Examples of algebraic values from "Minimization of Differential Equations and Algebraic Values of E-Functions"

requires algvalues and a recent version of gfun, to be downloaded from

https://perso.ens-lyon.fr/bruno.salvy/software/the-gfun-package/

```
> restart;st:=time():
with(gfun,minimizediffeq):
if gfun:-version()<3.95 then error "this worksheet requires a recent version of gfun" fi;
infolevel[minimizediffeq]:=4: # reduce to display less information
read "algvalues.mpl";
```

Bessel J_0' (section 4.2)

```
> deq:=gfun:-seriestodiffeq(series(diff(BesselJ(0,z),z$4),z,40),y(z),[ogf])[1];
```

$$deq := \left\{\left(z^5 - 10\, z^3 + 45\, z\right) y(z) + \left(z^4 - 18\, z^2 + 45\right) \left(\frac{d}{dz}\, y(z)\right) + \left(z^5 - 6\, z^3 + 9\, z\right) \left(\frac{d^2}{dz^2}\, y(z)\right), y(0) = \frac{3}{8}, D(y)(0) = 0\right\} \quad (1.1)$$

```
> algvalues(deq,y(z));
```

$$\{y(RootOf(\_Z^2 - 3)) = 0\} \quad (1.2)$$

Eq. (7.5) from Adamczewski-Rivoal 2018

```
> deq7_5:=diff(y(z),z$3)+(1-2*z-2*z^2)/z/(1+z)*diff(y(z),z,z)-(1+4*z+z^2)/z^2/(1+z)*diff(y(z),z);
```

$$deq7\_5 := \frac{d^3}{dz^3}\, y(z) + \frac{\left(-2\, z^2 - 2\, z + 1\right) \left(\frac{d^2}{dz^2}\, y(z)\right)}{z\, (1 + z)} - \frac{\left(z^2 + 4\, z + 1\right) \left(\frac{d}{dz}\, y(z)\right)}{z^2\, (1 + z)} \quad (2.1)$$

```
> sol:=series(add(n^2*binomial(2*n,n)/(n+1)^2*(z/2)^(n+1)/n!,n=0..
  Order),z);
```

$$sol := \frac{1}{8} z^2 + \frac{1}{6} z^3 + \frac{15}{128} z^4 + \frac{7}{120} z^5 + O(z^6) \tag{2.2}$$

```
> series(eval(deq7_5,y(z)=sol),z);
```

$$O(z^3) \tag{2.3}$$

```
> deq7_5:={deq7_5,y(0)=0,(D@@2)(y)(0)=1/4};
```

$$deq7\_5 := \left\{ \frac{d^3}{dz^3} y(z) + \frac{(-2z^2 - 2z + 1)\left(\frac{d^2}{dz^2} y(z)\right)}{z(1+z)} - \frac{(z^2 + 4z + 1)\left(\frac{d}{dz} y(z)\right)}{z^2(1+z)}, y(0) = 0, D^{(2)}(y)(0) = \frac{1}{4} \right\} \tag{2.4}$$

```
> algvalues(deq7_5,y(z));
```

$$\left\{ y(1) = \frac{1}{2} \right\} \tag{2.5}$$

```
> dsolve(deq7_5,y(z));
```

$$y(z) = \frac{1}{2} + \frac{e^z (z-1)\, \mathrm{BesselI}(0, z)}{2} \tag{2.6}$$

```
> eval(op(2,%),z=1);
```

$$\frac{1}{2} \tag{2.7}$$

Special cases of Prop. 4.1

```
> deq:=gfun:-holexprtodiffeq(hypergeom([d+1],[a+d+1],-z),y(z));
```

$$deq := \left\{ \left(\frac{d^2}{dz^2} y(z)\right) z + (z + a + d + 1)\left(\frac{d}{dz} y(z)\right) + y(z)\, d + y(z), y(0) = 1 \right\} \tag{3.1}$$

The implementation does not deal with parameters, but can handle special cases:

```
> algvalues(eval(deq,[d=5,a=-32/5]),y(z));
```

$$\left\{ y\left(\frac{12}{5}\right) = \frac{1309}{625}, y\big(RootOf(625\,\_Z^4 - 18500\,\_Z^3 + 171600\,\_Z^2 - 538560\,\_Z + 323136)\big) = -212058 \Big/ \Big(125\,\big(250\,RootOf(625\,\_Z^4 - 18500\,\_Z^3 + 171600\,\_Z^2 - 538560\,\_Z + 323136)^3 - 4125\,RootOf(625\,\_Z^4 - 18500\,\_Z^3 + 171600\,\_Z^2 - 538560\,\_Z + 323136)^2 + 16830\,RootOf(625\,\_Z^4 - 18500\,\_Z^3 + 171600\,\_Z^2 - 538560\,\_Z + 323136) - 10098\big)\Big) \right\} \tag{3.2}$$

```
> algvalues(eval(deq,[d=5,a=-64/5]),y(z));
```

$$\left\{ y\left(\frac{84}{5}\right) = \frac{8437}{625}, y\big(RootOf(625\,\_Z^4 - 29500\,\_Z^3 + 448400\,\_Z^2 - 2662080\,\_Z + 5233536)\big) = -1135134 \Big/ \Big(125\,\big(25\,RootOf(625\,\_Z^4 - 29500\,\_Z^3 + 448400\,\_Z^2 - 2662080\,\_Z + 5233536)^2 - 390\,RootOf(625\,\_Z^4 - 29500\,\_Z^3 \tag{3.3}$$

$$+ 448400\_Z^2 - 2662080\_Z + 5233536) + 1386))\}$$

```
> algvalues(eval(deq,[d=4,a=-128/3]),y(z));
```

$$\left\{y\left(\frac{140}{3}\right) = -\frac{30073}{27}, y(RootOf(27\_Z^3 - 3348\_Z^2 + 131760\_Z - 1659200)) = -52627750 \Big/ \left(9\left(9\,RootOf(27\_Z^3 - 3348\_Z^2 + 131760\_Z - 1659200)^2 - 1830\,RootOf(27\_Z^3 - 3348\_Z^2 + 131760\_Z - 1659200) + 51850\right)\right)\right\} \quad (3.4)$$

```
> algvalues(eval(deq,[d=3,a=-27/7]),y(z));
```

$$\left\{y\left(\frac{6}{7}\right) = -\frac{65}{49}, y(RootOf(49\_Z^2 - 525\_Z + 1170)) = -\frac{104}{49\,(7\,RootOf(49\_Z^2 - 525\_Z + 1170) - 26)}\right\} \quad (3.5)$$

```
> algvalues(eval(deq,[d=3,a=-2197/725]),y(z));
```

$$\left\{y\left(\frac{312}{725}\right) = -\frac{20999}{525625}, y(RootOf(525625\_Z^2 - 4552275\_Z + 7742904)) = -\frac{350592}{105125\,(725\,RootOf(525625\_Z^2 - 4552275\_Z + 7742904) - 1992)}\right\} \quad (3.6)$$

```
> algvalues(eval(deq,[d=4,a=-32/5]),y(z));
```

$$\left\{y(RootOf(25\_Z^2 - 420\_Z + 1224)) = \frac{187}{25\,(5\,RootOf(25\_Z^2 - 420\_Z + 1224) - 17)}, y(RootOf(25\_Z^2 - 220\_Z + 264)) = -\frac{1377}{25\,(5\,RootOf(25\_Z^2 - 220\_Z + 264) + 3)}\right\} \quad (3.7)$$

```
> `union`(allvalues(%));
```

$$\left\{y\left(\frac{22}{5} - \frac{2\sqrt{55}}{5}\right) = -\frac{1377}{25\,(25 - 2\sqrt{55})}, y\left(\frac{22}{5} + \frac{2\sqrt{55}}{5}\right) = -\frac{1377}{25\,(25 + 2\sqrt{55})}, y\left(\frac{42}{5} - \frac{6\sqrt{15}}{5}\right) = \frac{187}{25\,(25 - 6\sqrt{15})}, y\left(\frac{42}{5} + \frac{6\sqrt{15}}{5}\right) = \frac{187}{25\,(25 + 6\sqrt{15})}\right\} \quad (3.8)$$

```
> map(evala@Expand,%);
```

$$\left\{y\left(\frac{22}{5} - \frac{2\sqrt{55}}{5}\right) = -\frac{17}{5} - \frac{34\sqrt{55}}{125}, y\left(\frac{22}{5} + \frac{2\sqrt{55}}{5}\right) = -\frac{17}{5} + \frac{34\sqrt{55}}{125}, y\left(\frac{42}{5} - \frac{6\sqrt{15}}{5}\right) = \frac{11}{5} + \frac{66\sqrt{15}}{125}, y\left(\frac{42}{5} + \frac{6\sqrt{15}}{5}\right) = \frac{11}{5} - \frac{66\sqrt{15}}{125}\right\} \quad (3.9)$$

```
> algvalues(eval(deq,[d=4,a=-8/3]),y(z));
```

$$\left\{y\left(\frac{2}{3}\right) = \frac{5}{27}, y(RootOf(27\_Z^3 - 270\_Z^2 + 540\_Z + 40)) = 2 \Big/ \left(9\left(9\,RootOf(27\_Z^3 - 270\_Z^2 + 540\_Z + 40)^2 - 30\,RootOf(27\_Z^3 - 270\_Z^2 \right.\right. \quad (3.10)$$

$+ 540\,\_Z + 40) - 2))\}$

```
> algvalues(eval(deq,[d=4,a=-1/3]),y(z));
```

$$\left\{y\left(-\frac{2}{3}\right) = \frac{55}{27}, y(RootOf(27\,\_Z^3 - 54\,\_Z^2 - 40)) = 110 \Big/ \left(9\left(18\,RootOf(27\,\_Z^3 - 54\,\_Z^2 - 40)^2 + 15\,RootOf(27\,\_Z^3 - 54\,\_Z^2 - 40) + 20\right)\right)\right\} \tag{3.11}$$

```
> algvalues(eval(deq,[d=4,a=-27/25]),y(z));
```

$$\left\{y\left(\frac{12}{25}\right) = \frac{1679}{3125}, y(RootOf(15625\,\_Z^3 - 60000\,\_Z^2 - 20700\,\_Z - 4968)) = -6716 \Big/ \left(125\left(1875\,RootOf(15625\,\_Z^3 - 60000\,\_Z^2 - 20700\,\_Z - 4968)^2 + 575\,RootOf(15625\,\_Z^3 - 60000\,\_Z^2 - 20700\,\_Z - 4968) + 92\right)\right)\right\} \tag{3.12}$$

```
> algvalues(eval(deq,[d=4,a=-24/25]),y(z));
```

$$\left\{y\left(-\frac{6}{25}\right) = \frac{4199}{3125}, y(RootOf(15625\,\_Z^3 - 63750\,\_Z^2 + 11700\,\_Z - 5304)) = 8398 \Big/ \left(125\left(4375\,RootOf(15625\,\_Z^3 - 63750\,\_Z^2 + 11700\,\_Z - 5304)^2 - 650\,RootOf(15625\,\_Z^3 - 63750\,\_Z^2 + 11700\,\_Z - 5304) + 442\right)\right)\right\} \tag{3.13}$$

Special cases of Prop. 4.3

```
> f:=exp(-c*z)*hypergeom([a],[b],z);
```

$$f := e^{-c\,z}\,\mathrm{hypergeom}([a],[b],z) \tag{4.1}$$

```
> deq:=gfun:-holexprtodiffeq(f,y(z));
```

$$deq := \left\{\left(\frac{d^2}{dz^2}y(z)\right)z + (2\,c\,z + b - z)\left(\frac{d}{dz}y(z)\right) + y(z)\,c^2\,z + y(z)\,b\,c - y(z)\,c\,z - a\,y(z), y(0) = 1\right\} \tag{4.2}$$

```
> deq:=gfun:-poltodiffeq(diff(y(z),z,z),[deq],[y(z)],y(z));
```

$$deq := \{(c^4 z^3 + 2\,b\,c^3 z^2 - 2\,c^3 z^3 - 2\,a\,c^2 z^2 + b^2 c^2 z - 2\,b\,c^2 z^2 + c^2 z^3 - 2\,a\,b\,c\,z + 2\,a\,c\,z^2 + b\,c^2 z + a^2 z - 2\,a\,c\,z + a\,z)\left(\frac{d^2}{dz^2}y(z)\right) - 6\,y(z)\,a\,b\,c^3 z + 6\,y(z)\,a\,b\,c^2 z + 3\,y(z)\,b\,c^5 z^2 - 3\,y(z)\,a\,c^4 z^2 + 3\,y(z)\,b^2 c^4 z - 6\,y(z)\,b\,c^4 z^2 + 6\,y(z)\,a\,c^3 z^2 - 3\,y(z)\,b^2 c^3 z + 3\,y(z)\,b\,c^4 z + 3\,y(z)\,b\,c^3 z^2 + 3\,y(z)\,a^2 c^2 z - 3\,y(z)\,a\,b^2 c^2 - 6\,y(z)\,a\,c^3 z - 3\,y(z)\,a\,c^2 z^2 - 2\,a\,y(z) - y(z)\,a^3 - 3\,y(z)\,a^2 + (2\,c^5 z^3 + 5\,b\,c^4 z^2 - 5\,c^4 z^3 - 4\,a\,c^3 z^2 + 4\,b^2 c^3 z - 8\,b\,c^3 z^2 + 4\,c^3 z^3 - 6\,a\,b\,c^2 z \tag{4.3}$$

$$+6ac^2z^2+b^3c^2-3b^2c^2z+4bc^3z+3bc^2z^2-c^2z^3+2a^2cz-2ab^2c$$

$$+4abcz-6ac^2z-2acz^2+3b^2c^2-3bc^2z+a^2b-a^2z-6abc+6acz$$

$$+2bc^2+2a^2+ab-4ac-az+2a)\left(\frac{d}{dz}y(z)\right)+y(z)c^6z^3-3y(z)c^5z^3$$

$$+3y(z)c^4z^3+y(z)b^3c^3-y(z)c^3z^3+3y(z)b^2c^3+2y(z)bc^3+6y(z)a^2c$$

$$-6y(z)ac^2+6y(z)ac-3y(z)bc^3z+3y(z)a^2bc-3y(z)a^2cz$$

$$-9y(z)abc^2+9y(z)ac^2z+3y(z)abc-3y(z)acz, y(0)$$

$$=\frac{b^2c^2-2abc+bc^2+a^2-2ac+a}{b(b+1)}\Big\}$$

```
> algvalues(eval(eval(deq,b=a*(2*c-1)/c^2),[c=1/3,a=1/2]),y(z));
```

$$\left\{y\left(-\frac{9}{2}\right)=0, y\left(-\frac{1}{2}\right)=0\right\} \quad (4.4)$$

```
> algvalues(eval(deq,[c=1,b=4/3,a=1/3]),y(z));
```

$$\left\{y(RootOf(9\_Z^2+6\_Z+4))=\frac{3}{4+3\,RootOf(9\_Z^2+6\_Z+4)}\right\} \quad (4.5)$$

```
> `union`(allvalues(%));
```

$$\left\{y\left(-\frac{1}{3}-\frac{I\sqrt{3}}{3}\right)=\frac{3}{3-I\sqrt{3}}, y\left(-\frac{1}{3}+\frac{I\sqrt{3}}{3}\right)=\frac{3}{3+I\sqrt{3}}\right\} \quad (4.6)$$

Special cases of Prop. 4.4

```
> a,b,c:=2/5, 3/5, 32/5:
> deq:=gfun:-holexprtodiffeq(hypergeom([a,b],[c],z),y(z));
```

$$deq := \left\{(25z^2-25z)\left(\frac{d^2}{dz^2}y(z)\right)+(50z-160)\left(\frac{d}{dz}y(z)\right)+6y(z), y(0)=1\right\} \quad (5.1)$$

```
> algvalues(deq,y(z));
Error, (in Beukersatalpha) not an apparent singularity, 1
|algvalues.mpl:146|
```

This is not an E-function, which explains why the code, which is supposed to work only for E-functions, does not work there.

```
> unassign('a','b','c');
```

A linear combination

```
> f:=a+b*(z-1)+(z-1)^5*sum(binomial(2*n,n)*(z/2)^n/n!,n=0..
  infinity);
```

$$f := a+b(z-1)+(z-1)^5\,e^z\,\mathrm{BesselI}(0,z) \quad (6.1)$$

```
> gfun:-holexprtodiffeq(f,y(z));
```

(6.2)

$$\left\{\left(z^3-2z^2+z\right)\left(\frac{d^2}{dz^2}y(z)\right)+\left(-2z^3-5z^2+6z+1\right)\left(\frac{d}{dz}y(z)\right)-7bz^3+9y(z)z^2-9az^2-3bz^2+17y(z)z-17az+7zb+4y(z)-4a+3b, y(0)=a-b-1\right\} \quad (6.2)$$

```
> deq:=eval(%,[a=1/3,b=1/7]);
```

$$deq := \left\{\left(z^3-2z^2+z\right)\left(\frac{d^2}{dz^2}y(z)\right)+\left(-2z^3-5z^2+6z+1\right)\left(\frac{d}{dz}y(z)\right)-z^3+9y(z)z^2-\frac{24z^2}{7}+17y(z)z-\frac{14z}{3}+4y(z)-\frac{19}{21}, y(0)=-\frac{17}{21}\right\} \quad (6.3)$$

```
> deq:=gfun:-diffeqtohomdiffeq(%,y(z));
```

$$deq := \left\{\left(3969z^4+14994z^3+12474z^2+4914z+1449\right)y(z)+\left(-3969z^5-20286z^4-32466z^3-21546z^2-8001z-1932\right)\left(\frac{d}{dz}y(z)\right)+\left(882z^6+5229z^5+6636z^4-1659z^3-7434z^2-2856z-798\right)\left(\frac{d^2}{dz^2}y(z)\right)+\left(-441z^6-630z^5+525z^4+2205z^3-1260z^2-399z\right)\left(\frac{d^3}{dz^3}y(z)\right), y(0)=-\frac{17}{21}, D(y)(0)=\frac{29}{7}, D^{(2)}(y)(0)=-\frac{23}{2}\right\} \quad (6.4)$$

```
> algvalues(deq,y(z));
```

$$\left\{y(1)=\frac{1}{3}\right\} \quad (6.5)$$

```
> algvalues(deq,y(z),false);
```

$$\left\{-\frac{1}{3}+y(1)=0, -\frac{1}{7}+D(y)(1)=0\right\} \quad (6.6)$$

Eq (7.8) from Adamczewski-Rivoal 2018

At this stage, our code works only over Q(z).

```
> f:=eval(z^a*exp(a*z)+z^b*exp(b*z),[a=1,b=3]);
```

$$f := z\,e^{z}+z^3\,e^{3z} \quad (7.1)$$

```
> deq:=gfun:-holexprtodiffeq(f,y(z));
```

$$deq := \left\{\left(z^3+z^2\right)\left(\frac{d^2}{dz^2}y(z)\right)+\left(-4z^3-8z^2-3z\right)\left(\frac{d}{dz}y(z)\right)+3y(z)z^3+9y(z)z^2+9y(z)z+3y(z), D(y)(0)=1, D^{(3)}(y)(0)=9\right\} \quad (7.2)$$

```
> series(f,z,4);
```

(7.3)

$$z + z^2 + \frac{3}{2} z^3 + \mathrm{O}(z^4) \quad \textbf{(7.3)}$$

Workaround weakness in holexprtodiffeq

```
> deq:=deq union {(D@@3)(y)(0)=coeff(%,z,3)*3!};
```

$$deq := \left\{ (z^3 + z^2) \left(\frac{d^2}{dz^2} y(z) \right) + (-4 z^3 - 8 z^2 - 3 z) \left(\frac{d}{dz} y(z) \right) + 3 y(z) z^3 + 9 y(z) z^2 + 9 y(z) z + 3 y(z), \mathrm{D}(y)(0) = 1, \mathrm{D}^{(3)}(y)(0) = 9 \right\} \quad \textbf{(7.4)}$$

```
> algvalues(deq,y(z),false);
```

$$\{\mathrm{D}(y)(-1) = 0\} \quad \textbf{(7.5)}$$

A combination of exponentials

```
> deq:=gfun:-holexprtodiffeq(exp(x)+(x-1)^2*exp(2*x)+(x-1)^4*exp
  (3*x),y(x));
```

$$deq := \left\{ (x^4 - 2 x^2 + 1) \left(\frac{d^3}{dx^3} y(x) \right) + (-6 x^4 - 6 x^3 + 12 x^2 + 6 x - 6) \left(\frac{d^2}{dx^2} y(x) \right) + (11 x^4 + 20 x^3 - 12 x^2 - 20 x + 13) \left(\frac{d}{dx} y(x) \right) - 6 y(x) x^4 - 14 y(x) x^3 + 2 y(x) x^2 + 14 y(x) x - 8 y(x), y(0) = 3, \mathrm{D}(y)(0) = 0, \mathrm{D}^{(2)}(y)(0) = -4 \right\} \quad \textbf{(8.1)}$$

```
> algvalues(deq,y(x),false);
```

$$\{y(-1) - \mathrm{D}(y)(-1) = 0, y(1) - \mathrm{D}(y)(1) = 0\} \quad \textbf{(8.2)}$$

```
> algvalues(deq,y(x));
```

$$\varnothing \quad \textbf{(8.3)}$$

An example with a high valuation

```
> deq:=gfun:-holexprtodiffeq(exp(x)+(x-1)^100*exp(2*x),y(x));
```

$$deq := \left\{ (x^2 + 98 x - 99) \left(\frac{d^2}{dx^2} y(x) \right) + (-3 x^2 - 394 x - 9503) \left(\frac{d}{dx} y(x) \right) + 2 y(x) x^2 + 296 y(x) x + 9602 y(x), y(0) = 2, \mathrm{D}(y)(0) = -97 \right\} \quad \textbf{(9.1)}$$

```
> algvalues(deq,y(x),false);
```

$$\{y(-99) - \mathrm{D}(y)(-99) = 0, y(1) - \mathrm{D}(y)(1) = 0\} \quad \textbf{(9.2)}$$

```
> time()-st;
```

$$5.779 \quad \textbf{(1)}$$