54\,\alpha^2 - 201\,\alpha\,k + 141\,k^2 - 800\,t^2 - 54\,\alpha + 116\,k + 36\right)u(k+1) - 2\,(2k+11)\,(6+k)\,(k+5)\,(k+4)\left(8\,\alpha^4\,k^4 - 8\,\alpha^2\,k^6 - 80\,\alpha^2\,k^2\,t^4 - 16\,k^4\,t^4 + 104\,\alpha^4\,k^3 - 16\,\alpha^3\,k^4 - 44\,\alpha^3\,k^2\,t^2 - 144\,\alpha^2\,k^5 - 520\,\alpha^2\,k\,t^4 + 8\,\alpha\,k^6 + 44\,\alpha\,k^4\,t^2 + 80\,\alpha\,k^2\,t^4 + 64\,\alpha\,t^6 - 296\,k^3\,t^4 + 490\,\alpha^4\,k^2 - 208\,\alpha^3\,k^3 - 266\,\alpha^3\,k\,t^2 - 1054\,\alpha^2\,k^4 + 66\,\alpha^2\,k^2\,t^2 - 800\,\alpha^2\,t^4 + 144\,\alpha\,k^5 + 582\,\alpha\,k^3\,t^2 + 520\,\alpha\,k\,t^4 + 4\,k^6 - 22\,k^4\,t^2 - 1776\,k^2\,t^4 - 32\,t^6 + 994\,\alpha^4\,k - 980\,\alpha^3\,k^2 - 390\,\alpha^3\,t^2 - 4016\,\alpha^2\,k^3 + 399\,\alpha^2\,k\,t^2 + 1062\,\alpha\,k^4 + 2922\,\alpha\,k^2\,t^2 + 800\,\alpha\,t^4 + 72\,k^5 - 291\,k^3\,t^2 - 4216\,k\,t^4 + 732\,\alpha^4 - 1988\,\alpha^3\,k - 8400\,\alpha^2\,k^2 + 585\,\alpha^2\,t^2 + 4120\,\alpha\,k^3 + 6557\,\alpha\,k\,t^2 + 532\,k^4 - 1472\,k^2\,t^2 - 3440\,t^4 - 1464\,\alpha^3 - 9142\,\alpha^2\,k + 8890\,\alpha\,k^2 + 5505\,\alpha\,t^2 + 2064\,k^3 - 3345\,t^2\,k - 4044\,\alpha^2 + 10136\,\alpha\,k + 4432\,k^2 - 2850\,t^2$$

$$+ 4776\,\alpha + 4992\,k + 2304\big)\,u(k+2) + 2\,\big(96\,\alpha^4\,k^4\,t^2 - 64\,\alpha^2\,k^6\,t^2 - 128\,\alpha^2\,k^2\,t^6 - 32\,k^8\,t^2$$
$$- 128\,k^4\,t^6 + 24\,\alpha^5\,k^4 + 1536\,\alpha^4\,k^3\,t^2 - 48\,\alpha^3\,k^6 - 192\,\alpha^3\,k^4\,t^2 - 224\,\alpha^3\,k^2\,t^4 - 1536\,\alpha^2\,k^5\,t^2$$
$$- 1024\,\alpha^2\,k\,t^6 + 24\,\alpha\,k^8 + 64\,\alpha\,k^6\,t^2 + 224\,\alpha\,k^4\,t^4 + 128\,\alpha\,k^2\,t^6 - 1024\,k^7\,t^2 - 2048\,k^3\,t^6$$
$$+ 384\,\alpha^5\,k^3 - 60\,\alpha^4\,k^4 + 8936\,\alpha^4\,k^2\,t^2 - 1152\,\alpha^3\,k^5 - 3072\,\alpha^3\,k^3\,t^2 - 1792\,\alpha^3\,k\,t^4 + 72\,\alpha^2\,k^6$$
$$- 15344\,\alpha^2\,k^4\,t^2 + 336\,\alpha^2\,k^2\,t^4 - 1760\,\alpha^2\,t^6 + 768\,\alpha\,k^7 + 1536\,\alpha\,k^5\,t^2 + 3584\,\alpha\,k^3\,t^4$$
$$+ 1024\,\alpha\,k\,t^6 - 12\,k^8 - 14072\,k^6\,t^2 - 112\,k^4\,t^4 - 12000\,k^2\,t^6 + 2250\,\alpha^5\,k^2 - 960\,\alpha^4\,k^3$$
$$+ 22336\,\alpha^4\,k\,t^2 - 11364\,\alpha^3\,k^4 - 17872\,\alpha^3\,k^2\,t^2 - 3368\,\alpha^3\,t^4 + 1728\,\alpha^2\,k^5 - 81664\,\alpha^2\,k^3\,t^2$$
$$+ 2688\,\alpha^2\,k\,t^4 + 10650\,\alpha\,k^6 + 15440\,\alpha\,k^4\,t^2 + 21240\,\alpha\,k^2\,t^4 + 1760\,\alpha\,t^6 - 384\,k^7 - 108352\,k^5\,t^2$$
$$- 1792\,k^3\,t^4 - 30464\,k\,t^6 + 5712\,\alpha^5\,k - 5625\,\alpha^4\,k^2 + 20168\,\alpha^4\,t^2 - 58944\,\alpha^3\,k^3 - 44672\,\alpha^3\,k\,t^2$$
$$+ 17106\,\alpha^2\,k^4 - 243920\,\alpha^2\,k^2\,t^2 + 5052\,\alpha^2\,t^4 + 83568\,\alpha\,k^5 + 83200\,\alpha\,k^3\,t^2 + 55232\,\alpha\,k\,t^4$$
$$- 5337\,k^6 - 510552\,k^4\,t^2 - 10676\,k^2\,t^4 - 28160\,t^6 + 5280\,\alpha^5 - 14280\,\alpha^4\,k - 169452\,\alpha^3\,k^2$$
$$- 40336\,\alpha^3\,t^2 + 89376\,\alpha^2\,k^3 - 386688\,\alpha^2\,k\,t^2 + 405708\,\alpha\,k^4 + 252856\,\alpha\,k^2\,t^2 + 52652\,\alpha\,t^4$$
$$- 42072\,k^5 - 1504640\,k^3\,t^2 - 28064\,k\,t^4 - 13200\,\alpha^4 - 255840\,\alpha^3\,k + 259803\,\alpha^2\,k^2$$
$$- 253256\,\alpha^2\,t^2 + 1247424\,\alpha\,k^3 + 409024\,\alpha\,k\,t^2 - 205707\,k^4 - 2701472\,k^2\,t^2 - 27168\,t^4$$
$$- 158400\,\alpha^3 + 398040\,\alpha^2\,k + 2371200\,\alpha\,k^2 + 273424\,\alpha\,t^2 - 638640\,k^3 - 2692352\,t^2\,k$$
$$+ 250800\,\alpha^2 + 2546688\,\alpha\,k - 1229088\,k^2 - 1135104\,t^2 + 1182720\,\alpha - 1340160\,k - 633600\big)$$
$$(6+k)\,(k+5)\,u(3+k) + 2\,(2\,k+5)\,(6+k)\,(\alpha - k - 5)\,(\alpha + 4 + k)\,\big(8\,\alpha^4\,k^4 - 8\,\alpha^2\,k^6$$
$$- 80\,\alpha^2\,k^2\,t^4 - 16\,k^4\,t^4 + 152\,\alpha^4\,k^3 - 16\,\alpha^3\,k^4 - 44\,\alpha^3\,k^2\,t^2 - 240\,\alpha^2\,k^5 - 760\,\alpha^2\,k\,t^4 + 8\,\alpha\,k^6$$
$$+ 44\,\alpha\,k^4\,t^2 + 80\,\alpha\,k^2\,t^4 + 64\,\alpha\,t^6 - 216\,k^3\,t^4 + 1066\,\alpha^4\,k^2 - 304\,\alpha^3\,k^3 - 438\,\alpha^3\,k\,t^2$$
$$- 2974\,\alpha^2\,k^4 + 66\,\alpha^2\,k^2\,t^2 - 1760\,\alpha^2\,t^4 + 240\,\alpha\,k^5 + 826\,\alpha\,k^3\,t^2 + 760\,\alpha\,k\,t^4 + 4\,k^6 - 22\,k^4\,t^2$$
$$- 816\,k^2\,t^4 - 32\,t^6 + 3262\,\alpha^4\,k - 2132\,\alpha^3\,k^2 - 1078\,\alpha^3\,t^2 - 19472\,\alpha^2\,k^3 + 657\,\alpha^2\,k\,t^2$$
$$+ 2982\,\alpha\,k^4 + 5850\,\alpha\,k^2\,t^2 + 1760\,\alpha\,t^4 + 120\,k^5 - 413\,k^3\,t^2 - 136\,k\,t^4 + 3660\,\alpha^4 - 6524\,\alpha^3\,k$$
$$- 70992\,\alpha^2\,k^2 + 1617\,\alpha^2\,t^2 + 19624\,\alpha\,k^3 + 18563\,\alpha\,k\,t^2 + 1492\,k^4 - 2936\,k^2\,t^2 + 2640\,t^4$$
$$- 7320\,\alpha^3 - 136522\,\alpha^2\,k + 72058\,\alpha\,k^2 + 22297\,\alpha\,t^2 + 9840\,k^3 - 9391\,t^2\,k - 108060\,\alpha^2$$
$$+ 139784\,\alpha\,k + 36304\,k^2 - 11418\,t^2 + 111720\,\alpha + 71040\,k + 57600\big)\,u(k+4) - (2\,k$$
$$+ 5)\,(\alpha + 5 + k)\,(\alpha - k - 6)\,(\alpha - k - 5)\,(\alpha + 4 + k)\,\big(48\,\alpha^2\,k^3\,t^2 - 16\,k^5\,t^2 - 64\,k\,t^6$$
$$+ 4\,\alpha^3\,k^3 + 712\,\alpha^2\,k^2\,t^2 - 4\,\alpha\,k^5 - 48\,\alpha\,k^3\,t^2 - 80\,\alpha\,k\,t^4 - 472\,k^4\,t^2 - 224\,t^6 + 66\,\alpha^3\,k^2$$
$$- 6\,\alpha^2\,k^3 + 3424\,\alpha^2\,k\,t^2 - 118\,\alpha\,k^4 - 712\,\alpha\,k^2\,t^2 - 440\,\alpha\,t^4 + 2\,k^5 - 5520\,k^3\,t^2 + 40\,k\,t^4$$
$$+ 350\,\alpha^3\,k - 99\,\alpha^2\,k^2 + 5280\,\alpha^2\,t^2 - 1374\,\alpha\,k^3 - 3424\,\alpha\,k\,t^2 + 59\,k^4 - 31864\,k^2\,t^2 + 220\,t^4$$
$$+ 588\,\alpha^3 - 525\,\alpha^2\,k - 7877\,\alpha\,k^2 - 5280\,\alpha\,t^2 + 688\,k^3 - 90400\,t^2\,k - 882\,\alpha^2 - 22169\,\alpha\,k$$
$$+ 3955\,k^2 - 100320\,t^2 - 24402\,\alpha + 11172\,k + 12348\big)\,u(k+5) + 2\,t^2\,\big(8\,t^2 + 2\,\alpha - 1\big)\,(\alpha$$

$$-k-5)\,(\alpha+4+k)\,(\alpha-k-6)\,(\alpha+5+k)\,(\alpha-7-k)\,(\alpha+6+k)\,(2k+5)\,(7+2k)\,u(6+k)$$

```
> LREtools[hypergeomsols](rec,u(k),{},output=basis);
```

$$\left[\frac{(2t^2)^k}{\Gamma(\beta+k+1)\,\Gamma(\alpha+1+k)},\left(\frac{1}{2t^2}\right)^k\Gamma(k+\alpha+\beta+1)\,\Gamma(k+1)\right]$$

5.14.1.8

```
> F:=(t+1)/(1-t)^(alpha+beta+2)*hypergeom([(alpha+beta+2)/2,(alpha+
  beta+3)/2],[alpha+1],2*t*(x-1)/(t-1)^2);
```

$$F := \frac{(t+1)\,\text{hypergeom}\left(\left[\frac{\alpha}{2}+\frac{\beta}{2}+1,\frac{\alpha}{2}+\frac{\beta}{2}+\frac{3}{2}\right],[1+\alpha],\frac{2t(x-1)}{(t-1)^2}\right)}{(1-t)^{\alpha+\beta+2}}$$

```
> rec:=FunctiontoRec(F,JacobiP(k,alpha,beta,x),u) assuming alpha>-1,
  beta>-1;
FunctiontoRec: time: 0.651 sec.
```

$$rec := -t(\alpha+\beta+2k+5)\,(k+2+\alpha+\beta)\,(\alpha+\beta+2k+3)\,(k+\alpha+\beta+1)\,u(k)+(\alpha+1+k)\,(\alpha+\beta+2k+1)\,(t^2+1)\,(\alpha+\beta+2k+5)\,(k+2+\alpha+\beta)\,u(k+1)-(\alpha+2+k)\,(\alpha+1+k)\,(\alpha+\beta+2k+1)\,(\alpha+\beta+2k+3)\,t\,u(k+2)$$

Special case for alpha+beta=-1

```
> FunctiontoRec(op(subs(beta=-1-alpha,[F,JacobiP(k,alpha,beta,x)])),
  u) assuming alpha>-1,beta>-1;
FunctiontoRec: time: 0.171 sec.
```

$$-t(k+2)\,(k+1)\,u(k)+(\alpha+1+k)\,(k+2)\,(t^2+1)\,u(k+1)-(\alpha+1+k)\,(\alpha+2+k)\,t\,u(k+2)$$

```
> LREtools[hypergeomsols](rec,u(k),{},output=basis);
```

$$\left[\frac{t^k\,\Gamma(k+\alpha+\beta+1)\left(k+\frac{1}{2}+\frac{\beta}{2}+\frac{\alpha}{2}\right)}{\Gamma(\alpha+1+k)},\frac{\left(\frac{1}{t}\right)^k\Gamma(k+\alpha+\beta+1)\left(k+\frac{1}{2}+\frac{\beta}{2}+\frac{\alpha}{2}\right)}{\Gamma(\alpha+1+k)}\right]$$

5.14.1.11

```
> F:=hypergeom([gamma,alpha+beta-gamma+1],[alpha+1],(1-t-rho)/2)*
  hypergeom([gamma,alpha+beta-gamma+1],[beta+1],(1+t-rho)/2);
```

$$F := \text{hypergeom}\left([\gamma,\alpha+\beta-\gamma+1],[1+\alpha],\frac{1}{2}-\frac{t}{2}-\frac{\sqrt{t^2-2xt+1}}{2}\right)\text{hypergeom}\left([\gamma,\alpha\right.$$

$$+\beta-\gamma+1\Big],\Big[1+\beta\Big],\frac{1}{2}+\frac{t}{2}-\frac{\sqrt{t^2-2xt+1}}{2}\Bigg)$$

```
> eval(F,[alpha=1/3,beta=2/7,gamma=11/13,t=1/2]);
```

$$\text{hypergeom}\left(\left[\frac{211}{273},\frac{11}{13}\right],\left[\frac{4}{3}\right],\frac{1}{4}-\frac{\sqrt{\frac{5}{4}-x}}{2}\right)\text{hypergeom}\left(\left[\frac{211}{273},\frac{11}{13}\right],\left[\frac{9}{7}\right],\frac{3}{4}-\frac{\sqrt{\frac{5}{4}-x}}{2}\right)$$

```
> op(1,gfun:-holexprtodiffeq(%,y(x)));
```

[Length of output exceeds limit of 50000]

```
> DEtools[de2diffop](%,y(x),[Dx,x]);
```

[Length of output exceeds limit of 50000]

```
> factor(lcoeff(%,Dx));
```

$$146[...13\text{ digits}...]061\,(311[...53\text{ digits}...]239\,x^{20}+133[...56\text{ digits}...]434\,x^{19}-528[...58\text{ digits}...]032\,x^{18}+411[...59\text{ digits}...]926\,x^{17}+220[...61\text{ digits}...]557\,x^{16}-668[...62\text{ digits}...]120\,x^{15}+963[...63\text{ digits}...]304\,x^{14}-820[...64\text{ digits}...]352\,x^{13}+453[...65\text{ digits}...]086\,x^{12}-175[...66\text{ digits}...]468\,x^{11}+243[...66\text{ digits}...]952\,x^{10}+169[...67\text{ digits}...]716\,x^{9}-107[...68\text{ digits}...]038\,x^{8}+295[...68\text{ digits}...]144\,x^{7}-479[...68\text{ digits}...]312\,x^{6}+501[...68\text{ digits}...]600\,x^{5}-342[...68\text{ digits}...]829\,x^{4}+141[...68\text{ digits}...]482\,x^{3}-215[...67\text{ digits}...]216\,x^{2}-818[...66\text{ digits}...]362\,x+316[...66\text{ digits}...]289)\,(4x-5)^{4}\,(x-1)^{4}\,(1+x)^{4}$$

```
> rec:=FunctiontoRec(op(eval([F,JacobiP(k,alpha,beta,x)],[alpha=1/3,
  beta=2/7,gamma=11/13,t=1/2])),u):
FunctiontoRec: time: 524.786 sec.
> eval(rec,u=proc(t) X^(t-k) end):
> degree(%,X),degree(%,k);
```

$$50,\ 102$$

This is a recurrence of order 50 with coefficients of degrees around 100.

5.14.1.14

```
> F:=(1-x)^c;
```

$$F:=(1-x)^{c}$$

```
> rec:=FunctiontoRec(F,JacobiP(k,alpha,beta,x),u);
FunctiontoRec: time: 0.372 sec.
```

$$rec:=-(\alpha+\beta+2k+3)\,(k+\alpha+\beta+1)\,(c-k)\,u(k)-(\alpha+1+k)\,(\alpha+\beta+2k+1)\,(c$$

$$+ k + 2 + \alpha + \beta)\, u(k+1)$$

5.14.5.1

```
> ini:='ini':
> F:=sin(z*x);
```

$$F := \sin(z\,x)$$

```
> rec:=FunctiontoRec(F,JacobiP(k,alpha,alpha,x),u);
FunctiontoRec: time: 0.096 sec.
```

$$rec := (2\alpha + 2k + 9)(k + 4 + 2\alpha)(2\alpha + 2k + 7)(k + 3 + 2\alpha)(k + 2 + 2\alpha)(k + 2\alpha + 1)\, z^2\, u(k) + (\alpha + 2 + k)(\alpha + 1 + k)(2\alpha + 2k + 1)(2\alpha + 2k + 9)(k + 4 + 2\alpha)(k + 3 + 2\alpha)\left(4\alpha^2 + 8\alpha k + 4k^2 - 2z^2 + 20\alpha + 20k + 21\right) u(k+2) + (\alpha + 2 + k)(\alpha + 1 + k)(2\alpha + 2k + 1)(\alpha + 4 + k)(2\alpha + 2k + 3)(\alpha + 3 + k)\, z^2\, u(k+4)$$

Initial conditions

```
> for i from 0 to 3 do ini[i]:=int(expand(JacobiP(i,alpha,alpha,x))*
  F*(1-x^2)^(alpha),x=-1..1)/eval(hnJ,[beta=alpha,n=i]) od;
```

$$ini_0 := 0$$

$$ini_1 := \frac{1}{\Gamma(\alpha+2)\, 2^{2\alpha+1}}\left(z^{-\frac{3}{2}-\alpha}\sqrt{\pi}\; 2^{\frac{1}{2}+\alpha}\left(-\text{BesselJ}\left(-\frac{1}{2}+\alpha, z\right) z + 2\,\text{BesselJ}\left(\frac{1}{2}+\alpha, z\right)\alpha + \text{BesselJ}\left(\frac{1}{2}+\alpha, z\right)\right)(2\alpha+3)\,\Gamma(2\alpha+2)\right)$$

$$ini_2 := 0$$

$$ini_3 := -\frac{1}{\Gamma(\alpha+4)\, 2^{2\alpha+1}}\left(2^{\frac{1}{2}+\alpha}\sqrt{\pi}\left(-4\,\text{BesselJ}\left(-\frac{1}{2}+\alpha, z\right)\alpha^2 z + \text{BesselJ}\left(-\frac{1}{2}+\alpha, z\right) z^3 + 8\,\text{BesselJ}\left(\frac{1}{2}+\alpha, z\right)\alpha^3 - 4\,\text{BesselJ}\left(\frac{1}{2}+\alpha, z\right)\alpha z^2 - 16\,\text{BesselJ}\left(-\frac{1}{2}+\alpha, z\right)\alpha z + 36\,\text{BesselJ}\left(\frac{1}{2}+\alpha, z\right)\alpha^2 - 6\,\text{BesselJ}\left(\frac{1}{2}+\alpha, z\right) z^2 - 15\,\text{BesselJ}\left(-\frac{1}{2}+\alpha, z\right) z + 46\,\text{BesselJ}\left(\frac{1}{2}+\alpha, z\right)\alpha + 15\,\text{BesselJ}\left(\frac{1}{2}+\alpha, z\right)\right) z^{-\frac{7}{2}-\alpha}(2\alpha+7)\,\Gamma(2\alpha+4)\right)$$

Simplification:

```
> for i from 0 to 3 do ini[i]:=factor(simplify(ini[i]/BesselJ(alpha+
  i+1/2,z)))*BesselJ(alpha+i+1/2,z) od:
> sol:=(-1)^k*2^(alpha+2*k+1/2)*z^(-alpha-1/2)*(2*alpha+4*k+3)*GAMMA
  (alpha+k+1)*GAMMA(alpha+k+3/2)/GAMMA(alpha+2*k+2)*BesselJ(2*k+
  alpha+3/2,z);
```

$$sol := \frac{1}{\Gamma(\alpha+2k+2)}\left((-1)^k\, 2^{\alpha+2k+\frac{1}{2}}\, z^{-\frac{1}{2}-\alpha}\,(2\alpha+4k+3)\,\Gamma(\alpha+1+k)\,\Gamma\left(\alpha+k+\frac{3}{2}\right)\mathrm{BesselJ}\left(2k+\alpha+\frac{3}{2},z\right)\right)$$

```
> eval(sol,k=0):
> simplify(%/ini[1]) assuming alpha>-1;
```

$$1$$

```
> eval(sol,k=1):
> simplify(%/ini[3]) assuming alpha>-1;
```

$$1$$

Check solution:

```
> recodd:=subs(k=2*m+1,rec):
> subs(seq(u(2*m+1+2*i)=eval(sol,k=m+i),i=0..2),recodd):
> simplify(%) assuming alpha>-1;
```

$$0$$

5.14.5.2

```
> F:=hypergeom([(alpha+beta)/2+1],[alpha+1],x*z-z);
```

$$F := \mathrm{hypergeom}\left(\left[\frac{\alpha}{2}+\frac{\beta}{2}+1\right],[1+\alpha],zx-z\right)$$

```
> rec:=FunctiontoRec(F,JacobiP(k,alpha,alpha,x),u) assuming alpha>-1;
FunctiontoRec: time: 1.181 sec.
```

$$rec := -(k+2\alpha+1)\,(2k+2+\alpha+\beta)\,(2\alpha+2k+5)\,(k+2+2\alpha)\,z\,(\alpha+2+k)\,u(k) + (2\alpha+2k+5)\,(k+2+2\alpha)\,(2\alpha^2+4\alpha k+\alpha z-z\beta+2k^2+6\alpha+6k+4)\,(\alpha+1+k)\,(2\alpha+2k+1)\,u(k+1)+(\alpha+1+k)^2\,(2\alpha+2k+1)\,(\alpha+2+k)\,z\,(3\alpha-\beta+2k+4)\,u(k+2)$$

Special case when alpha=-1/2

```
> rec:=FunctiontoRec(op(subs(alpha=1/2,[F,JacobiP(k,alpha,alpha,x)])
  ),u);
FunctiontoRec: time: 0.156 sec.
```

$$rec := -4\,(k+2)\,(4k+2\beta+5)\,(3+k)^2\,z\,(2k+5)\,u(k)-8\,(3+k)^2\,(2z\beta-4k^2-16k-z-15)\,(2k+3)\,(k+1)\,u(k+1)-(2k+3)^2\,(k+1)\,(2k+5)\,z\,(2\beta-4k-11)\,u(k+2)$$

```
> time()-T0;
```

$$741.987$$

This is the total time for this session, with more than half of it spent on the recurrence of order 50 from Eq. 5.14.1.11 in Prudnikov-Brychkov-Marichev.